Math. Proc. Camb. Phil. Soc. (1992), **111**, 299
Printed in Great Britain

The characterization of orders for regular functions

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(*Received* 5 *March* 1991; *revised* 13 *May* 1991)

1. *Introduction*

For a given function f, regular in the unit disk $D(0,1)$, logarithmic means of order p may be defined in terms of integrals by the formulae

$$M(p,r,f) = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |\log|f(re^{i\theta})||^p \, d\theta\right\}^{1/p} \tag{1·1}$$

for $0 < p < \infty$, and by

$$M(\infty,r,f) = \sup_{|z|=r} \log|f(z)| \tag{1·2}$$

when $0 < r < 1$. Associated p-orders are subsequently defined by

$$\lambda(p,f) = \limsup_{r\to 1} \frac{\log M(p,r,f)}{\log(1/(1-r))} \tag{1·3}$$

for $0 < p \leqslant \infty$.

Well-known classical inequalities ([**4**], pp. 12–15) show that

(I) $\lambda(p,f)$ is a monotonic increasing function of p;

(II) $p\lambda(p,f)$ is convex with respect to p on $(0,\infty)$.

Thus if f has finite order then

$$\frac{t(\lambda(p,f)-\lambda(t,f))}{q(p-t)} \leqslant \frac{\lambda(q,f)-\lambda(p,f)}{q-p} \leqslant \frac{s(\lambda(s,f)-\lambda(q,f))}{p(s-q)} \tag{1·4}$$

for $0 < t < p < q < s$. By taking limits as $t \to p$ and $s \to q$ we obtain

$$\frac{p}{q}\lambda'(p,f) \leqslant \frac{\lambda(q,f)-\lambda(p,f)}{q-p} \leqslant \frac{q}{p}\lambda'(q,f) \tag{1·5}$$

for $0 < p < q < \infty$. Here $\lambda'(p,f)$ may be interpreted either as a right- or left-hand derivative: both exist for all p in $(0,\infty)$, and they are equal to each other except for at most a countable set of values p. In what follows it matters little which interpretation we give to $\lambda'(p,f)$ but to be definite we make the particular choice of right-hand derivative.

It is easily shown ([**2**], p. 19) that $\lambda(1,f)$, equivalent to the Nevanlinna order of f, and $\lambda(\infty,f)$, the maximum modulus order of f, satisfy the inequalities

$$\lambda(1,f) \leqslant \lambda(\infty,f) \leqslant \lambda(1,f)+1,$$

and these inequalities are best possible. Further, when $\lambda(\infty,f) \geqslant 1$, it can be shown ([**3**], and also implicit in [**1**]) that

$$\lambda(\infty,f) = \lim_{p\to\infty} \lambda(p,f)$$

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and $$p(\lambda(\infty,f)-\lambda(p,f)) \leqslant 1 \quad (p > 0).$$

In this case it follows, from (1·5), that

$$p\lambda'(p,f) \leqslant \lambda(\infty,f)-\lambda(p,f),$$

and

(III) $p^2\lambda'(p,f) \leqslant 1$ $(p > 0)$.

It is the intention to show here that, when $\lambda(1,f)+\lambda'(1,f) \geqslant 1$, the conditions (I), (II), and (III) characterize $\lambda(p,f)$ on the interval $[1,\infty)$. The formal statement of this result is as follows.

THEOREM 1. *Let $\mu(p)$ be a monotonic increasing function for which $p\mu(p)$ is convex and*

$$p^2\mu'(p) \leqslant 1 \tag{1·6}$$

when $p \geqslant 1$. Then there is a function f regular in $D(0,1)$ such that $\lambda(p,f) = \mu(p)$ whenever $\mu(p)+p\mu'(p) \geqslant 1$ and $p \geqslant 1$.

The convexity condition shows that $\mu(p)+p\mu'(p)$ increases as p increases. Hence the assumption $\mu(1)+\mu'(1) \geqslant 1$ would imply our conclusion for all p in $[1,\infty)$. Further, since $p^2\mu'(p)$ increases as p increases (see Lemma 4), (1·6) could be replaced by the condition $\lim_{p\to\infty} p^2\mu'(p) \leqslant 1$.

