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MODAL STRUCTURALISM AND REFLECTION

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Abstract. Modal structuralism promises an interpretation of set theory that avoids commitment to abstracta. This article investigates its underlying assumptions. In the first part, I start by highlighting some shortcomings of the standard axiomatisation of modal structuralism, and propose a new axiomatisation I call MSST (for *Modal Structural Set Theory*). The main theorem is that MSST interprets exactly Zermelo set theory plus the claim that every set is in some inaccessible rank of the cumulative hierarchy. In the second part of the article, I look at the prospects for supplementing MSST with a modal structural *reflection principle*, as suggested in Hellman (2015). I show that Hellman's principle is inconsistent (Theorem 5.32), and argue that modal structural reflection principles in general are either incompatible with modal structuralism or extremely weak.

§1. Introduction. What counts as evidence for a mathematical statement? This is a central question in the philosophy of mathematics. On some accounts, it looks like nothing could count as enough evidence to justify the statements of accepted mathematics. After all, those statements appear to be about abstract objects, disconnected from us in space and time.[1] A common response to this problem is to deny that mathematics is about abstracta after all. *Modal structuralism*—the view that mathematics is about logically possible *structures*—is one of the leading examples of this response.[2]

In set theory, a structure is a pair of sets: one set as its *domain* together with another set of ordered pairs as its *relation*. This is not the notion employed by the modal structuralist, however, since sets are abstract objects. Rather, they use the resources of *plural quantification* and *mereology* to define a similar notion without appeal to abstracta. The thought is that a structure can consist of some things as its domain together with some mereological fusions that behave suitably like ordered pairs as its relation.[3] Moreover, it is natural to think that neither pluralities nor fusions incur ontological commitments over and above the things they are pluralities and fusions of.[4] If that is right, then structures constituted by

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[1] This is the *Benacerraf problem*. See Benacerraf (1973), and Clarke-Doane (2017) for an illuminating and up-to-date discussion.

[2] The locus classicus for modal structuralism is Hellman (1989), with Hellman (1996) adding plural quantification and mereology. Putnam (1967) was the first to suggest the general strategy.

[3] To make things easier, I will frequently talk of possible objects, pluralities, structures, and worlds. For example, I will say "there is a possible plurality containing an object o ..." instead of the strictly correct "there could have been some things such that o is among them ...". Nothing I say will depend on misspeaking in this way, and can always be reformulated using the primitive modal operator, plural quantification, and mereology introduced below.

[4] See Hellman (1996) and Lewis (1991).

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nonabstract objects will also be nonabstract. The core idea of modal structuralism is that this is indeed right and that mathematics is about logically possible structures constituted by nonabstract objects.

To support this, the modal structuralist provides a systematic translation of mathematical statements, which appear to be about abstracta, as statements merely about possible structures. For example, the claim that there is a non-self-membered set is translated as the claim that there could have been a structure containing an object x in its domain such that the mereological fusion coding the ordered pair $\langle x, x\rangle$ is not contained in its relation. To ensure that the statements of accepted mathematics come out true under the translation, the modal structuralist restricts their attention to a particular class of structures: namely, those satisfying the axioms of second-order Zermelo–Fraenkel set theory with the axiom of choice (ZFC2).[5]

Avoiding abstracta may be necessary to solve the epistemological problem we started with, but it is not sufficient. That problem also arises for the modal structural translations: it is not obvious that possible ZFC2 structures are more epistemically tractable than the abstract objects they are used to avoid.[6] As a first step to assessing the evidence for the translations of accepted mathematics, we have to get clear on the assumptions needed to prove them. That will be the primary goal of this article.

Here's the plan. In §2, I start by outlining the standard axioms of modal structuralism. I show that they fail to interpret even the logical axioms of set theory (Theorem 5.40), and propose a new axiomatisation I call MSST (for *Modal Structural Set Theory*). I show that MSST exactly interprets Zermelo set theory plus the claim that every set is in some inaccessible rank of the cumulative hierarchy the main theorem. An immediate upshot is that MSST fails to interpret the axiom schema of Collection of ZFC (Lemma 5.13). In §3, I look at the prospects for supplementing MSST with a modal structural *reflection principle*, as suggested in Hellman (2015). I show that Hellman's principle is inconsistent (Theorem 5.32), and argue that modal structural reflection principles in general are either incompatible with modal structuralism or extremely weak. §5 is a technical appendix.

§2. Axiomatising modal structuralism.

2.1. The language. The modal structuralist wants to interpret set theory using logically possible structures satisfying the axioms of ZFC2, where a structure is a pair of pluralities: some things as a domain together with some mereological fusions that behave suitably like ordered pairs as a relation. Their language will thus have to contain a modal operator, $\Diamond$, expressing logical possibility, the usual resources of first-order logic, and suitable plural and mereological resources. In this article, I will use capital letters $X, Y, Z \ldots$ etc to range over pluralities; $x \in X$ to express that x is among the Xs; and first-order terms $\langle x, y\rangle$ for the ordered pair of x and y.[7] I will also take the claim that pluralities X and Y are identical, $X = Y$, to be well-formed. Let $\mathcal{L}_\Diamond$ denote this language.

[5] See §5.1 for a definition of ZFC2.

[6] See, for example, Hale (1996).

[7] I have taken ordered pairing as a primitive because it allows for a simpler and more general theory. My results can then be extended to a wider range of approaches to coding ordered pairs. In fact, it will turn out that even more minimal resources will do. My results go through when the language contains just the primitive relation $\langle x, y\rangle \in X$ for ordered pairs. See the remarks after the lower bound theorem in §5.3.3.

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2.2. Structures and satisfaction. A *structure* will simply be any pluralities X, Y, where X is taken as its domain and Y its relation.[8] For simplicity, and where it won't cause confusion, I will identify the structure X, Y with its relation Y, write $\mathsf{dom}(Y)$ for X, and write $x \in Y$ for $x \in \mathsf{dom}(Y)$. The notion of satisfaction in a structure is straightforward. For formulas φ in the language of second-order set theory, $\mathcal{L}^2_\in$,[9] we say that Y *satisfies* φ (in symbols, $Y \vDash \varphi$) just in case φ is true when its membership relation is interpreted according to Y, its first-order quantifiers are interpreted as ranging over $\mathsf{dom}(Y)$, and its second-order quantifiers are interpreted as ranging over the subpluralities of $\mathsf{dom}(Y)$. Formally: $Y \vDash \varphi$ abbreviates the result of replacing each occurrence of $x \in y$ in φ with $\langle x, y\rangle \in Y$, each occurrence of $\exists x$ with $\exists x \in \mathsf{dom}(Y)$, and each occurrence of $\exists X$ with $\exists X \subseteq \mathsf{dom}(Y)$ (where $X \subseteq Y$ abbreviates $\forall x(x \in X \rightarrow x \in Y)$). I will use variables M, M', M'', etc for structures satisfying $\mathsf{ZFC2}$.

2.3. The translation schema. The modal structural translation schema provides a way to systematically interpret claims about sets as claims about possible $\mathsf{ZFC2}$ structures. It is motivated by a now standard result in ZFC: a set-theoretic structure satisfies $\mathsf{ZFC2}$ just in case it is isomorphic to some V_α, for α an inaccessible cardinal.[10,11] Each $\mathsf{ZFC2}$ structure thus contains isomorphic copies of all and only the sets in some such V_α.[12] Moreover, we can show that if there are arbitrarily large V_α, for α inaccessible, then any $\mathsf{ZFC2}$ structure can be extended to contain an isomorphic copy of any particular set. This means that we can talk about the sets via their isomorphic copies in $\mathsf{ZFC2}$ structures. In particular, suppose that $\vec{x}$ are isomorphic copies in M of some sets $\vec{y}$. Then, if we want to say that there is a $\varphi(\vec{y})$, we just say that M can be extended to a $\mathsf{ZFC2}$ structure M' containing a $\varphi(\vec{x})$.[13] The translation schema attempts to replicate this using possible $\mathsf{ZFC2}$ structures and thereby capture talk that is ostensibly about sets without appeal to sets.

DEFINITION 2.1. *Y' is an end-extension of Y, $Y' \sqsupseteq Y$, if Y is a transitive substructure of Y'. Formally: $Y' \sqsupseteq Y$ if (i) EY, Y'; (ii) $Y \subseteq Y'$; (iii) for any $x, y \in \mathsf{dom}(Y)$, $\langle x, y\rangle \in Y$ iff $\langle x, y\rangle \in Y'$; and (iv) for any $x \in \mathsf{dom}(Y)$ and $y \in \mathsf{dom}(Y')$, if $\langle y, x\rangle \in Y'$, then $\langle y, x\rangle \in Y$ (where EX abbreviates $\exists Z(Z = X)$).*[14]

DEFINITION 2.2. *Let $^{pt}_{Y}$ be the following translation from the language of first-order set theory, $\mathcal{L}_\in$,[15] to $\mathcal{L}_\Diamond$.*[16]

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[8] Whether or not Y contains nonpairs, it can play the role of a relation in virtue of the ordered pairs it does contain: Y relates x and y just in case $\langle x, y\rangle \in Y$.

[9] See §5.1 for the definition of $\mathcal{L}^2_\in$.

[10] See Zermelo (1996) and Theorem 6 of Uzquiano (1999). The V_α's are defined by transfinite recursion on the ordinals: $V_0 = \emptyset$, $V_{\alpha+1} = \mathcal{P}(V_\alpha)$, and $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ (where $\mathcal{P}(x)$ is the powerset of x, and λ is a limit cardinal). An ordinal α is *inaccessible* if it is uncountable and regular, and $2^\beta < \alpha$ whenever $\beta < \alpha$.

[11] For simplicity, I will use "structure" for the notion of structure in first-order set theory, second-order set theory, and modal structuralism. Context will make clear which is intended.

[12] Formally, we can say that element x of some structure $\langle D, R\rangle$ is an *isomorphic copy* of a set y if the structure we get by restricting $\langle D, R\rangle$ to the elements in x's transitive closure according to $\langle D, R\rangle$ is isomorphic to the membership relation on y's transitive closure.

[13] See Lemma 5.29 for a precise statement and proof.

[14] Similarly, for Ex and $E\langle x, y\rangle$.

[15] See §5.1 for the definition of $\mathcal{L}_\in$.

[16] This translation closely follows the semantics given in Hellman (1989, p. 76). The "*pt*" stands for "Putnam translation", since it was first outlined in Putnam (1967), with structures satisfying

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- $(x = y)^{pt}_Y = x = y$
- $(x \in y)^{pt}_Y = Y \vDash x \in y$
- $^{pt}_Y$ *commutes with the connectives*
- $(\exists x \varphi)^{pt}_Y = \Diamond \exists M \sqsupseteq Y \exists x \in M \varphi^{pt}_M$ (*making sure to avoid clashes of variables*).[17]

When φ is a sentence, I will let φ^{pt} denote $\varphi^{pt}_\emptyset$ and call it the *ms-translation* of φ (where $\emptyset$ is the empty plurality).[18]

2.4. The standard theory. Given the language of modal structuralism, its theory will have to consist of four components: a modal logic, general axioms governing pluralities and ordered pairs, and specific axioms governing ZFC2 structures. I will now outline the standard articulation of these components.[19]

2.4.1. Logic. The logic of modal structuralism is a positive free version of S5 modal logic. This is just the modal logic sound and complete for Kripke models with variable domains and a universal accessibility relation.[20]

2.4.2. Pluralities. The general axioms governing pluralities are the instances of a comprehension schema which says that every condition determines a plurality. Formally:

$$\exists X \forall x (x \in X \leftrightarrow \varphi), \qquad \text{(comp)}$$

where $\varphi \in \mathcal{L}_\Diamond$ and X is not free in φ.[21]

2.4.3. Pairs. There are two general axioms governing ordered pairs. The first is a defining axiom: it says that ordered pairs $\langle x, y \rangle$ and $\langle x', y' \rangle$ are equal just in case $x = x'$

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second-order Zermelo set theory replacing those satisfying ZFC2. See §2.8.4 for a discussion of the use of structures that don't satisfy ZFC2.

17 It is helpful to contrast this with the modal structural translation schema used for arithmetic. In ZFC, a structure satisfies the axioms of second-order arithmetic (PA2) just in case it is isomorphic to the natural numbers. So, any PA2 structure contains isomorphic copies of all and only the natural numbers. This means that instead of talking about the natural numbers directly, we can talk about their isomorphic copies in any or all PA2 structures. For this reason, a simpler translation schema is used: namely, $\varphi^{tr} = \Box \forall Y (Y \vDash PA2 \rightarrow Y \vDash \varphi)$. In principle, this kind of translation is available for set theory. In ZFC2, there *are* structures isomorphic to the sets: trivially, the sets together with their membership relation is such a structure. Moreover, Zermelo's (1930) results extend to show that in ZFC2, a structure satisfies ZFC2 just in case it is *either* isomorphic to the sets or to some V_α, for α inaccessible. This can then be used to provide a characterisation of the structures isomorphic to the sets: they are exactly the ZFC2 structures that cannot be end-extended by other ZFC2 structures. Call these *maximal* ZFC2 structures. The modal structuralist could thus translate claims about the sets as claims about what is true in any or all possible maximal ZFC2 structures. However, as we will see in §2.4.4, they have good reason to deny that there could have been maximal ZFC2 structures.

18 The axioms below guarantee that an empty plurality necessarily exists, and that there is at most one possible empty plurality. More precisely, comp implies $\Box \exists X \forall x (x \notin X)$, and PL1 and PL2 imply that $\Box \forall X (\forall x (x \notin X) \rightarrow \Box \forall Y (\forall x (x \notin Y) \rightarrow X = Y))$. It is therefore legitimate, given those axioms, to definitionally expand $\mathcal{L}_\Diamond$ with $\emptyset$.

19 They can be found in Hellman (1989) and Hellman (2005).

20 See §5.2.1 for an explicit version of the logic, and §2.8.2 for why the modal structuralist needs a free logic. See Hughes & Cresswell (1996) Chapter 16 for the soundness and completeness results.

21 For simplicity, I will use "comp" to denote this comprehension schema in various languages. It will be clear from context which is intended.

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and $y = y'$. The second is an existence axiom: it says that the pair of x and y exists whenever x and y exist.[22] Formally:

$$\forall x, x', y, y'(\langle x, y\rangle = \langle x', y'\rangle \leftrightarrow (x = x') \wedge (y = y')) \tag{P1}$$

$$\forall x, y E\langle x, y\rangle. \tag{P2}$$

Arguably, these axioms are false when $\Diamond$ expresses logical possibility. After all, P1 and P2 jointly imply that there are infinitely many objects if there are at least two,[23] and it seems logically possible that there be exactly three objects. Nonetheless, they are harmless. The modal structuralist can simply restrict their attention to worlds where P1 and P2 hold: without loss, they can read claims of the form $\Diamond\varphi$ as $\Diamond(\mathsf{P1} \wedge \mathsf{P2} \wedge \varphi)$. On this reading, P1 and P2 become necessary, and it is straightforward to check that the rest of the theory remains as plausible as it was on the original reading.

2.4.4. Structures. There are two axioms governing the existence and behaviour of ZFC2 structures. The first says that there could have been at least one ZFC2 structure. Formally:

EXISTENCE (E)

$$\Diamond\exists M(M = M).$$

It is easy to see that E is equivalent to the ms-translation of the claim that there is at least one set: formally, $\Diamond\exists M\exists x \in M(x = x)$. So, E is non-negotiable.

The second axiom embodies the modal structuralist's response to paradox. Briefly, we can see the set-theoretic paradoxes as arising from a tension between two plausible claims: namely, that any condition determines a plurality and that any plurality determines a set. In other words, it can be seen as a tension between comp and:

$$\forall X\exists x(x \equiv X), \tag{collapse}$$

where $x \equiv X$ abbreviates $\forall y(y \in x \leftrightarrow y \in X)$.[24] As usual, by considering a plurality of all and only the non-self-membered sets, we are quickly led to a contradiction. The modal structuralist proposes to resolve this tension by first observing that comp in $\mathcal{L}_\Diamond$ *is* consistent with a natural modal structural analogue of collapse: namely, that any subplurality of any possible ZFC2 structure *could* have determined a set in some end-extension. Formally:

THE EXTENDABILITY PRINCIPLE (EP).

$$\Box\forall M\forall X \subseteq M\Diamond\exists M' \sqsupseteq M\exists x \in M'(M' \vDash x \equiv X).$$

[22] The mereological principles underlying these axioms are those of classical mereology together with the claim that there are infinitely many mereological *atoms* (that is, objects with no proper parts). See (Hellman, 2005, p. 554–555).

[23] *Proof:* Let $x_0, \ldots, x_n$ be distinct existing objects. By P2, $\langle x_i, x_i\rangle$ exists for $i \leq n$. By P1, they are all distinct. Finally, by P2, $\langle x_0, x_1\rangle$ exists, and by P1, it is distinct from each $\langle x_i, x_i\rangle$.

[24] This formulation of the paradoxes relies heavily on Linnebo (2010), but the resolution is arguably implicit in Hellman (1989, 2002), Putnam (1967), and Zermelo (1996). See Linnebo (2010) for an extended argument in favour of collapse and a similar resolution in the modal nonstructural setting.

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They then claim that our reasons for accepting collapse are at most reasons for accepting EP.[25,26]

Say that a formula φ is a *closure* of ψ if it is the result of prefixing φ with a string of universal quantifiers and necessity operators in any order. Over the modal logic, let *the standard theory* consist of comp, P1, P2, E, and EP together with their closures.

2.5. The new theory.

2.5.1. Invariance. The standard theory faces an immediate problem: its plurality and pairing axioms tell us how pluralities and pairs behave within worlds, but ms-translations concern their behaviour across worlds. In particular, the ms-translations of simple theorems of ZFC require that pluralities and pairs are *invariant* between worlds: that pluralities cannot change the things they comprise, and that pairs cannot change the things they pair. For example, consider the ms-translation of the claim that there is an empty set (formally, $\exists x \forall y (y \notin x)$):

$$\Diamond \exists M \exists x \in M \Box \forall M' \sqsupseteq M \forall y \in M' (\langle y, x \rangle \notin M'). \tag{1}$$

Now, suppose pluralities can comprise different things in different worlds. Then, M may fail to contain a pair $\langle y, x \rangle$ in some world, but contain it in another: so, x could go from being empty in M in some world to being nonempty in M in another, leading to failures of (1). Similarly, if pairs can change the things they pair.

Is this kind of invariance plausible? For pluralities, it seems to be implied by a natural conception according to which a plurality is nothing over and above the things it comprises. To see this, consider the following formalisation of that conception. It has three principles. The first says that pluralities are *sufficient* for the things they comprise: that pluralities cannot exist without them, and without continuing to comprise them. Formally:

$$x \in X \rightarrow \Box (EX \rightarrow Ex \wedge x \in X). \tag{PL1}$$

The second says that pluralities are *necessary* for the things they comprise: that individual things cannot co-exist without a plurality of them. Formally, this gives us the comprehension schema comp. Finally, there is an extensionality principle: it says that pluralities comprising the same things are identical. Formally:

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[25] Once the ms-translation schema is extended to the language of second-order set theory in §2.8.3, EP will be equivalent to the ms-translation of collapse, and the inconsistency of collapse and comp will be preserved under ms-translation. So, they will have to reject the ms-translation of some instance of comp, and consequently claim that our reasons for accepting comp do not extend to those ms-translations. This seems plausible if comp is motivated as I suggest in the next section.

