Glasgow Math. J. **63** (2021) 584–621. © The Author(s), 2020. Published by Cambridge University Press on behalf of Glasgow Mathematical Journal Trust.
doi:10.1017/S0017089520000397.

LINEAR RELATIONS AND INTEGRABILITY FOR CLUSTER ALGEBRAS FROM AFFINE QUIVERS

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(Received 14 January 2020; revised 8 July 2020; accepted 15 July 2020; first published online 13 August 2020)

Abstract. We consider frieze sequences corresponding to sequences of cluster mutations for affine D- and E-type quivers. We show that the cluster variables satisfy linear recurrences with periodic coefficients, which imply the constant coefficient relations found by Keller and Scherotzke. Viewing the frieze sequence as a discrete dynamical system, we reduce it to a symplectic map on a lower dimensional space and prove Liouville integrability of the latter.

2010 *Mathematics Subject Classification*. Primary: 13F60; Secondary: 37J35, 11B37

1. Introduction. A cluster algebra is an algebra with a distinguished set of generators, cluster variables, that appear in sets called seeds [**6**]. Each seed is obtained recursively, by a process called mutation, from an initial seed. In our work, there is a quiver associated with each seed, with $N+1$ vertices, without loops or 2-cycles. It is a theorem of [**7**] that there are finitely many cluster variables if and only if the quiver associated to one of the seeds is of simply laced Dynkin type.

From a quiver without cycles, we construct a frieze [**1**] by a sequence of mutations at only sinks or sources. These mutations give a coupled system of $N+1$ recurrences

$$X_{n+1}^k X_n^k = 1 + \left(\prod_{i\to k} X_n^i\right)\left(\prod_{i\leftarrow k} X_{n+1}^i\right), \tag{1}$$

which define the frieze sequence. Here, the products are taken over the arrows in to and out of k, respectively, in the initial quiver. The superscript runs over each $k = 1, \ldots, N+1$, and the subscript n denotes the iterates of the recurrence. We consider the frieze variables with superscript k to be "living" at vertex k. To see the frieze as a discrete dynamical system, we set initial variables X_0^k for each vertex k and define recursively X_{n+1}^k by equation (1). The map

$$\varphi : \begin{pmatrix} X_n^1 \\ X_n^2 \\ \vdots \\ X_n^{N+1} \end{pmatrix} \mapsto \begin{pmatrix} X_{n+1}^1 \\ X_{n+1}^2 \\ \vdots \\ X_{n+1}^{N+1} \end{pmatrix}, \tag{2}$$

is called the cluster map. The form of cluster mutation means that each new variable can be written as a birational function of the previous ones, and since (1) is obtained as

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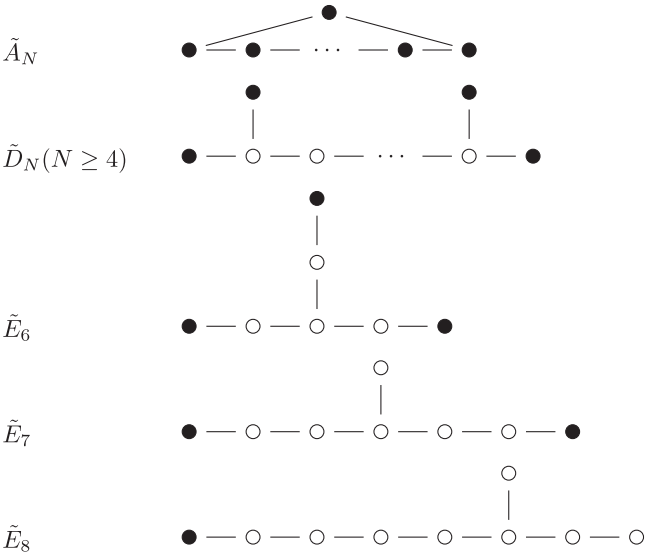


Figure 1. The simply laced affine diagrams with extending vertices colored black.

a composition of mutations the map (2) is birational. This induces an automorphism φ^*, a "shift," on the field of rational functions

$$\mathbb{C}(\{X_0^k\}_{k=1,\dots,N+1}),$$

by $\varphi^*(X_n^k) = X_{n+1}^k$ for each $n \in \mathbb{Z}$. We say an element of this field is period p if it is fixed by $(\varphi^*)^p$. Nonconstant period one elements are called invariants or first integrals.

It was proved in [**1**] that if there are linear relations between the variables in the frieze sequence, then the corresponding diagram is either affine or Dynkin and is simply laced. These are shown in Figure 1 where, and throughout this paper, we have used a tilde to denote the affine version of the diagram or quiver. The converse for $\tilde{A}$ and $\tilde{D}$ types was also proved in [**1**] that the associated frieze sequences satisfy linear relations. In [**13**], using a representation theoretic approach, the authors proved this for all $\tilde{A}\tilde{D}\tilde{E}$ quivers, with some of these linear relations given explicitly. We adopt a different method of proof here, allowing us a uniform expression of these results:

THEOREM 1.1. *For our choice of orientation for the affine diagrams (given in Section 4), the variables that live at each extending vertex k satisfy the constant coefficient linear relations*

$$X_{n+2b}^k - \mathcal{K}(Q) X_{n+b}^k + X_n^k = 0. \tag{3}$$

Here, b depends on the quiver, the values of which are given in Figure 2, and $\mathcal{K}(Q)$ depends on the quiver, each of which is invariant under the shift φ^*. The extending vertices for an affine quiver are those labelled with a 1 *in Figure 1.*

For succinctness, we will write $\mathcal{K}$ instead of $\mathcal{K}(Q)$. As stated, these results hold for one choice of orientation per quiver. We have not yet attempted to find the corresponding results for other choices. We remark that in [**13**], for $\tilde{D}$ and $\tilde{E}$ types, $\mathcal{K}$ is given by X_δ, or a function thereof, defined as the image of any module with dimension δ under the Caldero–Chapoton map [**2**], where δ gives the radical of the symmetrized Euler form [**3**].

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Quiver		b
	$\tilde{A}_{p,q}$	$\mathrm{lcm}(p,q)$
$\tilde{D}_N$	N even	$N-2$
$\tilde{D}_N$	N odd	$2N-4$
	$\tilde{E}_6$	6
	$\tilde{E}_7$	12
	$\tilde{E}_8$	30

Figure 2. Values of b for the $\tilde{A}\tilde{D}\tilde{E}$ quivers.

Quiver	Period	Quantity	Explicit Expression
$\tilde{A}_{p,q}$	p	$J_n(\tilde{A}_{p,q})$	Lemma 4.1 of [10]
	q	$\tilde{J}_n(\tilde{A}_{p,q})$	
$\tilde{D}_N$	$N-2$	$J_n(\tilde{D}_N)$	Equation 22
	2	X_n^1/X_n^2	Lemma 4.1
	2	X_n^N/X_n^{N+1}	Lemma 4.1
$\tilde{E}_6$	3	$J_n(\tilde{E}_6)$	Lemma 4.9
	3	$\tilde{J}_n(\tilde{E}_6)$	Lemma 4.9
	2	$K_n(\tilde{E}_6)$	Lemma 4.12
$\tilde{E}_7$	4	$J_n(\tilde{E}_7)$	Theorem 4.20
	3	$K_n(\tilde{E}_7)$	Theorem 4.19
	2	$\tilde{K}_n(\tilde{E}_7)$	Equation 42
$\tilde{E}_8$	5	$J_n(\tilde{E}_8)$	Lemma 4.26
	3	$K_n(\tilde{E}_8)$	Theorem 4.28
	2?	$\tilde{K}_n(\tilde{E}_8)$	Conjecture 4.30

Figure 3. Periodic quantities found for the $\tilde{A}\tilde{D}\tilde{E}$ quivers.

We take the approach the authors of [10, 11] used in the $\tilde{A}_{p,q}$ case. This is constructed by taking the $\tilde{A}$ diagram and orienting p arrows consecutively clockwise and q arrows consecutively anticlockwise. Due to the uniform nature of the mutations used, the frieze (1) can be written as the single recurrence

$$x_{n+N+1}x_n = x_{n+p}x_{n+q} + 1, \tag{4}$$

where $p+q = N+1$. The authors of [10, 11] found the two periodic quantities

$$J_n := \frac{x_{n+2q}+x_n}{x_{n+q}}, \qquad \tilde{J}_n := \frac{x_{n+2p}+x_n}{x_{n+p}},$$

which we display in the first row of Figure 3. These immediately give the linear relations

$$x_{n+2q} - J_n x_{n+q} + x_n = 0, \qquad x_{n+2p} - \tilde{J}_n x_{n+p} + x_n = 0, \tag{5}$$

which are used to prove the $\tilde{A}$ case of (3) in [10, 11]. In order to prove Theorem 1.1 for the $\tilde{D}$ and $\tilde{E}$ quivers, we first find periodic quantities for their frieze sequences which give linear relations with periodic coefficients for the extending vertices.

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Quiver	a	p
$\tilde{D}_N$ N even	1	$N-2$
$\tilde{D}_N$ N odd	1	$2N-4$
$\tilde{E}_6$	3	2
$\tilde{E}_7$	4	3
	6	5
$\tilde{E}_8$	10	3
	15?	2?

Figure 4. Values of a and p for the $\tilde{D}\tilde{E}$ quivers.

THEOREM 1.2. *The frieze sequences for the $\tilde{A}\tilde{D}\tilde{E}$ quivers have the periodic quantities given in Figure 3.*

Each periodic quantity is given as a function of its quiver, though we suppress this information later as each quiver appears in distinct subsections. Here, we find only the period 5 and 3 quantities, J_n and K_n, respectively, for the $\tilde{E}_8$ quiver. The question mark is Conjecture 4.30 below. We remark the similarities between the widths of the tubes for the Auslander–Reiten quivers of $\tilde{A}\tilde{D}\tilde{E}$ type (see, e.g., [**3**]) and our Figure 3. This is why we conjecture the missing period for $\tilde{E}_8$.

For each quiver at least one of the quantities in Figure 3 gives a linear relation of the form (5) between the variables X_n^k at only one vertex k. The rest of the quantities give linear relations that involve variables that live at different extending vertices. Focussing on the former, we derive relations that are remarkably similar to (4), two for $\tilde{E}_8$ (plus a third conjectured relation), and one for each of the remaining quivers.

THEOREM 1.3. *For the $\tilde{D}_N$, with N even, and $\tilde{E}$ quivers, we have the following $\tilde{A}$-type recurrence:*

$$X_{n+a+p}^k X_n^k = X_{n+a}^k X_{n+p}^k + \gamma_n^k. \tag{6}$$

Here, k is an extending vertex and γ_n^k is period a. For $\tilde{D}_N$ with N odd, we have instead

$$X_{n+a+p}^k X_n^k = (\lambda_{n+1}^k)^2 X_{n+a}^1 X_{n+p}^1 + \gamma_n^k, \tag{7}$$

where λ_{n+1}^k is period 2 and γ_n^k is period a. The values of a and p are given in Figure 4.

Since a frieze is a system of recurrences, we can consider it a discrete dynamical system. In [**12**], it was shown that a cluster algebra has a pre-symplectic form that is compatible with mutation. In our cases, however, this will be degenerate. We instead follow [**10**] and project to a lower dimensional space with a reduced cluster map and a symplectic form. We are interested in the integrability of this reduction, the definition of which we take from [**14**, **16**].

DEFINITION 1.4. Let (M, ω) be a $2m$-dimensional symplectic manifold and $\hat{\varphi} : M \to M$ a symplectic map. The map $\hat{\varphi}$ is called Liouville integrable if there exists m functionally independent first integrals $\mathcal{I}_i$, $i = 1, \ldots, m$, in involution, i.e $\{\mathcal{I}_i, \mathcal{I}_j\} = 0$ for all i, j.

THEOREM 1.5. *The reduced cluster map for the $\tilde{D}_N$ quivers, where N is odd, and for the $\tilde{E}$-type quivers is Liouville integrable.*

We provide an example of integrability for $\tilde{D}_6$ and leave the proof for general even N open.

The structure of this paper is as follows: in Section 2, we describe how the frieze can be obtained from cluster mutation and discuss our notion of integrability for cluster maps.

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In Section 3, we explain our process for finding periodic quantities for the friezes and how we use these to construct linear relations. In Section 4, we apply this to the $\tilde{D}$- and $\tilde{E}$-type quivers. In Section 5, we prove integrability of the reduced cluster maps for these quivers.

2. Background material on cluster algebras and integrability. Here, we briefly recall some notions from cluster algebras and discrete integrable systems. The following definitions largely follow [**5**] and [**10**]. Here, by a cluster algebra, we mean a cluster algebra without coefficients or frozen variables.

A quiver Q is a directed graph where multiple edges are allowed. We disallow loops or 2-cycles. Quiver mutation μ_k at some vertex k is defined in three steps:

1. for each length two path $i \to k \to j$ add a new arrow $i \to j$;
2. reverse the direction of all arrows entering or exiting k;
3. delete all 2-cycles that have appeared.

The exchange matrix for a quiver Q is the skew-symmetric matrix B with entries b_{ij} which is the number of arrows from i to j, with $b_{ji} = -b_{ij}$. In addition to this, we will also attach a cluster variable x_i at each vertex i. Cluster mutation at vertex k, also denoted μ_k, fixes all variables x_i with $i \neq k$ but transforms x_k as follows:

$$x_k' := \mu_k(x_k) := \frac{1}{x_k}\left(\prod_{i \to k} x_i + \prod_{i \leftarrow k} x_i\right). \tag{8}$$

Here, the two products run over the arrows into and out of k, respectively, e.g., in the first product we have an x_i for every arrow from i to k. We consider μ_k as a mutation both of the quiver and of the cluster variables. One can see that if there is no edge between vertices k and k', then μ_k and $\mu_{k'}$ commute.

