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The conservation theorem for differential nets

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We prove the conservation theorem for differential nets – the graph-theoretical syntax of the differential extension of Linear Logic (Ehrhard and Regnier's DiLL). The conservation theorem states that the property of having infinite reductions (here infinite chains of cut elimination steps) is preserved by non-erasing steps. This turns the quest for strong normalisation (SN) into one for non-erasing weak normalisation (WN), and indeed we use this result to prove SN of simply typed DiLL (with promotion).
Along the way to the theorem we achieve a number of additional results having their own interest, such as a standardisation theorem and a slightly modified system of nets, $\mathsf{DiLL}_{\partial!}$.

1. Introduction

In λ-calculus, a β-reduction is *non-erasing* whenever it does not erase the argument involved in the redex, i.e. whenever it results in the substitution of a variable actually occurring in the body of the function. The conservation theorem states that non-erasing β-reduction preserves the property of having infinite reductions (Theorem 13.4.12, p. 339 of Barendregt (1981)). In this paper, we prove an analogous theorem for *differential nets* – the graph-theoretical syntax of the differential extension of Linear Logic (DiLL (Ehrhard and Regnier 2006)).

The conservation property is a powerful tool in the pursuit of SN, as the existence of a maximal non-erasing reduction sequence entails that *all* non-erasing reduction sequences are finite. Moreover, since erasing reductions are obviously finite and can be postponed after the non-erasing ones, a term having a maximal non-erasing reduction sequence enjoys SN for the whole reduction. Indeed another way of wording the theorem is to say that non-erasing reduction is *perpetual*, i.e. when starting from a non-strongly normalising term non-erasing steps will necessarily go on forever. One can say that the conservation theorem turns the quest for SN into one for non-erasing WN. Indeed a wealth of SN proofs for various typed calculi have been achieved through conservation theorems, e.g. Gandy (1980), Nederpelt (1973) and Sørensen (1997).

Various techniques have been proposed for achieving SN, alternative to the ones passing through a conservation theorem. Among them let us mention the proofs based on inductions on a suitable ordinal and the ones based on notions of computability or of reducibility candidates. The former usually consists in defining a measure on terms which decreases at any reduction step, with respect to a suitable well-ordering. Such a proof is worthwhile since it gives an appraisal of the complexity of the reduction by producing

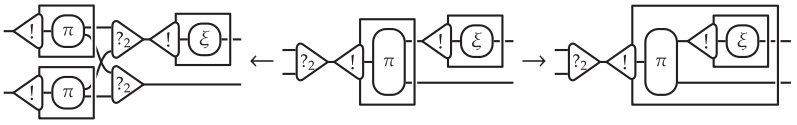
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explicitly an ordinal on which the induction relies. However, it is quite hard to show a decreasing measure even for rather simple calculi: for example no proof of this kind has been given for the simply typed λ-calculus[1].

On the other hand, the techniques based on computability (e.g. Tait (1967)) or on reducibility candidates (e.g. Girard (1972)) are very versatile and have been applied to various calculi, even quite complex such as Girard's System F. As Girard underlines in Girard *et al.* (1989), this versatility depends on the fact that the complexity of the reduction can be kept implicit, 'hidden' under the logical complexity of the definition of computability/reducibility candidate (basically an alternation of the quantifiers $\forall$ and $\exists$, as deep as the order of the type assigned to the term). Indeed, there is a common belief that such a logical complexity makes these techniques adaptable enough to be able to turn a proof of WN into one of SN (when it holds) just by minor changes in the definition of computability/reducibility candidate, namely without the need of some additional, complex lemmas such as a conservation theorem.

Although this is undoubtedly the case for typing systems of the λ-calculus together with many of its variants, arguably it might not be the case in general. For example, in some typed λ-calculi with explicit substitution, applying the technique of reducibility candidates to achieve SN needs some additional lemmas stating a property similar to conservation, e.g. Bonelli (2001). We suspect that this might be due to the absence of orthogonality, i.e. the presence of reduction rules overlapping and leading to non-trivial critical pairs. For a presentation of (weak) orthogonality in higher order rewriting systems we refer to Terese (2003, Section 11.6).

The proof nets of Linear Logic (LL (Girard 1987a)) yield another example of non-orthogonal rewriting system for which the reducibility candidates technique needs, as far as we know, a conservation theorem to achieve SN. Proof nets are graph-theoretical structures representing proofs of LL, both abstracting away unessential rule permutations and providing more local cut-elimination rules. On the one hand, proof nets avoid several commutative steps required in sequent calculus, cutting down the bureaucracy needed to 'unearth' a cut to be fired from under a pile of rules unrelated to the cut itself. On the other hand their cut-elimination is more local than in natural deduction, dividing in smaller steps the substitution of hypotheses. Proof nets yield a non-orthogonal rewriting since they have non-trivial critical pairs, as in



where we have two redexes overlapping at a box $!\pi$ (which represents the promotion rule of LL applied to a proof net π): one with a contraction $?_2$ and the other with another box $!\xi$, leading to different reductions. The reader unfamiliar with proof nets will find the definitions in Section 2.

[1] Providing such a proof is RTA's 19th open problem, proposed by Jean-Jacques Lévy in 1991 (http://www.win.tue.nl/rtaloop/problems/19.html), and still unsolved at the time of writing.

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As hinted above, though the reducibility candidate method can be applied to LL proof nets to get WN, the additional ingredient of a conservation theorem is needed to get to SN. In the original paper by Girard, this theorem is given the name of *standardisation*[2] (Theorem 4.25 in page 72 of Girard (1987a)), however its statement is incomplete and its proof is omitted altogether (see the discussion in Pagani and Tortora de Falco (2010)). Danos gives a satisfactory proof of it for MELL, the multiplicative exponential fragment of LL, baptising the result as *striction lemma* (théorème 8.31 p. 64 of Danos (1990)). Finally the first author and Tortora de Falco show it in Pagani and Tortora de Falco (2010), where it dons again Girard's name: the result is for a syntax for full LL (*sliced* proof nets) which is quite liberal with respect to the representation of the additive structure. Again this theorem is used to infer the SN property from the WN one, that one obtains by reducibility candidates, leading in turn to SN of full second order LL proof nets of Girard (1987a) via a simulation in sliced proof nets. It must be noted though that the conservation theorem is not directly obtained for the LL proof nets of Girard (1987a).

Here, apart from turning back to the *conservation* nomenclature, we concentrate on the syntax of differential linear logic, DiLL's *differential nets* (Ehrhard and Regnier 2006), and prove the conservation theorem (Theorem 32) in this system. DiLL extends MELL both with new rules (*codereliction*, *cocontraction* and *coweakening*, symmetric to the correspondingly named rules of MELL) and with a formal sum modelling a form of internal nondeterminism, somewhat similar, syntactically, to how slices are aggregated in sliced proof nets. In order to properly state the theorem we need to precisely define what a non-erasing reduction is (Definition 2): in fact reductions to 0 introduce new ways to erase parts of the nets, with the particularity of extending their erasing capabilities outside of the redex originating them.

As already explained, the conservation theorem will enable to transfer results of WN to SN, notably for simply typed and second order DiLL. While we put the latter aside, we will show how this is carried on for the former at the end of the paper (Section 5), by integrating the proof of WN given by the first author in Pagani (2009) with the finite development theorem given by the second one in Tranquilli (2009a).

The interest of DiLL lies in several aspects. First, it is based on a solid and interesting semantics, namely *finiteness spaces* (Ehrhard 2005). These spaces arise by relaxing the notion of duality proper to coherent spaces (the archetypal semantics of LL (Girard 1987a)). However, they give rise to a novel interpretation of LL as they can be equivalently seen as a special kind of topological vector spaces, where all the operations of LL are mapped to the corresponding operations of linear algebra. Most importantly, the usual factorisation of the intuitionistic arrow/function space $A \to B \cong !A \multimap B$ gets modelled as a space of *analytic functions*. Namely, finiteness spaces allow for a notion of differentiation, which was then lifted to the syntax as shown in Ehrhard and Regnier (2003) (for λ-calculus) and Ehrhard and Regnier (2006) (for LL, giving rise to the above mentioned symmetrisation of its rules).

[2] This nomenclature is quite misleading when related to λ-calculus. We reserve the name 'standardisation' for a result (Theorem 23) much closer to the renowned standardisation theorem of λ-calculus.