[26] EP has an alternative formulation in Hellman (1989): it says that any ZFC2 structure M has a *proper* end-extension M', which is to say $M \sqsubseteq M'$ and $\mathsf{dom}(M') \not\subseteq \mathsf{dom}(M)$. Over the other axioms of the theory to be proposed below, these two formulations are equivalent. *Proof:* If we let $X = \mathsf{dom}(M)$, then the M' in EP will have to be a proper end-extension of M. Now, suppose that M' is a proper end-extension of M. In ZFC2 we can show that any transitive X satisfying ZFC2 either contains all sets or all and only the sets in some V_α. (This is a simple generalisation of Theorem 6 in Uzquiano (1999). See also (Drake, 1974, p. 112).) Since $M \sqsubseteq M'$, $\mathsf{dom}(M)$ will be transitive in M' and thus contain all and only the sets in some V_α in M', because $\mathsf{dom}(M') \not\subseteq \mathsf{dom}(M)$. Thus, all of M's subpluralities will form sets in $V_{\alpha+1}$ in M'. □ This alternative formulation was first proposed in Zermelo (1996), and independently in Putnam (1967).

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$$\Box\forall x[\Diamond(x \in X) \leftrightarrow \Diamond(x \in Y)] \rightarrow X = Y.^{27} \tag{PL2}$$

It is often claimed that mereological fusions are also nothing over and above the things they fuse. But it is unclear whether this supports the relevant kind of invariance. At a minimum, we seem to need a principle which says that if the pair $\langle x, y\rangle$ is among the Xs, then it is logically impossible that X exists without $\langle x, y\rangle$ being among the Xs. Formally:

$$\langle x, y\rangle \in X \rightarrow \Box(EX \rightarrow \langle x, y\rangle \in X). \tag{P3}$$

This essentially requires that the mereological fusions playing the role of ordered pairs cannot logically change their parts. But parthood does not appear to be a logical relation.[28] Let me briefly discuss one way around this problem.

Suppose we enrich the language of modal structuralism with the resources to cross-reference worlds: to say of things in one world what they are like in another.[29] Then, we could say of a plurality in one world that it contains the same pairs as it does in another, and thus of a structure in one world that it has the same structure it has in another. The ms-translation schema could be modified accordingly. For example, we could translate the claim that every set is contained in another set along the lines of: for any M in any world w, and any $x \in M$, there is some $M' \sqsupseteq M$ in a world w' where M has the same structure as it does in w, and M' contains a y for which $\langle x, y\rangle \in M'$. Using such a translation, P3 could be dropped.

Nonetheless, I will work with P3 as it allows for a simpler overall theory. The results I prove can then be adapted to more complicated theories and translations schemas.

How much set theory do these new axioms allow us to interpret? It turns out not very much at all. In fact, the standard theory together with PL1, PL2, and P3 fails to interpret even the logical axioms of ZFC. In particular, it fails to prove ms-translation of the logical axiom for vacuous quantification (Theorem 5.40):

$$\forall\vec{x}(\varphi \leftrightarrow \forall y\varphi), \tag{L3}$$

where $\varphi \in \mathcal{L}_{\in}$ with free variables among $\vec{x} - \{y\}$.

2.5.2. Stability. What went wrong? The problem is that different structures can have radically different kinds of end-extensions, and thus have access to radically different kinds of sets. For example, it is consistent with the theory considered so far that some M only has end-extensions containing finitely many inaccessible cardinals whereas another structure M' has end-extensions containing infinitely many. From the perspective of M, it will look like there are only finitely many inaccessible cardinals; whereas, from the perspective of M', it will look like there are infinitely many.[30]

This is precisely what the ms-translation of L3 rules out: it says that the same kinds of sets are accessible from all structures. In particular, it implies that if $(\exists x\varphi)^{pt}_{M}$ for some M,

[27] See Uzquiano (2011) and Linnebo (2013) for further discussion of the interaction between plural and modal logic, and see §5.2.2 for a discussion of this particular formulation of the extensionality principle.

[28] See Uzquiano (2014), Sid, and the articles in Cotnoir & Baxter (2014) for discussion.

[29] See, for example, Pettigrew (2012). He uses a new pair of modal operators to express the claim that the physical part of a world w has the same structure as the physical part of some other world w'.

[30] See the proof of Theorem 5.40.

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then $(\exists x\varphi)^{pt}_{M'}$ for any other possible M' when $\exists x\varphi$ is a sentence.[31] Similarly, it implies that if $(\exists x\varphi)^{pt}_{M}$ for some M, then $(\exists x\varphi)^{pt}_{M'}$ for any possible end-extension M' of M when $\exists x\varphi$'s parameters are in M. So, just as ms-translations require that pluralities and pairs are invariant between worlds, they also require that ms-translations themselves are invariant or *stable* between structures.

STABILITY (S).

$$[\forall \vec{x}(\varphi \leftrightarrow \forall y\varphi)]^{pt},$$

where $\varphi \in \mathcal{L}^2_{\in}$'s free variables are among $\vec{x} - \{y\}$.

2.5.3. Summary. The ms-translations of basic theorems impose significant constraints on the uniformity of modal space beyond the standard theory: they require that pluralities and pairs are invariant between worlds, and that ms-translations are stable between structures. But, as I will now show, imposing those constraints suffices to interpret a significant fragment of ZFC plus a large cardinal hypothesis.

2.6. The main theorem.

DEFINITION 2.3. *Let* MSST (*for Modal Structural Set Theory*) *be the standard theory together with* PL1, PL2, P3, *and* S; *let* In *be the claim that there are arbitrarily large inaccessible cardinals* (*formally,* $\forall\alpha\exists\beta > \alpha(\beta$ *is an inaccessible cardinal*)); *and, for any theory* T, *let* T* *be* T *plus the claim that every set is in some* V_α (*formally,* $\forall x\exists\alpha(x \in V_\alpha)$). Z *is* ZFC *minus the axiom schema of Collection.*[32]

MAIN THEOREM 2.4. MSST *exactly interprets* Z* + In *via ms-translation. In other words,* MSST *proves* φ^{pt} *if and only if* Z* + In *proves* φ, *for sentences* $\varphi \in \mathcal{L}_{\in}$.

It follows immediately from the main theorem that MSST proves the ms-translations of all the axioms of ZFC other than instances of the axiom schema of Collection. Standardly, those instances are classified according to their syntactic complexity: the more alternations of quantifiers, the more complex. It turns out that although Z* + In proves all instances of Collection at the lowest level of complexity, it fails to prove all instances at the very next level (Claim 3 and Lemma 5.13). In the jargon, it proves all Π_0 instances, but not all Π_1 instances.[33] So, by the main theorem, MSST proves the ms-translations of all Π_0 instances of Collection, but not the ms-translations of all Π_1 instances.

2.7. Discussion. I will now look at some questions and issues raised by the main theorem.

2.7.1. Deriving S. Can we derive S from more obvious principles? In ZFC, the analogue of S holds because any two structures co-exist: when M' contains an isomorphic copy of some set, we can use it to construct an end-extension of M also containing such a copy.[34] So, in ZFC, the same kinds of sets are accessible from all structures. But this can fail in the modal setting: it may not be possible for M to co-exist with enough objects

[31] *Proof:* Suppose EM and φ^{pt}_{M}, for φ a sentence. The ms-translation of an instance of L3 implies that if $(\neg\varphi)^{pt}$, then $\Box\forall M\forall y \in M(\neg\varphi)^{pt}_{M}$ and thus $(\neg\varphi)^{pt}_{M}$. So, φ^{pt}. By another instance, we then get that $\Box\forall M\varphi^{pt}_{M}$.

[32] See §5.1 for details, including a presentation of the axiom schema of Collection.

[33] See §5.1.

[34] See the proof of Lemma 5.29.

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to construct the relevant kind of end-extension. So, perhaps the modal structuralist should require that any two possible structures can co-exist. Formally:

$$\Box\forall M\Box\forall M'\Diamond(EM, M')^{35}. \qquad \text{(s-compossible)}$$

Indeed, s-compossible seems plausible for logical possibility: there appears to be nothing logical stopping any two structures co-existing. However, s-compossible sets a dangerous precedent for the modal structuralist. If logical possibility is permissive enough that any two possible structures can co-exist, it seems as though it should be permissive enough that all possible objects can co-exist. Formally:

$$\Diamond\exists X\Box\forall x(x \in X). \qquad \text{(u-compossible)}$$

Although u-compossible is consistent with MSST (Theorem 5.42), it is inconsistent with a natural generalisation of EP that I will argue in the next section the modal structuralist should adopt. They should thus reject u-compossible, and with it s-compossible.

Hellman suggests an alternative to s-compossible in the case of arithmetic.[36] The idea is that even though it may not be possible for two structures M and M' to co-exist, structures satisfying the same sentences can. Formally:

$$\Diamond\exists M(M \vDash \varphi) \wedge \Diamond\exists M(M \vDash \psi) \rightarrow \Diamond\exists M, M'(M \vDash \varphi \wedge M' \vDash \psi), \qquad \text{(AP)}$$

where $\varphi, \psi \in \mathcal{L}^2_\in$ are sentences. Unfortunately, AP fails to prove S over the other axioms of MSST (Theorem 5.40).[37,38]

2.7.2. Paradox. Above, I took the set-theoretic paradoxes to arise from a tension between comp and collapse. The modal structuralist proposed to resolve this tension by replacing collapse with EP, which says that any *subplurality* of a possible structure determines a set in some possible end-extension. There are, however, other natural analogues of collapse in the modal structural setting. In particular, there is the principle which says that any possible plurality *whatsoever* forms a set in some structure. Formally:

$$\Box\forall X\Diamond\exists M\exists x \in M(EX \wedge M \vDash x \equiv X). \qquad \text{(EP*)}$$

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[35] It is routine to extend the proof of the analogue of S in ZFC to show that MSST - S + s-compossible proves S.

[36] See (Hellman, 1989, p. 43) and (Hellman, 1996, p. 106).

[37] Underlying the accumulation principle is a more general principle which says that isomorphic copies of M and M' can co-exist. Formally:

$$\Box\forall M\Box\forall M'\Diamond\exists M''(\exists i : M \approx M'' \wedge \Diamond\exists M'''(\exists i : M' \approx M''' \wedge \Diamond(EM'', M'''))),$$

where $\exists i : M \approx M'$ formalises the claim that there is a plurality of ordered pairs coding an isomorphism between M and M'. This principle also fails to prove S over the other axioms of MSST (see the remarks after Theorem 5.40). Thanks to Øystein Linnebo and Leon Horsten for suggesting this way of handling isomorphisms between structures that cannot co-exist.

[38] Hellman's justification for the accumulation principle is that "anything internal to a given structure cannot conflict with anything internal to another" (Hellman, 1996, p. 106), and that it is internal to a structure which sentences it satisfies. The problem is that ms-translation is not internal to a structure: it concerns not merely the structure itself, but also its end-extensions. We might say that satisfaction is a *local* property of structures, whereas ms-translation is a *global* property. It is precisely because ms-translation is a global property of structures that S is such a substantial assumption.

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Since the usual reasons for thinking that pluralities should determine sets are not sensitive to whether they are subpluralities of structures or not,[39] the modal structuralist should also adopt EP*.

EP* implies EP over the other axioms of MSST, and goes beyond it in at least one crucial way:[40] it is inconsistent with u-compossible. This forces the modal structuralist to adopt a free logic, since u-compossible is derivable in MSST over classical logic.[41,42,43] Since EP is easier to work with and since it will not affect the main results of the article,[44] however, I will leave MSST as it is.

2.7.3. Second-order set theory. So far, I have focused on the interpretation of first-order set theory. But set theorists frequently make use of the language of second-order set theory. For example, systematic connections have been discovered between large cardinal hypotheses by reformulating them in terms of second-order functions over the universe of sets.[45] For readability, I will refer to whatever second-order variables range over as *classes*.

So, what second-order set theory can the modal structuralist interpret? To answer this question, we first need to extend the ms-translation schema to its language. The most obvious way to do this is by interpreting second-order variables as ranging over subpluralities of structures. Formally:

DEFINITION 2.5. • $(x \in X)^{pt}_Y = (x \in X)$
• $(\exists X \varphi)^{pt}_Y = \Diamond \exists M \sqsupseteq Y \exists X \subseteq M \varphi^{pt}_M$.

Unfortunately, this results in an extremely weak second-order set theory according to which classes are completely redundant. Let Z^*_2 be Z^* with its logical axioms extended to $\mathcal{L}^2_\in$. The following minimal instance of comp says that *at least* every set determines a class.

$$\forall x \exists X (x \equiv X). \qquad \text{(min-comp)}$$

We can then take collapse to say that *at most* every set determines a class. Together, collapse and min-comp say that classes and sets are equivalent: any claim we can make

[39] See, for example, the arguments in Hellman (2002) and Linnebo (2010).

[40] *Proof sketch:* Suppose EM. By EP*, M determines a set in some M'. By the plurality axioms, M will exist and continue to be a ZFC2 structure. The results of Zermelo (1996) then show that M is isomorphic to a V_α in M', and the plurality and pairing axioms can be used to construct an end-extension of M isomorphic to M'. Since all of dom(M)'s subpluralities determine sets in $V_{\alpha+1}$ in M', they will also determine sets in such an end-extension, verifying EP.

[41] *Proof:* Ex and EX are axioms of classical logic (see §5.2.1). By comp, let X be a plurality of everything, and assume $\Diamond(Ex \wedge x \notin X)$. By classical logic, Ex and thus $x \in X$. Then, by classical logic and necessitation, we have $\Box EX$, and thus $\Box(x \in X)$ by PL1, contradicting our assumption.

[42] If the move from EP to EP* can plausibly be blocked, then that would open the way to an interesting nonmodal structuralism. MSST and thus EP can be true in Kripke models with a single world (Theorem 5.42). So, the theory that results by deleting the modal operators in MSST is consistent. I am sometimes tempted to read Zermelo (1996) as proposing a nonmodal structuralism of this kind.

[43] In some places, Hellman recognises something like the distinction between EP and EP*. See, for example, (Hellman, 2005, p. 544), where he distinguishes "the extendability principle" from "the general extendability principle". But in other places, this is less clear. See, for example, (Hellman, 2011, p. 636).

[44] In particular, the upper and lower bound theorems are easily seen to hold when MSST is replaced with MSST + EP*.

[45] See Kanamori (2003) for details, and Uzquiano (2003) for discussion.

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with classes, we can make with sets. More precisely, for $\varphi \in \mathcal{L}^2_\in$, let its *first-orderisation*, φ^*, be the result of replacing its second-order variables with first-order variables. Then, collapse and min-comp are jointly equivalent to the schema:

$$\varphi \leftrightarrow \varphi^*$$

for sentences $\varphi \in \mathcal{L}^2_\in$.[46] Finally, let MSST^2 be MSST together with the stability axiom S extended to all formulas in $\mathcal{L}^2_\in$. Then:

THEOREM 2.6. MSST^2 *exactly interprets* Z^*_2 + collapse + min-comp *via ms-translation. In other words,* MSST^2 *proves* φ^{pt} *if and only if* Z^*_2+ collapse + min-comp *proves* φ*, for sentences* $\varphi \in \mathcal{L}^2_\in$.[47]

Can the modal structuralist interpret a stronger second-order set theory using some other extension of the ms-translation schema? One standard way to measure the strength of a second-order set theory is by the instances of comp it proves: the more instances it proves, the stronger it is. So, the question is: how many instances of comp can the modal structuralist interpret?

Given EP^*, pluralities will only get us min-comp. But even if we expand the language of modal structuralism with new resources, there appear to be limits on the number of instances of comp the modal structuralist can interpret. For, whatever kind of collections the modal structuralist uses to interpret second-order variables, they must be nonabstract. The problem is that the more collections there are of certain kind, the less likely they are to be nonabstract.[48,49]

2.7.4. Using ZFC2 *structures.* The ms-translation schema was motivated by the fact that in ZFC + In, truth in the ZFC2 structures is equivalent to truth in the sets. But, it turns out that this holds for many other classes of structures. For example, in ZFC, truth throughout the well-founded extensional structures is equivalent to truth in the sets (see Lemma 5.29). Consequently, the main theorem extends to a similarly broad class of theories (Corollary 5.15). Let me mention one interesting example. Let Z2 be second-order

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[46] *Proof:* Given collapse and min-comp, we show by a simple induction on the complexity of φ that $\varphi(\vec{X}, \vec{y})$ is equivalent to $\varphi^*(\vec{x}, \vec{y})$ when φ's free variables are among $\vec{X}, \vec{y}$ and $\vec{x} \equiv \vec{X}$. Moreover, it is easy to see that collapse* and min-comp* are the same trivial logical truth. So, collapse and min-comp follow immediately from the schema.

[47] See Corollary 5.34 and the remarks following it.

[48] Though see Rayo & Yablo (2001) for a dissenting voice.

[49] Hellman makes the stronger point that any nonset sized collections are likely to be abstract. He says:

> Ordinary mathematical *abstracta* seem tame compared to such extravagances [like a collection of all possible objects]; indulging them would deprive [modal structuralism] of much of its interest as a distinctive program. (Hellman, 2005, p. 554)

See also (Hellman, 1989, p. 31). The force of this point relies heavily on how we understand "collection". Using a primitive satisfaction predicate, it is straightforward to code nonset sized collections as formulas. For example, the formula "$x = x$" (understood a particular natural number in any or all possible ZFC2 structures) codes a collection of all possible objects: in any world, it satisfies every object and thus "contains" all possible objects (see Parsons, 1974). Nonetheless, this strategy is limited, and only suffices to interpret a small number of instances of comp (see Fujimoto, 2012). So, it is not the *size* of collections of some kind that gives us a reason to they are abstract, but rather the *number* of collections of that kind.

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Zermelo set theory,[50] $\mathsf{MSST}_{\mathsf{Z2}^*}$ be MSST with $\mathsf{Z2}^*$ structures replacing ZFC2 structures, and Beth be the claim that there are arbitrarily large fixed points in the enumeration of the $\beth$ cardinals, defined by $\beth_0 = \omega$, $\beth_{\alpha+1} = 2^{\beth_\alpha}$, $\beth_\lambda = \bigcup_{\alpha<\lambda} \beth_\alpha$ (formally, Beth is $\forall\alpha\exists\beta > \alpha(\beta = \beth_\beta)$).[51] Then, we have:

THEOREM 2.7. $\mathsf{MSST}_{\mathsf{Z2}^*}$ *exactly interprets* Z^* + Beth *via ms-translation. In other words,* $\mathsf{MSST}_{\mathsf{Z2}^*}$ *proves* φ^{pt} *if and only if* Z^* + Beth *proves* φ*, for sentences* $\varphi \in \mathcal{L}_\in$.[52]

This is interesting because although $\mathsf{Z2}^*$ is a much weaker theory than ZFC2, there is a precise sense in which Z^* + Beth and Z^* + In prove the same amount of ZFC. In particular, they both prove all Π_0 instances of Collection, but not all Π_1 instances (Claim 1 and Lemma 5.13). So, the main theorem and Theorem 2.7 imply that $\mathsf{MSST}_{\mathsf{ZFC2}}$ and $\mathsf{MSST}_{\mathsf{Z2}^*}$ prove the ms-translations of the same amount of ZFC.

2.7.5. Incompleteness. The primary upshot of the main theorem is that MSST fails to interpret Collection.[53] The success of modal structuralism thus depends on whether we can find and justify principles beyond MSST that imply the ms-translations of its instances. But there are also other reasons to be interested in principles beyond MSST. First, many set theorists accept most of the so-called small large cardinal hypotheses.[54] Although the main theorem shows that MSST interprets the large cardinal hypothesis In, it also shows that it interprets no stronger hypotheses.[55] Second, even if they are not part of accepted mathematics, large cardinal hypotheses are interesting in their own right. Indeed, there are a huge number of questions independent of accepted mathematics which are settled by large cardinal hypotheses. So, it is independently interesting to see whether there are well-motivated principles beyond MSST that imply their ms-translations.[56] The rest of this article will look at the prospects for interpreting Collection and large cardinal hypotheses using *reflection principles*.