In the special case where the vertex k is a sink or a source, the mutation simply reverses the arrows at k and (8) takes a simpler form, with one of the products being empty.

2.1. Obtaining the frieze as a sequence of cluster mutations. To demonstrate the relationship between friezes and cluster algebras, we recall how to obtain each frieze variable by cluster mutation, as described in [**13**].

We first define initial cluster variables X_0^k for $k = 1, \ldots, N+1$ and take a labeling of the vertices such that k is a sink in the full subquiver with vertices labeled greater than or equal to k, which is possible since the quiver is acyclic. The composition of mutations $\mu := \mu_{N+1} \ldots \mu_2 \mu_1$ then gives the frieze (1) for $n = 1$, in the sense that X_0^k becomes X_1^k under this mutation. Moreover, each vertex k will be a sink when μ_k acts, so μ will flip each arrow twice, returning the original quiver. Further applications of μ will then give the frieze cluster variables X_n^k for each positive integer n. Since each μ_k is an involution, we can also define μ^n and X_n^k for negative n. We note that different labelings satisfying these conditions will give different friezes.

2.2. The bipartite belt as a frieze sequence. In the special case where the quiver is bipartite, one can take an orientation such that every vertex is either a sink or a source. We can then define the following compositions of mutations:

$$\mu_{\text{sink}} := \prod_{k \text{ a sink}} \mu_k, \qquad \mu_{\text{source}} := \prod_{k \text{ a source}} \mu_k,$$

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where the products are taken over vertices that are sinks (or sources) in the initial quiver. These products are well defined since no edge connects any pair of sinks (or sources). The composition $\mu := \mu_{\text{source}}\mu_{\text{sink}}$ will, after relabeling of vertices, satisfy the conditions of the previous subsection, and in this case, the frieze relations (1) become simpler: for each $k = 1, \ldots, N+1$, we have

$$\begin{aligned} X_{n+1}^k X_n^k &= \left(1 + \prod_{j \to k} X_n^j\right) \qquad \text{if } k \text{ is a sink,} \\ X_{n+1}^k X_n^k &= \left(1 + \prod_{j \leftarrow k} X_{n+1}^j\right) \qquad \text{if } k \text{ is a source.} \end{aligned} \tag{9}$$

The cluster variables constructed in this way form the bipartite belt of the cluster algebra associated to the quiver, defined in [8].

2.3. Integrability of dynamical systems from periodic quivers. The pre-symplectic form associated with the exchange matrix B is

$$\omega := \sum_{i<j} \frac{b_{ij}}{x_i x_j} dx_i \wedge dx_j$$

and was shown in [12] to be compatible with cluster mutation, in the sense that it transforms as

$$\mu_k^*(\omega) = \sum_{i<j} \frac{b'_{ij}}{x'_i x'_j} dx'_i \wedge dx'_j,$$

where x_i are the generic cluster variables, b_{ij} are the entries of the exchange matrix, and the prime denotes the images of these under the mutation μ_k. In our work, the cluster variables will be only those that are part of the frieze sequence, so the x_i will all be of the form X_n^i.

LEMMA 2.1. *The cluster map φ associated with a frieze sequence preserves the symplectic form, i.e., $\varphi^*\omega = \omega$.*

Proof. As mentioned above the composition, μ acts as the identity on the quiver so it fixes B, hence fixes ω. □

In the cases, we shall encounter the determinant of B will be zero, so ω will be degenerate. To remedy this, we project to a lower dimensional space where φ reduces to a symplectic map. For each n, we take reduced coordinates $\{y_n^j\}_{j=1,\ldots,2m}$, where $\operatorname{rank}(B) = 2m$ and

$$y_n^j := \prod_i (X_n^i)^{v_{j,i}}.$$

Here $v_{j,i}$ is the ith component of v_j and the vectors $v_1, \ldots, v_{2m}$ are a $\mathbb{Z}$-basis for $\operatorname{im} B$. We define this projections by

$$\pi : \{X_n^1, X_n^2, \ldots, X_n^{N+1}\} \to \{y_n^1, y_n^2, \ldots, y_n^{2m}\}.$$

As in Lemma 2.7 of [10], we have that for each n the set $\{y_n^j\}_{j=1,\ldots,2m}$ is a full set of invariants for the group of scaling transformations defined by

$$(X_n^1, X_n^2 \ldots) \mapsto (\lambda^{u_1} X_n^1, \lambda^{u_2} X_n^2, \ldots,), \tag{10}$$

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for any $\lambda \in \mathbb{C}^\star$ and $u = (u_1, \ldots, u_N) \in \ker B$, and that for each n the two form

$$\hat{\omega} = \sum_{i<j} \frac{\hat{b}_{ij}}{y_n^i y_n^j} dy_n^i \wedge dy_n^j, \tag{11}$$

is symplectic with $\pi^* \circ \hat{\omega} = \omega$. Here, the coefficients $\hat{b}_{ij}$ are given by the entries of $\hat{B}$ which satisfies

$$(A^T)^{-1} B A^{-1} = \begin{pmatrix} \hat{B} & 0 \\ 0 & 0 \end{pmatrix},$$

where the matrix A has the vectors $v_1, \ldots, v_{N+1}$ as rows, and the sets of vectors $\{v_1, \ldots, v_{2m}\}$ and $\{v_{2m+1}, \ldots, v_{N+1}\}$ are bases for $\operatorname{im} B$ and $\ker B$, respectively. The form of $\hat{B}$ and the reduced coordinates depend on the choice of basis for the image and kernel of B. The following theorem is the analogue of Theorem 2.6 in [**10**].

THEOREM 2.2. *For the $\tilde{D}\tilde{E}$-type quivers, the cluster map φ reduces to a map $\hat{\varphi}$:*

$$\hat{\varphi} : \begin{pmatrix} y_n^1 \\ y_n^2 \\ \vdots \\ y_n^{2m} \end{pmatrix} \mapsto \begin{pmatrix} y_{n+1}^1 \\ y_{n+1}^2 \\ \vdots \\ y_{n+1}^{2m} \end{pmatrix}, \tag{12}$$

such that $\pi \circ \varphi = \hat{\varphi} \circ \pi$. The map $\hat{\varphi}$ is birational and symplectic with respect to the two form (11).

Proof. The property $\pi \circ \varphi = \hat{\varphi} \circ \pi$ and birationality of the maps $\hat{\varphi}$ is shown as they are constructed, case by case, in Section 5. That $\hat{\varphi}$ is symplectic is the same as in [**10**] Lemma 2.9. □

Since $\hat{\omega}$ is symplectic, it defines a nondegenerate Poisson bracket:

$$\{y_n^i, y_n^j\} = c_{ij} y_n^i y_n^j, \qquad \text{where} \qquad C = (c_{ij}) := \hat{B}^{-1}. \tag{13}$$

Our aim is to prove Liouville integrability of the map (12), as in Definition 1.4, by finding enough involutive first integrals with respect to this bracket.

3. Method of proof for the $\tilde{D}$ and $\tilde{E}$ quivers. Here, we will discuss our general process for finding periodic quantities for the frieze sequences. We show how to find constant coefficient linear relations from these.

We orient the $\tilde{D}$ and $\tilde{E}$ quivers such that every vertex is either a sink or a source. This allows us to construct the bipartite belt frieze described in Section 2.2.

The key to our results, for $\tilde{D}$ type, is to write each of the mutation relations (9) as a 2×2 matrix with determinant -1:

$$-1 = \begin{vmatrix} \star & X_{n+1}^k \\ X_n^k & \star \end{vmatrix}, \tag{14}$$

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where the product of the stars will give the product inside the brackets in (9). For the $\tilde{E}$ quivers, we have taken the determinant in (14) to be $+1$, but the procedure is equivalent, up to swapping matrix columns. We construct a 3×3 matrix such that each connected 2×2 determinant inside is of the form (14). By the following lemma, part of the Dodgson condensation algorithm [**4**], the 3×3 matrix will have determinant zero.

LEMMA 3.1. *For general x_{ij}, we have*

$$\begin{vmatrix} x_{0,0} & x_{0,1} & x_{0,2} \\ x_{1,0} & x_{1,1} & x_{1,2} \\ x_{2,0} & x_{2,1} & x_{2,2} \end{vmatrix} x_{1,1} = \begin{vmatrix} x_{0,0} & x_{0,1} \\ x_{1,0} & x_{1,1} \end{vmatrix} \begin{vmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{vmatrix} - \begin{vmatrix} x_{0,1} & x_{0,2} \\ x_{1,1} & x_{1,2} \end{vmatrix} \begin{vmatrix} x_{1,0} & x_{1,1} \\ x_{2,0} & x_{2,1} \end{vmatrix}. \quad (15)$$

In our cases, the 3×3 matrix will have a kernel vector of the form $(1, -\alpha, 1)^T$. We then add rows to the bottom (or top) of this matrix that annihilate this vector. Our main method for doing this will be such that, using (14), the new row will create two new connected 2×2 matrices with equal determinant. By the following lemma, this newly formed 4×3 matrix will have the same kernel vector.

LEMMA 3.2. *Let M be a 3×3 matrix*

$$M := \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} \\ x_{1,0} & x_{1,1} & x_{1,2} \\ x_{2,0} & x_{2,1} & x_{2,2} \end{pmatrix},$$

such that $(1, -\alpha, 1)^T$ is in the kernel of the first two rows. We define the determinants $\delta_{i,j}$ by

$$\delta_{i,j} := \begin{vmatrix} x_{i,j} & x_{i,j+1} \\ x_{i+1,j} & x_{i+1,j+1} \end{vmatrix}.$$

If $x_{1,1} \neq 0$ and $\delta_{1,0} = \delta_{1,1} \neq 0$, then $(1, -\alpha, 1)^T$ is in the kernel for all three rows of M.

Proof. By (15), and since $\delta_{1,0} = \delta_{1,1} \neq 0$, we have

$$|M| = 0 \Leftrightarrow \delta_{0,0}\delta_{1,1} - \delta_{1,0}\delta_{0,1} = 0 \Leftrightarrow \delta_{0,0} - \delta_{0,1} = 0.$$

Now the matrix

$$\begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} \\ x_{1,0} & x_{1,1} & x_{1,2} \\ -1 & 0 & 1 \end{pmatrix},$$

has a kernel vector $(1, -\alpha, 1)^T$ and determinant zero. We expand this determinant along the bottom row to see that $\delta_{0,0} = \delta_{0,1}$. Hence, $|M|$ is zero. Taking a kernel vector $(A, B, 1)^T$ for M, we have $A = \frac{\delta_{1,1}}{\delta_{1,0}} = 1$ by Cramer's rule and $B = -\alpha$. □

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After adding one row, our 4×3 matrix will be of the form

$$\begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} \\ x_{1,0} & x_{1,1} & x_{1,2} \\ x_{2,0} & x_{2,1} & x_{2,2} \\ x_{3,0} & x_{3,1} & x_{3,2} \end{pmatrix},$$

so that the bottom 3×3 matrix satisfies the conditions of Lemma 3.2, hence the 4×3 matrix has $(1, -\alpha, 1)^T$ as a kernel vector. As mentioned in 14 for the $x_{i,j}$, we will take cluster variables X_n^k, or functions thereof. We continue adding rows in this way (except for a few exceptional rows in the $\tilde{D}$ cases) until we find that either α is periodic, in the $\tilde{D}$ cases, or one of the matrix entries is periodic, in the $\tilde{E}$ cases, with respect to the shift φ^*.

3.1. Linear relations with fixed coefficients. The periodic quantities we find immediately give linear relations with periodic coefficients, which in turn give linear relations with constant coefficients. In each case, we will have $\Psi_n L_n = \Psi_{n+q}$ for some integer q, where Ψ_n is a 2×2 matrix with cluster variable entries and L_n is a 2×2 matrix with periodic elements, i.e., $L_{n+p} = L_n$ for some p. Taking $m := \mathrm{lcm}(p, q)$, we have

$$\Psi_n M_n = \Psi_{n+m}, \qquad M_n := L_n L_{n+q} L_{n+2q} \dots L_{n+m-q}. \tag{16}$$

THEOREM 3.3. *The matrix Ψ_n satisfies*

$$\Psi_{n+2m} - \mathrm{tr}\,(M_n)\Psi_{n+m} + \det(M_n)\Psi_n = 0, \tag{17}$$

where $\mathrm{tr}\,(M_n)$ *is period* q.

Proof. Applying the Cayley–Hamilton theorem to M_n gives

$$M_n^2 - \mathrm{tr}(M_n)M_n + \det(M_n)I = 0.$$

Multiplying by Ψ_n from the left gives the result. □

In each of the cases we deal with, however, we find that $\mathcal{K} := \mathrm{tr}(M_n)$ is invariant (under shifts in n) and that $\det(M_n) = 1$. Hence, (17) reduces to $\Psi_{n+2m} - \mathcal{K}\Psi_{n+m} + \Psi_n = 0$ and any entry of this matrix equation will give us constant coefficient linear relations between cluster variables.

4. Linear relations in the $\tilde{D}$ and $\tilde{E}$ quivers. Here, we apply the ideas of the previous section to the $\tilde{D}$ and $\tilde{E}$ quivers. For $\tilde{D}_N$, we deal with different parities separately. As well as those stated in the introduction, we have sporadic appearances of other linear relations.