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Moreover, though finiteness spaces and DiLL allow for a sort of tamed nondeterminism, differential nets turn out to be a well-behaved rewriting system. After the first results stated in Ehrhard and Regnier (2006) but restricted to the promotion-free fragment, other works have strengthened this claim: as already mentioned, the first author showed WN of simply typed DiLL in Pagani (2009) (see also Gimenez (2011) for a different proof, based on reducibility candidates), the second showed confluence (or rather Church–Rosser modulo, as we will have occasion to explain) of the untyped case in Tranquilli (2009a), and we here add yet another brick to the solidity of the system. Confluence in presence of a formal sum modelling nondeterminism means that such a nondeterministic choice is completely internal and does not depend on the reduction strategy.

Finally, an enticing point of interest is the link with concurrency. Differentiation, as modelled by the codereliction rule, can be in fact seen as the introduction of a one-use entity that goes to only one among the possibly many who are asking for it. This has been employed by Ehrhard and Laurent (2007) to model concurrent communication, where only one among many may answer to a given signal. Their work establishes then a translation from a finitary fragment of π-calculus into (essentially) promotion-free differential nets, backed by a bisimulation. In this way a simple yet profound mathematical concept such as differentiation is linked to an important computer science one such as inter-process concurrent communication.

As for Pagani and Tortora de Falco (2010), our proof of the conservation theorem relies on what Girard refers to as Gandy's method (Girard (1987b), referring to Gandy (1980)), while called Nederpelt's lemma by Bezem and Klop (Terese (2003), referring to Nederpelt (1973)). Namely, we aim at defining an increasing measure which together with some form of confluence yields the implication $\mathsf{WN} \implies \mathsf{SN}$, via Gandy's/Nederpelt's lemma. We give an updated version of this result in the case of reduction modulo (Lemma 39 in Appendix A), as equivalences were already shown to be necessary in Tranquilli (2009a). As we will see confluence of non-erasing reduction fails in DiLL, so we will pass by a slightly modified system enjoying it, $\mathsf{DiLL}_{\partial\varrho}$. Once we will have defined DiLL in detail, we will spend some more words on the difficulties and the ideas motivating this choice (see subsection 2.3).

Outline. In Section 2, we will introduce all the syntax on which this paper is based. After a general introduction to the interaction net paradigm, both DiLL and $\mathsf{DiLL}_{\partial\varrho}$ are defined, together with the notion of non-erasing reduction. The geometric correctness criterion of switching acyclicity is also presented, which is necessary for most of the proofs to work.

Section 3 is devoted to proving the conservation theorem for $\mathsf{DiLL}_{\partial\varrho}$ (Theorem 19), which is followed in Section 4 by the main result of the paper, the same property for DiLL.

Section 5 gives an example of the application of this theorem, by showing how SN of simply typed DiLL can be deduced, after defining what a simple typing is.

Appendix A gives the necessary notions of rewriting and rewriting modulo an equivalence relation, and states the lemmas needed in the rest of the paper, proving those

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which were not found in the literature. Finally, Appendix B provides an index of the main notations and definitions introduced along the paper.

2. Differential nets

2.1. *Preliminaries*

Differential nets are defined on top of Lafont's interaction nets (Lafont 1990), which intuitively are circuits made of cells and wires connecting their ports. As we will see in subsection 2.3, the conservation theorem fails when considering truly untyped nets, hence we will deal with a very weak form of typing.

Suppose given an alphabet Σ of symbols and a set $\mathcal{T}$ of types endowed with an involutive negation $(\,.\,)^{\perp} : \mathcal{T} \to \mathcal{T}$. A **typed interaction net** α is given by the following data.

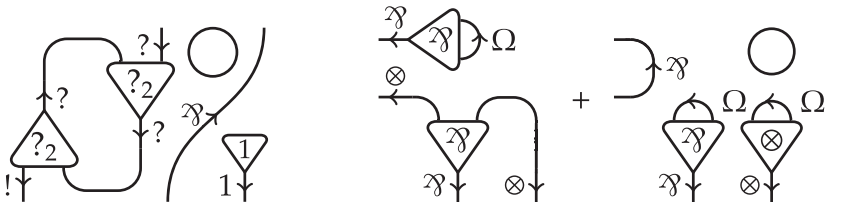
— A set $\mathsf{p}(\alpha)$ of **ports**.
— A set $\mathsf{c}(\alpha)$ of **cells**; to each cell c is assigned a **symbol** $\sigma(c)$ in Σ, a port in $\mathsf{p}(\alpha)$ called **principal**, and a number of other, **auxiliary** ones. How the latter are treated distinguishes between two kinds of cells: in **non commutative** ones, auxiliary ports are a finite sequence, in **commutative** ones they form a finite set. We denote by $\mathsf{deg}(c)$ the number of ports (principal and auxiliary) of the cell c, which is determined by the symbol $\sigma(c)$. Every port in $\mathsf{p}(\alpha)$ can be associated with at most one cell; a port associated with a cell is called **connected**, otherwise it is **free**. The set of free ports (also called conclusions) are denoted by $\mathsf{fp}(\alpha)$.
— A set $\mathsf{w}(\alpha)$ of **wires**, which can be either unordered pairs $\{p, q\}$ of ports, or **loops**, i.e. wires not connecting any port (intuitively short circuited wires). Each port is in exactly one wire. **Directed wires**, denoted by $\vec{\mathsf{w}}(\alpha)$, are ordered pairs (p, q) such that $\{p, q\}$ is a wire; an involutive duality is trivially defined by $(p, q)^{\perp} := (q, p)$. **Terminal wires** are the directed ones going to the free ports.
— A function $\ell : \vec{\mathsf{w}}(\alpha) \to \mathcal{T}$ (the **typing**) preserving duality, i.e. such that $\ell(d^{\perp}) = \ell(d)^{\perp}$. Without going too much into details, a typing discipline is enforced by imposing conditions which depend on the symbols of the cells.

Notice that a typing can be equivalently defined by assigning types to ports rather than wires, imposing that ports on the two ends of a wire get dual types.

In fact, in what follows we work with graphical representations of nets, which are much more intuitive to deal with. In such representations only cells and wires are drawn, and ports are left implicit, corresponding to wire endpoints. A cell is represented as a triangle, with its symbol depicted inside; the principal port is identified with an apex of the triangle while the auxiliary ones lay on the opposite side. Free ports appear as endpoints of dangling wires. Types may label wires once directions are chosen by drawing arrows on them. For example, the net in Figure 1a has five free ports, two cells of symbol $?_2$, one cell of symbol 1 and one loop. The set of types used in the figure will be defined in a while.

In order to introduce differential nets we also need a notion of addition between nets, intuitively modelling a non-deterministic superposition of nets. Hence we define our main

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(a) A simple net (b) A net

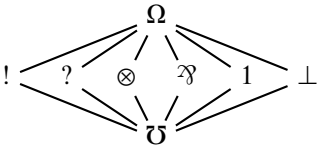
Fig. 1. Example of a simple net and of a net with two summands.

object of study, the **nets**, as formal finite sums of typed interaction nets, or equivalently finite multisets of nets, all sharing the same free ports and such that corresponding terminal wires have the same type. If we want to address a singleton net made by a single summand, we call it **simple net**. Figure 1b gives an example of a net with two summands. The correspondence between the free ports of the summands is given by the order of the terminal wires (from left to right and from up to bottom).

Initial Greek letters α, β, γ, δ (resp. middle Greek letters π, ξ, ρ, ν) range over simple nets (resp. nets). We use additive notation for multisets: 0 is the empty multiset, $\pi + \xi$ is the multiset union of π and ξ adding multiplicities.

2.2. *The system* DiLL

Lax types. We will use the following eight types, called **lax types**: ?, !, $\otimes$, $\parr$, 1, $\bot$, $\mathfrak{O}$ (**error**) and $\mathbf{\Omega}$ (**any**), i.e. the usual six type constructors of MELL (though all taken as *constant* types) together with two new ones. The involutive duality of lax types is defined by $!^\bot = ?$, $\otimes^\bot = \parr$, $1^\bot = \bot$ while $\mathfrak{O}$ and $\mathbf{\Omega}$ are auto-dual. Moreover, they are given an order $\sqsubseteq$ defined by just setting $\mathbf{\Omega}$ and $\mathfrak{O}$ as the top and bottom elements respectively, as depicted below.



It is not a subtyping relation as it is preserved, rather than inverted, by duality. Given two lax types A and B the type $A \sqcap B$ is defined as the meet of the two. So for example $? \sqcap ! = \mathfrak{O}$ and $\otimes \sqcap \mathbf{\Omega} = \otimes$. Notice that $(A \sqcap B)^\bot = A^\bot \sqcap B^\bot$.