§3. Reflection principles. Typically, reflection principles say that the universe of sets is indescribable: whatever is true in the sets is also true in some V_α.[57] Formally:

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[50] See §5.1 for a definition of Z2.

[51] In Z^*, the $\beth$ cardinals can be defined by $\beth_\alpha = |V_{\omega+\alpha}|$. Moreover, it is easy to see that the inaccessible cardinals are exactly the regular $\beth$ fixed points (i.e., the uncountable α for which $|V_\alpha| = \alpha$). So, in Z^*, Beth is a weakening of In.

[52] See Theorem 5.7.

[53] (Hellman, 1989, p. 78) is aware that E and EP fail to interpret Collection. In response, he proposes the following strengthened version of EP.

> Let $\varphi(x, y)$ be a formula "defining a function", where this is spelled out by writing out the [ms-translation] of the usual condition; further let a be any set in any full model such that, for any x in a, M_β is the least full model containing the unique y such that $\varphi(x, y)$. Then it is possible that there exists a common proper extension, M, of all such M_β. (p. 79, 1989)

The problem with this suggestion is that there will be many possible 'least' structures containing such a y, all isomorphic to one another, and no way to choose between them.

[54] See, for example, Maddy's contribution to Feferman, Friedman, Maddy, & Steel (2000).

[55] In particular, it is straightforward to verify that given the least upper bound κ_ω of the first ω inaccessible cardinals, V_{κ_ω} models Z^* + In. But, ZFC + In proves that κ_ω and thus V_{κ_ω} exist.

[56] See Gödel (1964) for a classic statement of this project in the non-modal-structural setting, and Koellner (2006) for an illuminating discussion in light of recent developments in set theory.

[57] See Koellner (2009) and the references therein.

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$$\varphi \rightarrow \exists \alpha \varphi^{V_\alpha}, \tag{R}$$

where φ^{V_α} formalises the claim that φ is true in V_α. We can obtain specific principles from R by specifying (i) a class of formulas for which it is to hold, and (ii) what it means for formulas in that class to be true in a V_α (that is, what φ^{V_α} means). For formulas in the language of first-order set theory, φ^{V_α} is usually taken to be the result of re-interpreting its quantifiers as ranging over V_α: that is, of replacing occurrences of $\exists x$ in φ with $\exists x \in V_\alpha$. Let R_1 denote this restriction of R. For formulas in the language of second-order set theory, it is usually taken to be the result of re-interpreting its first-order quantifiers as ranging over V_α and re-interpreting its second-order variables as ranging over subsets of V_α: that is, of replacing occurrences of $\exists x$ in φ with $\exists x \in V_\alpha$, $\exists X \psi(X)$ with $\exists x \subseteq V_\alpha \psi(x)$, and free variables X with $X \cap V_\alpha$. Let R_2 denote this restriction of R.[58]

Many find reflection principles like R_1 and R_2 compelling. Indeed, many take them to "follow from" the iterative conception of set that underlies the axioms of ZFC.[59] According to this conception, the sets occur in an unending series of stages: at each stage, there are sets of any sets occurring at some previous stage.[60] The thought is that it is part of the unending nature of the stages that whenever some claim is true, they extend far enough to make it true in some stage. Since each stage is co-extensive with a V_α, that gives us R.

Although R_1 is relatively weak,[61] R_2 is quite strong. Over Z2, it implies all instances of Collection and the existence of arbitrarily large inaccessible, Mahlo, weakly compact, and Π^1_n-indescribable cardinals.[62] Moreover, recently proposed reflection principles that generalise R_2 go much further. For example, the principle R_S in Roberts (2017) also implies the existence of arbitrarily large Ramsey, Measurable, Woodin, and 1-extendible cardinals.[63] It is therefore natural to ask whether there are modal structural versions of R_2 that are similarly strong and well-motivated.

3.1. Modal structural reflection principles. The most obvious version of R_2 in the modal structural setting is its ms-translation. However, this turns out to be inconsistent when we use the extension of the ms-translation schema from §2.7.3, where second-order variables are interpreted as ranging over subpluralities of structures. As I mentioned, that makes the ms-translation of collapse true (Theorem 2.6). But R_2 implies comp,[64] and thus its ms-translation will imply the ms-translation of comp. Moreover, as I pointed out, it is unclear in general whether the modal structuralist can interpret comp on any extension of the ms-translation schema whilst avoiding abstracta. So, they need a less obvious version of R_2.

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[58] Finding a suitable notion of truth in V_α for the language of third-order set theory has proved difficult. The most natural notion, for example, yields an inconsistent principle. See Tait (1998) and Koellner (2009) for discussion.

[59] See, for example, (Gödel, 1964, p. 258–259) footnote 16 and Tait (1998). See Paseau (2007) for discussion.

[60] See (Gödel, 1964, p. 259) and Boolos (1998).

[61] In particular, it is provable in ZFC and $\mathsf{Z}^* + \Pi_2$-Col proves there is a model of $\mathsf{Z}^* + \mathsf{R}_1$. See Lévy & Vaught (1961). The proof can be extracted from the proof of Lemma 5.38.

[62] See Bernays (1976) and Kanamori (2003) §6 for details.

[63] See Welch (2017) for an alternative generalisation of R_2 employing embeddings.

[64] *Proof:* For contradiction, suppose $\neg\exists X \forall x (x \in X \leftrightarrow \varphi)$. Then, by R_2, there would be a V_α with no subset of all and only the φ^{V_α}'s. But that is impossible. By Separation, there will always be a subset of V_α of all and only the φ^{V_α}'s.

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3.1.1. MSR. Hellman (2015) motivates an alternative version of R_2.

> The mathematical possibilities of ever larger structures are so vast as to be "indescribable": whatever condition we attempt to lay down to characterize that vastness fails in the following sense: if indeed it is accurate regarding the possibilities of mathematical structures, it is also accurate regarding a mere segment of them, where such a segment can be taken as the domain of a single structure. (p. 271, 2015)

There are two ideas here. The first is an indescribability idea: whatever is true in all possible structures is also true in a "segment" of them. I will assume for now that a segment of structures is just a suitably small collection of them, and that for φ to be true in all possible structures is for its ms-translation to be true. Then, we can formalise the idea as

$$\varphi^{pt} \to \exists \mathcal{X}(\varphi^{pt})^{\mathcal{X}}, \tag{S-indes}$$

where $(\varphi^{pt})^{\mathcal{X}}$ is the result of binding the structure quantifiers in φ^{pt} to the segment $\mathcal{X}$, and φ^{pt} is defined as in §2.7.3. The second idea is that a segment of structures $\mathcal{X}$ "can be taken as the domain of a single structure": whatever is true in $\mathcal{X}$ is also true in some particular structure. Formally:

$$(\varphi^{pt})^{\mathcal{X}} \to \Diamond \exists M(M \vDash \varphi). \tag{ident}$$

Together, S-indes and ident imply Hellman's principle:

$$\varphi^{pt} \to \Diamond \exists M(M \vDash \varphi). \tag{MSR}$$

As Hellman notes (p. 272), however, MSR is inconsistent. Just like the ms-translation of R_2, it implies the ms-translation of comp.[65]

In response, Hellman proposes a restriction of MSR "to sentences… that are consistent with… [ZFC2]" (p. 272, 2015). There are two ways to implement this restriction, corresponding to two notions of consistency: semantic and syntactic.

If φ is semantically consistent with ZFC2, i.e., $\Diamond \exists M(M \vDash \varphi)$, then: ($\mathsf{MSR}_{sem}$)

$$\varphi^{pt} \to \Diamond \exists M(M \vDash \varphi)$$

and:

If φ is syntactically consistent with ZFC2, i.e., $(\mathsf{ZFC2} \nvdash \neg\varphi)^{pt}$, then: ($\mathsf{MSR}_{syn}$)

$$\varphi^{pt} \to \Diamond \exists M(M \vDash \varphi).$$

Since it is trivially true, MSR_{sem} cannot be what Hellman has in mind. But, it turns out that MSR_{syn} is inconsistent (Theorem 5.32).

3.1.2. Saving MSR *from inconsistency.* It might be tempting at this point to look for other restrictions of MSR. But this strategy is unpromising. Any restriction should be well-motivated, and it is unclear whether there are any well-motivated restrictions of MSR that are strong and consistent. Indeed, even if MSR_{syn} were consistent, it would still have been entirely mysterious why MSR held for sentences syntactically consistent with ZFC2, but not for all sentences.

Once we give up on trying to find restrictions, it is easy to see that the problem with Hellman's suggestion is ident. Just as collapse is true in all possible ZFC2 structures, it can also be true in a segment of them. In fact, it will be true in any segment of structures

[65] Since each instance of comp is true in every possible M.

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without a greatest structure by end-extension. But, collapse is trivially false in any particular structure. So, ident is false. Nonetheless, I think S-indes suggests a crucial insight for implementing reflection in the modal structural setting. Both S-indes and R_2 are instances of a much more general indescribability idea: namely, that whatever is true in all entities of some kind, is true in a small collection of them. Call this the *general reflection principle*. For S-indes, the entities in question are structures; for R_2, they are classes and sets. In contrast, neither MSR nor the ms-translation of R_2 are instances of the general reflection principle. For example, MSR says that when φ^{pt} is true in all possible ZFC2 structures, then the distinct claim φ is true in the subpluralities and sets of some particular ZFC2 structure.[66] So, it is natural to take our question to be whether there are strong and consistent instances of the general reflection principle in the modal structural setting. To answer this question, I will start by formalising S-indes and calibrating its strength, and then move on to look at other possible instances.

3.1.3. $\mathsf{R}_\Diamond$. Formalising S-indes is just a matter of formalising the notion of a segment of structures. What constraints should we impose on such a formalisation? By analogy with the V_αs used in R_2, we might require that the segment be set-sized. Similarly, since the V_αs are transitive, we might require that the segment be downward closed under structures in the sense that whenever $M \in \mathcal{X}$ and $M' \sqsubseteq M$, then $M' \in \mathcal{X}$. The most natural way to satisfy these constraints is by taking a segment of structures to be the structures in some V_α, which in turn will be in some possible structure. So, the formalisation of S-indes will say that if the ms-translation of φ is true in all possible ZFC2 structures, then it is true in the ZFC2 structures in some V_α of some possible ZFC2 structure. For simplicity, I will further assume that V_α satisfies the claim that every set is in some transitive set satisfying ZFC2 (which I'll abbreviate $\mathsf{Trans}_{\mathsf{ZFC2}}$).[67]

For $\varphi \in \mathcal{L}_\Diamond$, let φ^* be the result of deleting φ's modal operators, and replacing its second-order variables with first-order variables. Then, we can state the principle more precisely as follows.

$$\varphi^{pt} \rightarrow \Diamond \exists M \exists \alpha \in M (M \vDash (\mathsf{Trans}_{\mathsf{ZFC2}} \wedge (\varphi^{pt})^*)^{V_\alpha}) \tag{1}$$

for sentences $\varphi \in \mathcal{L}^2_\in$. Moreover, we can extend (1) to arbitrary formulas in $\mathcal{L}^2_\in$ by relativising it to a structure.

$$\text{If } E M, \vec{Y} \subseteq M, \vec{x} \in M, \text{ and } \varphi^{pt}_M, \text{ then:} \tag{$\mathsf{R}_\Diamond$}$$

$$\Diamond \exists M' \sqsupseteq M \exists \vec{y}, z, \alpha \in M' (M' \vDash \vec{y}, z \equiv \vec{Y}, \mathsf{dom}(M) \wedge z \in V_\alpha \wedge (\mathsf{Trans}_{\mathsf{ZFC2}} \wedge (\varphi^{pt}_Z)^* (\vec{y}, z))^{V_\alpha}),$$

where $\varphi \in \mathcal{L}^2_\in$'s free variables are among $\vec{x}, \vec{Y}$.

Unfortunately, $\mathsf{R}_\Diamond$ is extremely weak. In particular, it is equivalent to the ms-translation of R_1 (Lemma 5.35). So, the main theorem extends to show that:

THEOREM 3.1. $\mathsf{MSST} + \mathsf{R}_\Diamond$ *exactly interprets* $\mathsf{Z}^* + \mathsf{In} + \mathsf{R}_1$ *via ms-translation. In other words,* $\mathsf{MSST} + \mathsf{R}_\Diamond$ *proves* φ^{pt} *if and only if* $\mathsf{Z}^* + \mathsf{In} + \mathsf{R}_1$ *proves* φ.[68]

It can be shown that $\mathsf{Z}^* + \mathsf{In} + \mathsf{R}_1$ proves all Π_1 instances of Collection, but not all Π_2 instances (Lemmas 5.37 and 5.38). So, Theorem 3.1 implies that $\mathsf{MSST} + \mathsf{R}_\Diamond$ only proves the ms-translations of instances of Collection of the two lowest levels of complexity. It can also be shown that $\mathsf{Z}^* + \mathsf{In} + \mathsf{R}_1$ proves the existence of the least upper bound of the first

[66] Of course, φ^{pt} and φ are related; but they are also very different claims.

[67] See §5.3.1 Definition 5.

[68] See Theorem 5.36.

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ω inaccessible cardinals, but not an inaccessible cardinal with arbitrarily large inaccessible cardinals below it.[69]

3.1.4. Generalising $\mathsf{R}_\Diamond$. Are there stronger instances of the general reflection principle? There is reason to think not. In particular, there is a principle that appears to subsume all such instances, but which is no stronger than $\mathsf{R}_\Diamond$. The principle says that whatever is true, is true in some possible world structure. Formally,

$$\varphi \rightarrow \Diamond \exists M \exists K \in M (M \vDash \exists w \in K (w \vDash \varphi)), \tag{2}$$

where $\varphi \in \mathcal{L}_\Diamond$ is a sentence and K is an S5 Kripke model in M. For simplicity, I will assume that K's worlds are V_αs, that plural quantifiers at a world range over its subsets, and that pairing terms are interpreted in the obvious way by set-theoretic ordered pairs. Given this assumption, it is straightforward to verify that (2) implies the corresponding instance of $\mathsf{R}_\Diamond$, because φ can be an ms-translation. In general, if we extend (2) to arbitrary formulas as we did with $\mathsf{R}_\Diamond$, then that extension will imply all instances of $\mathsf{R}_\Diamond$. Moreover, it is straightforward to modify the proof of Theorem 5.30 to show that this extension of (2) is interpretable in $\mathsf{Z}^* + \mathsf{In} + \mathsf{R}_1$. It follows that its addition to MSST would exactly interpret $\mathsf{Z}^* + \mathsf{In} + \mathsf{R}_1$, just like $\mathsf{MSST} + \mathsf{R}_\Diamond$.

3.1.5. Summary. Given the failure of Hellman's MSR_{syn}, I suggested that the prospects for reflection principles in the modal structural setting turn on whether there are strong and consistent instances of the general reflection principle. I then argued that there are not: that instances of the general reflection principle in $\mathcal{L}_\Diamond$ are extremely weak. In particular, that they fail to interpret Collection.

§4. Conclusion. Modal structuralism promises an epistemology of mathematics. The results in this article give us reason to be cautious about its success. In the first instance, they show that the standard axioms need to be supplemented with something like the stability principle S (Theorem 5.40), whose justification is unclear. Once S is added to those axioms, a significant fragment of ZFC becomes interpretable, but many instances of the axiom schema of Collection remain out of reach (the main theorem).[70] In the second instance, they show that one of the most promising ways to justify the axiom schema of Collection and many of the small large cardinal hypotheses—namely, using reflection principles—is unavailable to the modal structuralist (Theorem 3.1 and §3.1.4). Finally, they show that the translations of second-order set theories involving a large number of instances of the comprehension schema comp may simply be incompatible with modal structuralism (Theorem 2.6 and §2.7.3). Although there is little consensus among set theorists concerning second-order set theory, there is a growing interest in such theories.[71] If they become accepted, this would be a serious problem for the modal structuralist.

§5. Technical appendix. This appendix contains proofs of the results mentioned in the main text. I start with an axiomatisation of MSST. I then establish some results concerning the ms-translations provable in a broad class of theories like MSST. Finally, I establish similar results for reflection principles.

5.1. Preliminaries. The language of first-order set theory, $\mathcal{L}_\in$, has in addition to the usual resources of first-order logic, the nonlogical membership relation $\in$. It takes $x \in y$

[69] It is straightforward to verify that if κ is such a cardinal, then V_κ models ZFC + In and thus $\mathsf{Z}^* + \mathsf{In} + \mathsf{R}_1$.

[70] Moreover, these instances continue to be uninterpretable when we replace ZFC2 structures with T structures for any plausible set theory T (Lemma 5.13 and the upper bound theorem).

[71] For example, by Joel Hamkins and Victoria Gitmam.

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and $x = y$ to be well-formed. The language of second-order set theory, $\mathcal{L}^2_\in$, extends $\mathcal{L}_\in$ with second-order variables $X, Y, Z, \ldots$ etc. It takes $x \in X$ to be well-formed, although not $X = Y$.

I will use the following modified Levy hierarchy to measure the complexity of formulas in $\mathcal{L}_\in$. If φ's quantifiers are all of the form $\exists x \in y$, then it is Π_0, Σ_0, and Δ_0. If its quantifiers are all of the form $\exists x \in y$ or $\exists x \subseteq y$, then it is Π^*_0, Σ^*_0, and Δ^*_0. In general, if φ is $\Pi^{(*)}_n$, then $\exists \vec{x}\varphi$ is $\Sigma^{(*)}_{n+1}$, and if φ is $\Sigma^{(*)}_n$, then $\forall \vec{x}\varphi$ is $\Pi^{(*)}_{n+1}$. A formula is $\Pi^{(*)T}_n$, $\Sigma^{(*)T}_n$, or $\Delta^{(*)T}_n$ if it is equivalent in the theory T to a $\Pi^{(*)}_n$, $\Sigma^{(*)}_n$, or to both a $\Pi^{(*)}_n$ and a $\Sigma^{(*)}_n$, formula, respectively.[72]

Let ZFC be the $\mathcal{L}_\in$ theory consisting of Extensionality, Infinity, Pairing, Union, Power-set, Foundation, Separation, Choice,[73] and:

(Collection) $$\forall x \exists y \varphi(x, y, \vec{z}) \rightarrow \forall u \exists v (\forall x \in u)(\exists y \in v)\varphi(x, y, \vec{z}),$$

where φ's free variables are among $x, y, \vec{z}$ and where $x, y, \vec{z}, u, v$ are all distinct. $\Pi^{(*)}_n$-Col and $\Sigma^{(*)}_n$-Col denote the restriction of Collection to $\Pi^{(*)}_n$ and $\Sigma^{(*)}_n$ formulas, respectively. Let $\mathsf{ZFC2}$ denote the conjunction of the axioms of ZFC with Separation, Collection, and Foundation replaced by their second-order formulations.[74] Zermelo set theory ($\mathsf{Z(2)}$) is $\mathsf{ZFC(2)}$ minus Collection. T^* is T plus the claim that every set is in some V_α.

5.2. An axiomatisation of MSST. MSST consists of four groups of axioms: a modal logic, general axioms governing pluralities and ordered pairs, and specific axioms governing $\mathsf{ZFC2}$ structures.