4.1. Quivers of $\tilde{D}$ type. We will orient the $\tilde{D}_N$ diagram as in Figure 5. The two-sided arrows are used on the right-hand side to indicate that the direction depends on the parity of N, i.e., for a given N one of the arrowheads needs to be deleted on each double arrow, ensuring that each vertex is a sink or source. We deal with the case $N = 4$ separately, in Section 4.1.2.

We use the sequence of mutations described in Section 3. For example, mutating at vertex 3 gives

$$X_{n+1}^3 = \frac{1}{X_n^3}(1 + X_n^1 X_n^2 X_n^4), \tag{18}$$

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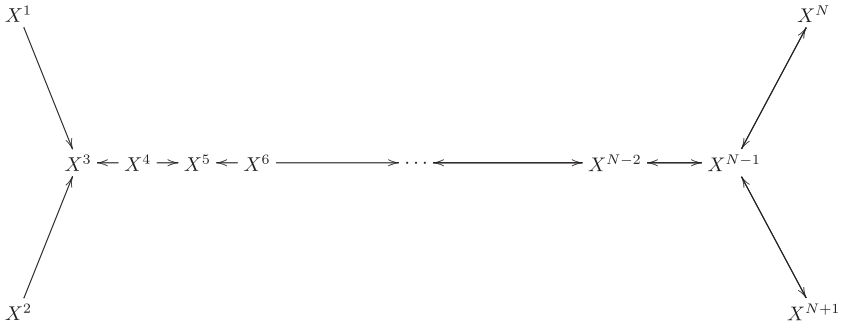


Figure 5. The $\tilde{D}_N$ quiver.

and then at vertex 1

$$X^1_{n+1} = \frac{1}{X^1_n}\left(1 + X^3_{n+1}\right) = \frac{1}{X^1_n}\left(1 + \frac{1}{X^3_n}\left(1 + X^1_n X^2_n X^4_n\right)\right).$$

In general, the relations are

$$\begin{aligned}
&X^1_{n+1} = \frac{1}{X^1_n}(1 + X^3_{n+1}), && X^2_{n+1} = \frac{1}{X^2_n}(1 + X^3_{n+1}), && X^3_{n+1} = \frac{1}{X^3_n}(1 + X^1_n X^2_n X^4_n),\\
&X^i_{n+1} = \frac{1}{X^i_n}(1 + X^{i-1}_{n+1} X^{i+1}_{n+1}), && 3 < i < N-1 && \text{for } i \text{ even},\\
&X^i_{n+1} = \frac{1}{X^i_n}(1 + X^{i-1}_n X^{i+1}_n), && 3 < i < N-1 && \text{for } i \text{ odd}.
\end{aligned} \tag{19}$$

Toward the right end of the diagram, we need to distinguish two cases, depending on the parity of N. When N is even, we have

$$\begin{aligned}
&X^{N-1}_{n+1} = \frac{1}{X^{N-1}_n}(1 + X^{N-2}_n X^N_n X^{N+1}_n),\\
&X^N_{n+1} = \frac{1}{X^N_n}(1 + X^{N-1}_{n+1}), && X^{N+1}_{n+1} = \frac{1}{X^{N+1}_n}(1 + X^{N-1}_{n+1}),
\end{aligned} \tag{20}$$

and when N is odd, we have

$$\begin{aligned}
&X^{N-1}_{n+1} = \frac{1}{X^{N-1}_n}(1 + X^{N-2}_{n+1} X^N_{n+1} X^{N+1}_{n+1}),\\
&X^N_{n+1} = \frac{1}{X^N_n}(1 + X^{N-1}_n), && X^{N+1}_{n+1} = \frac{1}{X^{N+1}_n}(1 + X^{N-1}_n).
\end{aligned} \tag{21}$$

LEMMA 4.1. *We note that from (19) at the extending vertices* 1 *and* 2*, we have*

$$\frac{X^1_{n+1}}{X^2_{n+1}} = \frac{X^2_n}{X^1_n} = \ldots = \left(\frac{X^1_0}{X^2_0}\right)^{(-1)^n},$$

as well as a corresponding result for vertices N *and* $N+1$ *from equations (20) and (21). In particular,* X^1_n/X^2_n *and* X^{N+1}_n/X^N_n *are period* 2.

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The 3×3 matrix from Lemma 3.2 that we will use is the following:

LEMMA 4.2. *The matrix*

$$M := \begin{pmatrix} X^1_{n+1}X^2_{n+1} & X^3_{n+1} & X^4_n \\ X^3_{n+2} & X^4_{n+1} & X^5_{n+1} \\ X^4_{n+2} & X^5_{n+2} & X^6_{n+1} \end{pmatrix}$$

has kernel vector $(1, -J_n, 1)^T$*, where*

$$J_n := \frac{X^1_{n+1}X^2_{n+1} + X^4_n}{X^3_{n+1}}.$$

Proof. Each of the cluster relations, e.g. (18), can be written as

$$-1 = \begin{vmatrix} \star & \star \\ \star & \star \end{vmatrix},$$

for appropriate 2×2 matrices, so each connected 2×2 determinant inside M is equal to -1. We see from (19) that

$$\left(1, -\frac{X^1_{n+1}X^2_{n+1} + X^4_n}{X^3_{n+1}}, 1\right)^T$$

is in the kernel of the first two rows of M. Hence by Lemma 3.2, this vector is in the kernel of M. □

We can extend M to involve the variables at each vertex, ensuring that each connected 2×2 matrix has determinant -1, as described in Section 3. By repeated applications of Lemma 3.2, the vector $(1, -J_n, 1)$ will be in the kernel of each newly added row. For example, after the first application, we have

$$M := \begin{pmatrix} X^1_{n+1}X^2_{n+1} & X^3_{n+1} & X^4_n \\ X^3_{n+2} & X^4_{n+1} & X^5_{n+1} \\ X^4_{n+2} & X^5_{n+2} & X^6_{n+1} \\ X^5_{n+3} & X^6_{n+2} & X^7_{n+2} \end{pmatrix},$$

with the newly formed connected 2×2 matrices

$$\begin{pmatrix} X^4_{n+2} & X^5_{n+2} \\ X^5_{n+3} & X^6_{n+2} \end{pmatrix}, \qquad \begin{pmatrix} X^5_{n+2} & X^6_{n+1} \\ X^6_{n+2} & X^7_{n+2} \end{pmatrix},$$

each having determinant -1, which follows from the mutation relations (19). The kernel vector $(1, -J_n, 1)$ for this 4×3 matrix gives us a new equation

$$\begin{pmatrix} X^5_{n+3} & X^6_{n+2} & X^7_{n+2} \end{pmatrix} \begin{pmatrix} 1 \\ -J_n \\ 1 \end{pmatrix} = 0.$$

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This particular relation is not so important, we simply state it to stress that we arrive at a new equation for each row we add. We can use equations (19) to add $N-6$ rows to M in the same way:

$$M := \begin{pmatrix} X^1_{n+1}X^2_{n+1} & X^3_{n+1} & X^4_n \\ X^3_{n+2} & X^4_{n+1} & X^5_{n+1} \\ X^4_{n+2} & X^5_{n+2} & X^6_{n+1} \\ X^5_{n+3} & X^6_{n+2} & X^7_{n+2} \\ \vdots & \vdots & \vdots \\ X^{N-3}_{n+l} & X^{N-2}_{n+l-h} & X^{N-1}_{n+l-1} \\ X^{N-2}_{n+l+1-h} & X^{N-1}_{n+l} & X^N_{n+l-h}X^{N+1}_{n+l-h} \end{pmatrix}.$$

Here, the subindex in the leftmost column increases by 1 only at each sink, so l is 1 greater than the number of sinks from X^3 to X^{N-3} inclusive. The value of h is 0 if N is odd and 1 if N is even. To proceed, we can use the identities:

$$J_n = \frac{X^1_{n+1}+X^1_{n-1}}{X^2_n} = \frac{X^1_{n+1}X^2_{n+1}+X^4_n}{X^3_{n+1}} = \frac{X^1_{n-1}X^2_{n-1}+X^4_n}{X^3_n}, \tag{22}$$

which follow from some simple manipulation of (19), to add two rows (shown in black) to the top of the following matrix while preserving our kernel vector. Note that these new connected 2×2 minors do not have determinant -1. This gives us

$$M := \begin{pmatrix} X^4_n & X^3_n & X^1_{n-1}X^2_{n-1} \\ X^1_{n+1} & X^2_n & X^1_{n-1} \\ X^1_{n+1}X^2_{n+1} & X^3_{n+1} & X^4_n \\ \vdots & \vdots & \vdots \\ X^{N-2}_{n+l+1-h} & X^{N-1}_{n+l} & X^N_{n+l-h}X^{N+1}_{n+l-h} \end{pmatrix}.$$

In order to extend the bottom in this way, we will need to use similar identities to (22) for the vertices at the right end of the diagram, but these depend on the parity of N, so now we split the problem into to two cases.

Before we do this, we note that for small N the above process may be not defined. We deal with the case $N=4$ in Section 4.1.2. For $N=5$, one should start with the matrix

$$\begin{pmatrix} X^1_{n+1}X^2_{n+1} & X^3_{n+1} & X^4_n \\ X^3_{n+2} & X^4_{n+1} & X^5_{n+1}X^6_{n+1} \end{pmatrix}$$

and use (22) to add rows above. Rows below should then be added as described in Section 4.1.3. For $N=6$, add rows above the matrix

$$\begin{pmatrix} X^1_{n+1}X^2_{n+1} & X^3_{n+1} & X^4_n \\ X^3_{n+2} & X^4_{n+1} & X^5_{n+1} \\ X^4_{n+2} & X^5_{n+2} & X^6_{n+1}X^7_{n+1} \end{pmatrix},$$

with (22) and add rows below as described in Section 4.1.1.

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4.1.1. The even N case with N > 4

Here, the vertices N and $N+1$ are sources, $l=\frac{N}{2}-1$ and $h=1$. We now establish why J_n is of such importance.

LEMMA 4.3. *The quantity of (22),* $J_n=\frac{X^1_{n+1}+X^1_{n-1}}{X^2_n}$, *is periodic with period* $2l=N-2$

Proof. The diagram is symmetric about the center vertex so the relations

$$\frac{X^{N+1}_{n+1}+X^{N+1}_{n-1}}{X^N_n}=\frac{X^{N+1}_{n+1}X^N_{n+1}+X^{N-2}_n}{X^{N-1}_{n+1}}=\frac{X^{N+1}_{n-1}X^N_{n-1}+X^{N-2}_n}{X^{N-1}_n}, \tag{23}$$

mirror (22). We use these to extend M:

$$\begin{pmatrix} X^4_n & X^3_n & X^1_{n-1}X^2_{n-1} \\ X^1_{n+1} & X^2_n & X^1_{n-1} \\ \vdots & \vdots & \vdots \\ X^{N-2}_{n+l} & X^{N-1}_{n+l} & X^N_{n+l-1}X^{N+1}_{n+l-1} \\ X^{N+1}_{n+l+1} & X^N_{n+l} & X^{N+1}_{n+l-1} \\ X^{N+1}_{n+l+1}X^N_{n+l+1} & X^{N-1}_{n+l+1} & X^{N-2}_{n+l} \end{pmatrix}.$$

Now the bottom row here is the counterpart to the top row of (4.1) from the opposite end of the quiver. Hence, we can add extra rows below in exactly the same way in which we started:

$$\begin{pmatrix} X^4_n & X^3_n & X^1_{n-1}X^2_{n-1} \\ X^1_{n+1} & X^2_n & X^1_{n-1} \\ X^1_{n+1}X^2_{n+1} & X^3_{n+1} & X^4_n \\ X^3_{n+2} & X^4_{n+1} & X^5_{n+1} \\ \vdots & \vdots & \vdots \\ X^{N+1}_{n+l+1} & X^N_{n+l} & X^{N+1}_{n+l-1} \\ X^{N+1}_{n+l+1}X^N_{n+l+1} & X^{N-1}_{n+l+1} & X^{N-2}_{n+l} \\ X^{N-1}_{n+l+2} & X^{N-2}_{n+l+1} & X^{N-3}_{n+l+1} \\ X^{N-2}_{n+l+2} & X^{N-3}_{n+l+2} & X^{N-4}_{n+l+1} \\ \vdots & \vdots & \vdots \\ X^5_{n+2l} & X^4_{n+2l-1} & X^3_{n+2l-1} \\ X^4_{n+2l} & X^3_{n+2l} & X^2_{n+2l-1}X^1_{n+2l-1} \\ X^1_{n+2l+1} & X^2_{n+2l} & X^1_{n+2l-1} \end{pmatrix},$$

where the final row follows from (22). Comparing the second and last rows, we can see that

$$J_n=\frac{X^1_{n+1}+X^1_{n-1}}{X^2_n}=\frac{X^1_{n+2l+1}+X^1_{n+2l-1}}{X^2_{n+2l}}=J_{n+2l}.$$

□

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THEOREM 4.4. *For even $N > 4$, the constant coefficient linear relation for the variables at the extending vertices is*

$$X^k_{n+2N-4} - \mathcal{K} X^k_{n+N-2} + X^k_n = 0, \tag{24}$$

by which we mean that $k \in \{1, 2, N, N+1\}$ and $\mathcal{K}$ is invariant.