Syntax. The promotion rule of linear logic corresponds in the syntax of nets to the **exponential box**, which is a special cell having a whole net as symbol. This means that the set DiLL of differential (poly)nets should be given by induction on the exponential depth: first we define DiLL_d for every $d \in \mathbb{N}$ and then we set $\mathsf{DiLL} := \bigcup_{d=0}^{\infty} \mathsf{DiLL}_d$.

The nets of DiLL_0 are built from all the symbols in Figure 2 but the box one. The typing of the ports should respect the rules shown in the figure, which must be intended

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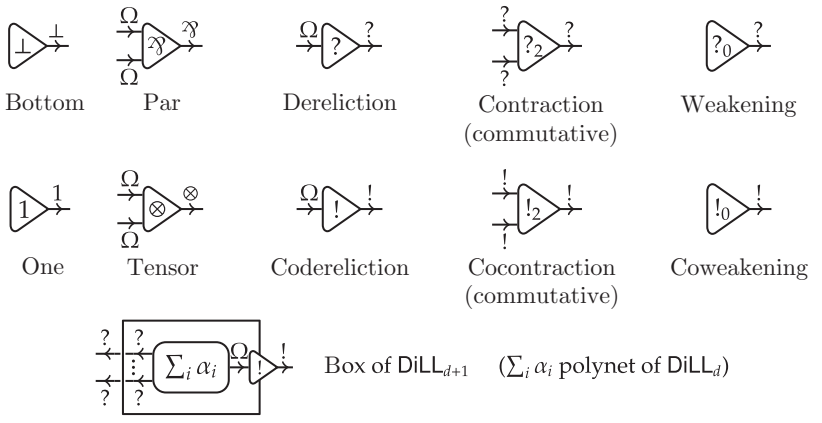
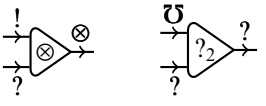


Fig. 2. The cells of differential interaction nets, together with their typing rules.

downward closed with respect to $\sqsubseteq$, so that for example



are both admissible typings. Notice in particular that every net is typable in this way, by just assigning $\mathfrak{O}$ to all wires. Lax typing will play a real role in the definition of context closure and reduction, as we will see in a while. Apart from types, DiLL_0 gives exactly the differential *interaction* nets presented in Ehrhard and Regnier (2006).

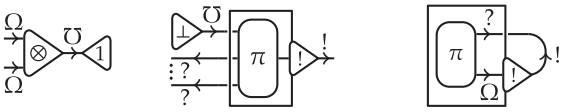
The nets and nets of DiLL_{d+1} are the ones built from the cells of Figure 2 where for each box c its symbol is a net $\sum \alpha_i$ in DiLL_d and there is a bijection between the ports of c and the free ports of $\sum \alpha_i$. The symbol $\sum \alpha_i$ of the box c is also called its **contents**. We will denote by $!(\sum \alpha_i)$ a generic box having $\sum \alpha_i$ as contents. As can be seen in the figure, boxes follow a special graphical convention with respect to other cells.

A DiLL net π is one of DiLL_d for any d; the **depth** of π is the minimal d such that $\pi \in \mathsf{DiLL}_d$ (in fact, the maximal number of nested boxes). Let $\mathsf{p}_!(\pi)$, $\mathsf{fp}_!(\pi)$, $\mathsf{c}_!(\pi)$, $\mathsf{b}_!(\pi)$ and $\mathsf{w}_!(\pi)$ be respectively the set of all occurrences of ports, free ports, cells, boxes, and wires occurring in π, including in all contents of the boxes in π. We can partition those sets by depth, so we will denote by $\mathsf{p}_i(\pi)$, $\mathsf{fp}_i(\pi)$, $\mathsf{c}_i(\pi)$, $\mathsf{b}_i(\pi)$ and $\mathsf{w}_i(\pi)$ the corresponding elements contained in i nested boxes, where i is called the depth of the element in π[3].

Operational semantics. A port is **active** if it is either a principal port, or an auxiliary port of a box. A wire linking two active ports is a **cut**. A **clash** is a cut that can be typed exclusively with $\mathfrak{O}$, a **deadlock** is a cut between two ports of the same cell (namely a box). In Figure 3 we give examples of deadlock and clash.

[3] All of this can be defined more formally by an inductive definition.

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(a) Clashes (b) Deadlock

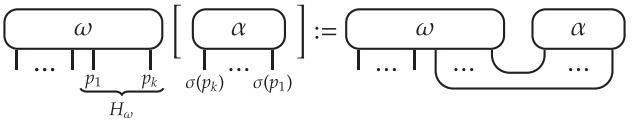
Fig. 3. Examples of clashes and deadlock.

We will now define some reductions on nets, among which cut-elimination is the chief one. Deadlocks and clashes are as usual non-reducible cuts, hence being a normal form does not entail being cut-free. This oddity with respect to a standard proof-theoretical framework is due to the wide freedom we authorise: in fact lax typing, as opposed to usual typing (see Section 5), allows clashes, while the absence of a logical correctness criterion allows deadlocks. One may wonder therefore what is the role of lax typing. In fact we set *all* $\parr$-typed cuts to be irreducible, and set in place a mechanism dynamically blocking certain cuts in this way. More details will be given in the following, while the reasons behind lax typing will be given while explaining the example in Figure 9.

As with various calculi, the reduction of differential nets can be presented as the context closure of a set of reduction rules, presented as pairs consisting of a simple net (the **redex**) and a net (the **contractum**).

A **linear context** $\omega[\,]$ is a simple net ω together with a subset H_ω of its free ports (the **hole** of $\omega[\,]$). The set of free ports of $\omega[\,]$ is defined as $\mathsf{fp}(\omega[\,]) := \mathsf{fp}(\omega) \setminus H_\omega$. The adjective *linear* underlines that the context is not a sum and the hole is not inside a box, so that the extension of plugging by linearity that we will present shortly makes indeed sense for the theory of DiLL.

Given a simple net α and a bijection σ between H_ω and the free ports of α, the plugging $\omega[\alpha]$ of α in the hole of $\omega[\,]$ amounts to identifying the ports according to σ and welding the wires that come together in this way[4].



The typing of $\omega[\alpha]$ will be obtained from the one of ω and the one of α by setting for each directed wire the type $A_1 \sqcap \cdots \sqcap A_n$ where A_i are all the types of the directed wires that got merged together during glueing. Notice that the typing of $\omega[\alpha]$ respects the condition of assigning dual types to opposite wire directions since $(A \sqcap B)^\perp = A^\perp \sqcap B^\perp$ holds; moreover it respects the typing discipline as typing rules are downwards closed for $\sqsubseteq$. The definition of plugging is then extended by linearity when we plug a net, by setting $\omega\left[\sum_i \alpha_i\right] := \sum_i \omega[\alpha_i]$.

Finally, **contexts** generalise the concept in the following way. A linear context $\omega[\,]$ is a context; furthermore if $\omega[\,]$ is a context then:

[4] For the quite delicate technical details the reader is referred to Vaux (2007); de Falco (2009).

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1. $\psi[\omega[\,]]$ for $\psi[\,]$ linear is a context[5] (notice $H_{\psi[\omega[\,]]} = H_\omega$ and $\mathsf{fp}(\psi[\omega[\,]]) = \mathsf{fp}(\psi[\,])$);
2. $\omega[\,] + \pi$ for π net is a context (recall that by definition $\mathsf{fp}(\omega[\,] + \pi) = \mathsf{fp}(\pi) = \mathsf{fp}(\omega[\,]) = \mathsf{fp}(\omega) \setminus H_\omega$);
3. $!\omega[\,]$, i.e. a box containing a context, is a context (again by definition the ports of the box will correspond to the free ports of $\omega[\,]$ as redefined).

The **depth of the hole** of $\omega[\,]$ is defined as the number of nested boxes containing the hole. Plugging is easily extended to all contexts. In particular notice that sums do not commute with boxes, i.e. $!\left(\sum_i \alpha_i\right) \neq \sum_i !(\alpha_i]$. Final Greek letters $\psi[\,], \omega[\,]$ range over contexts. The **context closure** $\tilde{R}$ of a relation R is then defined by $\pi\ \tilde{R}\ \xi$ iff $\pi = \omega[\pi']$, $\xi = \omega[\xi']$ and $\pi'\ R\ \xi'$. In this case, taking $\omega[\,]$ with maximal hole depth[6] we call the depth of the hole in ω the **depth of the step** $\omega[\pi']\ \tilde{R}\ \omega[\xi']$.