5.2.1. Logic. The underlying logic of MSST is a positive free $\mathsf{S5}$ modal logic. More precisely, its axioms are the instances in $\mathcal{L}_\Diamond$ of the truth-functional tautologies, the $\mathsf{S5}$ axioms,[75] and the following quantificational and identity axioms (where $\mathbf{x}, \mathbf{y}$ are either both first- or second-order variables):

($\mathsf{L1}$) $\forall \mathbf{y}(\forall \mathbf{x}\varphi \rightarrow \varphi[\mathbf{y}/\mathbf{x}])$, where $\mathbf{y}$ is free for $\mathbf{x}$ in φ
($\mathsf{L2}$) $\forall \mathbf{x}(\varphi \rightarrow \psi) \rightarrow (\forall \mathbf{x}\varphi \rightarrow \forall \mathbf{x}\psi)$
($\mathsf{L3}$) $\varphi \leftrightarrow \forall \mathbf{x}\varphi$, where $\mathbf{x}$ is not free in φ
($\mathsf{L4}$) $\mathbf{x} = \mathbf{x}$
($\mathsf{L5}$) $\mathbf{x} = \mathbf{y} \rightarrow (\varphi[\mathbf{x}/\mathbf{z}] \leftrightarrow \varphi[\mathbf{y}/\mathbf{z}])$, where $\mathbf{x}$ and $\mathbf{y}$ are free for $\mathbf{z}$ in φ.

The rules of inference are MP, from φ and $\varphi \rightarrow \psi$ infer ψ; GEN, if φ is a theorem, then so is $\forall \mathbf{x}\varphi$; and NEC, if φ is a theorem, then so is $\Box\varphi$.

REMARK 5.1. *This version of* $\mathsf{S5}$ *is sound and complete for Kripke models with variable domains and a universal accessibility relation.*[76] *Over the truth-functional tautologies,*

[72] Throughout the appendix I will claim that various notions have a certain complexity, giving a partial or full justification where necessary. Where a partial justification is given, Kunen (2011) can be used to fill it out.

[73] I will take Choice to be the claim that every set has an enumeration. Formally, $\forall x \exists f (f$ is a function $\wedge\ \mathsf{dom}(f)$ is an ordinal $\wedge\ \mathsf{rng}(f) = x)$.

[74] The second-order formulation of Foundation is $\forall X(\exists x(x \in X) \rightarrow \exists x \in X \forall y \in X(y \notin x))$.

[75] That is: K (i.e., $\Box(\varphi \rightarrow \psi) \rightarrow \Box\varphi \rightarrow \Box\psi$)—which is valid on all Kripke models—T (i.e., $\varphi \rightarrow \Diamond\varphi$)—which corresponds to the frame condition on Kripke models that accessibility be reflexive—and 5 (i.e., $\Diamond\varphi \rightarrow \Box\Diamond\varphi$)—which corresponds to the frame condition on Kripke models that accessibility be Euclidean (i.e., if xRy and xRz, then yRz).

[76] See Hughes & Cresswell (1996) Chapter 16. One useful feature of the logic is that it allows for existential instantiation within the scope of modal operators. In particular, if ψ is provable

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A1-5 is an axiomatisation of positive free logic. Adding the schema $E\mathbf{x}$ *results in an axiomatisation of classical logic.*[77]

5.2.2. Pluralities.

(comp) $\exists X \forall x (x \in X \leftrightarrow \varphi)$

(PL1) $x \in X \rightarrow \Box(EX \rightarrow Ex \wedge x \in X)$

(PL2) $\Box \forall x [\Diamond(x \in X) \leftrightarrow \Diamond(x \in Y)] \rightarrow X = Y$

(PL3) $EX, Y \wedge X \subseteq Y \rightarrow \Box(EY \rightarrow EX \wedge X \subseteq Y)$.[78,79]

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from premises Γ and $\Diamond(E\vec{\mathbf{x}} \wedge \varphi)$ (where $\vec{\mathbf{x}}$ are not free in Γ or ψ), then ψ is provable from Γ and $\Diamond\exists\vec{\mathbf{x}}\varphi$. (*Proof:* Suppose ψ is provable from Γ and $\Diamond(E\vec{\mathbf{x}} \wedge \varphi)$ as above. By the deduction theorem, $\Diamond(E\vec{\mathbf{x}} \wedge \varphi) \rightarrow (\bigwedge \Gamma' \rightarrow \psi)$ is a theorem for some finite $\Gamma' \subseteq \Gamma$. The S5 axioms then imply $(E\vec{\mathbf{x}} \wedge \varphi) \rightarrow \Box(\bigwedge \Gamma' \rightarrow \psi)$ (see Hughes & Cresswell (1996), p. 62) and thus $\exists\vec{\mathbf{x}}\varphi \rightarrow \Box(\bigwedge \Gamma' \rightarrow \psi)$. Similarly, S5 then implies that $\Diamond\exists\vec{\mathbf{x}}\varphi \rightarrow (\bigwedge \Gamma' \rightarrow \psi)$ (see Hughes & Cresswell (1996), p. 295). So ψ is provable from Γ and $\Diamond\exists\vec{\mathbf{x}}\varphi$ as required.)

77 The difference between classical and positive free logic can be ignored in many contexts. In particular, a simple induction on the length of proofs shows that φ is provable from premises Γ in classical S5 using MP and GEN just in case $E\vec{\mathbf{x}} \rightarrow \varphi$ is provable from Γ in positive free logic using MP and GEN (where φ's free variables are among $\vec{\mathbf{x}}$). So, we can reason classically as long as all the relevant parameters exist and we do not appeal to NEC. Moreover, it follows that classical and positive free logics agree on the sentences provable from any Γ, using just MP and GEN.

78 PL3 is redundant, but added for simplicity. *Proof:* The idea is to show that whenever Y exists, we can use comp to get a subplurality of Y which is co-extensive with X. PL1 and PL2 can then be used to show that Z is equal to X. More precisely, assume $\Diamond(EX, Y \wedge X \subseteq Y)$ and $\Diamond(EZ, Y \wedge Z = \{x \in Y : \Diamond(x \in X)\})$. Suppose $\Diamond(x \in X)$. By PL1 and S5, $\Box(EX \rightarrow Ex \wedge x \in X)$. So, $\Diamond(x \in Y)$. By PL1 and S5, $\Box(EY \rightarrow Ex \wedge x \in Y \wedge \Diamond(x \in X))$. So, $\Diamond(x \in Z)$. Conversely, let $\Diamond(x \in Z)$. Again, $\Box(EZ \rightarrow Ex \wedge x \in Z)$. So, $\Diamond\Diamond(x \in X)$ and thus $\Diamond(x \in X)$ by S5. Since "x" is not free in our assumptions and they are of the form $\Diamond\varphi$, we can conclude in S5 that:

$$\Box \forall x (\Diamond(x \in X) \leftrightarrow \Diamond(x \in Z))$$

and thus that $X = Z$ by PL2. So, $\Diamond(EX, Y \wedge X \subseteq Y)$ implies:

$$(EY, Z \wedge Z = \{x \in Y : \Diamond(x \in X)\}) \rightarrow X = Z \wedge X \subseteq Y$$

by S5. But $\exists Z(Z = \{x \in Y : \Diamond(x \in X)\})$, by comp. So:

$$\Diamond(EX, Y \wedge X \subseteq Y) \rightarrow (EY \rightarrow (EX \wedge X \subseteq Y))$$

and thus, by S5:

$$EX, Y \wedge X \subseteq Y \rightarrow \Box(EY \rightarrow EX \wedge X \subseteq Y).$$

It is worth noting that PL3 does not follow from the weaker but more standard extensionality principles for pluralities, like $EX, Y \wedge \forall x(x \in X \leftrightarrow x \in Y) \rightarrow X = Y$, $\Box\forall x(x \in X \leftrightarrow x \in Y) \rightarrow X = Y$, and $\Box\forall x\Box(x \in X \leftrightarrow x \in Y) \rightarrow X = Y$. (See Linnebo (2017) and Uzquiano (2011).) *Proof:* consider an S5 Kripke model K with two worlds w_0, w_1, both with the first-order domain {0, 1}. Let the pluralities at w_0 be ∅, {0}, {1}, and {0, 1} and the pluralities at w_1 be ∅, 3, {1}, and {0, 1}. At w_0 and w_1, we let ∅ contain nothing, {1} contain 1, and {0, 1} contain 0 and 1. At w_0, we let {0} contain 0, but at w_1 we make it contain nothing; and at w_1, we let 3 contain 0, but at w_0 we make it contain nothing. So, K validates $\Box\forall x(\Diamond(x \in \{0\}) \leftrightarrow \Diamond(x \in 3))$. It is straightforward to check that K validates comp, PL1, and the other versions of PL2. Moreover, at w_0, both {0} and {0, 1} exist and {0} is a subplurality of {0, 1}; but, at w_1, {0, 1} exists even though {0} does not. So, K does not validate PL3.

79 Hewitt (2012) argues that PL1 is false. As he points out, however, PL1 does hold for "rigid" pluralities. If necessary, the modal structuralist can re-interpret their second-order variables as ranging over rigid pluralities, without loss.

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5.2.3. Pairs.

(P1) $\forall x, x', y, y'(\langle x, y\rangle = \langle x', y'\rangle \leftrightarrow (x = x') \wedge (y = y'))$
(P2) $\forall x, y E\langle x, y\rangle$
(P3) $\langle x, y\rangle \in X \rightarrow \Box(EX \rightarrow \langle x, y\rangle \in X)$
(P4) $\langle x, y\rangle = \langle x, y\rangle$
(P5) $\tau = \tau' \rightarrow (\varphi[\tau/z] \leftrightarrow \varphi[\tau'/z])$, where τ, τ' are first-order terms free for z in φ.

REMARK 5.2. *I have separated the identity axioms* P4 *and* P5 *from the logic because they make substantial claims about the pairing operator. For example, on the mereological reading,* P5 *essentially implies that fusions coding pairs cannot change their parts: if* $x = \langle y, z\rangle$, *then* $\Box(x = \langle y, z\rangle)$. *As I will point out below,*[80] *however, the axioms other than* P3 *turn out to be redundant in the sense that* MSST *with those axioms proves exactly the same ms-translations as without them.*

5.2.4. Existence, Extendability, and Stability. EXISTENCE (E)

$$\Diamond\exists M(M = M)$$

THE EXTENDABILITY PRINCIPLE (EP)

$$\Box\forall M\forall X \subseteq M\Diamond\exists M' \sqsupseteq M\exists x \in M'(M' \vDash x \equiv X) \quad .$$

STABILITY (S)

$$[\forall\vec{x}(\varphi \leftrightarrow \forall y\varphi)]^{pt},$$

where $\varphi \in \mathcal{L}_\in$'s free variables are among $\vec{x} - \{y\}$.

In addition these axioms, MSST has as axioms the result of prefixing any of them with a sequence of $\forall$'s and $\Box$'s in any order. It follows that whenever φ is a theorem of MSST, so are $\forall x\varphi$ and $\Box\varphi$.

5.2.5. MSST_T. MSST can be easily modified for theories other than ZFC2. Let MSST_T denote the result of replacing ZFC2 structures in the MSST axioms with T structures (where T is a sentence in $\mathcal{L}^2_\in$). As I will show, many central results concerning MSST also hold for a much broader class of theories of the form MSST_T.

5.3. The ms-translations provable in MSST_T**.** There are two primary results concerning the ms-translations provable in MSST_T. The first sets a *lower bound*: it says that MSST_T *at least* interprets a certain set theory S_T via ms-translation. The second sets an *upper bound*: it says that for a broad class of theories T, MSST_T *at most* interprets S_T via ms-translation. Before proving these results, I will state them more precisely and draw out some of their consequences.

5.3.1. The lower bound theorem.

DEFINITION 5.3. *For* $\varphi \in \mathcal{L}^2_\in$, *let*:

(i) φ^x *abbreviate the claim that x satisfies* φ. *Formally: it is the result of replacing first-order quantifiers* $\exists y$ *in* φ *with* $\exists y \in x$, *occurrences of* $\exists Y\psi(Y)$ *with* $\exists y \subseteq x\psi(y)$, *and free second-order variables with first-order variables.*

[80] See the remarks at the end of §5.3.3.

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(ii) Trans_φ *abbreviate the claim that any sets* x, y *are in some transitive set satisfying* φ (*formally,* $\forall x, y \exists z(x, y \in z \wedge z$ *is transitive* $\wedge \varphi^z)$).

(iii) S_φ *denote Separation* + Trans_φ.

THEOREM 5.4 (The lower bound theorem). *Let* $\varphi \in \mathcal{L}_\in$ *be a sentence. If* S_T *proves* φ*, then* MSST_T *proves* φ^{pt}.

S_T is an extremely simple theory. Nonetheless, by adding various sentences to T, it can be made to prove increasingly large fragments of ZFC. Ultimately, it can be made to prove all of Z plus Π_0-Col. Let's look at some examples.

It is a standard result, provable in logic alone, that Δ_0 formulas are *absolute* for transitive sets,[81] and thus that Σ_1 formulas are *upward absolute* for transitive sets.[82] It follows that:

Claim 1. *If* φ *is a* Π_2 *sentence provable in* T*, then* S_T *proves* φ.

Proof. Suppose φ is a sentence of the form $\forall \vec{x} \psi$ where ψ is Σ_1. By $\mathsf{Trans}_\mathsf{T}$, any $\vec{x}$ are in some transitive set y satisfying T. Since φ is provable in T, y also satisfies φ and thus ψ. So, ψ is true by the upward absoluteness of Σ_1 formulas. □

EXAMPLE 5.5. *Extensionality and Foundation are* Π_1; *Infinity is* Σ_1; *and Pairing, Union, and Choice are all* Π_2*, as is the Mostowski collapse lemma, which says that every well-founded extensional structure is isomorphic to a transitive set.*[83]

So, by the lower bound theorem and Claim 1, MSST proves the ms-translations of all theorems of Z minus Powerset. But this also holds for MSST_T for theories T much weaker than $\mathsf{ZFC2}$. In particular, it is easy to see that Pairing and Union already follow from S_T. So, the lower bound theorem and Claim 1 imply that when T contains Extensionality, Foundation, Infinity, and Choice, MSST_T also proves the ms-translations of all theorems of Z minus Powerset.

REMARK 5.6. *Given that* S_T *proves Pairing and Union, it will prove that sequences of universal quantifiers are equivalent to single universal quantifiers: that is, for any* Σ_n *formula* φ*, it proves that* $\forall \vec{x} \varphi$ *is equivalent to* $\forall x \psi$ *for some* Σ_n *formula* ψ*. Similarly, for existential quantifiers.*[84] *Consequently, it also proves that* Σ_{n+1}-Col *is equivalent to* Π_n-Col.[85]

We can extend Claim 1 by requiring that the sets satisfying T in $\mathsf{Trans}_\mathsf{T}$ are *supertransitive*: that, in addition to being transitive, they contain any subset of any set they contain. In the presence of Extensionality and Separation, it is straightforward to show that this is equivalent to requiring that they satisfy second-order Separation in addition to being transitive. Then, just as Δ_0 formulas are absolute for transitive sets, it is easy to see that Δ_0^* formulas

[81] We say that φ is *absolute* for x when $\forall \vec{y} \in x(\varphi^x \leftrightarrow \varphi)$, where φ's free variables are among $\vec{y}$.

[82] We say that φ is *upward absolute* for x when $\forall \vec{y} \in x(\varphi^x \rightarrow \varphi)$, where φ's free variables are among $\vec{y}$.

[83] Formally, the Mostowski collapse lemma is: $\forall x, y((\text{Extensionality} \wedge \text{second-order Foundation})^{x,y} \rightarrow \exists f(f$ is a one–one function $\wedge\ \mathsf{dom}(f) = x \wedge \mathsf{rng}(f)$ is transitive $\wedge \forall z, w \in x(\langle z, w\rangle \in y \leftrightarrow f(z) \in f(w)))$.

[84] See Devlin (1984), Lemma 8.9.

[85] See Devlin (1984), Lemma 11.3, of which the mentioned result is a simple generalisation.

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are absolute for supertransitive sets, and thus that Σ_1^* formulas are upward absolute for supertransitive sets. It follows as before that:

Claim 2. *If φ is a Π_2^* sentence provable in* T *and* T *contains second-order Separation, then* $\mathsf{S_T}$ + *Extensionality proves* φ.[86]

EXAMPLE 5.7. *$\exists x(x$ is transitive $\wedge\ \varphi^x)$ is Σ_1^* and thus so are $\exists x(x$ is transitive $\wedge$ $(\mathsf{ZFC2})^x)$ and $\exists x(x$ is transitive $\wedge\ (\mathsf{Z2}^*)^x)$. $\forall x \exists \alpha(x \in V_\alpha)$ is Π_2^*,*[87] *as are Powerset and* $\mathsf{Trans_T}$.

So, by the lower bound theorem and Claims 1 and 2, MSST proves the ms-translations of all theorems of Z. But again this also holds for theories T much weaker than $\mathsf{ZFC2}$. It is easy to see that Powerset already follows from $\mathsf{S_T}$ + Extensionality when T contains second-order Separation. So, the lower bound theorem and Claim 1 already imply that when T contains Extensionality, Foundation, Infinity, Choice, and second-order Separation, $\mathsf{MSST_T}$ proves the ms-translations of all theorems of Z.

REMARK 5.8. *Recall that, in Z^*, an uncountable ordinal κ is a fixed point in the enumeration of the $\beth$ cardinals just in case $\kappa = |V_\kappa|$, and that κ is an inaccessible cardinal just in case it is also regular. Now, in Z^*, we can show that a transitive set satisfies $\mathsf{Z2}^*$ minus Choice just in case it is of the form V_λ for $\lambda > \omega$ a limit ordinal.*[88] *It is then straightforward to verify that V_λ satisfies Choice just in case $\lambda = |V_\lambda|$, and that it satisfies Choice and Collection just in case $\lambda = |V_\lambda|$ and λ is regular. So, in Z^*, the claim* (Beth) *that there are arbitrarily large fixed points in the $\beth$ enumeration has the Π_2^* formulation $\forall x \exists y(y$ is transitive $\wedge\ x \in y \wedge (\mathsf{Z2}^*)^y)$, and the claim (*In*) that there are arbitrarily large inaccessible cardinals has the Π_2^* formulation $\forall x \exists y(y$ is transitive $\wedge\ x \in y \wedge (\mathsf{ZFC2})^y)$.*

---

[86] Claims 1 and 2 are optimal: there are Σ_2 sentences φ which are unprovable in $\mathsf{S_T}$ for some T containing φ, Extensionality, and second-order Separation. *Proof:* Let T contain just Extensionality, second-order Separation, and the Σ_2 sentence which says that there is a greatest ordinal. Working in ZFC, first note that every V_n is supertransitive and contains a greatest ordinal. Since "x is an ordinal" is Δ_0, each V_n will satisfy T. Moreover, since "x is transitive" is Δ_0, and "φ^x" is Δ_0^*, they will be absolute for the supertransitive V_ω. So, $V_\omega \vDash \mathsf{S_T}$. But, $V_\omega \vDash$ "there is no greatest ordinal".