Proof. Defining

$$\Psi_n := \begin{pmatrix} X^1_n & X^2_{n+1} \\ X^1_{n+N-2} & X^2_{n+N-1} \end{pmatrix}, \quad \tilde{L}_n := \begin{pmatrix} 0 & -1 \\ 1 & J_n \end{pmatrix}, \quad \text{then} \quad \Psi_n \tilde{L}_n = \begin{pmatrix} X^2_{n+1} & X^1_{n+2} \\ X^2_{n+N-1} & X^1_{n+N} \end{pmatrix},$$

so $\Psi_n \tilde{L}_n \tilde{L}_{n+1} = \Psi_{n+2}$. Calling $L_n := \tilde{L}_n \tilde{L}_{n+1}$ allows us to apply Theorem 3.3 with $q = 2$ and $p = m = N-2$. Explicitly $M_n = L_n L_{n+2} \dots L_{n+N-4}$. Since $\tilde{L}_{n+N-2} = \tilde{L}_n$ and the trace is fixed under cyclic permutations, we have that $\mathrm{tr}(M_n) = \mathcal{K}$ is invariant. We also have $|M_n| = 1$. The theorem gives $\Psi_{n+2N-4} - \mathcal{K}\Psi_{n+N-2} + \Psi_n = 0$, and the top left entry of this matrix equation gives the linear relation (24) for $k = 1$. By symmetry, this holds for the other extending vertices. □

We now give the following result, one of the cases in Theorem 1.3.

COROLLARY 4.5. *For even $N > 4$, the X^1_n variables satisfy the $\tilde{A}$-type recurrence*

$$X^1_{n+N-1} X^1_n = X^1_{n+N-2} X^1_{n+1} + \gamma^1,$$

where γ^1 is invariant.

Proof. Taking determinants in $\Psi_n \tilde{L}_n = \Psi_{n+1}$, we have that

$$X^1_n X^2_{n+N-1} - X^2_{n+1} X^1_{n+N-2} = X^2_{n+1} X^1_{n+N} - X^1_{n+2} X^2_{n+N-1},$$

and by Lemma 4.1, we can write X^2_n as $X^1_n \lambda_n$, where $\lambda_n := \left(\frac{X^2_0}{X^1_0}\right)^{(-1)^n}$ which is period 2. When replacing each X^2_n, each λ will appear with the same subscript and we cancel them to give the result. □

4.1.2. The case $N = 4$

In this case, the mutation relations are given by

$$X^i_{n+1} = \frac{1}{X^i_n}(1 + X^3_{n+1}), \qquad \text{for } i = 1, 2, 4, 5,$$
$$X^3_{n+1} = \frac{1}{X^3_n}(1 + X^1_n X^2_n X^4_n X^5_n).$$

The analogue of (22) is

$$J_n := \frac{X^1_{n+1} + X^1_{n-1}}{X^2_n} = \frac{X^1_{n+1} X^2_{n+1} + X^4_n X^5_n}{X^3_{n+1}} = \frac{X^1_{n-1} X^2_{n-1} + X^4_n X^5_n}{X^3_n},$$

and the analogous result

$$\sigma(J_n) = \frac{X^4_{n+1} + X^4_{n-1}}{X^5_n} = \frac{X^4_{n+1} X^5_{n+1} + X^1_n X^2_n}{X^3_{n+1}} = \frac{X^4_{n-1} X^5_{n-1} + X^1_n X^2_n}{X^3_n}$$

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is given by the permutation $\sigma := (14)(25)$. A simple calculation shows that $J_n = \sigma(J_{n-1})$ hence J_n is period $N-2=2$. Since the expression $J_n = \frac{X^1_{n+1}+X^1_{n-1}}{X^2_n}$ is the same as the other cases, Theorem 4.4 and Corollary 4.5 hold for $N=4$.

4.1.3. The odd N case

Here, vertices N and $N+1$ are sinks, $l = \frac{N-3}{2}$, and $h=0$. We can start with the matrix

$$\begin{pmatrix} X^4_n & X^3_n & X^1_{n-1}X^2_{n-1} \\ X^1_{n+1} & X^2_n & X^1_{n-1} \\ X^1_{n+1}X^2_{n+1} & X^3_{n+1} & X^4_n \\ X^3_{n+2} & X^4_{n+1} & X^5_{n+1} \\ X^4_{n+2} & X^5_{n+2} & X^6_{n+1} \\ X^5_{n+3} & X^6_{n+2} & X^7_{n+2} \\ \vdots & \vdots & \vdots \\ X^{N-3}_{n+l} & X^{N-2}_{n+l} & X^{N-1}_{n+l-1} \\ X^{N-2}_{n+l+1} & X^{N-1}_{n+l} & X^N_{n+l}X^{N+1}_{n+l} \end{pmatrix}.$$

The analogue for (23) is

$$\frac{X^{N+1}_{n+1}+X^{N+1}_{n-1}}{X^N_n} = \frac{X^{N+1}_{n+1}X^N_{n+1}+X^{N-2}_n}{X^{N-1}_n} = \frac{X^{N+1}_{n-1}X^N_{n-1}+X^{N-2}_n}{X^{N-1}_{n-1}},$$

which we use to add the first two black rows to (25). The following rows are again constructed in the same way we started:

$$\begin{pmatrix} X^4_n & X^3_n & X^1_{n-1}X^2_{n-1} \\ X^1_{n+1} & X^2_n & X^1_{n-1} \\ X^1_{n+1}X^2_{n+1} & X^3_{n+1} & X^4_n \\ X^3_{n+2} & X^4_{n+1} & X^5_{n+1} \\ X^4_{n+2} & X^5_{n+2} & X^6_{n+1} \\ X^5_{n+3} & X^6_{n+2} & X^7_{n+2} \\ \vdots & \vdots & \vdots \\ X^{N-3}_{n+l} & X^{N-2}_{n+l} & X^{N-1}_{n+l-1} \\ X^{N-2}_{n+l+1} & X^{N-1}_{n+l} & X^N_{n+l}X^{N+1}_{n+l} \\ X^{N+1}_{n+l+2} & X^N_{n+l+1} & X^{N+1}_{n+l} \\ X^{N+1}_{n+l+2}X^N_{n+l+2} & X^{N-1}_{n+l+1} & X^{N-2}_{n+l+1} \\ X^{N-1}_{n+l+2} & X^{N-2}_{n+l+2} & X^{N-3}_{n+l+1} \\ \vdots & \vdots & \vdots \\ X^5_{n+2l+1} & X^4_{n+2l} & X^3_{n+2l} \\ X^4_{n+2l+1} & X^3_{n+2l+1} & X^2_{n+2l}X^1_{n+2l} \\ X^1_{n+2l+2} & X^2_{n+2l+1} & X^1_{n+2l} \end{pmatrix}. \tag{25}$$

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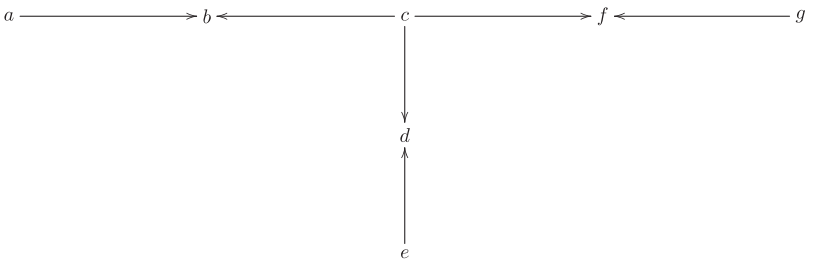


Figure 6. The $\tilde{E}_6$ quiver.

From the kernel for this matrix, we see that

$$J_n := \frac{X^1_{n+1} + X^1_{n-1}}{X^2_n}$$

has period $2l + 1 = N - 2$, the same as in the even N case.

THEOREM 4.6. *For odd N, the linear relation for the extending vertex variables is*

$$X^k_{n+4N-8} - \mathcal{K} X^k_{n+2N-4} + X^k_n = 0. \tag{26}$$

Proof. As above, we have

$$\Psi_n := \begin{pmatrix} X^1_n & X^2_{n+1} \\ X^1_{n+N-2} & X^2_{n+N-1} \end{pmatrix}, \quad \tilde{L}_n := \begin{pmatrix} 0 & -1 \\ 1 & J_{n+1} \end{pmatrix}, \quad \text{and} \quad \Psi_n \tilde{L}_n = \begin{pmatrix} X^2_{n+1} & X^1_{n+2} \\ X^2_{n+N-1} & X^1_{n+N} \end{pmatrix}.$$

Once again we define $L_n := \tilde{L}_n \tilde{L}_{n+1}$ so $\Psi_n L_n = \Psi_{n+2}$. Applying Theorem 3.3 with $q = 2$, $p = N - 2$ and $m = 2N - 4$ yields $\Psi_{n+4N-8} - \mathcal{K}\Psi_{n+2N-4} + \Psi_n = 0$, from which we can extract the linear relation. The arguments that $\mathrm{tr}(M_n)$ is invariant and $|M_n| = 1$ are identical to the ones from the N even case. □

The following result is equation (7) in Theorem 1.3.

COROLLARY 4.7. *The X^1_n variables satisfy the A-type recurrence*

$$X^k_{n+N-1} X^k_n = (\lambda^k_{n+1})^2 X^k_{n+N-2} X^k_{n+1} + \gamma^k_n,$$

where both γ^k_n and λ^k_n are period 2.

Proof. By taking determinants in $\Psi_n L_n = \Psi_{n+2}$, we see that

$$X^1_n X^2_{n+N-1} - X^2_{n+1} X^1_{n+N-2}$$

is period 2. By Lemma 4.1, we can replace X^2_{n+N-1} and X^2_{n+1} to give that

$$X^1_n X^1_{n+N-1} \lambda_n - X^1_{n+1} X^1_{n+N-2} \lambda_{n+1} \tag{27}$$

is period 2, where we have again defined $\lambda_n := \left(\frac{X^2_0}{X^1_0}\right)^{(-1)^n}$, which is also period 2. We define expression (27) as $\gamma^1_n / \lambda_{n+1}$ and rearrange to give the corollary for $k = 1$. The other extending vertices are proved similarly. □

4.2. The $\tilde{E}_6$ quiver. The $\tilde{E}_6$ diagram is given in Figure 6 with a source–sink orientation. Since we have a fixed number of vertices, we can label each using different letters,

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making calculations clearer. In our previous notation, we would have had, for example, $a_n = X^1_n$. The sequence of mutations, μ, gives the following recurrence relations.

$$\begin{aligned} &a_{n+1}a_n = 1 + b_n, && b_{n+1}b_n = 1 + a_{n+1}c_{n+1}, && c_{n+1}c_n = 1 + b_n d_n f_n, \\ &d_{n+1}d_n = 1 + c_{n+1}e_{n+1}, && e_{n+1}e_n = 1 + d_n, && \\ &f_{n+1}f_n = 1 + c_{n+1}g_{n+1}, && g_{n+1}g_n = 1 + f_n. && \end{aligned} \tag{28}$$

The diagram has S_3 symmetry generated by (for example) the reflections $(bf)(ag)$ and $(bd)(ae)$. It will be useful to name some of the group elements.

$$\sigma_1 := (bf)(ag), \qquad \sigma_2 := (bd)(ae), \qquad \sigma_3 := (bdf)(aeg).$$

4.2.1. Periodic quantities

Using the mutation equations, (28), we form the following matrix, which has determinant 0 by (15). We add labels to the right of the matrix, so we may refer to each row:

$$M := \begin{pmatrix} 1 & a_{n-1} & b_{n-1} \\ a_n & 1 & 0 \\ b_n & a_{n+1} & 1 \end{pmatrix} \begin{matrix} -1 \\ 0 \\ 1 \end{matrix}.$$

After scaling we take a kernel vector $(1, -a_n, 1)^T$ since, as before, we can see that the third entry of this vector is equal to the first by (28). We can add rows to M satisfying the conditions of Theorem 3.2:

$$M := \begin{pmatrix} \star & d_{n-3}/f_{n-2} & e_{n-2} \\ b_{n-3} & c_{n-2} & d_{n-2}f_{n-2} \\ a_{n-2} & b_{n-2} & c_{n-1} \\ 1 & a_{n-1} & b_{n-1} \\ a_n & 1 & 0 \\ b_n & a_{n+1} & 1 \\ c_{n+1} & b_{n+1} & a_{n+2} \\ d_{n+1}f_{n+1} & c_{n+2} & b_{n+2} \\ e_{n+2} & d_{n+2}/f_{n+1} & \star \end{pmatrix} \begin{matrix} -4 \\ -3 \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix}.$$

until we reach rows ± 4 where, in order to use the d mutation relation, we have to divide in our middle entry. The starred entries of M can be filled, preserving the kernel, by insisting that the right 2×2 minor of rows 3 and 4 have determinant 1, ditto for the left 2×2 minor of rows -4 and -3.