We are now able to define **cut-elimination** $\xrightarrow{\text{cut}}$ as the context closure of the rules in Figure 4. Notice that each rule fires on a unique cut, and it is identified by the ordered pair of the symbols of the two interacting cells, so e.g. $1/\bot$, $?_0/!_2$, etc. will denote a specific rule. In particular, $!\pi/!\xi$ refers to the rule firing on a cut between the principal port of $!\pi$ and an auxiliary port of $!\xi$. We also call such pairs the **kind** of either the reduction rule, the fired cut or the redex.

Remark 1. Let us point out some features specific to lax typing.

1. Lax typing can change during reduction, and its point is exactly that: the aim of lax typing is to memorise those cuts that are or *have been* clashes, using the $\mathfrak{V}$ type. Indeed the condition on the type of the cut of each redex (i.e. not being $\mathfrak{V}$) blocks the reduction of any cut that is a descendant of a clash. This is our reason for including typing as an integral part of the definition of net in Section 2.1: lax typing is not a certificate that can be discarded once the type assignment has been done, as it has an active role in defining reduction.
2. Thanks to the fact that typing rules are downward closed with respect to $\sqsubseteq$, it can be seen that each rule preserves the lax typing modulo $\sqsubseteq$, in the sense that if $A_1, \ldots, A_n$ are the types of the terminal wires of a redex, and if $B_1, \ldots, B_n$ are the types of the corresponding terminal wires of its contractum, then $B_i \sqsubseteq A_i$, for every $i \leqslant n$.
3. Given a net and a redex in it, due to the glueing operation changing the types on the two interfaces (rather than requiring them to match as is usual with such an operation), there may be different typings given to the redex and the context. However, it is straightforward to see that the result of reducing the redex does not depend on such differences (a fact boiling down to associativity of the $\sqcap$ operator).

More discussion on lax typing will be done while discussing Figure 9.

In addition to cut-elimination, we consider the **equivalences** and **canonical reductions** of Figure 5. The reason for taking them into consideration is that cut-elimination in DiLL fails to give a confluent rewriting, not even locally. Indeed, we cannot ignore

[5] Composition of contexts should be defined, but it is trivial once plain plugging is defined.
[6] In general the depth may vary, though in the cases of the reduction and conversion steps we will define such a depth will be unique.

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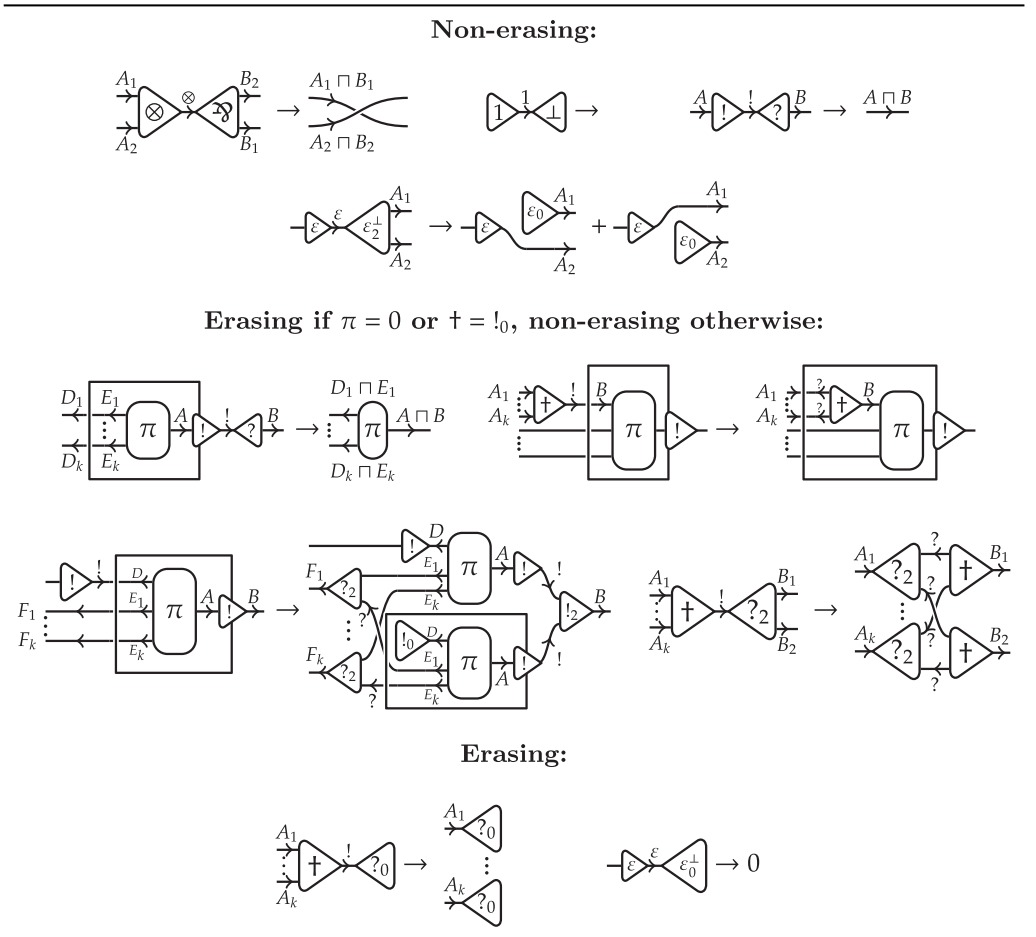


Fig. 4. Reductions defining the cut-elimination of DiLL. The symbol † denotes either !ξ (i.e. a box), !2 or !0; the symbol ε denotes either ! or ?. Notice that both 1 on ⊥ and ?0 on !0 reduce to the empty net, not to be confused with 0. Notice also that the lax type on the cut is a requirement for the reduction, equivalent to it not being $\mathfrak{V}$. Some lax types are omitted as they do not participate and are unchanged by the reduction.

associativity of (co)contractions and neutrality of (co)weakening over (co)contraction (see Tranquilli (2009a) and the discussion of subsection 2.3 regarding Figure 7). This prompts us to introduce the former as an equivalence and the latter as a reduction. As we need to consider reduction modulo an equivalence anyway, we also study other useful equivalences which are optional though must be taken together. Each equivalence is accompanied by a reduction which in a sense settles a zeroary case of the equivalence. Reversing each of these gives unwanted looping reductions. Again, the interested reader will find more details in Tranquilli (2009a). The **associative**, **push** and **bang sum** equivalences (denoted by $\overset{\mathtt{a}}{\sim}$, $\overset{\mathtt{p}}{\sim}$ and $\overset{\mathtt{s}}{\sim}$ respectively), together with the **neutral**, **pull** and **bang zero** reductions (which *do not* reduce cuts and are denoted by $\overset{\mathtt{n}}{\to}$, $\overset{\mathtt{p}}{\to}$ and $\overset{\mathtt{z}}{\to}$), are

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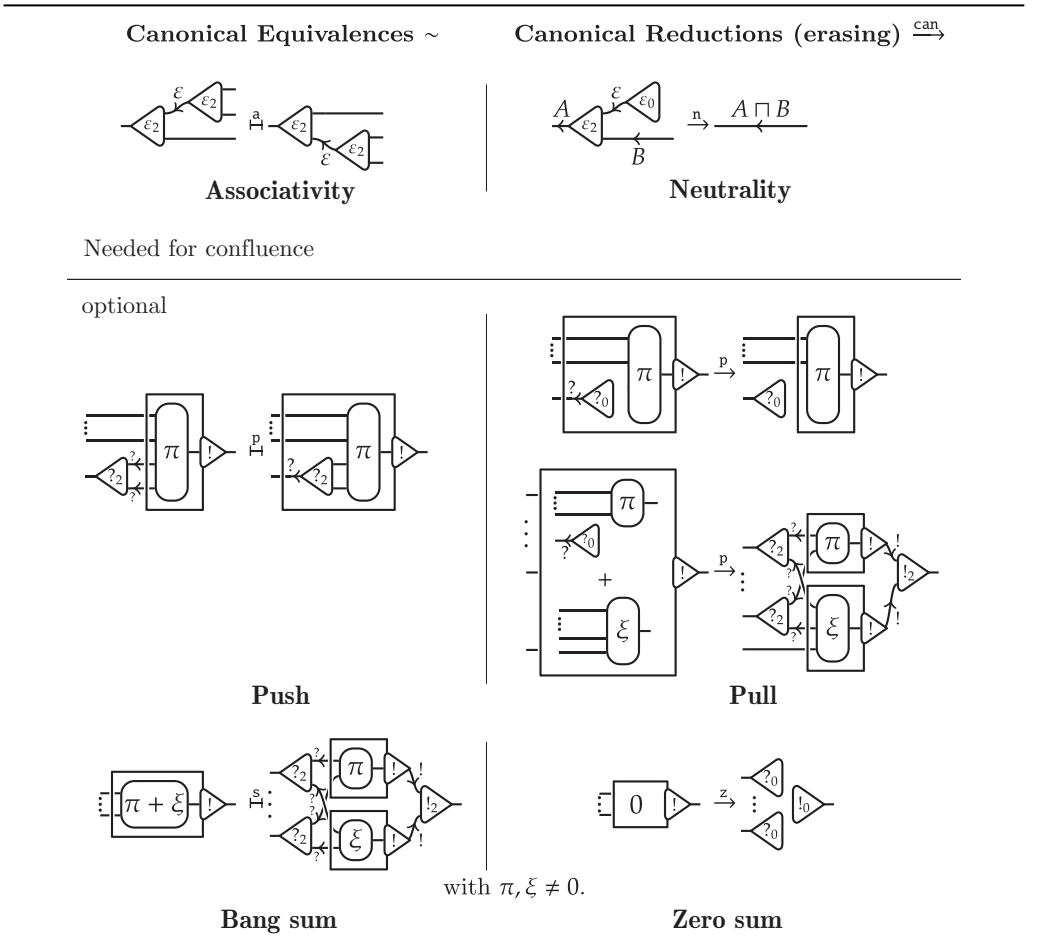