We observe, also, that $p^2\mu'(p)$ has range contained in $[0,1]$. In the extreme case where $p^2\mu'(p) \equiv 1$ for $1 \leqslant p < \infty$ integration gives $\mu(p) = \mu(\infty)-p^{-1}$, and if $\mu(\infty) \geqslant 1$ it is easily shown that for any positive constant C

$$f(z) = \exp\{-C(1-z)^{-\mu(\infty)}\} \quad (|z| < 1)$$

satisfies $\lambda(p,f) = \mu(\infty)-p^{-1} = \mu(p)$ for $p \geqslant 1$. In the other extreme case, where $p^2\mu'(p) \equiv 0$ for $p \geqslant 1$, we have $\mu(p) = \mu(\infty)$ for $p \geqslant 1$. Examples of canonical products can be generated to satisfy $\lambda(p,f) = \mu(p) \geqslant 1$ in this case; see [**2**].

Our method of proving Theorem 1 depends on the construction of suitable canonical products B satisfying the condition $\lambda(\infty,B) \geqslant 1$, and interpolating $\lambda(p,B)$ from $\lambda(\infty,B)$ and $\lambda(1,B)$ when $p \geqslant 1$. It is this interpolation that requires the conditions $\mu(p)+p\mu'(p) \geqslant 1$ and $p \geqslant 1$ in Theorem 1.

2. *Canonical products with basic order properties*

In this section we find p-orders for a certain type of canonical product with zeros having a uniform sort of distribution over prescribed arcs of a set of circles with centre at 0. The basic order properties are given as follows.

LEMMA 1. *Let $\alpha \geqslant 1$, $0 \leqslant \beta \leqslant 1$, $A > 0$ and $0 < r_0 < 1$. Then there exists a function B regular in $D(0,1)$ such that*

$$\lambda(p,B) = \lambda(p) = \alpha+(\beta-1)/p \tag{2·1}$$

for $1 \leqslant p < \infty$, while $|B(z)-1| < A$ for $|z| \leqslant r_0$.

The cases where $\beta = 0$ and $\beta = 1$ are covered by the special examples mentioned in Section 1. For other values of β the result will be verified by considering a sequence $\{a_m\}$ of complex numbers $(1-2^{-k})e^{ij2^{-k}}$, each counted $[2^{\alpha k}]$ times for $k \geqslant k_0 \geqslant 1$,

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$0 \leqslant j \leqslant 2^{\beta k}$ for a constant k_0 which is specified below. This sequence defines a canonical product (see Tsuji[5], v, §10)

$$B(z,\{a_m\}) = B(z) = \prod_m \left(1-\frac{1-|a_m|^2}{1-z\bar{a}_m}\right)\exp\sum_{j=1}^{q}\frac{1}{j}\left(\frac{1-|a_m|^2}{1-z\bar{a}_m}\right)^j,$$

where q is the least positive integer satisfying $q > \alpha+\beta-1$.

We first note ([2], theorem v) that

$$\lambda(\infty,B) = \alpha$$

and, since the Nevanlinna counting function $n(r,B)$ satisfies

$$(2(1-r))^{-\alpha-\beta} \leqslant n(r,B) \leqslant 2(1-r)^{-\alpha-\beta}, \tag{2·2}$$

then

$$\lambda(1,B) = \alpha+\beta-1. \tag{2·3}$$

It remains to interpolate $\lambda(p,B)$ appropriately on $(1,\infty)$. A suitable inequality is easily obtained in one direction since the convexity property (1·4) yields

$$\lambda(p,B) \leqslant \frac{\lambda(1,B)}{p}+\frac{\lambda(\infty,B)(p-1)}{p} = \alpha+\frac{\beta-1}{p} \tag{2·4}$$

for $p \geqslant 1$ and all positive integers k_0.

The main remaining task will be to show that equality holds in (2·4). However, before doing this, we note that

$$\frac{1-|a_m|^2}{|1-z\bar{a}_m|} < \frac{2(1-|a_m|)}{1-r_0}$$

for $|z| \leqslant r_0 < 1$. Thus, if we choose k_0 initially so that

$$\frac{2^{-k_0}}{1-r_0} < \tfrac{1}{4},$$

then

$$|\log|B(z)|| < \sum_{k=k_0}^{\infty}\frac{2^{(\alpha+\beta-q-1)k+q+2}}{(q+1)(1-r_0)^{q+1}} = \frac{2^{(\alpha+\beta-q-1)k_0+q+2}}{(q+1)(1-r_0)^{q+1}(1-2^{\alpha+\beta-q-1})}$$

for $|z| \leqslant r_0$. It follows that for the given number A

$$|B(z)-1| < A$$

for a suitably chosen value of k_0.