LEMMA 4.8. *Setting the rightmost entry of row* 4 *to* $\frac{g_{n+4}}{f_{n+1}}$ *and the leftmost entry of row* -4 *to* $\frac{g_{n-4}}{f_{n-2}}$ *will preserve the kernel.*

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Proof. We just prove the statement for row 4. In order for the 2×2 minor to have determinant 1, we need to set the blank entry to

$$\frac{1 + b_{n+2}d_{n+2}/f_{n+1}}{c_{n+2}} = \frac{f_{n+1} + b_{n+2}d_{n+2}}{c_{n+2}f_{n+1}},$$

but we have

$$\begin{aligned}\frac{f_{n+1} + b_{n+2}d_{n+2}}{c_{n+2}} &= \frac{f_{n+1} + \frac{c_{n+3}c_{n+2}-1}{f_{n+2}}}{c_{n+2}} = \frac{f_{n+1}f_{n+2} - 1 + c_{n+3}c_{n+2}}{c_{n+2}f_{n+2}} \\ &= \frac{c_{n+2}g_{n+2} + c_{n+3}c_{n+2}}{c_{n+2}f_{n+2}} = \frac{g_{n+2} + c_{n+3}}{f_{n+2}}.\end{aligned}$$

Now from the kernel of row -2, we have $a_{n-2} - a_n b_{n-2} + c_{n-1} = 0$, to which we apply the permutation σ_1 and shift to see $\frac{g_{n+2}+c_{n+3}}{f_{n+2}} = g_{n+4}$. □

Using this Lemma, we can fill in rows ± 4. We then use vertex e mutation to partially add rows ± 5:

$$M := \begin{pmatrix} \star & e_{n-3} & f_{n-2} & -5 \\ g_{n-4}/f_{n-2} & d_{n-3}/f_{n-2} & e_{n-2} & -4 \\ b_{n-3} & c_{n-2} & d_{n-2}f_{n-2} & -3 \\ \vdots & \vdots & \vdots & \\ d_{n+1}f_{n+1} & c_{n+2} & b_{n+2} & 3 \\ e_{n+2} & d_{n+2}/f_{n+1} & g_{n+4}/f_{n+1} & 4 \\ f_{n+1} & e_{n+3} & \star & 5 \end{pmatrix}.$$

Now again we require that both

$$\begin{vmatrix} \star & e_{n-3} \\ g_{n-4}/f_{n-2} & d_{n-3}/f_{n-2} \end{vmatrix} = 1, \qquad \begin{vmatrix} d_{n+2}/f_{n+1} & g_{n+4}/f_{n+1} \\ e_{n+3} & \star \end{vmatrix} = 1.$$

We simply define quantities that satisfy these relations

$$J_n := \frac{f_{n+1} + e_{n+3}g_{n+4}}{d_{n+2}}, \qquad \tilde{J}_n := \frac{f_{n-2} + e_{n-3}g_{n-4}}{d_{n-3}}$$

and fill the blank entries with them. The reason for naming these J_n and $\tilde{J}_n$ will be made apparent in the following lemma. We will need the equations from the kernel for rows ± 5:

$$f_{n+1} - a_n e_{n+3} + J_n = 0, \qquad \tilde{J}_n - a_n e_{n-3} + f_{n-2} = 0. \tag{29}$$

LEMMA 4.9. *We have the following expressions for J_n and $\tilde{J}_n$ in terms of the extending variables:*

$$J_n = \frac{a_n + g_{n+4}}{e_{n+2}}, \qquad \tilde{J}_n = \frac{a_n + g_{n-4}}{e_{n-2}}, \tag{30}$$

and each of these has period 3.

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Proof. From the kernel for row 4 and vertex e mutation, we have

$$f_{n+1} = \frac{a_n d_{n+2} - g_{n+4}}{e_{n+2}} = \frac{a_n(e_{n+3}e_{n+2} - 1) - g_{n+4}}{e_{n+2}} = a_n e_{n+3} - \frac{a_n + g_{n+4}}{e_{n+2}}.$$

Comparing this with the first equation of (29) gives $J_n = \frac{a_n+g_{n+4}}{e_{n+2}}$. The corresponding $\tilde{J}_n$ result is proved similarly. Comparing the expressions in (30), one sees that $\sigma_1(J_n) = \tilde{J}_{n+4}$ and applying σ_2 to (29) we have $\sigma_2(J_n) = \tilde{J}_{n+3}$. Composing these gives $\sigma_2\sigma_1(J_n) = J_{n+1}$. The observation that $\sigma_2\sigma_1$ has order 3 gives the periodicity of J_n and $\tilde{J}_n$. □

4.2.2. Constant coefficient linear relations

THEOREM 4.10. *The recurrences for the variables attached at the extending vertices satisfy the constant coefficient linear relations*

$$x_{n+12} - \mathcal{K}x_{n+6} + x_n = 0, \tag{31}$$

i.e., $x \in \{a, e, g\}$ and $\mathcal{K}$ is invariant.

Proof. Defining

$$\Psi_n := \begin{pmatrix} e_{n+5} & a_{n+3} \\ e_{n+2} & a_n \end{pmatrix}, \qquad \tilde{L}_n := \begin{pmatrix} J_n & 1 \\ -1 & 0 \end{pmatrix},$$

we have $\Psi_n\tilde{L}_n = \sigma_3(\Psi_{n+2})$ which gives $\Psi_{n+6} = \Psi_n\tilde{L}_n\tilde{L}_{n+2}\tilde{L}_{n+4}$. We define $L_n := \tilde{L}_n\tilde{L}_{n+2}\tilde{L}_{n+4}$ and apply Theorem 3.3 with $p = 3$ and $q = 6$. This gives, since $|L_n| = 1$,

$$\Psi_{n+12} - tr(L_n)\Psi_{n+6} - \Psi_n = 0.$$

Note that here $L_n = M_n$. Any entry of this matrix equation will give (31), since $\mathcal{K} := \mathrm{tr}(L_n)$ is invariant as $\mathrm{tr}(L_{n+1}) = \mathrm{tr}(\tilde{L}_{n+1}\tilde{L}_n\tilde{L}_{n+2}) = \mathrm{tr}(\tilde{L}_n\tilde{L}_{n+1}\tilde{L}_{n+2}) = \mathrm{tr}(L_n)$. □

In this case, we have slightly more information. We have a period 2 quantity K_n which we use to find a linear relation for the b, d, and f vertices. We also prove a relation between K_n and $\mathcal{K}$.

LEMMA 4.11. *The following expressions, each in terms of variables at only one vertex, are fixed by all permutations:*

$$\frac{a_{n-3} + a_{n+3}}{a_n} = \frac{e_{n-3} + e_{n+3}}{e_n} = \frac{g_{n-3} + g_{n+3}}{g_n}. \tag{32}$$

Proof. From (29), we have $J_n - \tilde{J}_{n+3} = a_n e_{n+3} - a_{n+3}e_n$, but the left-hand side is period 3 so $a_n e_{n+3} - a_{n+3}e_n = a_{n+3}e_{n+6} - a_{n+6}e_{n+3}$, which we rearrange for the result. □

The following lemma is equivalent to the $\tilde{E}_6$ part of Theorem 1.3.

LEMMA 4.12. *The quantity*

$$K_n := \frac{a_{n-3} + a_{n+3}}{a_n} \tag{33}$$

has period 2.

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Proof. Since $J_n = \frac{a_n+g_{n+4}}{e_{n+2}}$ is period 3, we have

$$a_n e_{n+5} + e_{n+5} g_{n+4} = a_{n+3} e_{n+2} + e_{n+2} g_{n+7}.$$

Applying $(\varphi^*)^3 + 1$ to this equation, we get

$$e_{n+5}(a_n + a_{n+6} + g_{n+4} + g_{n+10}) = a_{n+3}(e_{n+2} + e_{n+8}) + g_{n+7}(e_{n+2} + e_{n+8}),$$

which we divide by $a_{n+3} e_{n+5} g_{n+7}$, noting that by (32) K_n is fixed by permutations,

$$\frac{K_{n+3}}{g_{n+7}} + \frac{K_{n+7}}{a_{n+3}} = \frac{K_{n+5}}{g_{n+7}} + \frac{K_{n+5}}{a_{n+3}}.$$

Assuming that we may divide by $K_{n+5} - K_{n+3}$ this is equivalent to

$$\frac{g_{n+7}(K_{n+7} - K_{n+5})}{K_{n+5} - K_{n+3}} = a_{n+3},$$

but here the left-hand side is fixed under σ_2, so we have $a_{n+3} = e_{n+3}$. This is a contradiction since a_{n+3} and e_{n+3} are independent variables, since setting $n = -3$ means that a_{n+3} and e_{n+3} would appear in a set of arbitrary initial variables. Hence, the division by $K_{n+5} - K_{n+3}$ was invalid, giving $K_{n+5} - K_{n+3} = 0$. □

LEMMA 4.13. *The period* 2 *quantity* K_n *and the invariant* $\mathcal{K}$ *are related via* $\mathcal{K} = K_0 K_1 - 2$.

Proof. From (33), we have

$$\begin{pmatrix} a_{n+5} & a_{n+2} \\ a_{n+3} & a_n \end{pmatrix} \begin{pmatrix} K_1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} a_{n+8} & a_{n+5} \\ a_{n+6} & a_{n+4} \end{pmatrix}.$$

Considering the product $\begin{pmatrix} K_1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} K_0 & 1 \\ -1 & 0 \end{pmatrix}$, we apply an argument similar to Theorem 3.3 and compare with (31) to arrive at the result. □

We can use K_n to find a constant coefficient linear relation for the vertices adjacent to the extending ones.

THEOREM 4.14. *We have the linear relation* $x_n - (\mathcal{K}+1)x_{n+3} + (\mathcal{K}+1)x_{n+6} - x_{n+9} = 0$, *for* $x \in \{b, d, f\}$.

Proof. From (28), we have

$$\frac{b_n - b_{n+9}}{b_{n+3} - b_{n+6}} = \frac{a_n a_{n-1} - a_{n+9} a_{n+8}}{a_{n+3} a_{n+2} - a_{n+6} a_{n+5}}.$$

Using (33) to replace everything in the numerator on the right-hand side, i.e.,

$$a_{n-1} = K_{n+1} a_{n+2} - a_{n+5}, \qquad a_n = K_n a_{n+3} - a_{n+6},$$
$$a_{n+8} = K_n a_{n+5} - a_{n+2}, \qquad a_{n+9} = K_{n+1} a_{n+6} - a_{n+3},$$

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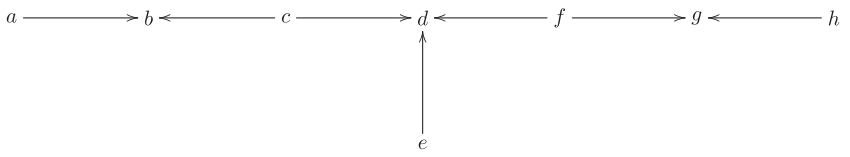


Figure 7. The $\tilde{E}_7$ quiver.

we can rearrange to give

$$\frac{b_n - b_{n+9}}{b_{n+3} - b_{n+6}} = K_n K_{n+1} - 1 = \mathcal{K} + 1.$$

This proves the theorem for $x = b$. The other two cases follow by symmetry. □

4.3. The $\tilde{E}_7$ quiver. The mutation relations for the $\tilde{E}_7$ quiver, Figure 7, are

$$\begin{aligned}
&a_{n+1}a_n = 1 + b_n, && b_{n+1}b_n = 1 + a_{n+1}c_{n+1}, && c_{n+1}c_n = 1 + b_n d_n,\\
&d_{n+1}d_n = 1 + c_{n+1}e_{n+1}f_{n+1}, && e_{n+1}e_n = 1 + d_n, && f_{n+1}f_n = 1 + d_n g_n,\\
&g_{n+1}g_n = 1 + f_{n+1}h_{n+1}, && h_{n+1}h_n = 1 + g_n. &&
\end{aligned} \tag{34}$$

This diagram has S_2 symmetry, the generator of which we denote $\sigma := (ah)(bg)(cf)$.

4.3.1. Periodic quantities

We begin with the matrix centered at vertex a with kernel vector $(1, -a_n, 1)^T$:

$$\begin{pmatrix}
\star & f_{n-3}/e_{n-2} & g_{n-3} & -5\\
c_{n-3} & d_{n-3} & e_{n-2}f_{n-2} & -4\\
b_{n-3} & c_{n-2} & d_{n-2} & -3\\
a_{n-2} & b_{n-2} & c_{n-1} & -2\\
1 & a_{n-1} & b_{n-1} & -1\\
a_n & 1 & 0 & 0\\
b_n & a_{n+1} & 1 & 1\\
c_{n+1} & b_{n+1} & a_{n+2} & 2\\
d_{n+1} & c_{n+2} & b_{n+2} & 3\\
e_{n+2}f_{n+2} & d_{n+2} & c_{n+3} & 4\\
g_{n+2} & f_{n+3}/e_{n+2} & \star & 5
\end{pmatrix}.$$

The construction of which is routine until we reach the stars in rows ± 5.

LEMMA 4.15. *Setting the lower right and upper left entries to be*

$$\frac{e_{n+2} + c_{n+3}f_{n+3}}{d_{n+2}e_{n+2}} = \frac{e_{n+4}}{e_{n+2}}, \qquad \frac{e_{n-2} + f_{n-3}c_{n-3}}{d_{n-3}e_{n-2}} = \frac{e_{n-4}}{e_{n-2}},$$

respectively, preserves the kernel.

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Proof. The proof of these equalities is the same argument as that used for Lemma 4.8. □

We can now replace the stars, adding to the rows marked ± 5 below. The rows labeled ± 6 are to be filled using the mutation relation for g, giving us another two starred entries:

$$\begin{pmatrix} \star & g_{n-4} & e_{n-2}h_{n-3} \\ e_{n-4}/e_{n-2} & f_{n-3}/e_{n-2} & g_{n-3} \\ \vdots & \vdots & \vdots \\ g_{n+2} & f_{n+3}/e_{n+2} & e_{n+4}/e_{n+2} \\ e_{n+2}h_{n+3} & g_{n+3} & \star \end{pmatrix} \begin{matrix} -6 \\ -5 \\ \\ 5 \\ 6 \end{matrix}.$$

which we can fill with the following result.

LEMMA 4.16. *Due to the relations*

$$\frac{e_{n+2}+e_{n+4}g_{n+3}}{f_{n+3}e_{n+2}} = \frac{a_{n+6}}{e_{n+2}}, \qquad \frac{e_{n-2}+e_{n-4}g_{n-4}}{e_{n-2}f_{n-3}} = \frac{a_{n-6}}{e_{n-2}},$$

we can replace the stars with a_{n-6} and a_{n+6}, respectively.