Fig. 5. The rules defining equivalences (left) and canonical reductions (right): the symbol $\overset{a}{\mapsto}$ (resp. $\overset{p}{\mapsto}$, $\overset{s}{\mapsto}$) denotes the one-step symmetric conversion generating the associative equivalence $\overset{a}{\sim}$ (resp. the push one $\overset{p}{\sim}$ and the bang sum one $\overset{s}{\sim}$). At the right we have the rules for neutral reduction $\overset{n}{\to}$, the two pull ones $\overset{p}{\to}$ (total above and partial below) and the bang zero one $\overset{z}{\to}$. The condition $\pi, \xi \neq 0$ applies to all rules. The symbol ε denotes either ! or ?. Most of the lax types are omitted as they are left unchanged.

the context closure of the rules shown in Figure 5. The $\pi, \xi \neq 0$ condition is needed, lest one would be able to spawn trees of contractions from nothing, giving dummy infinite reductions.

The push equivalence[7] has already been studied in the literature on proof nets and explicit substitutions (Di Cosmo and Guerrini 1999; Di Cosmo *et al.* 2003). The pull

[7] The name may be misleading, as an equivalence is not directed. It comes from Di Cosmo and Kesner (1997) and Tranquilli (2011) where it was treated as a reduction, and we felt like keeping it because of the good name pairing with the pull reduction.

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reduction comes here in two versions, the total (pulling away weakenings from inside boxes) and the partial one (which in fact does the same but from just a part of the sum inside, doing a sort of on-the-fly $\mathtt{s}$-conversion). The total pull reduction, which is the one making sense also in LL, was indeed considered in Di Cosmo and Kesner (1997) and Di Cosmo and Guerrini (1999). The partial one is needed to give local coherence and is thus necessary to give Church–Rosser modulo (see Appendix A for the notions relative to confluence in presence of an equivalence relation).

It was already known that the bang sum and zero are valid semantically and observationally. As the second author has shown in Tranquilli (2009a), we can (and must, if we are to use the push and pull rules) also treat them syntactically. It is interesting to note that these rules implement the well known exponential isomorphisms $!A \otimes !B \cong !(A \,\&\, B)$ and $!\top = 1$ from linear logic (see Ehrhard (2005)).

From now on $\mapsto$, $\sim$ and $\xrightarrow{\mathtt{can}}$ (symmetrical relation, equivalence and canonical reduction, respectively) will denote either $\overset{\mathtt{a}}{\mapsto}$, $\overset{\mathtt{a}}{\sim}$ and $\xrightarrow{\mathtt{n}}$ or the union of the $\mathtt{a}$, $\mathtt{p}$ and $\mathtt{s}$ conversions and the $\mathtt{npz}$-reduction respectively. The union of $\xrightarrow{\mathtt{cut}}$ and $\xrightarrow{\mathtt{can}}$ is denoted simply by $\to$, as it is *the* reduction of the system. Similarly, we denote by $\rightsquigarrow$ the concatenation $\sim\!\!\to\!\!\sim$.

In what follows we partition the reduction $\to$ in *erasing* and *non erasing*, giving the main ingredient for the statement of the conservation theorem (Theorem 32). For some of the rules the distinction is clear: weakening on box and reductions to 0 are clearly erasing whole parts of the net, possibly containing the source of an infinite reduction. For others the decision is less clear (like for canonical reductions, or the harmless ones involving (co)weakenings) and has been guided by more technical reasons, usually to make some proofs easier.

Definition 2. DiLL's **non-erasing** reduction $\xrightarrow{\neg\epsilon}$ is the context closure of the following cut-elimination steps (see Figure 4):

$$\neg\epsilon := \otimes/\parr + 1/\bot + !/? + !_2/? + !\pi/? + \dagger/!\pi + \dagger/?_2,$$

with $\pi \neq 0$ and $\dagger$ denoting either a box (possibly $!0$), a codereliction or a cocontraction.

DiLL's **erasing** reduction $\xrightarrow{\epsilon}$ is the context closure of the other cut-elimination steps and the canonical reduction ones (see Figures 4, 5):

$$\epsilon := !0/? + \dagger/!0 + \dagger/?_0 + !_0/\ddagger + \mathtt{n} + \mathtt{p} + \mathtt{z},$$

with $\dagger$ denoting either a box (possibly $!0$), $!_0$, $!$, or $!_2$, and $\ddagger$ denoting either a box, $?_0$, $?$ or $?_2$.

Notice that $\xrightarrow{\neg\epsilon}$ and $\xrightarrow{\epsilon}$ are disjoint and their union yields the whole set of reduction rules introduced so far for DiLL. Moreover, non-erasing reduction is a subset of cut-elimination.

2.3. *Examples and remarks*

Let us get to know DiLL by discussing some examples.

2.3.1. *An example of reduction.* Consider the simple net π_1 which is at the top left corner of Figure 6. The depth of π_1 is 1, it has three cuts at depth 0 (a clash, typed by

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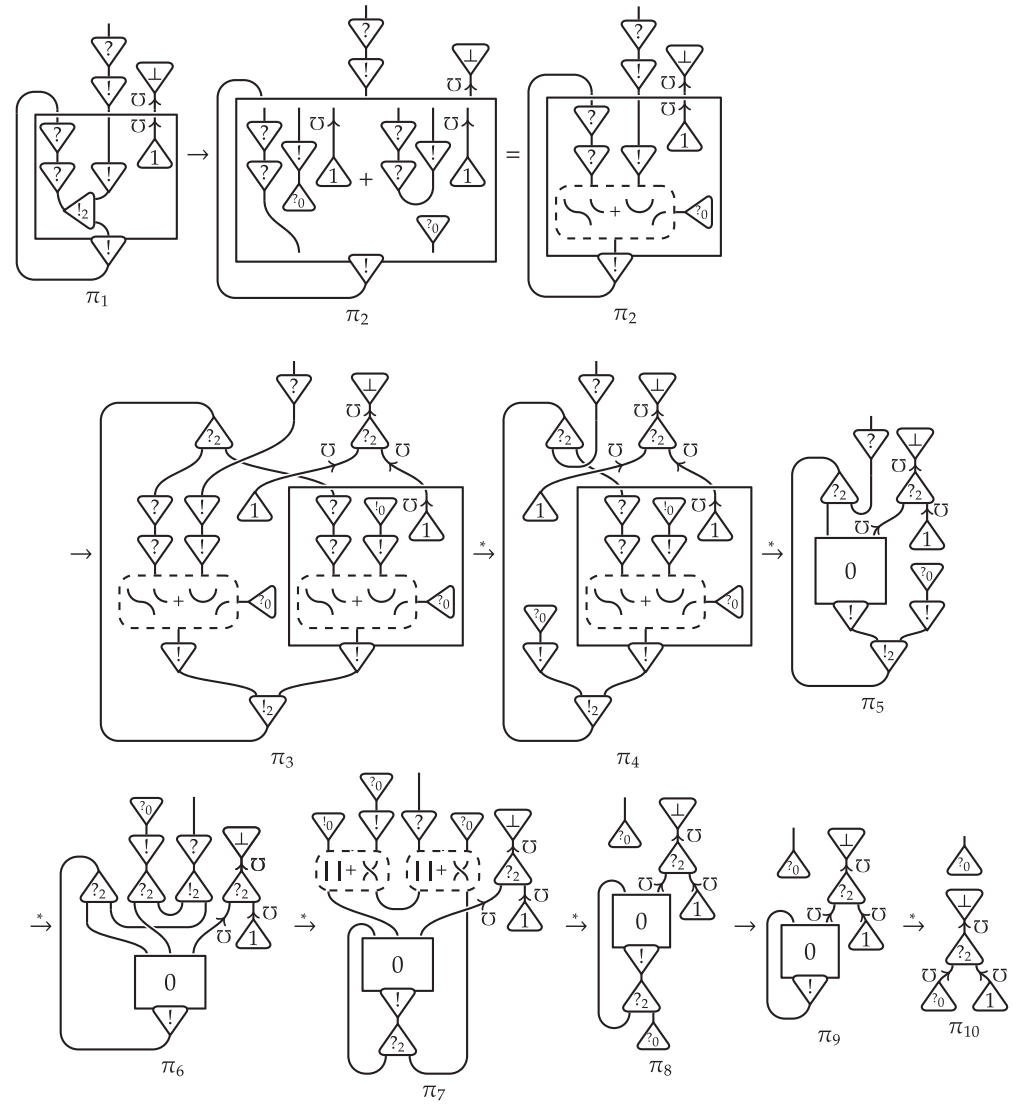