Next we make some estimates regarding the integral

$$\int_{-\pi}^{\pi}\log|B(re^{i\theta})|\,d\theta.$$

The property (2·2) can be used with the Poisson–Jensen formula to show that for any given positive number ϵ we can find r' in $(0,1)$ so that

$$\int_{-\pi}^{\pi}\log|B(re^{i\theta})|\,d\theta < (1-r)^{1-\alpha-\beta-\epsilon}, \tag{2·5}$$

$$\int_{-\pi}^{\pi}\log|B(re^{i\theta})|\,d\theta > (1-r)^{1-\alpha-\beta+\epsilon} \tag{2·6}$$

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for $r' \leqslant r < 1$. These two inequalities give a rough estimate of the size of the integral on $(r', 1)$. We show that in such estimates the main contribution to the magnitude of the integral arises from only a small part of the range of integration. In doing this we suppose that $0 < \epsilon < \frac{1}{2}(\alpha+\beta-1)$, and then choose δ so that

$$\frac{\epsilon(1-\beta)}{\alpha+\beta-1-\epsilon} < \delta < 1-\beta. \tag{2·7}$$

For any given r in $(\frac{1}{2}, 1)$ let $G(r)$ denote the set $\{\theta : |\theta| \leqslant (1-r)^{1-\beta-\delta}\}$.

In the first instance suppose that $k\,(\geqslant k_0)$ satisfies

$$2^{k(\beta-1)+1} \geqslant |\theta| > 0. \tag{2·8}$$

Put $z = re^{i\theta}$ and $a_m = r_m e^{i\theta_m} = (1-2^{-k})e^{ij2^{-k}}$ for given m and associated values of j and k. Then

$$|1-z\bar{a}_m|^2 = (1-rr_m)^2 + 4rr_m \sin^2(\tfrac{1}{2}(\theta - j2^{-k})) > 2^{-2k} + \pi^{-2}(\theta - j2^{-k})^2,$$

which leads to

$$\frac{1-|a_m|^2}{|1-z\bar{a}_m|} < \frac{2}{\{1+\pi^{-2}(\theta 2^k - j)^2\}^{\frac{1}{2}}}.$$

Let $B_1(z)$ denote the subproduct of $B(z)$ associated with those values of k for which (2·8) holds. A theorem of Tsuji ([5], p. 222) yields

$$\log|B_1(re^{i\theta})| < K \sum_{2^{k(\beta-1)+1} \geqslant |\theta|} 2^{k\alpha} \sum_{0 \leqslant j \leqslant 2^{k\beta}} (1+j^2)^{-(q+1)/2} < K|\theta|^{-\alpha/(1-\beta)} \tag{2·9}$$

for some constant K depending on α and β. As here, the symbol K will also be used later to denote a constant which depends on parameters indicated, but note that the particular value ascribed to K may vary from time to time.

Secondly, if (2·8) does not hold for a given non-zero θ then

$$|1-z\bar{a}_m|^2 > (1-rr_m)^2 + \left(\frac{\theta}{2\pi}\right)^2,$$

and

$$\log|B/B_1(re^{i\theta})| < K \sum_{2^{k(\beta-1)+1} < |\theta|} 2^{(\alpha+\beta)k} \left(\frac{2^{-k}}{|\theta|}\right)^{q+1} < K \sum_{2^{k(\beta-1)+1} < |\theta|} 2^{(\alpha+\beta-q-1)k} |\theta|^{-q-1}$$
$$< K|\theta|^{((q+1-\alpha-\beta)/(1-\beta))-q-1} = K|\theta|^{(\beta q-\alpha)/(1-\beta)}. \tag{2·10}$$

Since $\beta q \geqslant 0$ the inequalities (2·9) and (2·10) show that

$$\int_{[-\pi,\pi]\setminus G(r)} \log|B(re^{i\theta})|\,d\theta < K(1-r)^{(1-\alpha-\beta)(1-\delta/(1-\beta))}. \tag{2·11}$$