Proof. From the kernel for row -5, we have $e_{n-4} - f_{n-3}a_n + g_{n-3}e_{n-2} = 0$, which we shift upwards by 6 and substitute into our expression to get the first equality. The second is proved similarly. □

We now use the mutation relation for h to add most of rows ± 7:

$$\begin{pmatrix} & h_{n-4}/e_{n-2} & 1 \\ a_{n-6} & g_{n-4} & e_{n-2}h_{n-3} \\ \vdots & \vdots & \vdots \\ e_{n+2}h_{n+3} & g_{n+3} & a_{n+6} \\ 1 & h_{n+4}/e_{n+2} & \end{pmatrix} \begin{matrix} -7 \\ -6 \\ \\ 6 \\ 7 \end{matrix}.$$

This is the last row we need to add. We define the quantities

$$J_n := \frac{e_{n+2}+a_{n+6}h_{n+4}}{g_{n+3}}, \qquad \tilde{J}_n := \frac{e_{n-2}+a_{n-6}h_{n-4}}{g_{n-4}}$$

that sit in the blank corners, such that the kernel is persevered. As we shall see, these J_n and $\tilde{J}_n$ are also periodic.

THEOREM 4.17. *J_n and $\tilde{J}_n$ satisfy*

$$J_n = \tilde{J}_{n+6}. \tag{35}$$

Proof. Multiplying the numerator and denominator of J_n by g_{n+2}, we get

$$J_n = \frac{g_{n+2}e_{n+2}+g_{n+2}a_{n+6}h_{n+4}}{g_{n+2}g_{n+3}}. \tag{36}$$

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From the kernel for row -6, we have

$$a_n g_{n-4} = a_{n-6} + e_{n-2} h_{n-3},$$

which we shift upwards by 6 then multiply by h_{n+4} to give

$$a_{n+6} g_{n+2} h_{n+4} = a_n h_{n+4} + e_{n+4} h_{n+3} h_{n+4} = a_n h_{n+4} + e_{n+4}(1 + g_{n+3}), \tag{37}$$

where the second equality uses h mutation. We also have, from the kernel for row 5,

$$e_{n+2} g_{n+2} = a_n f_{n+3} - e_{n+4}. \tag{38}$$

Substituting (37) and (38) into (36) gives

$$J_n = \frac{a_n(f_{n+3} + h_{n+4}) + e_{n+4} g_{n+3}}{g_{n+2} g_{n+3}}. \tag{39}$$

From a shift of row 2, we have $c_{n+3} - a_{n+2} b_{n+3} + a_{n+4} = 0$ to which we can apply σ to get $f_{n+3} - h_{n+2} g_{n+3} + h_{n+4} = 0$. This is used to replace the bracketed term in (39), giving

$$J_n = \frac{a_n h_{n+2} + e_{n+4}}{g_{n+2}},$$

which is $\tilde{J}_{n+6}$. □

LEMMA 4.18. *The permutation σ acts on J_n and $\tilde{J}_n$ by $\sigma(J_n) = \tilde{J}_{n+4} = J_{n+2}$, therefore J_n and $\tilde{J}_n$ are period* 4.

Proof. From rows ± 7, we have

$$J_n = a_n h_{n+4} - e_{n+2}, \qquad \tilde{J}_n = a_n h_{n-4} - e_{n-2},$$

which gives the first equality. Using this, we can apply σ to (35) to get $J_{n+2} = \tilde{J}_{n+4}$. □

LEMMA 4.19. *We have the following expression, fixed under σ, in terms of the variables at only one extending vertex:*

$$K_n := \frac{a_{n+8} + a_n}{a_{n+4}} = \frac{h_{n+8} + h_n}{h_{n+4}}. \tag{40}$$

Proof. Since $J_n = a_n h_{n+4} - e_{n+2}$ and $J_{n+2} = \sigma(J_n) = a_{n+4} h_n - e_{n+2}$, we have

$$J_n - J_{n+2} = a_n h_{n+4} - a_{n+4} h_n.$$

The left-hand side of this is period 4 so $a_n h_{n+4} - a_{n+4} h_n$ is also period 4, which is equivalent to (40). □

We have the following more useful expression for J_n which we will use to calculate linear relations.

THEOREM 4.20. *The quantity J_n may be be written*

$$J_n = \frac{a_{n+6} + a_n}{h_{n+3}} = \frac{h_{n+8} + h_{n+2}}{a_{n+5}}. \tag{41}$$

Proof. From row 6, we see

$$\begin{aligned} 0 = e_{n+2} h_{n+3} - a_n g_{n+3} + a_{n+6} &= e_{n+2} h_{n+3} - a_n h_{n+4} h_{n+3} + a_n + a_{n+6} \\ &= -h_{n+3} J_n + a_n + a_{n+6}. \end{aligned}$$

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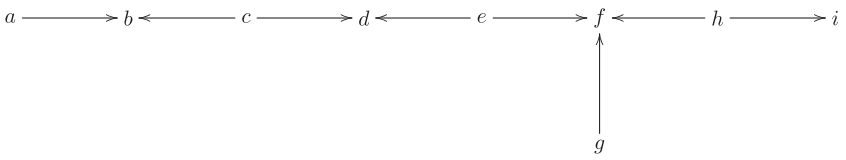


Figure 8. The $\tilde{E}_8$ quiver.

We apply σ to this equation, noting that by Lemma 4.18 $\sigma(J_n) = J_{n+2}$, to get the result. □

PROPOSITION 4.21. *The quantity K_n, defined in (40), is period* 3*, hence the a variables satisfy the following linear relation $a_{n+8} - K_n a_{n+4} + a_n = 0$ and the $\tilde{E}_7$ part of Theorem 1.3. We also have that the determinant*

$$\tilde{K}_n := \begin{vmatrix} a_n & h_{n+3} \\ a_{n+4} & h_{n+7} \end{vmatrix}, \tag{42}$$

is period 2 *with $\sigma(\tilde{K}_n) = \tilde{K}_{n+1}$.*

Proof. The proof of this statement is not very illuminating, so we omit it here. It will appear in the author's thesis. □

Finally, we conjecture an alternate expression for $\tilde{K}_n$.

CONJECTURE 4.22. The period 2 quantity $\tilde{K}_n$ can be expressed as

$$\tilde{K}_n = \frac{a_{n+12} + a_n}{h_{n+6}}.$$

4.3.2. Constant coefficient linear relations

THEOREM 4.23. *We have the constant coefficient linear relation $x_{n+24} - \mathcal{K} x_{n+12} + x_n = 0$, where $x \in \{a, h\}$.*

Proof. We define the matrices

$$\Psi_n := \begin{pmatrix} a_{n+5} & h_{n+2} \\ h_{n+3} & a_n \end{pmatrix}, \qquad \tilde{L}_n := \begin{pmatrix} J_n & 1 \\ -1 & 0 \end{pmatrix},$$

then, due to (41), we have

$$\Psi_n \tilde{L}_{n+2} = \sigma(\Psi_{n+3}), \qquad \Psi_n \tilde{L}_{n+2} \tilde{L}_{n+3} = \Psi_{n+6}.$$

Now we can set $L_n := \tilde{L}_{n+2}\tilde{L}_{n+3}$ and use Theorem 3.3 with $M_n = L_n L_{n+6}$. Again we call $\mathcal{K} := tr(M_n)$, which, by the same argument used for $\tilde{E}_6$, is invariant. □

4.4. The $\tilde{E}_8$ quiver. The $\tilde{E}_8$ is given in Figure 8. The recurrence relations are

$$\begin{aligned} &a_{n+1}a_n = 1 + b_n, && b_{n+1}b_n = 1 + a_{n+1}c_{n+1}, && c_{n+1}c_n = 1 + b_n d_n, \\ &d_{n+1}d_n = 1 + c_{n+1}e_{n+1}, && e_{n+1}e_n = 1 + d_n f_n, && f_{n+1}f_n = 1 + e_{n+1}g_{n+1}h_{n+1}, \\ &g_{n+1}g_n = 1 + f_n, && h_{n+1}h_n = 1 + f_n i_n, && i_{n+1}i_n = 1 + h_{n+1}. \end{aligned} \tag{43}$$

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4.4.1. Periodic quantities

In this subsection, we omit some proofs that are similar to previous calculations and will appear in the author's thesis. We begin with the matrix centered at vertex a, denoted M_a, with kernel vector $(1, -a_n, 1)^T$:

$$M_a := \begin{pmatrix} \star & h_{n-4}/g_{n-3} & i_{n-4} \\ e_{n-4} & f_{n-4} & g_{n-3}h_{n-3} \\ d_{n-4} & e_{n-3} & f_{n-3} \\ c_{n-3} & d_{n-3} & e_{n-2} \\ b_{n-3} & c_{n-2} & d_{n-2} \\ a_{n-2} & b_{n-2} & c_{n-1} \\ 1 & a_{n-1} & b_{n-1} \\ a_n & 1 & 0 \\ b_n & a_{n+1} & 1 \\ c_{n+1} & b_{n+1} & a_{n+2} \\ d_{n+1} & c_{n+2} & b_{n+2} \\ e_{n+2} & d_{n+2} & c_{n+3} \\ f_{n+2} & e_{n+3} & d_{n+3} \\ g_{n+3}h_{n+3} & f_{n+3} & e_{n+4} \\ i_{n+3} & h_{n+4}/g_{n+3} & \star \end{pmatrix} \begin{matrix} -7 \\ -6 \\ -5 \\ -4 \\ -3 \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{matrix}.$$

LEMMA 4.24. *The stars in rows* ± 7 *are found via the relations*

$$\frac{g_{n+3} + e_{n+4}h_{n+4}}{f_{n+3}g_{n+3}} = \frac{g_{n+5}}{g_{n+3}}, \qquad \frac{g_{n-3} + e_{n-4}h_{n-4}}{g_{n-3}f_{n-4}} = \frac{g_{n-5}}{g_{n-3}}.$$

The proof of this is analogous to Lemma 4.8.

We can now fill in rows ± 7 and add some of rows ± 8 via i mutation:

$$M_a = \begin{pmatrix} \star & i_{n-5} & g_{n-3} \\ g_{n-5}/g_{n-3} & h_{n-4}/g_{n-3} & i_{n-4} \\ e_{n-4} & f_{n-4} & g_{n-3}h_{n-3} \\ \vdots & \vdots & \vdots \\ g_{n+3}h_{n+3} & f_{n+3} & e_{n+4} \\ i_{n+3} & h_{n+4}/g_{n+3} & g_{n+5}/g_{n+3} \\ g_{n+3} & i_{n+4} & \star \end{pmatrix} \begin{matrix} -8 \\ -7 \\ -6 \\ \\ 6 \\ 7 \\ 8 \end{matrix}.$$

LEMMA 4.25. *The starred entries can be replaced by* $a_{n\pm 8}$*, respectively, preserving the kernel, because we have*

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$$\frac{g_{n+3}+g_{n+5}i_{n+4}}{h_{n+4}}=a_{n+8},\qquad \frac{g_{n-3}+g_{n-5}i_{n-5}}{h_{n-4}}=a_{n-8}.$$

Proof. From row -7, we have $g_{n-5}-a_nh_{n-4}+g_{n-3}i_{n-4}=0$, which we shift up 8 and substitute into our expression to get the first result. □

We have extended M_a as much as necessary. In this case, we will also need to construct a matrix centered at vertex i:

$$M_i:=\begin{pmatrix}
\tilde{J}_n & b_{n-4} & a_{n-3}g_{n-1} & -7\\
a_{n-8}/g_{n-1} & c_{n-3}/g_{n-1} & b_{n-3} & -6\\
i_{n-5} & d_{n-3} & c_{n-2}g_{n-1} & -5\\
g_{n-3}/g_{n-1} & e_{n-2}/g_{n-1} & d_{n-2} & -4\\
h_{n-2} & f_{n-2} & e_{n-1}g_{n-1} & -3\\
i_{n-2} & h_{n-1} & f_{n-1} & -2\\
1 & i_{n-1} & h_n & -1\\
i_n & 1 & 0 & 0\\
h_{n+1} & i_{n+1} & 1 & 1\\
f_{n+1} & h_{n+2} & i_{n+2} & 2\\
e_{n+2}g_{n+2} & f_{n+2} & h_{n+3} & 3\\
d_{n+2} & e_{n+3}/g_{n+2} & g_{n+4}/g_{n+2} & 4\\
c_{n+3}g_{n+2} & d_{n+3} & i_{n+5} & 5\\
b_{n+3} & c_{n+4}/g_{n+2} & a_{n+9}/g_{n+2} & 6\\
a_{n+4}g_{n+2} & b_{n+4} & J_n & 7
\end{pmatrix}.$$

The entries of rows ±4 and ±5 were simplified by results similar to Lemmas 4.24 and 4.25, respectively. For rows ±6, we have used the following equalities:

$$\frac{g_{n+2}+c_{n+4}i_{n+5}}{d_{n+3}g_{n+2}}=\frac{a_{n+9}}{g_{n+2}},\qquad \frac{g_{n-1}+c_{n-3}i_{n-5}}{d_{n-3}g_{n-1}}=\frac{a_{n-8}}{g_{n-1}}.$$

We have also defined

$$J_n:=\frac{g_{n+2}+a_{n+9}b_{n+4}}{c_{n+4}},\qquad \tilde{J}_n:=\frac{g_{n-1}+a_{n-8}b_{n-4}}{c_{n-3}}.$$

The following lemma proves one of the $\tilde{E}_8$ relations given in Theorem 1.3.