Fig. 6. Example of a reduction in DiLL.

$\mathfrak{V}$, a deadlock and a reducible cut of kind $!/!\rho$) and one cut at depth 1, of kind $?/!_2$. We have explicitly labelled only the wires with type $\mathfrak{V}$, as the other types are irrelevant to the example. This policy will be taken almost everywhere in the paper: types are left implicit whenever irrelevant or deducible. We will now describe each of the depicted reduction steps.

$\pi_1 \to \pi_2$**.** The first step fires on the cut inside the box. It is non-deterministic, creating a sum of two summands, depending on which premise of the cocontraction gets the dereliction (the other premise receiving a weakening instead). Here is a major difference with respect to Linear Logic, in which nets have an essentially deterministic

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reduction[8]: DiLL's cut-elimination can *create* sums from simple nets. Notice that the sum duplicates the context at its same exponential depth but it does not spread outside the box: this is akin to the additive duplication as modelled by sliced proof nets of LL (Girard 1987a; Pagani and Tortora de Falco 2010). In this example, however we will depict the nets by gathering the context shared by their summands, so that π_2 is equivalently written as in the top right corner of Figure 6.

$\pi_2 \to \pi_3$**.** This step fires on the codereliction against box cut, which is a very delicate reduction step of DiLL. Indeed this kind of step affects the cuts on all the ports of the box. In particular in this example the deadlock has been changed into a reducible cut of kind $?_2/!_2$, and the clash has changed its kind even if staying a clash. This is another strong difference with respect to Linear Logic: in LL only the cuts on the principal port of a box can affect the cuts on the other ports, while in this example the cut is on an auxiliary port. It is because of this very difference that we have had to introduce lax typing, necessary in DiLL to get conservation while LL does not need it. We will discuss in detail this point later, in Figure 9.

$\pi_3 \xrightarrow{*} \pi_4 \xrightarrow{*} \pi_5$**.** The simple net π_4 is obtained from π_3 performing a $?_0/!$ step, eliminating (i.e. reducing to 0) one of the two summands of π_3, and (in the other summand) performing two consecutive !/? steps. We then get π_5 by reducing the cuts in the box of π_4: such a reduction yields an empty box !0, since both the summands of the contents of the box of π_4 are erased. The attentive reader has certainly noticed that π_5 has been pictured by swapping the premises of its (co)contractions, benefiting from the commutative nature of these cells.

$\pi_5 \xrightarrow{*} \pi_6 \xrightarrow{*} \pi_7 \xrightarrow{*} \pi_8$**.** The reduction sequence $\pi_5 \xrightarrow{*} \pi_6$ fires on the $?_2/!_2$ cut and on the $!_2/!0$ one created in the first step. The $!_2/!0$ step shoves a copy of the cocontraction inside the empty box, making it disappear. The net π_7 is the result of the reduction of the two cuts in π_6 of kind $!/?_2$ and $?/!_2$ respectively. Since both kinds are non-deterministic, π_7 is a sum of four summands. Notice that the number of steps needed to reach π_7 from π_6 is not two but three, due to the context duplication generated by the first step. Finally, the simple net π_8 is obtained by reducing three summands of π_7 to 0, by steps of kind $!_0/?$, $!/?_0$ and $!/!0$, respectively (this last one resulting in 0 since it opens an empty box), and by performing an !/? and an $!_0/!0$ step on the remaining summand.

$\pi_8 \to \pi_9 \xrightarrow{*} \pi_{10}$**.** The net π_9 is the result of a canonical step `n`. Notice that π_9 is a normal form with respect to the reduction without the optional rules of Figure 5. Be aware that π_9 contains however two irreducible cuts (a deadlock and a clash): since nets are just laxly typed and might contain vicious cycles, being a normal form does not imply being cut-free. Besides, if we take into account the whole set of rules defined so far, π_9 can be further reduced by one `z` and then one $!_0/?_0$ step, achieving the normal form π_{10}. Notice that π_8 is not SN, as an infinite reduction starts from it (by reducing the $?_2/!0$ cut, then those created by that reduction and so on...) while π_9 and π_{10} are.

[8] Actually, some side non-determinism is given by the additive unit $\top$ and the commutative additive step in presence of additive boxes (Tortora de Falco 2003).

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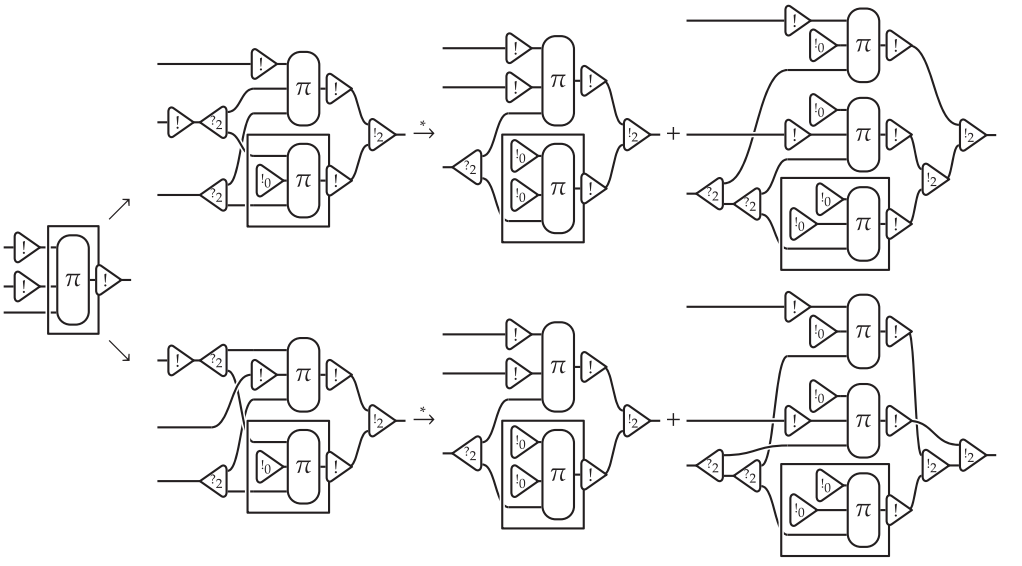


Fig. 7. Counterexample to local confluence in the absence of associativity.

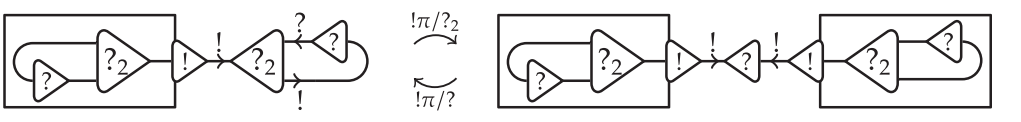


Fig. 8. Example of laxly typed and switching acyclic nets without normal form. The types are written explicitly only for 0-depth wires.