However (2·7) implies that

$$(\alpha+\beta-1)(1-\delta/(1-\beta)) < \alpha+\beta-1-\epsilon,$$

so that (2·6) yields

$$\int_{G(r)} \log|B(re^{i\theta})|\,d\theta > K(1-r)^{(1-\alpha-\beta+\epsilon)}$$

when $r' \leqslant r < 1$. On the other hand, Hölder's inequality leads to

$$\int_{G(r)} \log|B(re^{i\theta})|\,d\theta < 2(1-r)^{(1-\beta-\delta)(1-p^{-1})} \left(\int_{G(r)} |\log|B(re^{i\theta})||^p\,d\theta\right)^{p^{-1}}$$
$$\leqslant 2(1-r)^{(1-\beta-\delta)(1-p^{-1})} M(p, r, B),$$

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and we deduce
$$\lambda(p,B)-(1-\beta-\delta)(1-p^{-1}) \geqslant \alpha+\beta-1-\epsilon$$
for all δ and ϵ that satisfy (2·7). As $\epsilon \to 0$ and $\delta \to 0$ we find that
$$\lambda(p,B) \geqslant \alpha+(\beta-1)/p$$
which, together with (2·4), completes the proof of Lemma 1.

3. *Further preliminaries to the proof of Theorem* 1

The proof of Theorem 1 will be effected by combining functions of the type considered in Lemma 1. We collect here some elementary results which will be useful for our further calculations. The first can be obtained by standard application of the Hölder inequality and is stated thus.

LEMMA 2. *If f and g are regular in $D(0,1)$ then the product fg has p-order* $\sup\{\lambda(p,f),\lambda(p,g)\}$ *whenever the latter two orders are different.*

Secondly, the definition of p-order indicates that for a given function f regular and of finite order in $D(0,1)$, and $\epsilon > 0$, there exists $r_0(\epsilon,p)$ in $(0,1)$ such that
$$M(p,r,f) < (1-r)^{-\lambda(p,f)-\epsilon} \tag{3·1}$$
for $r_0(\epsilon,p) \leqslant r < 1$. The following lemma shows that (3·1) can be applied uniformly over bounded sets of values p.

LEMMA 3. *Let f be regular in $D(0,1)$ with $\lambda(\infty,f)$ finite. Then given $\epsilon > 0$ and ρ in $(0,\infty)$ there is a number $R_0(\epsilon,\rho)$ in $(0,1)$ such that* (3·1) *holds for $R_0(\epsilon,\rho) \leqslant r < 1$ and $0 \leqslant p < \rho$.*

Without loss of generality we may suppose that $0 < \epsilon < 1$, and set $\rho_0 = 0$ and $\rho_n = (1-\frac{1}{2}\epsilon)^{1-n}$ for $n \in \mathbb{N}$. Then $\rho_N \leqslant \rho < \rho_{N+1}$ for some N in $\mathbb{N}$. If $1 \leqslant p < \rho$ then $\rho_n \leqslant p < \rho_{n+1}$ for some n in $\{0,1,\ldots,N\}$. Since $M(p,r,f)$ is an increasing function of p on $(0,\infty)$ it follows that we can find $R(\epsilon,\rho_{n+1})$ such that
$$M(p,r,f) \leqslant M(\rho_{n+1},r,f) \leqslant (1-r)^{-\lambda(\rho_{n+1},f)-\frac{1}{2}\epsilon} \tag{3·2}$$
for $R(\epsilon,\rho_{n+1}) \leqslant r < 1$. But the integration of (III) gives
$$\lambda(\rho_{n+1},f) \leqslant \lambda(\rho_n,f)+\frac{1}{\rho_n}-\frac{1}{\rho_{n+1}} = \lambda(\rho_n,f)+\tfrac{1}{2}\epsilon(1-\tfrac{1}{2}\epsilon)^{n-1} \leqslant \lambda(p,f)+\tfrac{1}{2}\epsilon.$$
Hence, we have, by (3·2),
$$M(p,r,f) \leqslant (1-r)^{-\lambda(p,f)-\epsilon}$$
when $1 > r \geqslant \sup\{R(\epsilon,\rho_{n+1}) : 0 \leqslant n \leqslant N\}$, and Lemma 3 is proved.

Finally we note some simple properties of the function μ as specified in Theorem 1, properties which apply to p-orders generally and some of which we have noted previously in that context.

LEMMA 4. *If μ satisfies the conditions of Theorem* 1 *then $p^2\mu'(p)$ and $\mu(p)+p\mu'(p)$ are monotonic increasing functions of p, and*
$$\tau(q) = \mu(q)+q\mu'(q)\left(1-\frac{q}{p}\right) \tag{3·3}$$
defines a decreasing function of q for $q \geqslant p$.