[87] In particular, it can be formulated as: $\forall x \exists y, z(y = V_z \wedge x \in y)$, where "$x = V_y$" is the Δ_0^* (and Σ_2) formula "$\exists f \subseteq x(f$ is a function $\wedge\ \mathsf{dom}(f)$ is an ordinal $\wedge\ y$ is an ordinal $\wedge\ \forall \alpha \in \mathsf{dom}(f)(f(\alpha) = \bigcup\{\mathcal{P}(f(\beta)) : \beta \in \mathsf{dom}(f)\}) \wedge$:

(i) $y = \mathsf{dom}(f) \wedge x = \bigcup\{\mathcal{P}(f(\beta)) : \beta \in \mathsf{dom}(f)\} \vee$
(ii) $y = \mathsf{dom}(f) + 1 \wedge x = \mathcal{P}(\bigcup\{\mathcal{P}(f(\beta)) : \beta \in \mathsf{dom}(f)\}) \vee$
(iii) $y = \mathsf{dom}(f) + 2 \wedge x = \mathcal{P}(\mathcal{P}(\bigcup\{\mathcal{P}(f(\beta)) : \beta \in \mathsf{dom}(f)\}))$,

where "$x = \bigcup\{\mathcal{P}(f(\beta)) : \beta \in \mathsf{dom}(f)\}$" is the Δ_0^* (and Π_1) formula:

$$\forall y \in x \exists \beta \in \mathsf{dom}(f)(y \subseteq f(\beta)) \quad \wedge \quad \forall \beta \in \mathsf{dom}(f) \forall z \in \bigcup\bigcup f(z = f(\beta) \rightarrow \forall y \subseteq z(y \in x))$$

and "$x = \mathcal{P}(\bigcup\{\mathcal{P}(f(\beta)) : \beta \in \mathsf{dom}(f)\})$" is the Δ_0^* (and Σ_2) formula:

$$\exists z \in x(z = \bigcup\{\mathcal{P}(f(\beta)) : \beta \in \mathsf{dom}(f)\} \quad \wedge \quad x = \mathcal{P}(z)),$$

where "$x = \mathcal{P}(y)$" is the Δ_0^* (and Π_1) formula "$\forall z(z \in x \leftrightarrow z \subseteq x)$". Similarly, for (iii). (See (Kanamori, 2003, p. 314)).

[88] See Theorem 6 in Uzquiano (1999) and (Drake (1974), p. 112).

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It follows from Claims 1 and 2 that $\mathsf{S}_{\mathsf{ZFC2}}$ *proves* $\forall x \exists \alpha (x \in V_\alpha)$ *and* In. *So,* $\mathsf{S}_{\mathsf{ZFC2}} = \mathsf{Z}^* +$ In. *Similarly, it follows that* $\mathsf{S}_{\mathsf{Z2}^*}$ *proves* $\forall x \exists \alpha (x \in V_\alpha)$ *and* Beth. *So,* $\mathsf{S}_{\mathsf{Z2}^*} = \mathsf{Z}^* +$ Beth.

Finally, we can extend Claim 2 by requiring that Σ_1 formulas are absolute for the sets satisfying T in $\mathsf{Trans}_{\mathsf{T}}$.

DEFINITION 5.9. *Let* $\mathsf{T} \prec_1 V$ *be the schema which says that* Σ_1 *formulas are absolute for transitive sets satisfying* T. *Formally:*

$$\forall x ((x \text{ is transitive } \wedge \mathsf{T}^x) \rightarrow \forall \vec{y} \in x (\varphi^x \leftrightarrow \varphi)),$$

where φ *is* Σ_1 *with free variables among* $\vec{y}$.

Claim 3. *If* φ *is a* Π_3 *sentence provable in* T, *then* $\mathsf{S}_{\mathsf{T}} + \mathsf{T} \prec_1 V$ *proves* φ.[89]

EXAMPLE 5.10. *All instances of* Π_0-Col *are* Π_3, *as are Powerset,* $\forall x \exists \alpha (x \in V_\alpha)$, *and* $\mathsf{Trans}_{\mathsf{ZFC2}}$. *Note also that each instance of* $\mathsf{Z2}^* \prec_1 V$ *and* $\mathsf{ZFC2} \prec_1 V$ *are* Π_2.

So, by the lower bound theorem and Claims 1 and 3, MSST proves the ms-translations of all theorems of $\mathsf{Z} + \Pi_0$-Col (because each instance of $\mathsf{ZFC2} \prec_1 V$ is provable in ZFC).[90] But this also holds for theories weaker than ZFC2. It is easy to see that all instances of Π_0-Col already follow from $\mathsf{S}_{\mathsf{T}} + \mathsf{T} \prec_1 V$. And there are relatively weak theories T for which S_{T} proves $\mathsf{T} \prec_1 V$. The simplest example is $\mathsf{Z2}^*$.

LEMMA 5.11. $\mathsf{S}_{\mathsf{Z2}^*}$ *proves* $\mathsf{Z2}^* \prec_1 V$, *and thus all instances of* Π_0-Col.

Proof. As usual, let $H_\kappa = \{x : |\mathsf{tc}(x)| < \kappa\}$, where $\mathsf{tc}(x)$ is the transitive closure of x. It is a standard result in ZFC that Σ_1 formulas are absolute for H_κ when $\kappa > \omega$.[91] I will reprove that result in $\mathsf{S}_{\mathsf{Z2}^*} = \mathsf{Z}^*$ and then show that when x is a transitive set satisfying $\mathsf{Z2}^*$, $x = H_\kappa$ for some $\kappa > \omega$.

Working in Z^*, we can prove the Mostowski collapse lemma. To see this, suppose that $\langle D, R \rangle$ is a well-founded extensional structure with $|D| = \kappa$. (Recall that, on the formulation I am employing, Choice says that every set is equinumerous with an ordinal. This means κ^+ exists for any κ.) A simple induction shows that the range of any collapsing function from $\langle D, R \rangle$—i.e., a function f for which $\mathsf{dom}(f) = D$, $\mathsf{rng}(f)$ is transitive, and $\forall x, y \in D (\langle x, y \rangle \in R \leftrightarrow f(x) \in f(y))$—will be contained in H_{κ^+} (which exists because $H_{\alpha^+} \subseteq V_{\alpha^+}$, for any α). So, we can construct such a function by transfinite recursion using Separation on $D \times H_{\kappa^+}$. Now, let $\vec{x} \in H_\kappa$ for $\kappa > \omega$ and suppose there is some y for which $\varphi(\vec{x})$ (where φ is Δ_0 with free variables among $y, \vec{x}$). Let V_α contain y and $\vec{x}$. Then, because φ is Δ_0, $(\exists y \varphi(\vec{x}))^{V_\alpha}$. Let M be an elementary substructure of V_α with

[89] Claim 3 is optimal: there are T containing Σ_3 sentences φ which are unprovable in $\mathsf{S}_{\mathsf{T}} + \mathsf{T} \prec_1 V$. *Proof:* Let T be ZFC2 plus the claim that there are at most finitely many transitive sets satisfying ZFC2. Formally: $\exists f, n (f$ is a function $\wedge \mathsf{dom}(f) = n \wedge \forall x ((x$ is transitive $\wedge (\mathsf{ZFC2})^x) \rightarrow x \in \mathsf{rng}(f)))$ (which we can abbreviate as Φ). It is easy to see that Φ is Σ_3, since "$(\mathsf{ZFC2})^x$" is Π_1. Working in ZFC plus the claim that there are ω inaccessibles, let $\langle \kappa_n : n < \omega \rangle$ enumerate the first ω inaccessibles and let κ_ω be their least upper bound. Since the V_κs for κ inaccessible, are exactly the transitive sets satisfying ZFC2, there are precisely n many transitive sets satisfying ZFC2 in V_{κ_n}: namely, the V_{κ_m}s for $m < n$. Moreover, the function f enumerating these V_{κ_m}s will exist in V_{κ_n}. But "x is transitive" is Δ_0 and "φ^x" is Δ_0^*. So, they will be absolute for supertransitive V_α. Thus, $V_{\kappa_n} \models \Phi$ but $V_{\kappa_\omega} \models \neg\Phi$. Finally, by Lemma 5.11 below, S_{T} proves $\mathsf{T} \prec_1 V$.

[90] See, for instance, (Kanamori, 2003, p. 299).

[91] This was first established by Lévy (1965). See also (Kanamori, 2003, p. 299).

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$\mathsf{tc}(\{\vec{x}\}) \subseteq M$ and $|M| = |\mathsf{tc}(\{\vec{x}\}) \times \omega| < \kappa$, and let i be a collapsing function from M. Then, $\mathsf{rng}(i)$ is transitive, of size $< \kappa$, and thus in H_κ. Moreover, $(\exists y \varphi(i(\vec{x})))^{\mathsf{rng}(i)}$. A simple induction shows that i is the identity on $\mathsf{tc}(\{\vec{x}\}) \subseteq M$. So, $(\exists y \varphi(\vec{x}))^{\mathsf{rng}(i)}$ and thus $(\exists y \varphi(\vec{x}))^{H_\kappa}$ (because φ is Δ_0 and so absolute between $\mathsf{rng}(i)$ and H_κ).

Recall that a transitive set satisfies $\mathsf{Z2}^*$ just in case it is of the form V_κ for $\kappa > \omega$ with $\kappa = |V_\kappa|$. It follows that κ is a limit cardinal (because it is $\bigcup_{\alpha<\kappa} |V_\alpha|$). So, $H_\kappa = \bigcup_{\lambda<\kappa} H_\lambda$. Thus, because $H_{\lambda^+} \subseteq V_{\lambda^+}$ for any λ, it follows that $H_\kappa \subseteq V_\kappa$. Conversely, if $x \in V_\kappa$, then $x \in V_\alpha$ for some $\alpha < \kappa$. But, $|\mathsf{tc}(x)| \leq |V_\alpha| < |V_\kappa| = \kappa$. So, $x \in H_\kappa$. Thus, $V_\kappa = H_\kappa$. □

So, by the lower bound theorem, Claims 1 and 2, and Lemma 5.11, $\mathsf{MSST}_{\mathsf{Z2}^*}$ proves the ms-translations of all theorems of $\mathsf{Z}^* + \mathsf{Beth} + \Pi_0\text{-}\mathsf{Col}$.

Instances of Π_0-Col signal an insuperable limit on the amount of Collection provable in S_T. In particular, it turns out that for any T, S_T either contradicts a theorem of ZFC or fails to prove all instances of Π_1-Col. To show this, I need the following simple lemma.

LEMMA 5.12. *Suppose that any sets are in some supertransitive set. Then, Δ_0^* formulas have Δ_2 formulations.*

Proof. Let φ be Δ_0^* with free variables among $\vec{y}$, and assume that any sets are in some supertransitive set. Then, since Δ_0^* formulas are absolute for supertransitive sets, φ is equivalent to both $\exists x(x \text{ is supertransitive} \wedge \vec{y} \in x \wedge \varphi^x)$ and $\forall x((x \text{ is supertransitive} \wedge \vec{y} \in x) \rightarrow \varphi^x)$, which are Σ_2 and Π_2, respectively (because "x is supertransitive" is Π_1). □

LEMMA 5.13. *If S_T is consistent with Z^*, then it fails to prove all instances of Π_1-Col.*

Proof. Let Z_T^* be $\mathsf{Z}^* + \mathsf{Trans}_\mathsf{T}$. I will start by showing that if Z_T^* is consistent, then it fails to prove that there is a supertransitive set satisfying all of its axioms other than Separation.[92] Formally, $\exists x(x \text{ is supertransitive} \wedge (\mathsf{Z}_\mathsf{T} - \text{Separation})^x)$, which we can abbreviate as $\exists x \Phi(x)$. I will then show that $\mathsf{Z}_\mathsf{T}^* + \Pi_1\text{-}\mathsf{Col}$ does prove $\exists x \Phi(x)$. It follows immediately from these two claims that Z_T^* fails to prove all instances of Π_1-Col, if consistent.

So, suppose Z_T^* proves $\exists x \Phi(x)$. Working in Z_T^*, let x be a least set for which $\Phi(x)$. Since x is supertransitive, it satisfies each instance of Separation and thus each axiom of Z_T^*. So, $(\exists y \Phi(y))^x$. Now, "x is supertransitive" and "$(\mathsf{Z}_\mathsf{T}^* - \text{Separation})^x$" are both absolute for x, since they are Δ_0^* and Δ_0, respectively. So, $\exists y \in x \Phi(y)$, contradicting the minimality of x.

Since every set is in some V_α, a simple induction shows that for each n there is an n-length sequence f of V_αs such that both $f(n)$ and a function enumerating $f(n)$ are in a transitive set satisfying T in $f(n+1)$. Formally: $\forall n \exists f(f \text{ is a function} \wedge \mathsf{dom}(f) = n+1 \wedge$:

- $\forall m \leq n \exists x \subseteq f(n)(f(n) = V_x) \wedge$
- $\forall m < n \exists y \in f(m+1)(y \text{ is transitive} \wedge \mathsf{T}^y \wedge f(m) \in y \wedge \exists f', z \in y(z \text{ is an ordinal} \wedge f' \text{ is a function} \wedge \mathsf{dom}(f') = z \wedge \mathsf{rng}(f') = f(m)))$.

Abbreviate this as $\forall n \exists f \Psi(n, f)$. Since "$x = V_y$" and "$\mathsf{T}^x$" are Δ_0^*, $\Psi(n, f)$ is Δ_0^* and thus $\Sigma_2^{\mathsf{Z}^*}$ by Lemma 5.12 (because the V_αs are supertransitive). So, because Π_1-Col is equivalent to Σ_2-Col, it follows that there is a set containing such a function for each n. Using

[92] This could also be shown using Gödel's second incompleteness theorem.

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these functions, we can finally construct an ω-sequence of the same kind: namely, for which $f(n)$ is a V_α, and both $f(n)$ and a function enumerating $f(n)$ are in some transitive set satisfying T in $f(n+1)$. It is then straightforward to check that when f is such a sequence, $\bigcup_{n<\omega} f(n)$ is a V_λ satisfying Z^*_T. In particular, we get that V_λ satisfies Choice because $f(n+1)$ contains a function enumerating $f(n)$. Thus, V_λ is a witness to $\exists x \Phi(x)$. □

5.3.2. The upper bound theorem. Can MSST_T go beyond S_T? In particular, can it prove the ms-translations of all theorems of ZFC? It turns out that in a broad class of cases, it cannot. Let S^+_T be S_T plus $\mathsf{Trans}_{Pairing}$ and the Mostowski collapse lemma.

THEOREM 5.14 (The upper bound theorem). *Let $\varphi \in \mathcal{L}_\in$ be a sentence, and suppose T proves Extensionality and second-order Foundation. If MSST_T proves φ^{pt}, then S^+_T proves φ.*

By Claim 1, S_T proves the Mostowksi collapse lemma when it is provable in T. So, when T proves Pairing and the Mostowski collapse lemma, $\mathsf{S}^+_\mathsf{T} = \mathsf{S}_\mathsf{T}$. It follows from the lower and upper bound theorems that:

COROLLARY 5.15. *Let $\varphi \in \mathcal{L}_\in$ be a sentence, and suppose T proves Extensionality, second-order Foundation, Pairing, and the Mostowski collapse lemma. MSST_T proves φ^{pt} if and only if S^T proves φ.*

The main theorem is then immediate from this corollary and the fact that $\mathsf{S}_{\mathsf{ZFC2}} = \mathsf{Z}^* + \mathsf{In}$.

THEOREM 5.16 (The main theorem). *Let $\varphi \in \mathcal{L}_\in$ be a sentence. MSST proves φ^{pt} if and only if $\mathsf{Z}^* + \mathsf{In}$ proves φ.*

Similarly, it follows that:

THEOREM 5.17. *Let $\varphi \in \mathcal{L}_\in$ be a sentence. $\mathsf{MSST}_{\mathsf{Z2}^*}$ proves φ^{pt} if and only if $\mathsf{Z}^* + \mathsf{Beth}$ proves φ.*

Since $\mathsf{Z}^* + \mathsf{In}$ and $\mathsf{Z}^* + \mathsf{Beth}$ both prove all instances of Π_0-Col but both fail to prove some instance of Π_1-Col, there is a sense in which MSST and $\mathsf{MSST}_{\mathsf{Z2}^*}$ prove the ms-translations of the maximum and same amount of ZFC.

5.3.3. Proof of the lower bound theorem. A number of questions in MSST_T turn on whether notions like $Y \vDash \varphi$, $Y \sqsubseteq Y'$, and φ^{pt}_Y, are invariant between possible worlds. For example, in §2.5.1, we saw that the truth of:

$$(\exists x \forall y (y \notin x))^{pt}$$

relied on the invariance of $M \vDash y \notin x$ (which is to say, $\langle y, x \rangle \notin M$) between worlds where M exists and $x, y \in M$. The following lemma shows that in MSST_T a broad class of notions are invariant between worlds.

DEFINITION 5.18. *Say that $\varphi \in \mathcal{L}_\Diamond$ with free variables among $\vec{\mathbf{x}}$ is invariant if*:

$$\Box \forall \vec{\mathbf{x}} (\varphi \to \Box (E\vec{\mathbf{x}} \to \varphi)).$$

DEFINITION 5.19. *Say that $\varphi \in \mathcal{L}_\Diamond \cup \mathcal{L}^2_\in$ is quasi-modalised if its quantifiers are either bounded—i.e., of the form $\exists x \in y$, $\exists x \in Y$, or $\exists X \subseteq Y$—or modalised—i.e., of the form $\Diamond \exists \vec{x}$ or $\Diamond \exists \vec{X}$. (When a formula contains only bounded quantifiers, I will call it bounded.)*

LEMMA 5.20 (MSST_T). *All quasi-modalised formulas in $\mathcal{L}_\Diamond$ are invariant.*

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Proof. By induction on the complexity of φ, I will show that $\mathsf{MSST_T}$ proves:

$$E\vec{\mathbf{x}} \wedge \varphi \rightarrow \Box(E\vec{\mathbf{x}} \rightarrow \varphi)$$

whenever $\varphi \in \mathcal{L}_\Diamond$ is quasi-modalised with free variables among $\vec{\mathbf{x}}$. For $x \in X$ and $\langle x, y\rangle \in X$, this is immediate from PL1 and P3; for $\mathbf{x} = \mathbf{y}$ and $\tau = \tau'$, from L4 and L5, and P4 and P5. The conjunction, negation, $\Diamond$, and $\Diamond\exists\vec{\mathbf{x}}$ cases are trivial in S5, given the induction hypothesis. For $\exists y \in Y\varphi$, suppose $E\vec{\mathbf{x}}, Y, y \wedge y \in Y \wedge \varphi$. By the induction hypothesis and P3, it follows that $\Box(E\vec{\mathbf{x}}, Y \rightarrow Ey \wedge y \in Y \wedge \varphi)$ and thus that $\Box(E\vec{\mathbf{x}}, Y \rightarrow \exists y \in Y\varphi)$. The case for $\exists Z \subseteq Y$ is proved similarly using P4. □

So, $\mathsf{MSST_T}$ proves that $Y \vDash \varphi$ and φ^{pt}_Y are invariant, since each is quasi-modalised. Strictly speaking, $Y \sqsubseteq Y'$ is not quasi-modalised, because it involves $\exists Z(Z = X)$. However, it is easy to see that $\exists Z(Z = X)$ is equivalent to $\exists Z \subseteq X(Z = X)$. So, if we let EX abbreviate that formula instead, as I will from now on, $Y \sqsubseteq Y'$ becomes quasi-modalised, and thus invariant.

Using Lemma 5.20, a simple induction shows that in $\mathsf{MSST_T}$, $(\forall\vec{\mathbf{z}}\varphi)^{pt}_M$ is equivalent to $\Box\forall M' \sqsubseteq M\forall\vec{\mathbf{z}} \in M'\varphi^{pt}_{M'}$ (when EM, $\vec{x} \in M$, and $\vec{Y} \subseteq M$, and where $\varphi \in \mathcal{L}^2_\in$ with free variables among $\vec{\mathbf{z}}$).

In the previous section, I pointed out that Δ_0 formulas are absolute for transitive sets. The next lemma establishes an analogue of this result for the ms-translations of bounded formulas.

LEMMA 5.21 ($\mathsf{MSST_T}$). *Suppose EM, $\vec{x} \in M$, and $\vec{X} \subseteq M$. Then:*

$$(M \vDash \varphi) \leftrightarrow \varphi^{pt}_M,$$

where $\varphi \in \mathcal{L}^2_\in$ is bounded, with free variables among $\vec{x}$, $\vec{X}$.