2.3.2. *The need for associativity and neutrality.* Figure 7 explains the need for explicit associativity and rules in DiLL modelling the bialgebraic nature of (co)contractions, and (co)weakenings. Unlike LL, where such semantic equivalences can be added and studied *a posteriori* (as, for example, in Di Cosmo *et al.* (2003)), in DiLL they must be included in order to assure confluence (Tranquilli 2009a). Other confluence diagrams show how the neutral n-reduction is needed as well. One such example can be deduced from Figure 9.

2.3.3. *Infinite reductions.* As in Linear Logic (Pagani and Tortora de Falco 2010), infinite reductions can be produced in DiLL for two distinct reasons: because of the presence of a sort of *vicious* cycle (i.e. the switching cycles as defined in subsection 2.5) or because of the very liberal typing discipline. The infinite reductions starting from the net π_8 of Figure 6 (as well as those from the nets $\pi_3, \ldots, \pi_7$) are in fact examples of the first sort. Indeed, by replacing the one and bottom by a weakening and an empty box, respectively, we get a net which is simply typable (see Section 5) but still has the same infinite reductions. As an example of infinite reduction due to lax typing we have the net in Figure 8, where the reduction makes a loop between two laxly typed simple nets that have no switching cycles. This paper deals mainly with the latter sort of infinite reductions, as our main results are restricted to switching acyclic nets.

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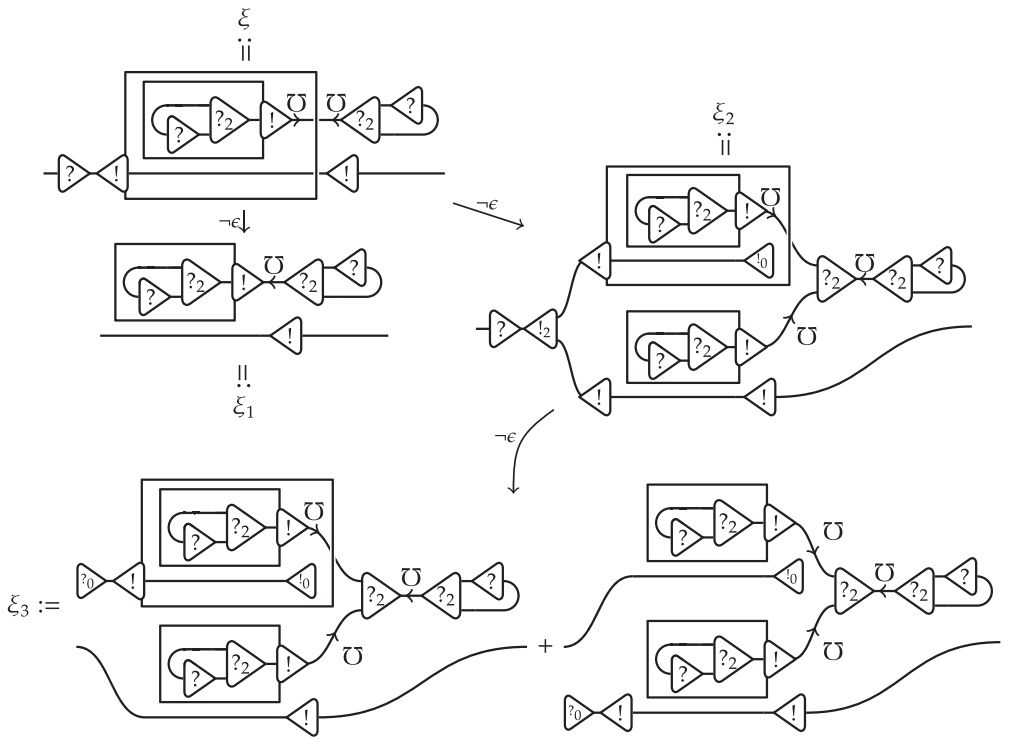


Fig. 9. Example of the importance of the lax typing in DiLL. Only the $\mathfrak{V}$ type is written explicitly. Notice that this is also a counterexample to the confluence of non-erasing reduction. Finally, it also works as a counterexample of confluence in the absence of neutral reduction, as in order to close the diagram the $?_0$ cell obtained from erasing the box needs to be absorbed by the $?_2$ one.

2.3.4. *The reason for lax typing.* Besides switching acyclicity, the other basic notion we adopt in this paper is the one of lax typing. Let us discuss it with the example in Figure 9. The net ξ at the top left corner of the figure has two reducible cuts resp. of kind $?/!\rho$ and $!/!\rho$ and one clash, which is a contraction against an auxiliary port of the box. By reducing the codereliction against box cut and then all the non-erasing cuts created at each step, we reach ξ_3 at the bottom of the figure, which is a $\neg\epsilon$-normal form. This means ξ is $\mathsf{WN}_{\neg\epsilon}$. On the other hand, the reduction of the dereliction against box cut in ξ results in the net ξ_1, which is a normal form also, since the contraction/box cut has $\mathfrak{V}$ type. Notice that this $\mathfrak{V}$ cut is the residue of a clash in ξ that has become *a priori* reducible in ξ_1, but lax typing prevents us from reducing it. Indeed, if we could reduce this cut in ξ_1 we would obtain the infinite reduction of Figure 8, getting therefore a counterexample to conservation, as $\xi \in \mathsf{WN}_{\neg\epsilon}$. Lax typing stands here precisely to forbid such a counterexample.

The need for lax typing marks another difference with respect to LL, where conservation can be proved (at least for the sliced nets) without introducing any sort of typing whatsoever (Pagani and Tortora de Falco 2010). In a completely untyped setting, both LL

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and DiLL have clashes becoming reducible after some non-erasing steps. In LL, though these 'resurrections' of clashes are independent of which reduction is performed: if a particular chain of non-erasing reductions resurrects a clash, then the same can be achieved from any other reduction, as *non-erasing reduction is confluent in* LL. As we explain next this is not the case in DiLL: in Figure 9, for example, if we ignore lax typing the clash in ξ becomes reducible in ξ_1 but remains irreducible in ξ_2 and in all of its reducts.

2.3.5. DiLL*'s non-erasing reduction is not confluent.* As already written in the Introduction, the main idea behind our proof of conservation is to follow (Pagani and Tortora de Falco 2010), inferring Conservation through Gandy/Nederpelt's Lemma (Gandy 1980; Nederpelt 1973). This lemma states that a rewriting system which is confluent and has an increasing measure enjoys WN = SN. The statement has been improved by Bezem and Klop Terese (2003) replacing confluence by local confluence. The goal would then be to achieve Conservation through local confluence of non-erasing reduction. However this plan fails when applied directly to DiLL. A first obstacle, which can however be overcome, is that local confluence already fails for general reduction in differential nets: one needs an equivalence $\sim$ (at least $\overset{\mathtt{a}}{\sim}$) and a canonical reduction (at least $\overset{\mathtt{n}}{\rightarrow}$) to achieve confluence modulo $\sim$ of DiLL's reduction, as is shown in Tranquilli (2009a)[9]. This problem is overcome extending DiLL with the canonical conversions and reductions and upgrading Bezem and Klop's lemma in a rewriting modulo setting, as we do in Lemma 39. In particular, this new setting needs to replace the local confluence condition with the local confluence modulo $\sim$ and local coherence with $\mapsto$ (the symmetric relation generating $\sim$).

However the next obstacle we stumble upon is that the introduction of the equivalence and canonical rules fixes the problem for DiLL's reduction but not for its non-erasing restriction. An example is given by the same net in Figure 9: the non-erasing reduction reduces ξ to two distinct $\neg\epsilon$-normal forms. The critical pair is made from a dereliction against a box on one side and a codereliction against the same box on the other. In order to make it converge, one needs to perform some evidently erasing steps on ξ_2's side. There seems to be a mismatch in the way dereliction and codereliction behave when cut against a box, as the latter leaves behind an unopened copy of it while the former does not. This copy will necessarily have to be erased to close the confluence diagrams.

For these reasons, we introduce $\mathsf{DiLL}_{\partial\varrho}$ in the next subsection, a new system essentially replacing the dereliction and codereliction with two cells, the linear query and the linear substitution respectively (Figure 10). The two new cells have a more symmetrical behaviour when cut against a box, as *both* leave an unopened copy behind. $\mathsf{DiLL}_{\partial\varrho}$ enjoys non-erasing local confluence modulo $\sim$[10]. We will thus apply Lemma 39 to $\mathsf{DiLL}_{\partial\varrho}$,

[9] Indeed (Tranquilli 2009a) proves DiLL enjoys Church–Rosser modulo, which is a property stronger than confluence modulo.

[10] Indeed it also enjoys full non-erasing Church–Rosser modulo, though we will not show it as it is beyond our objective.