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As in Section 1, where we considered $\lambda(p,f)$, the convexity of $p\mu(p)$ implies that

$$\frac{p}{q}\mu'(p) \leqslant \frac{\mu(q)-\mu(p)}{q-p} \leqslant \frac{q\mu'(q)}{p} \quad (0 < p < q < \infty). \tag{3·4}$$

The monotonicity of $p^2\mu'(p)$ follows immediately, and the monotonicity of $\mu(p)+p\mu'(p)$ is subsequently obtained from the inequality

$$\mu(q)-\mu(p) \geqslant p\mu'(p)-\frac{p^2\mu'(p)}{q} \quad (q > p).$$

The proof of the lemma is completed by noting that $Q > q \geqslant p$ implies

$$\tau(Q)-\tau(q) = \mu(Q)-\mu(q)+Q\mu'(Q)\left(1-\frac{Q}{p}\right)-q\mu'(q)\left(1-\frac{q}{p}\right)$$

$$\leqslant \mu(Q)-\mu(q)+Q\mu'(Q)\left(1-\frac{Q}{q}\right) \leqslant 0.$$

4·1. *The proof of Theorem* 1 *when* $\mu(p)+p\mu'(p) > 1$. In this section we construct a function

$$f(z) = \prod_{n=1}^{\infty} B_n(z),$$

where $\{B_n(z)\}$ is a sequence of products obtained by application of Lemma 1.

Suppose that $\mu(p)+p\mu'(p) > 1$ on a subinterval (a, ∞) of $(1, \infty)$, and let $\{p_n\}$ be a sequence of numbers dense in (a, ∞). Let r_1 be any given number in $(0, 1)$, and let ϵ_1 satisfy

$$0 < \epsilon_1 < \inf\{1, \tfrac{1}{2}(\mu(p_1)+p_1\mu'(p_1)-1)\}. \tag{4·1}$$

Then Lemma 1 shows that there is a function $B_1(z)$ for which

$$|B_1(z)-1| < \tfrac{1}{2} \quad (|z| \leqslant r_1),$$

while

$$\lambda(p, B_1) = \mu(p_1)+p_1\mu'(p_1)\left(1-\frac{p_1}{p}\right)-2\epsilon_1 \quad (p \geqslant 1). \tag{4·2}$$

An application of Lemma 3 shows that there is a number R_1 in $(r_1, 1)$ such that

$$M(p, r, B_1) < (1-r)^{-\lambda(p, B_1)-\epsilon_1} \quad (R_1 \leqslant r < 1) \tag{4·3}$$

for $a \leqslant p \leqslant a+p_1$, and for this R_1 we can find a number r_2 in $(R_1, 1)$ such that

$$M(p_1, r_2, B_1) > (1-r_2)^{-\lambda(p_1, B_1)+\epsilon_1}. \tag{4·4}$$

Next let

$$0 < \epsilon_2 < \inf\{\tfrac{1}{4}\epsilon_1, \tfrac{1}{2}(\mu(p_2)+p_2\mu'(p_2)-1)\}.$$

Then there exists a product $B_2(z)$ such that

$$|B_2(z)-1| < \frac{1}{2^2} \quad (|z| \leqslant r_2),$$

and

$$\lambda(p, B_2) = \mu(p_2)+p_2\mu'(p_2)\left(1-\frac{p_2}{p}\right)-2\epsilon_2 \quad (p \geqslant 1). \tag{4·5}$$

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Note that the right-hand sides of (4·2) and (4·5) are unequal unless

$$p(\mu(p_1)+p_1\mu'(p_1)-2\epsilon_1-\mu(p_2)-p_2\mu'(p_2)+2\epsilon_2) = p_1^2\mu'(p_1)-p_2^2\mu'(p_2). \tag{4·6}$$

If $p_1^2\mu'(p_1) \neq p_2^2\mu'(p_2)$ there is at most one solution p to (4·6). On the other hand, if $p_1^2\mu'(p_1) = p_2^2\mu'(p_2)$ then

$$t^2\mu'(t) = p_1^2\mu'(p_1)$$

for all t between p_1 and p_2. So integration gives

$$\mu(p_2)-\mu(p_1) = p_1\mu'(p_1)\left(1-\frac{p_1}{p_2}\right) = p_1\mu'(p_1)-p_2\mu'(p_2),$$

which shows that (4·6) has only the solution $p = 0$ in this case, and this is not being considered here.