LEMMA 4.26. *The expressions J_n and $\tilde{J}_n$ satisfy $J_n=\tilde{J}_n=a_{n-3}a_{n+4}-i_n$ and $J_n=J_{n+5}$. We also have the following expression which we will use to apply Theorem 3.3:*

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$$J_{n+3} = \frac{a_{n+12} + a_n}{a_{n+6}}. \tag{44}$$

Proof. We prove only the last relation. We have

$$\begin{aligned} a_{n+12} &= c_{n+7}i_{n+3} - b_{n+3}g_{n+2} = a_n b_{n+6} - a_{n+6}i_{n+3} = a_n(-1 + a_{n+6}a_{n+7}) - a_{n+6}i_{n+3} \\ &= -a_n + a_{n+6}(a_n a_{n+7} - i_{n+3}) = -a_n + a_{n+6}J_{n+3}. \end{aligned}$$

The first equality comes from row 6 of M_i and the second from rows -2 and -8 of M_a. The third uses a mutation and the fifth the first result of this lemma. □

To prove the existence of the period 3 quantity, we will use the following matrix, also beginning at vertex a, but with alternate rows 7 and 8.

$$\tilde{M}_a := \begin{pmatrix} a_n & 1 & 0 \\ b_n & a_{n+1} & 1 \\ c_{n+1} & b_1 & a_{n+2} \\ d_{n+1} & c_{n+2} & b_{n+2} \\ e_{n+2} & d_{n+2} & c_{n+3} \\ f_{n+2} & e_{n+3} & d_{n+3} \\ g_{n+3}h_{n+3} & f_{n+3} & e_{n+4} \\ 1 & g_{n+4}/h_{n+3} & i_{n+5}/h_{n+3} \\ h_{n+3} & g_{n+4} & i_{n+5} \end{pmatrix} \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix}.$$

LEMMA 4.27. *We have*

$$a_n a_{n+13} - i_{n+6} = J_{n+3}J_{n+4} - 1.$$

Proof. From (44), we have the first equality of the following:

$$a_n a_{n+13} = J_{n+4}a_n a_{n+7} - a_n a_{n+1} = J_{n+4}(J_{n+3} + i_{n+3}) - 1 - b_n. \tag{45}$$

The second equality uses $J_{n+3} = a_n a_{n+7} - i_{n+3}$ and a mutation. Now from row -7 of M_i, we have $i_{n+3}J_{n+4} - i_{n+3}i_{n+4}b_n + i_{n+3}a_{n+1}g_{n+3} = 0$. We replace the $i_{n+3}i_{n+4}$ term to get

$$i_{n+3}J_{n+4} - b_n = b_n h_{n+4} - i_{n+3}a_{n+1}g_{n+3}.$$

The right-hand side of (45) now becomes

$$J_{n+3}J_{n+4} - 1 + b_n h_{n+4} - i_{n+3}a_{n+1}g_{n+3}. \tag{46}$$

Row 7 of M_a yields

$$i_{n+3}a_{n+1}g_{n+3} = a_n a_{n+1}h_{n+4} - a_{n+1}g_{n+5} = h_{n+4} + b_n h_{n+4} - a_{n+1}g_{n+5},$$

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so (46) is equal to

$$J_{n+3}J_{n+4} - 1 + a_{n+1}g_{n+5} - h_{n+4}. \tag{47}$$

Finally, we have $h_{n+4} - a_{n+1}g_{n+5} + i_{n+6}$ from row 8 of $\tilde{M}_a$ so (47) becomes $J_{n+3}J_{n+4} - 1 - i_{n+6}$ which completes the proof. □

The following theorem proves the other $\tilde{E}_8$ relation given in Theorem 1.3.

THEOREM 4.28. *The quantity*

$$K_n := \frac{a_{n+20} + a_n}{a_{n+10}}$$

is period 3.

Proof. By Lemma 4.27, the expression $a_n a_{n+13} - i_{n+6}$ is period 5. In particular,

$$a_n a_{n+13} - i_{n+6} = a_{n+10}a_{n+23} - i_{n+16}.$$

We also have $J_{n+1} = a_{n+3}a_{n+10} - i_{n+6} = a_{n+13}a_{n+20} - i_{n+16}$ so

$$a_n a_{n+13} - a_{n+3}a_{n+10} = a_{n+10}a_{n+23} - a_{n+13}a_{n+20},$$

which can be factored to give the theorem. □

4.4.2. Constant coefficient linear relations

THEOREM 4.29. *The constant coefficient relation for the a variables is* $a_{n+60} - \mathcal{K}a_{n+30} + a_n = 0$*, where* $\mathcal{K}$ *is invariant.*

Proof. Defining

$$\Psi_n := \begin{pmatrix} a_{n+11} & a_{n+5} \\ a_{n+6} & a_n \end{pmatrix}, \qquad L_{n-3} := \begin{pmatrix} J_n & 1 \\ -1 & 0 \end{pmatrix}$$

gives $\Psi_n L_n = \Psi_{n+6}$ so again we apply Theorem 3.3 with $p = 5$, $q = 6$ and $M_n := L_n L_{n+6} \ldots L_{n+24}$ to arrive at the above result. □

Finally, we state that we were unable to prove the following conjecture to complete the table in Figure 3, but we give an ansatz.

CONJECTURE 4.30. The following expression is period 2:

$$\tilde{K}_n := \frac{a_{n+30} + a_n}{a_{n+15}}.$$

5. Integrability for $\tilde{D}$ and $\tilde{E}$ quivers. In this section, we construct reduced cluster maps (12) and find log-canonical Poisson structures for the reduced variables (13). By examining the Poisson subalgebras generated by the J_n, and in some cases K_n, we find enough commuting first integrals, defined in terms of these periodic quantities, to prove the integrability of the reduced systems. We do this for $\tilde{D}_N$ with N odd and for each $\tilde{E}$-type system. The proof for $\tilde{D}_N$ with N even, however, was too complicated so the problem remains open.

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5.1. Integrability for $\tilde{D}_N$. The B matrix for the $\tilde{D}$-type diagrams, oriented as above, is

$$B := \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & & & & & \\ 0 & 0 & 1 & 0 & 0 & 0 & & & & & \\ -1 & -1 & 0 & -1 & 0 & 0 & & & & & \\ 0 & 0 & 1 & 0 & 1 & 0 & & & & & \\ 0 & 0 & 0 & -1 & 0 & -1 & \ddots & & & & \\ & & & & & \ddots & \ddots & & & & \\ & & & & & & & \pm 1 & 0 & \pm 1 & 0 & 0 \\ & & & & & & & 0 & \mp 1 & 0 & \mp 1 & \mp 1 \\ & & & & & & & 0 & 0 & \pm 1 & 0 & 0 \\ & & & & & & & 0 & 0 & \pm 1 & 0 & 0 \end{pmatrix}.$$

Where the signs depend on the parity of N. The first two rows show that B is singular, so we first project to a lower dimensional space. Since the new coordinates are given in terms of the image of B, which is different for different parities of N, we will deal with the two cases separately.

Since the Poisson bracket on the reduced variables commutes with the shift operator φ^*, calculating the brackets between variables with the same subscripts is straightforward. When calculating the brackets between J variables, however, different subscripts will appear.

5.2. The N odd case. Let $\{e_i\}$ be standard basis vectors. Here, the kernel of B is spanned by $e_1 - e_2$ and $e_N - e_{N+1}$, and the image is spanned by

$$e_1 + e_2, \quad e_3, \quad e_4, \quad \ldots \quad , e_{N-1}, \quad e_N + e_{N+1}.$$

We take reduced variables $p_n := X_n^1 X_n^2$, $q_n := X_n^N X_n^{N+1}$ and leave $X_n^3, \ldots X_n^{N-1}$ fixed. This gives a reduced cluster map

$$\varphi : \begin{pmatrix} p_n \\ X_n^3 \\ \vdots \\ X_n^{N-1} \\ q_n \end{pmatrix} \mapsto \begin{pmatrix} p_{n+1} \\ X_{n+1}^3 \\ \vdots \\ X_{n+1}^{N-1} \\ q_{n+1} \end{pmatrix}, \tag{48}$$

where the X_{n+1}^i are defined as in (19) and (21) for $i = 4, \ldots, N-2$. The remaining relations are

$$p_{n+1} = \frac{1}{p_n}\left(1 + X_{n+1}^3\right)^2, \qquad X_{n+1}^3 = \frac{1}{X_n^3}\left(1 + p_n X_n^4\right),$$
$$q_{n+1} = \frac{1}{q_n}\left(1 + X_n^{N-1}\right)^2, \qquad X_{n+1}^{N-1} = \frac{1}{X_n^{N-1}}\left(1 + q_{n+1} X_{n+1}^{N-2}\right).$$

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Our goal is to prove the integrability of the map (48). As mentioned at the end of Section 2.3, the kernel and image of B are used to define A:

$$A := \begin{pmatrix} e_1 + e_2 \\ e_3 \\ \vdots \\ e_{N-1} \\ e_N + e_{N+1} \\ e_1 - e_2 \\ e_N - e_{N+1} \end{pmatrix},$$

which gives $\hat{B}$ and then $C = (c_{ij})$, the Poisson matrix for the reduced variables:

$$c_{ij} = \begin{cases} (-1)^{(j-i+1)/2} & j \text{ even, } i \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

This gives the Poisson bracket, as in (13), between y variables, where

$$y_n^1 := p_n, \quad y_n^2 := X_n^3, \quad y_n^3 := X_n^4, \quad \dots \quad y_n^{N-2} := X_n^{N-1}, \quad y_n^{N-1} := q_n.$$

For our calculations, however, we will stick to using X, p, and q. Since each J_n is preserved by scaling (10) for each kernel vector of B, we can express them in terms of the reduced variables, hence we can calculate brackets between the J_n.

THEOREM 5.1. *The Poisson structure for the J variables, defined in Lemma 4.3, is given by*

$$\{J_0, J_{-k}\} = (-1)^k J_0 J_{-k} + \delta_{1,k} - \delta_{N-3,k}, \tag{49}$$

for $k = 1, \dots, N-3$, where $\delta_{i,j}$ is the Kronecker delta. We can apply the shift φ^*, a Poisson algebra homomorphism, to this relation to obtain the brackets between the remaining J variables.

Proof. We use (25) to express J_n in plenty of different ways, i.e., for each row

$$J_n = \frac{1^{\text{st}} \text{ entry} + 3^{\text{rd}} \text{ entry}}{2^{\text{nd}} \text{ entry}}.$$

Shifting these expressions by appropriate amounts allows us to write each J_n in terms of reduced variables with subscript 0 or 1, simplifying calculations. We have

$$J_0 = \frac{p_1 + X_0^4}{X_1^3}, \qquad J_{\frac{-N+3}{2}} = \frac{X_1^{N-2} + q_0}{X_0^{N-1}},$$

and for $k = 1, \dots, \frac{N-5}{2}$

$$J_{-k} = \frac{X_1^{2k+2} + X_0^{2k+4}}{X_1^{2k+3}}.$$

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We first calculate

$$\{J_0, J_{-k}\} = \left\{\frac{p_1 + X_0^4}{X_1^3}, \frac{X_1^{2k+2} + X_0^{2k+4}}{X_1^{2k+3}}\right\} = \frac{1}{X_1^3 X_1^{2k+3}}\{p_1 + X_0^4, X_1^{2k+2} + X_0^{2k+4}\}$$
$$- \frac{J_{-k}}{X_1^3 X_1^{2k+3}}\{p_1 + X_0^4, X_1^{2k+3}\} - \frac{J_0}{X_1^{2k+3} X_1^3}\{X_1^3, X_1^{2k+2} + X_0^{2k+4}\}.$$

The issue now is calculating the brackets between elements with unequal subscripts. For example,

$$\{p_1, X_0^{2k+4}\} = \left\{p_1, \frac{1 + X_1^{2k+3} X_1^{2k+5}}{X_1^{2k+4}}\right\} = \frac{1}{X_1^{2k+4}}\{p_1, X_1^{2k+3} X_1^{2k+5}\}$$
$$= \frac{X_1^{2k+3}}{X_1^{2k+4}}\{p_1, X_1^{2k+5}\} + \frac{X_1^{2k+5}}{X_1^{2k+4}}\{p_1, X_1^{2k+3}\} = 0.$$

After some work, we have

$$\{J_0, J_{-k}\} = \delta_{k,1} + (-1)^k J_0 J_{-k},$$

while the bracket

$$\left\{J_0, J_{\frac{N-3}{2}}\right\} = (-1)^{(N-3)/2} J_0 J_{\frac{N-3}{2}},$$

is found similarly. □

THEOREM 5.2. *The reduced cluster map (48) is Liouville integrable.*

Proof. Equation (49) can be written as the sum of the two Poisson brackets

$$\{J_0, J_{-k}\}_0 := \delta_{1,k} - \delta_{N-3,k}, \qquad \{J_0, J_{-k}\}_2 := (-1)^k J_0 J_{-k}.$$

This bi-Hamiltonian structure [15] allows for the construction of $(N-1)/2$ commuting first integrals. The Poisson subalgebra (49) is the same, up to scaling, as that found for the $\tilde{A}_{1,N}$ quivers in [10], where the authors give a more complete proof. See [9] for further details. □

5.3. The case $N = 6$. For general even N, the calculations were too much to bear. Instead, we will work with an example with $N = 6$. The B matrix and our choice of A are

$$B = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}.$$

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This gives the Poisson matrix