Proof. By induction on the complexity of φ. The only difficult cases are the right-to-left directions for $\exists x \in y$, $\exists x \in Y$, and $\exists X \subseteq Y$. For the first, suppose EM, $\vec{x} \in M$, $\vec{X} \subseteq M$, and $(\exists z \in y\varphi)^{pt}_M$, that is:

$$\Diamond\exists M' \sqsupseteq M\exists z \in M'(z \in y \wedge \varphi)^{pt}_{M'},$$

where φ's free variables and y are among $\vec{x}$, $\vec{X}$. By PL1 and PL3, $\vec{x}$ and $\vec{X}$ exist and are elements and subpluralities of M whenever M exists. So, by the induction hypothesis:

$$\Diamond\exists M' \sqsupseteq M(E\vec{x}, \vec{X} \wedge \vec{x} \in M \wedge \vec{X} \subseteq M \wedge M' \vDash \exists z \in y\varphi).$$

It is easy to see that just as Δ_0 formulas are absolute for transitive sets, bounded formulas are absolute between structures M and M' when $M \sqsubseteq M'$.[93] Thus:

$$\Diamond(EM, \vec{x}, \vec{X} \wedge M \vDash \exists z \in y\varphi).$$

Finally:

$$M \vDash \exists x \in y\varphi$$

by Lemma 5.20. The cases for $\exists x \in Y$ and $\exists X \subseteq Y$ are proved similarly. □

[93] That is, $\forall\vec{x} \in M\forall\vec{X} \subseteq M(M \vDash \psi \leftrightarrow M' \vDash \psi)$ when $M \sqsubseteq M'$ and $\psi \in \mathcal{L}^2_\in$ is bounded with free variables among $\vec{x}$, $\vec{X}$.

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I also noted in the previous section that Δ_0^* formulas are absolute for supertransitive sets. It is easy to see that when T contains Extensionality and second-order Separation, Lemma 5.21 extends to formulas containing quantifiers of the form $\exists x \subseteq y$ in addition to bounded quantifiers. For in that case, M will be a *supertransitive* substructure of M' whenever $M \sqsubseteq M'$.[94]

It is an immediate consequence of S that the ms-translations of formulas in $\mathcal{L}_\in$ are stable between end-extensions: that is, $\varphi_M^{pt} \leftrightarrow \varphi_{M'}^{pt}$ whenever $M \sqsubseteq M'$ and $\vec{x} \in M$ for $\varphi \in \mathcal{L}_\in$ with free variables among $\vec{x}$. The next lemma extends this to bounded formulas in general.

LEMMA 5.22 ($\mathsf{MSST_T}$). *Suppose $M \sqsubseteq M'$, $\vec{x} \in M$, and $\vec{X} \subseteq M$. Then:*

$$\varphi_M^{pt} \leftrightarrow \varphi_{M'}^{pt},$$

where $\varphi \in \mathcal{L}_\in^2$ is bounded, with free variables among $\vec{x}, \vec{X}$.

Proof. This is immediate from Lemma 5.21 and the fact that bounded formulas are absolute between M and M' whenever $M \sqsubseteq M'$. □

When S is extended to second-order formulas, we can show that the ms-translation of a second-order formula is equivalent to the ms-translation of its first-orderisation (Theorem 5.33). The next lemma establishes this for bounded formulas.

LEMMA 5.23 ($\mathsf{MSST_T}$). *Suppose $EM, \vec{x}, \vec{y} \in M$, $\vec{Y} \subseteq M$, and $M \vDash \vec{y} \equiv \vec{Y}$. Then:*

$$\varphi_M^{pt} \leftrightarrow (\varphi^*)_M^{pt},$$

where $\varphi \in \mathcal{L}_\in^2$ is bounded, with free variables among $\vec{x}, \vec{Y}$.

Proof. By induction on the complexity of φ. The only difficult cases are those for the quantifiers. So, suppose $M, \vec{x}, \vec{y}, \vec{Y}$ are as in the lemma, which we can abbreviate as $\Psi(M, \vec{x}, \vec{y}, \vec{Y})$. Suppose also that $(\exists Z \subseteq Y \varphi)_M^{pt}$. By $\mathsf{PL1}$ and $\mathsf{PL3}$, $\vec{x}, \vec{y}, \vec{Y}$ exist whenever M does; and so, by Lemma 5.20, $\Psi(M, \vec{x}, \vec{y}, \vec{Y})$ holds whenever M exists (because Ψ is bounded and so quasi-modalised). Thus:

$$\Diamond \exists M' \sqsupseteq M \exists Z \subseteq M' ((Z \subseteq Y \wedge \varphi)_{M'}^{pt} \wedge E\vec{x}, \vec{y}, \vec{Y} \wedge \Psi(M, \vec{x}, \vec{y}, \vec{Y})).$$

Similarly, if $M' \sqsupseteq M$ and $EZ \wedge Z \subseteq M'$, then $\vec{x}, \vec{y}, \vec{Y}, Z$ will exist whenever M' exists; and so if $M' \sqsupseteq M$ and $EZ \wedge Z \subseteq M'$, then $Z \subseteq M'$, $(Z \subseteq Y \wedge \varphi)_{M'}^{pt}$, $M' \sqsupseteq M$, and $\Psi(M, \vec{x}, \vec{y}, \vec{Y})$ will hold whenever M' exists (because "$Z \subseteq M'$", "$(Z \subseteq Y \wedge \varphi)_{M'}^{pt}$", "$M' \sqsupseteq M$", and "$\Psi(M, \vec{x}, \vec{y}, \vec{Y})$" are quasi-modalised). Thus, by EP:

$$\Diamond \exists M' \sqsupseteq M \exists Z \subseteq M' \Diamond \exists M'' \sqsupseteq M' \exists z \in M'' (M'' \vDash z \equiv Z \wedge :$$

$$(Z \subseteq Y \wedge \varphi)_{M'}^{pt} \wedge E\vec{x}, \vec{y}, \vec{Y}, Z \wedge Z \subseteq M' \wedge M' \sqsupseteq M \wedge \Psi(M, \vec{x}, \vec{y}, \vec{Y})).$$

By the transitivity of $\sqsubseteq$, it will be the case that $M \sqsubseteq M''$, and thus that $\Psi(M'', \vec{x}, \vec{y}, z, \vec{Y}, Z)$ (because "$\vec{y} \equiv \vec{Y}$" is bounded). Moreover, it will follow from Lemma 5.22 that $(Z \subseteq Y \wedge \varphi)_{M''}^{pt}$. So, the induction hypothesis will imply that $(z \subseteq y \wedge \varphi^*)_{M''}^{pt}$. Thus:

$$\Diamond \exists M'' \sqsupseteq M \exists z \in M'' (z \subseteq y \wedge \varphi^*)_{M''}^{pt},$$

which is to say $((\exists Z \subseteq Y \varphi)^*)_M^{pt}$. The proof of the right-to-left direction uses comp to define Z instead of EP to get z. Other bounded quantifiers are handled similarly. □

[94] That is, in addition to $M \sqsubseteq M'$, for any $x \in M$ and $y \in M'$ such that $M' \vDash y \subseteq x$, $y \in M$.

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THEOREM 5.24 (The lower bound theorem). *Let $\varphi \in \mathcal{L}_\in$ be a sentence. If* $\mathsf{S_T}$ *proves* φ*, then* $\mathsf{MSST_T}$ *proves* φ^{pt}.

Proof. By induction on the length of proofs in $\mathsf{S_T}$, I will show that $\mathsf{MSST_T}$ proves

$$EM, \vec{x} \wedge \vec{x} \in M \rightarrow \varphi_M^{pt} \tag{*}$$

when $\mathsf{S_T}$ proves φ with free variables among $\vec{x}$. It follows that $\mathsf{MSST_T}$ proves $\varphi_\emptyset^{pt}$ when $\mathsf{S_T}$ proves φ, for sentences $\varphi \in \mathcal{L}_\in$. To see this, first note that since $\mathsf{MSST_T}$'s theorems are closed under GEN and NEC, if it proves $EM \rightarrow \varphi_M^{pt}$, it will also prove $\Box\forall M \sqsupseteq \emptyset\varphi_M^{pt}$. Then, working in $\mathsf{MSST_T}$, suppose $\neg\varphi_\emptyset^{pt}$. By S, $\Box\forall M \sqsupseteq \emptyset\neg\varphi_M^{pt}$. So, $\Diamond\exists M \sqsupseteq \emptyset(\varphi_M^{pt} \wedge \neg\varphi_M^{pt})$ by E, which is impossible.

Case 1: Logic.
Axioms. Since ms-translation commutes with the connectives, it is easy to see that (*) holds for the truth-functional tautologies. It is also straightforward to see that it holds for instances of $\mathsf{L1}$, $\mathsf{L2}$, $\mathsf{L4}$, $\mathsf{L5}$, and Ex. It holds for instances of $\mathsf{L3}$ because S is its ms-translation.
Rules of inference. Applications of MP are trivially preserved. So, suppose $\mathsf{MSST_T}$ proves $EM, \vec{x}, y \wedge \vec{x}, y \in M \rightarrow \varphi_M^{pt}$. Since $\mathsf{MSST_T}$'s theorems are closed under GEN and NEC, it also proves:

$$\Box\forall M' \sqsupseteq M(E\vec{x} \wedge \vec{x} \in M' \rightarrow \forall y \in M'\varphi_{M'}^{pt}). \tag{3}$$

$\mathsf{PL1}$ and the definition of end-extension imply $\vec{x} \in M \rightarrow \Box\forall M' \sqsupseteq M(E\vec{x} \wedge \vec{x} \in M')$. So, together with (3), that gives us:

$$EM, \vec{x} \wedge \vec{x} \in M \rightarrow (\forall y\varphi)_M^{pt}.$$

Case 2: Separation.
Suppose $EM, \vec{x}, y$ and $\vec{x}, y \in M$, where φ's free variables are among $\vec{x}, y$. Given comp, let EX and $X = \{z \in M : (z \in y \wedge \varphi)_M^{pt}\}$. By EP:

$$\Diamond\exists M' \sqsupseteq M\exists w \in M'(M' \vDash w \equiv X).$$

By $\mathsf{PL3}$, X will exist and be a subplurality of M whenever $M \sqsubseteq M'$. So, because "$w \equiv X$" is bounded, Lemma 5.21 implies:

$$\Diamond\exists M' \sqsupseteq M\exists w \in M'(w \equiv X)_{M'}^{pt},$$

which is to say:

$$\Diamond\exists M' \sqsupseteq M\exists w \in M'\Box\forall M'' \sqsupseteq M'\forall z \in M''(M'' \vDash z \in w \leftrightarrow z \in X).$$

Now, if $M' \sqsupseteq M$, then $E\vec{x}, y, X, \vec{x}, y \in M$, $X \subseteq M$, $M' \sqsupseteq M$, and $X = \{z \in M : (z \in y \wedge \varphi)_M^{pt}\}$ will all hold whenever M' exists by $\mathsf{PL1}$, $\mathsf{PL3}$, and Lemma 5.20 (because "$M' \sqsupseteq M$" and "$X = \{z \in M : (z \in y \wedge \varphi)_M^{pt}\}$" are quasi-modalised). So, if $M'' \sqsupseteq M'$, then $E\vec{x}, y, X, \vec{x}, y \in M$, $X \subseteq M$, $M'' \sqsupseteq M$, and $X = \{z \in M : (z \in y \wedge \varphi)_M^{pt}\}$. Moreover, if $E\vec{x}, y, X, \vec{x}, y \in M$, and $M \sqsubseteq M''$, then:

$$\{z \in M : (z \in y \wedge \varphi)_M^{pt}\} = \{z \in M'' : (z \in y \wedge \varphi)_{M''}^{pt}\}.$$

To see this, note that in this case S implies $(z \in y \wedge \varphi)_M^{pt}$ is equivalent to $(z \in y \wedge \varphi)_{M''}^{pt}$ for any $z \in M$; and if $M \sqsubseteq M''$ and $y \in M$, then $z \in M$ for any $z \in M''$ for which $(z \in y)_{M''}^{pt}$ (i.e., $M'' \vDash z \in y$).

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Putting these observations together, we get:

$$\Diamond\exists M' \sqsupseteq M \exists w \in M' \Box \forall M'' \sqsupseteq M' \forall z \in M'' ((M'' \vDash z \in w) \leftrightarrow (z \in y \wedge \varphi)^{pt}_{M''}),$$

which is to say $(\exists w(w = \{z \in y : \varphi\}))^{pt}_{M}$.

Case 3: $\mathsf{Trans_T}$.

Suppose that EM and $x, y \in M$. By EP:

$$\Diamond\exists M' \sqsupseteq M \exists z \in M' (M' \vDash z \equiv \mathsf{dom}(M) \wedge \ z \text{ is transitive}).$$

Since "$M \vDash \mathsf{T}$" is bounded, it follows from Lemma 5.20 that:

$$\Diamond\exists M' \sqsupseteq M \exists z \in M' (M' \vDash (z \equiv \mathsf{dom}(M) \wedge \ z \text{ is transitive } \wedge \mathsf{dom}(M) \vDash \mathsf{T})).$$

Then, because "z is transitive $\wedge \mathsf{dom}(M) \vDash \mathsf{T}$" is bounded, Lemma 5.21 implies:

$$\Diamond\exists M' \sqsupseteq M \exists z \in M' (M' \vDash z \equiv \mathsf{dom}(M) \wedge (z \text{ is transitive } \wedge \mathsf{dom}(M) \vDash \mathsf{T})^{pt}_{M'}).$$

Finally, Lemma 5.23 and the fact that "$(\mathsf{dom}(M) \vDash \mathsf{T})^*(z)$" is just "$\mathsf{T}^z$" imply:

$$\Diamond\exists M' \sqsupseteq M \exists z \in M' (z \text{ is transitive } \wedge \mathsf{T}^z)^{pt}_{M'}. \qquad \square$$

REMARK 5.25. *Let $\mathcal{L}^-_\Diamond$ be $\mathcal{L}_\Diamond$ minus the paring terms $\langle x, y\rangle$ plus the three-place relation $\langle x, y\rangle \in X$. Furthermore, let $\mathsf{MSST}^-_\mathsf{T}$ be $\mathsf{MSST_T}$ minus the pairing axioms* P1, P2, P4, P5. *It is easy to see that Lemma 5.20 is provable in $\mathsf{MSST}^-_\mathsf{T}$ for quasi-modalised formulas in $\mathcal{L}^-_\Diamond$; and since that lemma is only used in the subsequent results for notions in $\mathcal{L}^-_\Diamond$—like $M \sqsubseteq M'$, φ^{pt}_M, and $M \vDash \mathsf{T}$—it is straightforward to check that those results also hold for $\mathsf{MSST}^-_\mathsf{T}$. So, $\mathsf{MSST}^-_\mathsf{T}$ proves the ms-translations of all theorems of $\mathsf{S_T}$. It follows from the upper bound theorem and Claim 1 that when T proves Extensionality, second-order Foundation, Pairing, and the Mostowski collapse lemma, $\mathsf{MSST}^-_\mathsf{T}$ proves the ms-translations of all and only the theorems of $\mathsf{S_T}$. So, in that case, $\mathsf{MSST_T}$ and $\mathsf{MSST}^-_\mathsf{T}$ prove the same ms-translations.*

5.3.4. Proof of the upper bound theorem. I will first establish a natural interpretation of $\mathsf{MSST_T}$ into $\mathsf{S_T}$ + $\mathsf{Trans}_{Pairing}$ + the Mostowski collapse lemma when T contains Extensionality and second-order Foundation. As I will show, the most obvious translation from $\mathcal{L}_\Diamond$ to $\mathcal{L}_\in$, which takes possible worlds to be transitive sets closed under pairing, plural quantification over those sets to be first-order quantification over their subsets, and pairs $\langle x, y\rangle$ to be set-theoretic ordered pairs, is just such an interpretation.

DEFINITION 5.26. *Unless otherwise stated, I will assume that T contains Extensionality and second-order Foundation. Let S^+_T be $\mathsf{S_T}$ + $\mathsf{Trans}_{Pairing}$ + the Mostowski collapse lemma.*[95] *By Claim 1, S^+_T proves Extensionality. So, since it also proves Pairing, it can be definitionally expanded with the standard axioms for $\langle , \rangle$. Let a world be a nonempty set closed under pairing, and let $w, w', w'', \ldots$ etc range over them. Then, let $^{tr}_z$ be the following translation from $\mathcal{L}_\Diamond$ to the expanded language.*[96]

[95] $\mathsf{S_T}$ proves Pairing and Union, and by Claim 1, Extensionality and Foundation. See §5.3.1.

[96] This translation closely follows Linnebo (2013, p. 20).

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- $x_z^{tr} = x$
- $\langle x, y\rangle_z^{tr} = \langle x, y\rangle$
- $Y_z^{tr} = y$
- $^{tr}_z$ *commutes with the atomic predicates and the connectives*
- $(\exists x\varphi)_z^{tr} = \exists x \in z\varphi_z^{tr}$
- $(\exists Y\varphi)_z^{tr} = \exists y \subseteq z\varphi_z^{tr}$
- $(\Diamond\varphi)_z^{tr} = \exists w \supseteq z\varphi_w^{tr}$ (*making sure to avoid clashes of variables*).

So, the translation just replaces all quantifiers with first-order quantifiers and then restricts them appropriately to worlds. As the next lemma shows, this restriction is redundant for a broad class of formulas.

DEFINITION 5.27. *For $\varphi \in \mathcal{L}_\Diamond$, let φ^* be the result of deleting the modal operators in φ and replacing second-order variables with first-order variables.*

LEMMA 5.28 (S^+_T). *Suppose $\vec{y} \subseteq w$. Then*:

$$\varphi_w^{tr} \leftrightarrow \varphi^*$$

when $\varphi \in \mathcal{L}_\Diamond$ is quasi-modalised with free variables among $\vec{x}, \vec{Y}$.

Proof. By induction on the complexity of φ. The only difficult cases are those for the quantifiers. So, suppose w, $\vec{y}$, and $(\Diamond\exists Z\varphi)$ are as in the lemma. Suppose, also that $(\Diamond\exists Z\varphi)$; that is:

$$\exists w' \supseteq w \exists z \subseteq w' \varphi_{w'}^{tr}.$$

So, by the induction hypothesis, $\exists z\varphi^*$, which is to say $(\Diamond\exists Z\varphi)^*$. Now, suppose $\exists z\varphi^*$. By $\mathsf{Trans}_{Pairing}$, there is a $w' \supseteq w \cup \{z\}$. So, $\varphi_{w'}^{tr}$, by the induction hypothesis, and thus $(\Diamond\exists Z\varphi)_w^{tr}$. The cases for the other quantifiers are proved similarly. □

This lemma shows that the translations of notions like $X, Y \vDash \varphi$, $M \sqsubseteq M'$, and φ_Y^{pt} in $\mathcal{L}_\Diamond$ are equivalent to their obvious analogues in $\mathcal{L}_\in$. For example, it shows that $(X, Y \vDash \varphi)_w^{tr}$ is the result of binding φ's first-order quantifiers to x, replacing its second-order quantifiers with first-order quantifiers over subsets of x, replacing occurrences of $w \in z$ with $\langle w, z\rangle \in y$, and replacing free second-order variables with first-order variables. I will use the same notation for these notions in $\mathcal{L}_\in$.[97]

The next lemma shows that in S^+_T, truth throughout the T structures is equivalent to truth in the universe of sets.[98]

LEMMA 5.29 (S^+_T). *Let j be a collapsing function for M with $\vec{x} \in M$ and $\vec{y} \subseteq M$.*[99] *Then*:

$$(\varphi_M^{pt})^* \leftrightarrow \varphi^*(j(\vec{x}), j[\vec{y}]),$$

where $\varphi \in \mathcal{L}_\in$ with free variables among $\vec{x}, \vec{Y}$, and $j[y] = \{j(z) : z \in y\}$.