Lemma 2 shows that $F_2 = B_1B_2$ satisfies

$$\lambda(p, F_2) = \sup(\lambda(p, B_1), \lambda(p, B_2))$$

for $p \geqslant 1$. Thus we can find R_2 in $(r_2, 1)$ and r_3 in $(R_2, 1)$ so that

$$M(p, r, F_2) < (1-r)^{-\lambda(p, F_2)-\epsilon_2} \quad (R_2 \leqslant r < 1)$$

for $a \leqslant p \leqslant 2a+p_2$, and

$$M(p_2, r_3, F_2) > (1-r_3)^{-\lambda(p_2, F_2)+\epsilon_2}.$$

We proceed, by induction, to construct sequences of products $\{B_n(z)\}$ and $\{F_n(z)\}$, and monotonic sequences $\{r_n\}$, $\{R_n\}$ and $\{\epsilon_n\}$ in $(0, 1)$, as follows. For given $B_1(z), B_2(z), \ldots, B_{n-1}(z)$, r_n and ϵ_{n-1} choose ϵ_n so that

$$0 < \epsilon_n < \inf\{\tfrac{1}{4}\epsilon_{n-1}, \tfrac{1}{2}(\mu(p_n)+p_n\mu'(p_n)-1)\}.$$

Then, by Lemma 1, we obtain a product $B_n(z)$ for which

$$|B_n(z)-1| < \frac{1}{2^n} \quad (|z| \leqslant r_n), \tag{4·7}$$

and

$$\lambda(p, B_n) = \mu(p_n)+p_n\mu'(p_n)\left(1-\frac{p_n}{p}\right)-2\epsilon_n \quad (p \geqslant 1).$$

As in the case $n = 2$ it can be shown that the product $F_n(z) = \prod_{m=1}^{n} B_m(z)$ satisfies

$$\lambda(p, F_n) = \sup\{\lambda(p, B_m) : 1 \leqslant m \leqslant n\}.$$

Consequently there exist numbers R_n in $(r_n, 1)$ and r_{n+1} in $(R_n, 1)$ for which

$$M(p, r, F_n) < (1-r)^{-\lambda(p, F_n)-\epsilon_n} \quad (R_n \leqslant r < 1)$$

for $a \leqslant p \leqslant na+p_n$, and

$$M(p_n, r_{n+1}, F_n) > (1-r_{n+1})^{-\lambda(p_n, F_n)+\epsilon_n}. \tag{4.8}$$

The function

$$f(z) = \prod_{m=1}^{\infty} B_m(z) \tag{4·9}$$

will be shown to have all the required properties.

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The condition (4·7), for $n \in \mathbb{N}$, shows that $f(z)$ converges uniformly on all compact subsets of $D(0, 1)$. Hence $f(z)$ is regular in $D(0, 1)$. Suppose that $p \in (a, \infty)$. Choose any k such that $k > p$, and suppose that $R_k \leqslant r < R_{k+1}$. Then, since $p < ka + p_k$ we have

$$M(p, r, F_k) < (1-r)^{-\lambda(p, F_k)-\epsilon_k} = (1-r)^{-\lambda(p, B_q)-\epsilon_k} \tag{4·10}$$

for some q in $\{1, 2, \ldots, k\}$. But since $\epsilon_k \leqslant \epsilon_q$ it follows that

$$\lambda(p, B_q)+\epsilon_k = \mu(p_q)+p_q\mu'(p_q)\left(1-\frac{p_q}{p}\right)-2\epsilon_q+\epsilon_k \leqslant \mu(p)-\epsilon_q \tag{4·11}$$

by Lemma 4. Further, (4·7) and our qualification of r implies that

$$0 < \prod_{m=1}^{\infty}\left(1-\frac{1}{2^m}\right) \leqslant \prod_{m=k+1}^{\infty}\left(1-\frac{1}{2^m}\right) < |f(z)/F_k(z)| < \prod_{m=k+1}^{\infty}\left(1+\frac{1}{2^m}\right) \leqslant \prod_{m=1}^{\infty}\left(1+\frac{1}{2^m}\right). \tag{4·12}$$

The inequalities (4·10) and (4·11) together with the inequality of Minkowski now give

$$M(p, r, f) \leqslant M(p, r, F_k)+M\left(p, r, \frac{f}{F_k}\right) < (1-r)^{-\mu(p)}+K$$

for $R_k \leqslant r < R_{k+1}$ and $k > p$. It follows that

$$\lambda(p, f) \leqslant \mu(p) \tag{4·13}$$

for $p \geqslant a$.