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Our reduced variables are

$$y_n^1 := p_n := X_n^1 X_n^2 X_n^4, \quad y_n^2 := X_n^3, \quad y_n^3 := X_n^5, \quad y_n^4 := q_n := X_n^4 X_n^6 X_n^7,$$

giving a reduced cluster map

$$\varphi : \begin{pmatrix} p_n \\ X_n^3 \\ X_n^5 \\ q_n \end{pmatrix} \mapsto \begin{pmatrix} p_{n+1} \\ X_{n+1}^3 \\ X_{n+1}^5 \\ q_{n+1} \end{pmatrix}, \tag{50}$$

where

$$p_{n+1} = \frac{1}{p_n}\left(1 + X_{n+1}^3\right)^2 \left(1 + X_{n+1}^3 X_{n+1}^5\right), \qquad X_{n+1}^3 = \frac{1}{X_n^3}(1 + p_n),$$
$$X_{n+1}^5 = \frac{1}{X_n^5} = (1 + q_n), \qquad q_{n+1} = \frac{1}{q_n}\left(1 + X_{n+1}^3 X_{n+1}^5\right)\left(1 + X_{n+1}^5\right)^2.$$

Now we can express the J variables as, for example,

$$J_n = \frac{X_{n+1}^1 X_{n+1}^2 + X_n^4}{X_{n+1}^3}, \qquad J_{n-1} = \frac{X_{n+1}^4 + X_n^6 X_n^7}{X_{n+1}^5}. \tag{51}$$

The scaling (10), for the kernel vector $(1, 0, 0, -1, 0, 0, 1)^T$, acts as

$$X_{n+1}^1 \mapsto \lambda X_{n+1}^1, \quad X_{n+1}^4 \mapsto \lambda^{-1} X_{n+1}^4, \quad X_{n+1}^7 \mapsto \lambda X_{n+1}^7$$

and fixes the remaining X_{n+1}^i. From the expression $X_{n+1}^4 X_n^4 = 1 + X_{n+1}^3 X_{n+1}^5$ and that of J_n in (51), we see that

$$X_n^4 \mapsto \lambda X_n^4, \qquad J_n \mapsto \lambda J_n.$$

Since J_n is not preserved by this scaling, it cannot be written in terms of the reduced variables. However, one sees from the second expression in (51), that J_{n-1} transforms to $\lambda^{-1} J_{n-1}$. We instead define $J'_n := J_n J_{n-1}$ and look at the Poisson subalgebra generated by $\{J'_{n-i}\}_{i=0}^3$. One can check that these generators are preserved by the scaling associated with the other two kernel vectors of B, so can be written in terms of the reduced variables. Indeed, for example, we have

$$J'_n = \frac{1}{X_{n+1}^3 X_{n+1}^5}\left(p_{n+1} + q_n + 1 + X_{n+1}^3 X_{n+1}^5 + \frac{p_{n+1}}{q_{n+1}}(1 + X_{n+1}^5)^2\right).$$

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We find similar expressions for the other J' and rewrite so that the same subscript appears throughout. With the aid of a computer, we arrive at

$$\{J'_0, J'_{-1}\} = -J'_0 J'_{-1} + J'_0 + J'_{-1}, \qquad \{J'_0, J'_{-2}\} = J'_{-3} - J'_{-1}.$$

We remark that this Poisson subalgebra structure also appears in the six-dimensional reduced system from the $\tilde{E}_7$ quiver. Here however, unlike the $\tilde{E}_7$ case, the J_n generated enough independent, commuting first integrals to prove integrability:

$$J'_0 + J'_1 + J'_2 + J'_3, \qquad J'_0 J'_1 J'_2 J'_3.$$

5.4. Integrability for $\tilde{E}_6$. For the $\tilde{E}_6$ quiver, we have the B matrix

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

For which we can take image vectors

$$e_1 + e_3, e_2, e_3 + e_5, e_4, e_3 + e_6, e_7,$$

giving reduced coordinates

$$y_n^1 := a_n c_n, \qquad y_n^2 := b_n, \qquad y_n^3 := c_n e_n,$$
$$y_n^4 := d_n, \qquad y_n^5 := c_n g_n, \qquad y_n^6 := f_n.$$

The reduced cluster map

$$\varphi : \begin{pmatrix} y_n^1 \\ y_n^2 \\ \vdots \\ y_n^6 \end{pmatrix} \mapsto \begin{pmatrix} y_{n+1}^1 \\ y_{n+1}^2 \\ \vdots \\ y_{n+1}^6 \end{pmatrix} \tag{52}$$

is given by

$$y_{n+1}^1 = \frac{1}{y_n^1}\left(1 + y_n^2\right)\left(1 + y_n^2 y_n^4 y_n^6\right), \qquad y_{n+1}^2 = \frac{1}{y_n^2}\left(1 + y_{n+1}^1\right),$$
$$y_{n+1}^3 = \frac{1}{y_n^3}\left(1 + y_n^2 y_n^4 y_n^6\right)\left(1 + y_n^4\right) \qquad y_{n+1}^4 = \frac{1}{y_n^4}\left(1 + y_{n+1}^3\right),$$
$$y_{n+1}^5 = \frac{1}{y_n^5}\left(1 + y_n^2 y_n^4 y_n^6\right)\left(1 + y_n^6\right), \qquad y_{n+1}^6 = \frac{1}{y_n^6}\left(1 + y_{n+1}^5\right).$$

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The image vectors and a kernel vector $(e_1 - e_3 + e_5 + e_7)$ are used as a basis for the matrix A. We may then calculate the Poisson matrix, after scaling,

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

We can explicitly write J_0 as

$$J_0 = \frac{y_0^1 y_0^5 + y_0^1 y_0^6 + y_0^1 + y_0^5 + (1 + y_0^6)(y_0^2 y_0^4 y_0^6 + 1)}{y_0^2 y_0^5 y_0^6}$$

and find similar expressions for each of $J_1, J_2, \tilde{J}_0, \tilde{J}_1, \tilde{J}_2$ from the S_3 action. The first row of the Poisson matrix is

$$\{J_0, J_1\} = J_0 J_1 - 1, \qquad \{J_0, J_2\} = -J_0 J_2 + 1, \qquad \{J_0, \tilde{J}_i\} = 0,$$

for $i = 1, 2, 3$. From this, we can find each bracket, noting that the tilde is an automorphism. The algebra is the direct sum of the subalgebras generated by the J_i and the $\tilde{J}_i$, and these summands are isomorphic. We take independent commuting first integrals

$$J_0 + J_1 + J_2, \qquad J_0 J_1 J_2, \qquad \tilde{J}_0 + \tilde{J}_1 + \tilde{J}_2,$$

proving the integrability of (52).

5.5. Integrability for $\tilde{E}_7$. In this case, we have the B matrix

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Our choice of image vectors is

$$e_1 + e_3,\ e_2,\ e_3 + e_5 + e_6,\ e_4,\ e_6 + e_8,\ e_7,$$

and reduced coordinates

$$y_n^1 := a_n c_n, \qquad y_n^2 := b_n, \qquad y_n^3 := c_n e_n f_n,$$
$$y_n^4 := d_n, \qquad y_n^5 := f_n h_n, \qquad y_n^6 := g_n.$$

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Here, the reduced cluster map is given by

$$y^1_{n+1} = \frac{1}{y^1_n}\left(1+y^2_n\right)\left(1+y^2_n y^4_n\right), \qquad y^2_{n+1} = \frac{1}{y^2_n}\left(1+y^1_{n+1}\right),$$
$$y^3_{n+1} = \frac{1}{y^3_n}\left(1+y^2_n y^4_n\right)\left(1+y^4_n\right)\left(1+y^4_n y^6_n\right), \qquad y^4_{n+1} = \frac{1}{y^4_n}\left(1+y^3_{n+1}\right),$$
$$y^5_{n+1} = \frac{1}{y^5_n}\left(1+y^4_n y^6_n\right)\left(1+y^6_n\right), \qquad y^6_{n+1} = \frac{1}{y^6_n}\left(1+y^5_{n+1}\right).$$

The matrix B has kernel vectors $e_1 - e_3 + e_6 - e_8$ and $e_1 - e_3 + e_5$, so we construct A as usual and find

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

The kernel vectors give the scaling symmetries

$$\lambda_1 : \begin{pmatrix} a_n \\ b_n \\ c_n \\ d_n \\ e_n \\ f_n \\ g_n \\ h_n \end{pmatrix} \mapsto \begin{pmatrix} \lambda^{(-1^n)} a_n \\ b_n \\ \lambda^{(-1^{n+1})} c_n \\ d_n \\ e_n \\ \lambda^{(-1^n)} f_n \\ g_n \\ \lambda^{(-1^{n+1})} h_n \end{pmatrix}, \qquad \lambda_2 : \begin{pmatrix} a_n \\ b_n \\ c_n \\ d_n \\ e_n \\ f_n \\ g_n \\ h_n \end{pmatrix} \mapsto \begin{pmatrix} \lambda^{(-1^n)} a_n \\ b_n \\ \lambda^{(-1^{n+1})} c_n \\ d_n \\ \lambda^{(-1^n)} e_n \\ f_n \\ g_n \\ h_n \end{pmatrix}.$$

Our periodic quantities are fixed by λ_1 but not by λ_2. We have

$$\lambda_2 : \begin{pmatrix} J_0 \\ J_1 \\ J_2 \\ J_3 \end{pmatrix} \mapsto \begin{pmatrix} \lambda J_0 \\ \lambda^{-1} J_1 \\ \lambda J_2 \\ \lambda^{-1} J_3 \end{pmatrix},$$

so we can instead take $J'_i := J_i J_{i+1}$ for $i = 0, 1, 2$ with each J'_i fixed under λ_1 and λ_2. We have, for example,

$$J'_0 = J_0 J_1 = \frac{1}{y^2_0 y^3_0 (y^4_0)^2 y^5_0 y^6_0}\left(y^1_0 y^3_0 + y^1_0 y^4_0 y^6_0 + y^3_0 + y^1_0 + (y^4_0 y^6_0 + 1)(y^2_0 y^4_0 + 1)\right)$$
$$\times \left(y^4_0 (y^4_0 ((y^6_0)^2 + y^6_0) + y^5_0 + (y^6_0)^2 + 2y^6_0 + 1) + (y^5_0 + y^6_0 + 1)(y^3_0 + 1)\right).$$

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Now we can calculate

$$\{J_0', J_1'\} = J_0' J_1' - J_0' - J_1', \qquad \{J_0', J_2'\} = J_1' - J_3',$$

from which the other brackets follow. We have commuting first integrals

$$J_0' + J_1' + J_2' + J_3', \qquad J_0' J_2' + J_1' J_3'.$$

Similar calculations give that the first integral $K_0 + K_1 + K_2$ is independent of, and commutes with, the previous two. This proves integrability of the reduced cluster map for $\tilde{E}_7$.

5.6. Integrability for $\tilde{E}_8$. Here the B matrix is

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

with image vectors

$$e_1 + e_3,\ e_2,\ e_3 + e_5,\ e_4,\ e_5 + e_7 + e_8,\ e_6,\ e_8,\ e_9,$$

and kernel vector $e_1 - e_3 + e_5 - e_7$. These give the matrix

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

The reduced coordinates are

$$y_n^1 := a_n c_n, \qquad y_n^2 := b_n, \qquad y_n^3 := c_n e_n, \qquad y_n^4 := d_n,$$
$$y_n^5 := e_n g_n h_n, \qquad y_n^6 := f_n, \qquad y_n^7 := h_n, \qquad y_n^8 := i_n,$$

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with reduced cluster map φ, given by

$$y^1_{n+1} = \frac{1}{y^1_n}\left(1+y^2_n\right)\left(1+y^2_n y^4_n\right), \quad y^2_{n+1} = \frac{1}{y^2_n}\left(1+y^1_{n+1}\right),$$
$$y^3_{n+1} = \frac{1}{y^3_n}\left(1+y^2_n y^4_n\right)\left(1+y^4_n y^6_n\right), \qquad y^4_{n+1} = \frac{1}{y^4_n}\left(1+y^3_{n+1}\right),$$
$$y^5_{n+1} = \frac{1}{y^5_n}\left(1+y^4_n y^6_n\right)\left(1+y^6_n\right)\left(1+y^6_n y^8_n\right), \qquad y^6_{n+1} = \frac{1}{y^6_n}\left(1+y^5_{n+1}\right),$$
$$y^7_{n+1} = \frac{1}{y^7_n}\left(1+y^6_n y^8_n\right), \qquad y^8_{n+1} = \frac{1}{y^8_n}\left(1+y^7_{n+1}\right).$$

The single kernel vector generates the scaling

$$\lambda : \begin{pmatrix} a_n \\ b_n \\ c_n \\ d_n \\ e_n \\ f_n \\ g_n \\ h_n \\ i_n \end{pmatrix} \mapsto \begin{pmatrix} \lambda^{(-1^n)} a_n \\ b_n \\ \lambda^{(-1^{n+1})} c_n \\ d_n \\ \lambda^{(-1^n)} e_n \\ f_n \\ \lambda^{(-1^n)} g_n \\ h_n \\ i_n \end{pmatrix},$$

for which the period 5 quantity

$$J_n = \frac{a_{n+12} + a_n}{a_{n+6}},$$

is fixed. Again we compute the brackets

$$\{J_0, J_1\} = J_0 J_1 - 1, \qquad \{J_0, J_2\} = -J_0 J_2$$

and find the three commuting first integrals

$$J_0 + J_1 + J_2 + J_3 + J_4, \quad J_0 J_1 J_2 J_3 J_4, \quad J_0 J_1 J_2 + J_1 J_2 J_3 + J_2 J_3 J_4 + J_3 J_4 J_0 + J_4 J_0 J_1.$$

From the period 3 quantities, we construct a final first integral $K_0 + K_1 + K_2$ which commutes with the other three, proving integrability.

ACKNOWLEDGMENTS. This work was supported by EPSRC grant EP/M004333/1.

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