Proof. By induction on the complexity of φ. The only difficult cases are those for the quantifiers. So, suppose $M, \vec{x}, \vec{y}$, j, and $\exists z\varphi$ are as in the lemma. Suppose also $(\exists z\varphi_M^{pt})^*$;

[97] I will also sometimes use $x \vDash \varphi$ to mean φ^x, which is just $x, y \vDash \varphi$ without the re-interpretation of $\in$ according to y. Context will make clear which notion is intended.

[98] See §2.3 and §2.7.4.

[99] A collapsing function for $M = x, y$ is a function f with $\mathsf{dom}(f) = x$, $\mathsf{rng}(f)$ is transitive, and $\forall z, z' \in x(\langle z, z'\rangle \in y \leftrightarrow f(z) \in f(z')))$.

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that is:

$$\exists M' \sqsupseteq M \exists z \in M' (\varphi^{pt}_{M'})^*.$$

The Mostowksi collapse lemma implies that there is a collapsing function i from M', since T contains Extensionality and second-order Foundation. It follows from the induction hypothesis that $\exists z \varphi^*(i(\vec{x}), i[\vec{y}])$. But, since $M \sqsubseteq M'$, a simple induction on M shows that i agrees with j on M. So, $(\exists z \varphi)^*(j(\vec{x}), j[\vec{y}])$. Now, suppose $(\exists z \varphi)^*(j(\vec{x}), j[\vec{y}])$. By $\mathsf{Trans}_{\mathsf{T}}$, there is a transitive set y satisfying T and containing M, z, and $\mathsf{rng}(j)$. We can use y to construct an isomorphic end-extension M' of M as follows. First, let t be any set not in y. Then, let $\mathsf{dom}(M') = \mathsf{dom}(M) \cup \{\langle t, x\rangle : x \in y \setminus \mathsf{rng}(j)\}$,[100] and let $i : \mathsf{dom}(M') \to y$ be such that $i(x) = j(x)$ when $x \in \mathsf{dom}(M)$ and $i(\langle w, x\rangle) = x$ otherwise. Moreover, let $M' = \{\langle x, y\rangle \in \mathsf{dom}(M') \times \mathsf{dom}(M') : i(x) \in i(y)\}$. It is easy to see that i is a collapsing function for M' which agrees with j on M. It follows from the induction hypothesis that $(\varphi^{pt}_{M'})^*(i^{-1}(z), \vec{x}, \vec{y})$. It is also easy to see that M' is a T structure end-extending M. So, $((\exists z \varphi)^{pt}_{M})^*$. The case for $\exists Z$ is handled similarly. □

THEOREM 5.30. $\mathsf{S}^+_{\mathsf{T}}$ *proves the* ^{tr}*-translations of all theorems of* $\mathsf{MSST}_{\mathsf{T}}$.

Proof. By induction on the length of proof, I will show that $\mathsf{S}^+_{\mathsf{T}}$ proves:

$$w \text{ is a world } \to \varphi^{tr}_{w} \qquad (*)$$

when φ is a theorem of $\mathsf{MSST}_{\mathsf{T}}$.

Case 1: Logic.

Axioms. It is straightforward but tedious to show that (*) holds for all the axioms of $\mathsf{MSST}_{\mathsf{T}}$'s underlying logic. The assumption that w is nonempty is used for right-to-left direction of the logical axiom for vacuous quantification, L3.

Rules of inference. It is also easy to see using GEN that applications of NEC, GEN, and MP are preserved.

Case 2: Axioms for pluralities and pairing.

It is again straightforward but tedious to show that (*) holds for all of the pairing and plurality axioms. We use the fact that w is transitive and satisfies Pairing for the pairing axioms, Separation for comp, Extensionality for PL2.

Case 3: E, EP, and S.

By Lemma 5.28, it suffices to show that (*) holds for E^*, EP^*, and φ^* for instances φ of S (because E, EP, and φ are quasi-modalised).

E^*. By $\mathsf{Trans}_{\mathsf{T}}$, there is a set x satisfying T. Thus, $x, \in \cap x \times x$ witnesses E^*.

EP^*. Let $x \subseteq \mathsf{dom}(M)$. By the Mostowski collapse lemma, there is a collapsing function j for M. By $\mathsf{Trans}_{\mathsf{T}}$, there is a transitive set y satisfying T and containing $\mathsf{dom}(M)$, $\mathsf{rng}(j)$, and $j[x]$. As in the proof of Lemma 5.29, we can use y to construct an isomorphic end-extension M' of M. Clearly, x will form a set in M' since $j[x]$ forms a set in y.

S^*. Let φ^{pt} be an instance of S. Since φ is a logical truth, it follows from Lemma 5.29 that $(\varphi^{pt})^*$. □

[100] The product $x \times y$ of any two sets exists by $\mathsf{Trans}_{Pairing}$.

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THEOREM 5.31 (The upper bound theorem). *Let $\varphi \in \mathcal{L}_{\in}$ be a sentence, and suppose* T *proves Extensionality and second-order Foundation. If* $\mathsf{MSST}_{\mathsf{T}}$ *proves* φ^{pt}*, then* $\mathsf{S}^{+}_{\mathsf{T}}$ *proves* φ.

Proof. Immediate from Theorem 5.30 and Lemmas 5.28 and 5.29. □

5.4. The ms-translations provable with reflection principles.

THEOREM 5.32. MSST + $\mathsf{MSR\text{-}syn}$ *is syntactically inconsistent.*

Proof. Let R be a Rosser sentence for $\mathsf{ZFC2}$ formulated so as to be Σ_1, and let R^* be a Σ_1 formulation of its negation in Z. (Since the Rosser sentence concerns the natural numbers, it can have the form $\exists x(x = \omega \wedge \varphi^x)$, and similarly for its negation (where "$x = \omega$" is Δ_0).) Since Z^* + In proves that $\mathsf{ZFC2}$ is syntactically consistent, it will prove that $\mathsf{ZFC2} + \neg\mathsf{R}$ and thus $\mathsf{ZFC2} + (\mathsf{R} \rightarrow \mathsf{collapse})$ are syntactically consistent (since $\neg R$ entails $R \rightarrow \mathsf{collapse}$).[101] From the main theorem it follows that MSST proves:

$$(\mathsf{ZFC2} + (\mathsf{R} \rightarrow \mathsf{collapse}) \text{ is syntactically consistent})^{pt}.$$

Now, working in MSST + $\mathsf{MSR\text{-}syn}$, we have

$$(R \rightarrow \mathsf{collapse})^{pt} \rightarrow \Diamond\exists M(M \vDash R \rightarrow \mathsf{collapse}). \tag{4}$$

Since $\mathsf{collapse}^{pt}$ is equivalent to EP[102] and ^{pt} commutes with $\rightarrow$, it follows that the antecedent of (4) is true. So, $\Diamond\exists M(M \vDash R^*)$ (because $\mathsf{collapse}$ is false in all possible M). Since R^* is Σ_1, Lemma 5.21 implies that $(R^*)^{pt}$ and thus $(\neg R)^{pt}$ by the main theorem. We can then run the preceding argument with $\neg R$ replacing R to get R^{pt}. So, $R^{pt} \wedge \neg R^{pt}$. □

The next lemma shows that when S is extended to $\mathcal{L}^2_{\in}$, the ms-translations second-order formulas are equivalent to the ms-translations of their first-orderisations. Let S_2 denote this extension, and let $\mathsf{MSST}^2_{\mathsf{T}} = \mathsf{MSST}_{\mathsf{T}} + \mathsf{S}_2$.

THEOREM 5.33 ($\mathsf{MSST}^2_{\mathsf{T}}$). *Suppose $\vec{x}, \vec{y} \in M$, $\vec{Y} \subseteq M$, and $M \vDash \vec{y} \equiv \vec{Y}$. Then*

$$\varphi^{pt}_{M} \leftrightarrow (\varphi^*)^{pt}_{M},$$

where $\varphi \in \mathcal{L}^2_{\in}$ with free variables among $\vec{x}, \vec{Y}$.

Proof. The proof is exactly the same as for Lemma 5.23, except that where we use Lemma 5.22 for bounded formulas we now use S_2 for arbitrary formulas in $\mathcal{L}^2_{\in}$.[103] □

COROLLARY 5.34. *Let $\varphi \in \mathcal{L}^2_{\in}$ be a sentence, and suppose* T *proves Extensionality, second-order Foundation, Pairing, and the Mostowski collapse lemma.* $\mathsf{MSST}^2_{\mathsf{T}}$ *proves* φ^{pt} *if and only if* S_{T} *proves* φ^*.

Proof. Assume T Extensionality and second-order Foundation. By the lower bound theorem, when S_{T} proves φ^*, $\mathsf{MSST}_{\mathsf{T}}$ proves φ^* and thus $\mathsf{MSST}^2_{\mathsf{T}}$ proves φ by Theorem 5.33, for any sentence $\varphi \in \mathcal{L}^2_{\in}$. Now, let φ^{pt} be an instance of S_2. In $\mathsf{S}^{+}_{\mathsf{T}}$, $(\varphi^{pt})^{tr}_{w}$ is equivalent to φ^* by Lemmas 5.28 and 5.29. But φ^* is just an instance of $\mathsf{L3}$. So, the proof of the upper bound theorem extends to show that $\mathsf{S}^{+}_{\mathsf{T}}$ proves the ^{tr}-translations of the theorems

[101] Indeed, it will prove that $\mathsf{ZFC2} + (\mathsf{R} \rightarrow \varphi)$ is syntactically consistent for any φ.

[102] See footnote 45.

[103] It is easy to see that Lemma 5.22 for arbitrary formulas in $\mathcal{L}^2_{\in}$ is a trivial consequence of S_2.

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of $\mathsf{MSST}^2_{\mathsf{T}}$. It follows from Lemmas 5.28 and 5.29 again that $\mathsf{MSST}^2_{\mathsf{T}}$ proves φ^{pt} only if S_{T} proves φ^*, for any sentence $\varphi \in \mathcal{L}^2_\in$. □

Let $\mathsf{S}^2_{\mathsf{T}}$ be S_{T} with its logical axioms and rules of inference extended to $\mathcal{L}^2_i n$. Since collapse and min-comp imply the schema $\varphi \leftrightarrow \varphi^*$, for sentences $\varphi \in \mathcal{L}^2_\in$,[104] $\mathsf{S}^2_{\mathsf{T}}$ proves φ whenever S_{T} proves φ^*. Moreover, a simple induction on the length of proofs shows that S_{T} proves φ^* whenever $\mathsf{S}^2_{\mathsf{T}}$ proves φ, because collapse* and min-comp* are the same trivial logical truth.

LEMMA 5.35 (MSST^2). $\mathsf{R}_\diamond$ *is equivalent to* ${\mathsf{R}_1}^{pt}$.

Proof. $\mathsf{R}_\diamond \Rightarrow {\mathsf{R}_1}^{pt}$. Suppose EM and $\vec{x} \in M$, where $\varphi \in \mathcal{L}_\in$ with free variables among $\vec{x}$. Suppose also that φ^{pt}_M. By $\mathsf{R}_\diamond$:

$$\diamond\exists M' \sqsupseteq M \exists y, w, z \in M'(M' \vDash y = V_w \wedge :$$

$$z \equiv \mathsf{dom}(M) \wedge z \in y \wedge (\mathsf{Trans}_{\mathsf{ZFC2}})^y \wedge (((\varphi)^{pt}_Z)^*)^y).$$

By Lemma 5.20, it will be the case that $M \vDash \mathsf{ZFC2}$. So, $M' \vDash z$ is transitive $\wedge\ \mathsf{ZFC2}^z$. Now, working in M', "z is transitive $\wedge\ \mathsf{ZFC2}^z$" will be absolute for y because it is Δ^*_0 and y is supertransitive. So, in y, z will be a transitive set satisfying ZFC2 and the identity function will be a collapsing function for it. Since y satisfies $\mathsf{S}_{\mathsf{ZFC2}}$, and since $E\vec{x}$ and $\vec{x} \in M$ whenever M exists, Lemma 5.29 implies that φ^y. So, $M' \vDash y = V_w \wedge \varphi^y$. But, since "$y = V_w \wedge \varphi^y$" is Δ^*_0, we know that $(y = V_w \wedge \varphi^y)^{pt}_{M'}$ by the extension of Lemma 5.21 Δ^*_0 formulas and theories T containing Extensionality and second-order Separation. So:

$$\diamond\exists M' \sqsupseteq M \exists y, w \in M'(y = V_w \wedge \varphi^y)^{pt}_{M'},$$

which is to say $(\exists\alpha\varphi^{V_\alpha})^{pt}_M$.

${\mathsf{R}_1}^{pt} \Rightarrow \mathsf{R}_\diamond$. Suppose EM, $\vec{x} \in M$, $\vec{Y} \subseteq M$, and φ^{pt}_M, where $\varphi \in \mathcal{L}^2_\in$ with free variables among $\vec{x}$, $\vec{Y}$. By EP:

$$\diamond\exists M' \sqsupseteq M \exists z, \vec{y} \in M'(M' \vDash z, \vec{y} \equiv \mathsf{dom}(M), \vec{Y})$$

because M' satisfies second-order Separation, and the $\vec{Y}$s will exist and be subpluralities of M whenever it exists, by PL3. Similarly, the $\vec{x}$s will exist and be elements of M whenever it exists. So, it will be the case that φ^{pt}_M by Lemma 5.20, and thus that $\varphi^{pt}_{M'}$ by S_2. Theorem 5.33 then implies that $(\varphi^*)^{pt}_{M'}$. Using R^{pt}_1 and the fact that $(\mathsf{Trans}_{\mathsf{ZFC2}})^{pt}_{M'}$ (by the main theorem and S_2), we would get $(\exists\alpha(\mathsf{Trans}_{\mathsf{ZFC2}} \wedge Ez \wedge \varphi^*)^{V_\alpha})^{pt}_{M'}$. In other words:

$$\diamond\exists M'' \sqsupseteq M' \exists t, w \in M''(t = V_w \wedge (\mathsf{Trans}_{\mathsf{ZFC2}} \wedge Ez \wedge \varphi^*)^t)^{pt}_{M''}.$$

Again, all of our parameters will continue to exist and be elements and subpluralities of M and M'. So, because "$t = V_w \wedge (\mathsf{Trans}_{\mathsf{ZFC2}} \wedge Ez \wedge \varphi^*)^t$" is Δ^*_0, we can use the extension of Lemma 5.21 to Δ^*_0 formulas to get:

$$\diamond\exists M'' \sqsupseteq M' \exists t, w \in M''(M'' \vDash t = V_w \wedge z \in t \wedge (\mathsf{Trans}_{\mathsf{ZFC2}} \wedge \varphi^*)^t).$$

As above, it will be the case that, in t, z is a transitive set satisfying ZFC2 and that the identity function is a collapsing function for it. So, since t satisfies $\mathsf{S}_{\mathsf{ZFC2}}$, Lemma 5.29 implies that $((\varphi^{pt}_Z)^*)^t$. Finally, it will remain the case that $M' \vDash z, \vec{y} \equiv \mathsf{dom}(M), \vec{Y}$ and

[104] See §2.7.3.

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$M \sqsubseteq M'$ by Lemma 5.20. Thus, $M'' \vDash z, \vec{y} \equiv \mathsf{dom}(M), \vec{Y}$, because "$x \equiv X$" is bounded, and so:

$$\Diamond \exists M'' \sqsupseteq M \exists t, w, \vec{y}, z \in M''(M'' \vDash t = V_w \wedge :$$
$$\vec{y}, z \equiv \vec{Y}, \mathsf{dom}(M) \wedge z \in t \wedge (\mathsf{Trans_{ZFC2}} \wedge (\varphi_Z^{pt})^*)^t). \qquad \square$$

THEOREM 5.36. $\mathsf{MSST}^2 + \mathsf{R}^\Diamond$ *exactly interprets* $\mathsf{Z}^* + \mathsf{In} + \mathsf{R}_1$ *via ms-translation.*

Proof. Let $\varphi \in \mathcal{L}_\in$ be a sentence. It is immediate from the main theorem and Lemma 5.35 that $\mathsf{MSST}^2 + \mathsf{R}^\Diamond$ proves φ^{pt} whenever $\mathsf{Z}^* + \mathsf{In} + \mathsf{R}_1$ proves φ. Now, suppose $\mathsf{MSST}^2 + \mathsf{R}^\Diamond$ proves φ^{pt}. So, for some instances ψ^{pt} and χ^{pt} of S_2 and R_1^{pt}, MSST proves $(\psi^{pt} \wedge \chi^{pt}) \rightarrow \varphi^{pt}$. It follows that $\mathsf{Z}^* + \mathsf{In}$ proves $[(\psi^{pt} \wedge \chi^{pt}) \rightarrow \varphi^{pt}]_w^{tr}$ by Theorem 5.30 and thus $(\psi^* \wedge \chi) \rightarrow \varphi$ by Lemmas 5.28 and 5.29. So, $\mathsf{Z}^* + \mathsf{In} + \mathsf{R}_1$ proves φ, since ψ^* is an instance of the logical axiom for vacuous quantification $\mathsf{L3}$ and χ is an instance of R_1. $\square$

Finally, the next two results show that $\mathsf{Z}^* + \mathsf{In} + \mathsf{R}_1$ goes beyond $\mathsf{Z}^* + \mathsf{In}$ in proving Π_1-Col, but that it goes no further: it does not prove all instances of Π_2-Col.

LEMMA 5.37. $\mathsf{Z}^* + \mathsf{In} + \mathsf{R}_1 \vdash \Pi_1$-$\mathsf{Col}$.

Proof. Working in Z^*, recall from Lemma 5.11 that Σ_1 formulas are absolute for H_κ, when κ is an uncountable cardinal. It is easy to check that $H_\kappa = V_\kappa$, when κ is inaccessible, and thus that $H_\kappa = V_\kappa$, when κ is a limit of inaccessibles. So, Σ_1 formulas are absolute for V_κ, when κ is a limit of inaccessibles. Now suppose $\forall x \exists y \varphi(x, y, \vec{z})$, where φ is Π_1 with free variables among $x, y, \vec{z}$. Applying R_1 to $\forall x \exists y \varphi(x, y, \vec{z}) \wedge E\vec{z} \wedge Eu \wedge \mathsf{Trans_{ZFC2}}$, we get a V_α which contains $\vec{z}$ and u, satisfies $\forall x \exists y \varphi(x, y, \vec{z})$, and for which α is a limit of inaccessibles (because "x is transitive $\wedge$ $\mathsf{ZFC2}^x$" is Δ_0^* and thus absolute for V_α, and because the transitive sets satisfying $\mathsf{ZFC2}$ are exactly the V_βs, for β inaccessible). Thus, $\forall x \in u \exists y \in V_\alpha \varphi(x, y, \vec{z})$. $\square$

LEMMA 5.38. $\mathsf{Z}^* + \mathsf{In} + \mathsf{R}_1 \nvdash \Pi_2$-$\mathsf{Col}$.