To obtain the complementary inequality to (4·13), let us fix p in (a, ∞) and consider a subsequence of $\{p_n\}$ which increases strictly to p and satisfies the inequality $\mu(p)-\mu(p_n) < 1/n$ for $n = 1, 2, 3, \ldots$. Let p_N be in this subsequence. Then

$$M(p, r_{N+1}, F_N) \geqslant M(p_N, r_{N+1}, F_N) > (1-r_{N+1})^{-\lambda(p_N, F_N)+\epsilon_N},$$

where

$$\lambda(p_N, F_N) \geqslant \lambda(p_N, B_N) = \mu(p_N)-2\epsilon_N,$$

$\lim_{N\to\infty}\mu(p_N) = \mu(p)$, and $\lim_{N\to\infty}\epsilon_N = 0$. Note, also that (4·12) can be extended to give

$$0 < \prod_{m=1}^{\infty}\left(1-\frac{1}{2^m}\right) \leqslant |f(z)/F_N(z)| \leqslant \prod_{m=1}^{\infty}\left(1+\frac{1}{2^m}\right)$$

for $|z| = r_{N+1}$. Hence, the Minkowski inequality gives

$$M(p, r_{N+1}, f) \geqslant (1-r_{N+1})^{-\mu(p_N)+3\epsilon_N}-K \geqslant (1-r_{N+1})^{-\mu(p)+3\epsilon_N+1/N}-K$$

for $N \in \mathbb{N}$. Thus $\lambda(p, f) \geqslant \mu(p)$ follows for $p \geqslant a$, and the proof of Theorem 1 is complete in this case.

4·2. *The proof of Theorem* 1 *when* $\mu(p)+p\mu'(p) = 1$. We have verified Theorem 1 in those cases where $\mu(p)+p\mu'(p) > 1$ and $p > a$. The continuity of $\mu(p)$ and $\lambda(p, f)$ shows that the result holds also when $p = a$. We have, therefore, to consider cases where

$$\mu(p)+p\mu'(p) = 1 \quad (b \leqslant p < a) \tag{4·14}$$

for some number b in $[1, \infty)$.

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The integration of (4·14) gives

$$\mu(p) = 1 - \frac{a(1-\mu(a))}{p} \quad (b \leqslant p \leqslant a). \tag{4·15}$$

We compare this with the p-orders of the function f constructed in Section 4·1 when $b \leqslant p \leqslant a$. For that function f we have

$$\lambda(p,f) = \sup_{p_n \geqslant a} \left(\mu(p_n) + p_n \mu'(p_n) \left(1 - \frac{p_n}{p}\right) - 2\epsilon_n \right) \quad (p \geqslant 1),$$

where $\{p_n\}$ is dense on (a, ∞). By Lemma 4 the terms

$$\mu(p_n) + p_n \mu'(p_n) \left(1 - \frac{p_n}{p}\right)$$

decrease as p_n increases when $a \geqslant p \geqslant 1$. Hence

$$\lambda(p,f) = \mu(a) + a\mu'(a) \left(1 - \frac{a}{p}\right) \quad (a \geqslant p \geqslant 1). \tag{4·16}$$

Recall that $\mu'(a)$ is a right-hand derivative.

If $\mu(a) + a\mu'(a) = 1$ then (4·15) and (4·16) yield $\lambda(p,f) = \mu(p)$ for $b \leqslant p \leqslant a$, and the proof of Theorem 1 is complete.

In the case where $\mu(a) + a\mu'(a) > 1$ we have

$$\mu(p) - \lambda(p,f) = \left(1 - \frac{a}{p}\right)(1 - \mu(a) - a\mu'(a)) > 0 \quad (b \leqslant p < a).$$

Here we apply Lemma 1 to obtain a function ϕ satisfying

$$\lambda(p, \phi) = 1 - \frac{a(1-\mu(a))}{p} = \mu(p) \quad (b \leqslant p \leqslant a),$$

and hence, by Lemma 2,

$$\lambda(p, \phi f) = \mu(p) \quad (p \geqslant b).$$

The proof of Theorem 1 is complete in all cases.

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https://doi.org/10.1017/S0305004100075393 Published online by Cambridge University Press