Proof. Working in $\mathsf{Z}^* + \mathsf{In} + \Pi_2$-$\mathsf{Col}$, I will build a model of $\mathsf{Z}^* + \mathsf{In} + \mathsf{R}_1$. Let "$x \vDash y$" be a $\Delta_1^{\mathsf{Z}^*}$ satisfaction relation (where x is a model and y a formula/finite variable assignment pair).[105] Suppose we could find a limit of inaccessibles κ such that:

$$\forall x \in V_\kappa(\exists \alpha(V_\alpha \vDash x) \rightarrow \exists \alpha < \kappa(V_\alpha \vDash x)). \tag{5}$$

Then, it would follow that $V_\kappa \vDash \mathsf{Z}^* + \mathsf{In} + \mathsf{R}_1$.[106] To see this, first note that the Δ_0^* formula "x is transitive $\wedge$ $\mathsf{ZFC2}^x$" will be absolute for V_κ, and so $V_\kappa \vDash \mathsf{S_{ZFC2}} = \mathsf{Z}^* + \mathsf{In}$. Second, note that if $\varphi(\vec{x})^{V_\kappa}$, then $\exists \alpha(V_\alpha \vDash \langle \text{“}\varphi\text{”}, \vec{x} \rangle)$ and $\langle \text{“}\varphi\text{”}, \vec{x} \rangle \in V_\kappa$, and so $\exists \alpha < \kappa(V_\alpha \vDash \langle \text{“}\varphi\text{”}, \vec{x} \rangle)$. Thus, $\varphi(\vec{x})^{V_\alpha}$ and because "$y = V_z \wedge \varphi^y$" is Δ_0^* and thus absolute for V_κ, it follows that $(\varphi(\vec{x})^{V_\alpha})^{V_\kappa}$.

I will now prove that such a κ exists in $\mathsf{Z}^* + \mathsf{In} + \Pi_2$-$\mathsf{Col}$. Let $\Phi(x, \alpha)$ abbreviate "$\exists \beta(V_\beta \vDash x) \rightarrow (V_\alpha \vDash x)$". Since "$x = V_y$" is Δ_0^*, it is $\Sigma_2^{\mathsf{Z}^*}$ by Lemma 5.12. So, since "$V_\alpha \vDash x$" is just "$\exists y(y = V_\alpha \wedge y \vDash x)$", and since "$x \vDash y$" is $\Sigma_1^{\mathsf{Z}^*}$, "$V_\alpha \vDash x$" is also $\Sigma_2^{\mathsf{Z}^*}$. So, $\Phi(x, \alpha)$ is $\Sigma_3^{\mathsf{Z}^*}$. Since Π_2-Col is equivalent to Σ_3-Col in Z^*,[107] and since trivially

[105] See, for instance, Kunen (2011) Definition I.15.5.

[106] The idea of using such an κ to get a model of $\mathsf{Z}^* + \mathsf{In} + \mathsf{R}_1$ is due to Lévy & Vaught (1961).

[107] See the remarks after Claim 1 in §5.3.1.

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$\forall x \exists \alpha \Phi(x, \alpha)$, it follows in Z^* + In + Π_2-Col that:

$$\forall \alpha \exists \beta [\beta \text{ is inaccessible } \wedge \forall x \in V_\alpha \exists \gamma \ < \beta \Phi(x, \gamma)]. \tag{6}$$

Let $\Psi(\alpha, \beta)$ abbreviate "$[\beta$ is inaccessible $\wedge \forall x \in V_\alpha \exists \gamma \ < \beta \Phi(x, \gamma)]$". Given (6), we can construct finite sequences of such α, β. Formally, $\forall n \exists f (f$ is a function $\wedge \mathsf{dom}(f) = n + 1 \wedge f(0) = 0 \wedge \forall m < n \Psi(f(m), f(m+1))$. Let $\forall n \exists f \mathrm{X}(n, f)$ abbreviate this.

In the presence of Π_2-Col, the bounded quantifiers "$\forall x \in V_\alpha$" and "$\exists \gamma < \beta$" can be absorbed into $\Phi(x, \alpha)$.[108] So, $\Psi(\alpha, \beta)$ is $\Sigma_3^{\mathsf{Z}^*}$ (because "β is inaccessible" has the $\Sigma_2^{\mathsf{Z}^*}$ formulation "$\exists y (y = V_\beta \wedge \mathsf{ZFC2}^y)$" by Lemma 5.12). Similarly, "$\forall m < n$" can be absorbed into $\Psi(\alpha, \beta)$. So, $\mathrm{X}(n, f)$ is $\Sigma_3^{\mathsf{Z}^*}$. It follows by Π_2-Col that there is some x such that $\forall n \exists f \in x \mathrm{X}(n, f)$. We can then use these functions in x to build an ω-sequence f such that $\mathrm{X}(f(n), f(n+1))$. Finally, it is straightforward to check that $\cup \mathsf{rng}(f)$ is a limit of inaccessibles for which (5) holds. □

5.4.1. Stability is unprovable in MSST - S. The next result shows that S is unprovable in MSST- S, even when it is supplemented with Hellman's accumulation principle.

DEFINITION 5.39. *Let* AP *denote:*

$$\Diamond \exists M (M \vDash \varphi) \wedge \Diamond \exists M' (M' \vDash \psi) \rightarrow \Diamond \exists M, M' (M \vDash \varphi \wedge M' \vDash \psi),$$

where $\varphi, \psi \in \mathcal{L}^2_\in$ *are sentences.*

THEOREM 5.40 (ZFC + In). MSST - S + AP *does not prove* S.

Proof. Let $\kappa_0 = 0$, $\kappa_{\alpha+1}$ be the least inaccessible greater than κ_α, and $\kappa_\lambda = \bigcup_{\alpha<\lambda} \kappa_\alpha$. Consider a Kripke model K with set of worlds $W = \{\langle \alpha, n \rangle : (\alpha < \omega \wedge n = 0) \vee (\alpha < \omega \dot{2} \wedge n = 1)\}$, where the first-order domain at $\langle \alpha, n \rangle$ is $\{\langle x, n \rangle : x \in V_{\kappa_\alpha}\}$ and the second-order domain is $\mathcal{P}(\{\langle x, n \rangle : x \in V_{\kappa_\alpha}\})$. Any worlds w, w' access each other. So we have two kinds of worlds. The 0-worlds are just 0-tagged copies of the first ω inaccessible ranks and the 1-worlds are just 1-tagged copies of the first $\omega \dot{2}$ inaccessible ranks. Thus, the 0-worlds and 1-worlds are completely disjoint. We interpret the pairing operator so that "$\langle \langle x, m \rangle, \langle y, n \rangle \rangle$" denotes $\langle \langle x, y \rangle, m \rangle$ if $m = n$ and otherwise 0 (since 0 is not in any of the domains). In other words, the pair of x, y behaves as expected when x, y can co-exist and is otherwise a dummy object. It is straightforward but tedious to verify that MSST - S is valid in K. I will now show that the following instance of S is false at some world in K.

$$(\exists \alpha (\mathsf{Trans}_{\mathsf{ZFC2}})^{V_\alpha})^{pt} \rightarrow \Box \forall M (\exists \alpha (\mathsf{Trans}_{\mathsf{ZFC2}})^{V_\alpha})^{pt}_M. \tag{7}$$

It is easy to check that V_α satisfies "$\exists \beta (\mathsf{Trans}_{\mathsf{ZFC2}})^{V_\beta}$" just in case $\alpha > \kappa_\omega$ (because "x is transitive $\wedge \mathsf{ZFC2}^x$" and "$x = V_y$" are Δ_0^*, and thus absolute for any V_β, and the V_γ s, for γ inaccessible, are precisely the transitive sets satisfying ZFC2). Let M be a ZFC2 structure in some 1-world w, where M is isomorphic to $V_{\kappa_{\omega+1}}$. So, $V_{\kappa_{\omega+1}} \vDash \exists \beta (\mathsf{Trans}_{\mathsf{ZFC2}})^{V_\beta}$, and thus $w \vDash M \vDash \exists \beta (\mathsf{Trans}_{\mathsf{ZFC2}})^{V_\beta}$. Since MSST-S is valid in the model, and since the proof of Lemma 5.21 and its extension to Δ_0^* formulas does not use S, it follows that $w \vDash (\exists \beta (\mathsf{Trans}_{\mathsf{ZFC2}})^{V_\beta})^{pt}$ (because "$x = V_y \wedge (\mathsf{Trans}_{\mathsf{ZFC2}})^x$" is Δ_0^*). Now, for contradiction, suppose that the consequent of (7) is true at w. Let M be any ZFC2 structure contained in any 0-world. It will follow that for some world w', $w' \vDash \exists M' \sqsupseteq M \exists \beta \in$

[108] See Devlin (1984), Lemma 11.6, which is easily generalised to show that in Z^* + Π_n-Col, $\exists x \in y \varphi$ has a Π_{n+1} formulation when φ is Π_{n+1}, and thus that $\forall x \in y \varphi$ has a Σ_{n+1} formulation when φ is Σ_{n+1}.

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$M'((\mathsf{Trans}_{\mathsf{ZFC2}})^{V_\beta})^{pt}_{M'}$. Again, by the extension of Lemma 5.21 to Δ_0^* formulas, it follows that $w' \vDash M' \vDash \exists\beta(\mathsf{Trans}_{\mathsf{ZFC2}})^{V_\beta}$. But that is impossible. Since the 0-worlds and 1-worlds are disjoint, w' will have to be a 0-world (because it contains M). So, $|M'| \leq |w'| = |V_{\kappa_n}|$, for some n. However, if $w' \vDash M' \vDash \exists\beta(\mathsf{Trans}_{\mathsf{ZFC2}})^{V_\beta}$, then M' is isomorphic to some V_α, for $\alpha > \kappa_\omega$. □

REMARK 5.41. *In §2.7.1, footnote 36, I claimed that there is a more general principle underlying* AP*, which says that isomorphic copies of any two structures can co-exist. Formally:*

$$\Box\forall M\Box\forall M'\Diamond\exists M''(\exists i : M \approx M'' \wedge \Diamond\exists M'''(\exists i : M' \approx M''' \wedge \Diamond(EM'', M'''))), \quad (*)$$

where $\exists i : M \approx M'$ formalises the claim that there is a plurality of ordered pairs coding an isomorphism between M and M'. It is straightforward to modify the construction above to make MSST - S + (*) *valid, but* S *fail. The idea is to take half of the things in the 0-worlds and add them to the 1-worlds. Then, any 0-world structure will be isomorphic to a 0-world structure which is also a 1-world structure. That will verify* (*)*. We can then run the argument above, picking the 0-world structure M so that it is disjoint from all the 1-worlds.*

5.5. **MSST *is satisfiable in a single world.*** The final result of this article shows that MSST is satisfiable in a Kripke model with a single world.

THEOREM 5.42 (ZFC + In). MSST^2 *is valid in an* S5 *Kripke model with one world.*

Proof. As in the proof of Theorem 5.40, let κ_ω be the least upper bound of the first ω inaccessibles. Let K be a Kripke model with set of worlds $W = \{0\}$. The first-order domain at 0 is just V_{κ_ω} and the second-order domain is $\mathcal{P}(V_{\kappa_\omega})$. We interpret $\langle x, y\rangle$ as the set-theoretic pair of x and y. It is straightforward but tedious to verify that MSST - E - EP - S is valid in K.

Now, suppose that $M \subseteq V_{\kappa_\omega}$ and $0 \vDash M \vDash \mathsf{ZFC2}$. Then, by absoluteness, M really is a ZFC2 structure. So, for some inaccessible κ, $|V_\kappa| = |M| \leq |V_{\kappa_\omega}|$. Thus, $\kappa = \kappa_n$. Using this fact, we can show by induction that:

$$(0 \vDash \varphi^{pt}_M) \leftrightarrow V_{\kappa_\omega} \vDash (\varphi^{pt}_{V_{\kappa_n}})^*(j\vec{(x)}, j[\vec{Y}]),$$

where j is an isomorphism between M and V_{κ_n}, and where $\varphi \in \mathcal{L}^2_\in$ with free variables among $\vec{x}, \vec{Y}$.[109] Now, it is easy to see that V_{κ_ω} satisfies $\mathsf{S}_{\mathsf{ZFC2}} = \mathsf{S}^+_{\mathsf{ZFC2}}$. Thus, by Lemma 5.29:

$$(0 \vDash \varphi^{pt}) \leftrightarrow V_{\kappa_\omega} \vDash \varphi^*$$

for sentences $\varphi \in \mathcal{L}^2_\in$. Now, EP is equivalent to $\mathsf{collapse}^{pt}$ in MSST - E - EP - S,[110] and that E is equivalent to $(\exists x(x = x))^{pt}$. We also know that $\mathsf{collapse}^*$ and φ^* are trivial

[109] The only difficult cases are those for the quantifiers. So, suppose $0 \vDash \exists M' \sqsupseteq M\exists z \in M'\varphi^{pt}_{M'}$. By the fact mentioned above, there is an isomorphism i from M' to some V_{κ_m}, for $m \geq n$. The induction hypothesis then implies that $(\varphi^{pt}_{V_{\kappa_m}})^*(i(z), i\vec{(x)}, i[\vec{Y}])$, and a simple induction shows that $i \restriction M' = j$. Moreover, V_{κ_m} is an end-extension of V_{κ_n}. So, $((\exists z\varphi)^{pt}_{V_{\kappa_n}})^*(j\vec{(x)}, j[\vec{Y}])$. In the other direction, we use any end-extension M' of V_{κ_n} to build an end-extension M'' of M, just as we did in the proof of Lemma 5.29. The second-order quantifier is handled similarly.

[110] To see this, note that Lemma 5.21 is provable without E, EP, or S.

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logical truths, where φ^{pt} is an instance of S_2. So, it follows that $0 \vDash \mathsf{E} \wedge \mathsf{EP} \wedge \varphi^{pt}$, where φ^{pt} is an instance of S_2. □

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BIBLIOGRAPHY

Benacerraf, P. (1973). Mathematical truth. *Journal of Philosophy*, **70**(19), 661–679.

Bernays, P. (1976). On the problem of schemata of infinity in axiomatic set theory. In Müller, G. H., editor. *Sets and Classes: On the Work by Paul Bernays*. Studies in Logic and the Foundations of Mathematics, Vol. 84. Amsterdam: North-Holland, pp. 121–172.

Boolos, G. (1998). The iterative conception of set. In Jeffrey, R., editor. *Logic, Logic, and Logic*. Cambridge, MA: Harvard University Press, pp. 13–29.

Boolos, G. (1998). *Logic, Logic, and Logic*. Cambridge, MA: Harvard University Press. Edited by Richard Jeffrey.

Clarke-Doane, J. (2017). What is the benacerraf problem? In Pataut, F., editor. *New Perspectives on the Philosophy of Paul Benacerraf: Truth, Objects, Infinity*. New York: Springer, pp. 93–125.

Cotnoir, A. J. & Baxter, D. L. M. (2014). *Composition as Identity*. Oxford: Oxford University Press.

Devlin, K. J. (1984). *Constructibility*. Perspectives in Mathematical Logic. Berlin: Springer-Verlag.

Drake, F. (1974). *Set Theory: An Introduction to Large Cardinals*. Studies in Logic and the Foundations of Mathematics. Amsterdam: North-Holland.

Feferman, S., Friedman, H. M., Maddy, P., & Steel, J. R. (2000). Does mathematics need new axioms? *The Bulletin of Symbolic Logic*, **6**(4), 401–446.

Fujimoto, K. (2012). Classes and truths in set theory. *Annals of Pure and Applied Logic*, **163**(11), 1484–1523.

Gödel, K. (1964). What is cantor's continuum problem? In Benacerraf, P., and Putnam, H., editors. *Philosophy of Mathematics*. Upper Saddle River, NJ: Prentice Hall, pp. 470–785.

Hale, B. (1996). Structuralism's unpaid epistemological debts. *Philosophia Mathematica*, **4**(2), 124–147.

Hellman, G. (1989). *Mathematics without Numbers: Towards a Modal-Structural Interpretation*. Oxford: Clarendon.

Hellman, G. (1996). Structuralism without structures. *Philosophia Mathematica*, **4**(2), 100–123.

Hellman, G. (2002). Maximality vs. extendability: Reflections on structuralism and set theory. In Malament, D., editor. *Reading Natural Philosophy*. Chicago: Open Court, pp. 335–361.

Hellman, G. (2005). Structuralism. In Shapiro, S., editor. *The Oxford Handbook of Philosophy of Mathematics and Logic*. Oxford: Oxford University Press, pp. 536–562.

Hellman, G. (2011). On the significance of the burali-forti paradox. *Analysis*, **71**(4), 631–637.

Hellman, G. (2015). Infinite possibilities and possibilities of infinity. In Auxier, R. E., Anderson, D. R., and Hahn, L. E., editors. *The Philosophy of Hilary Putnam*. La Salle, IL: Open Court, pp. 259–278.

https://doi.org/10.1017/S1755020318000138 Published online by Cambridge University Press

Hewitt, S. (2012). Modalising plurals. *Journal of Philosophical Logic*, **41**(5), 853–875.

Hughes, G. & Cresswell, M. (1996). *A New Introduction to Modal Logic*. London: Routledge.

Kanamori, A. (2003). *The Higher Infinite* (second edition). Berlin: Springer.

Koellner, P. (2006). On the question of absolute undecidability. *Philosophia Mathematica*, **14**(2), 153–188.

Koellner, P. (2009). On reflection principles. *Annals of Pure and Applied Logic*, **157**(2–3), 206–219.

Kunen, K. (2011). *Set Theory*. London: College Publications.

Lévy, A. (1965). *A Hierarchy of Formulas in Set Theory*. Memoirs of the American Mathematical Society, Vol. 57. Providence, RI: American Mathematical Society.

Lévy, A. & Vaught, R. (1961). Principles of partial reflection in the set theories of Zermelo and ackermann. *Pacific Journal of Mathematics*, **11**, 1045–1062.

Lewis, D. (1991). *Parts of Classes*. Oxford: Blackwell.

Linnebo, Ø. (2010). Pluralities and sets. *Journal of Philosophy*, **107**(3), 144–164.

Linnebo, Ø. (2013). The potential hierarchy of sets. *The Review of Symbolic Logic*, **6**, 205–228.

Linnebo, Ø. (2017). Plural quantification. In Zalta, E. N., editor. *The Stanford Encyclopedia of Philosophy* (Summer 2017 Edition). Available at https://plato.stanford.edu/archives/sum2017/entries/plural-quant/.

Parsons, C. (1974). Sets and classes. *Noûs*, **8**(1), 1–12.

Paseau, A. (2007). Boolos on the justification of set theory. *Philosophia Mathematica*, **15**(1), 30–53.

Pettigrew, R. (2012). Indispensability arguments and instrumental nominalism. *The Review of Symbolic Logic*, **5**(4), 687–709.

Putnam, H. (1967). Mathematics without foundations. *Journal of Philosophy*, **64**(1), 5–22.

Rayo, A. & Yablo, S. (2001). Nominalism through de-nominalization. *Noûs*, **35**(1), 74–92.

Roberts, S. (2017). A strong reflection principle. *The Review of Symbolic Logic*, **10**(4), 651–662.

Tait, W. W. (1998). Zermelo's conception of set theory and reflection principles. In Schirn, M., editor. *Philosophy of Mathematics Today*. New York: Oxford University Press, pp. 469–483.

Uzquiano, G. (1999). Models of second-order Zermelo set theory. *Bulletin of Symbolic Logic*, **5**(3), 289–302.

Uzquiano, G. (2003). Plural quantification and classes. *Philosophia Mathematica*, **11**(3), 67–81.

Uzquiano, G. (2011). Plural quantification and modality. *Proceedings of the Aristotelian Society*, **111**(2pt2), 219–250.

Uzquiano, G. (2014). Mereology and modality. In Kleinschmidt, S., editor. *Mereology and Location*. Oxford: Oxford University Press, pp. 33–56.

Welch, P. (2017). Global reflection principles. In Sober, E., Niiniluoto, I., and Leitgeb, H., editors. *Logic, Methodology and Philosophy of Science: Proceedings of the Fifteenth International Congress*. London: College Publications, pp. 18–36.

Zermelo, E. (1996). On boundary numbers and domains of sets: New investigations in the foundations of set theory. In Ewald, W., editor. *From Kant to Hilbert: A Source Book in the Foundations of Mathematics*, Vol. 2. New York: Oxford University Press, pp. 1219–1233.

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