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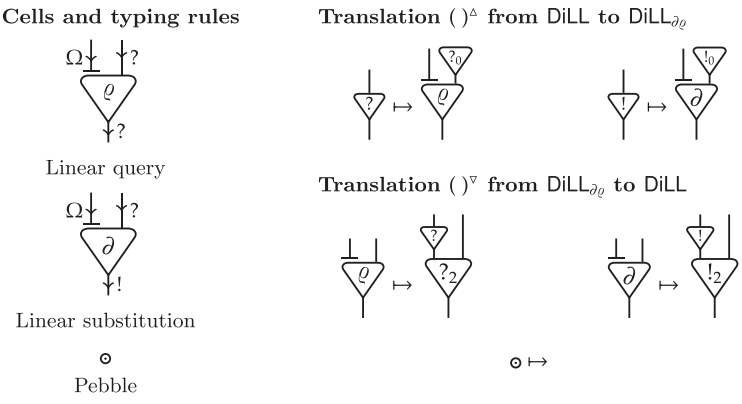


Fig. 10. The cells ∂ and ϱ of $\mathsf{DiLL}_{\partial\varrho}$, and the translation from and to DiLL. The cells are *not* commutative, but may be drawn with swapped inactive ports, hence the special marking of the left port.

inferring Conservation for that system (Section 3), and then we will transfer the result to DiLL (Section 4).

2.4. *The system* $\mathsf{DiLL}_{\partial\varrho}$

In $\mathsf{DiLL}_{\partial\varrho}$ the roles of dereliction and codereliction are replaced by new cells, allowing non-erasing reduction to be locally confluent[11] modulo $\sim$ (Definition 3, Proposition 8). Having a locally confluent non-erasing reduction is an essential ingredient for proving Conservation Theorem via Lemma 39. However, as discussed above, this fails in DiLL, so we pass via $\mathsf{DiLL}_{\partial\varrho}$. Actually $\mathsf{DiLL}_{\partial\varrho}$ has its own interest both for its computational behaviour (somewhat smoother than DiLL) and its tight link with a categorical interpretation of differential linear logic (Fiore 2007): ∂ and ϱ fill the roles of the annihilation and creation operators of Fiore (2007) respectively.

$\mathsf{DiLL}_{\partial\varrho}$ is obtained by removing from DiLL the dereliction and codereliction cells and adding in the two cells ∂ (**linear substitution**) and ϱ (**linear query**) shown together with their lax typing rules in Figure 10. As usual lax typing should be intended downward closed with respect to $\sqsubseteq$. We also add a cell, called **pebble**, with no ports (which is therefore a tolerated exception to each cell having a principal port), whose only role is to make the size always increasing under non-erasing reduction[12]. For the sake of clarity we define pebbles as an integral part of $\mathsf{DiLL}_{\partial\varrho}$; they can however be completely disregarded if $\mathsf{DiLL}_{\partial\varrho}$ is to be considered a stand-alone system. In the same figures, we also define translations from DiLL nets to $\mathsf{DiLL}_{\partial\varrho}$ ones and back, by giving constructs substituting

[11] The reduction is indeed Church–Rosser modulo $\sim$, see Tranquilli (2009b).

[12] Therefore the role of pebbles covers what is done by labelling terms with natural numbers in Sørensen (1997).

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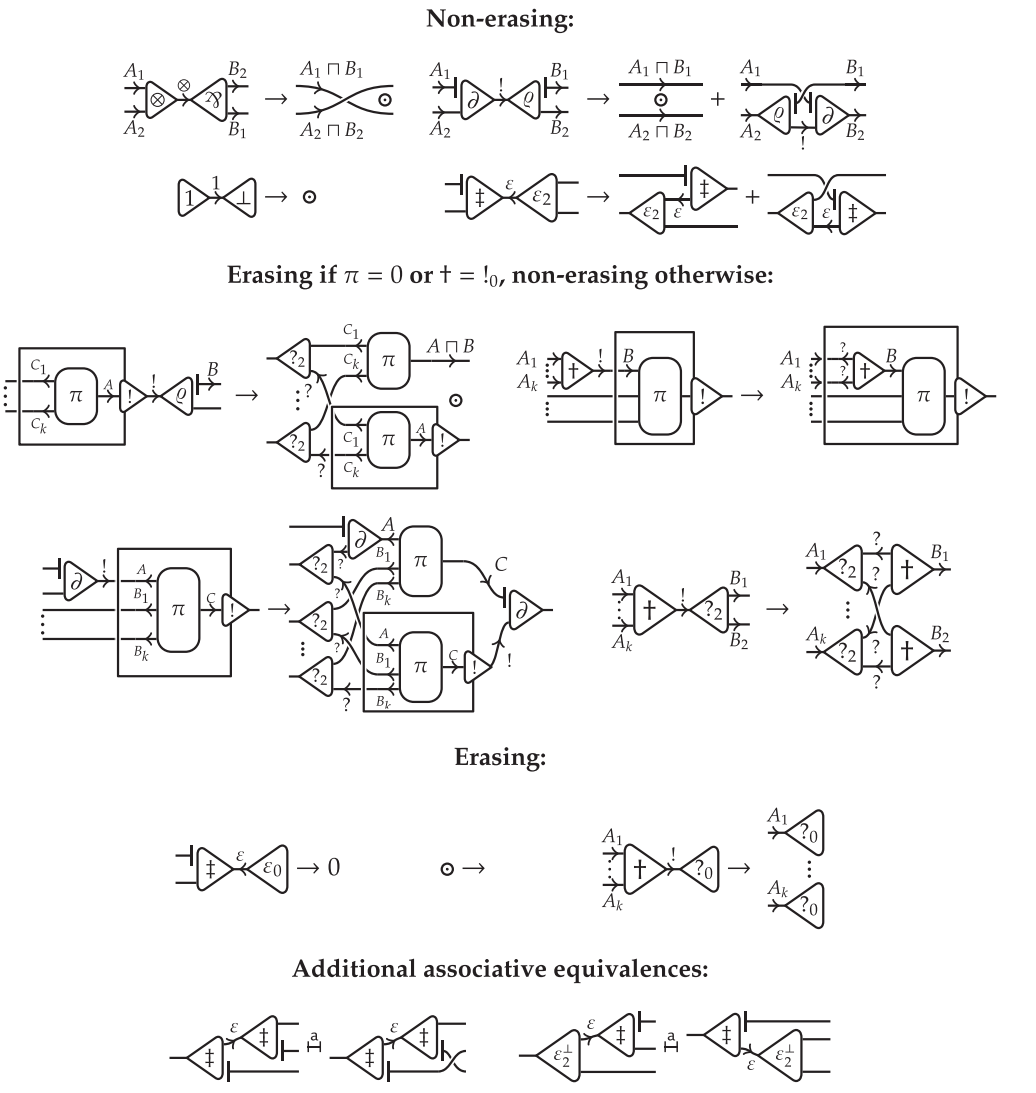


Fig. 11. Reduction and new conversion rules of $\mathsf{DiLL}_{\partial\varrho}$. The symbols ε and ‡ can be either ! and ϱ or ? and ∂ respectively. The symbol † denotes either $!\xi$ (i.e. a box) $!_2$ or $!_0$.

the missing cells. Given a net π of DiLL (resp. $\mathsf{DiLL}_{\partial\varrho}$), we denote by π^{Δ} (resp. π^{∇}) its translation into $\mathsf{DiLL}_{\partial\varrho}$ (resp. DiLL).

All the reduction and conversion rules of DiLL (Figures 4 and 5) but for the ones involving tensor, one, dereliction or codereliction are still valid in $\mathsf{DiLL}_{\partial\varrho}$. The new rules are listed in Figure 11. The main differences with respect to DiLL are in the definition of the steps dealing with ∂ and ϱ (easily deducible by the translation into DiLL), and in the addition of pebbles in those DiLL steps that, if we were in a simple type setting, would have decreased the type of cuts.

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Analogously to DiLL, we split $\mathsf{DiLL}_{\partial\varrho}$ reduction rules into erasing and non-erasing ones.

Definition 3. $\mathsf{DiLL}_{\partial\varrho}$'s **non-erasing** reduction $\xrightarrow{\neg\epsilon}$ is the context closure of the following cut-elimination steps (see Figures 4 and 11):

$$\neg\epsilon := \otimes \,/\, ⅋ + 1/\bot + \partial/\varrho + !_2/\varrho + !\pi/\varrho + \dagger/!\pi + \dagger/?_2,$$

with $\pi \neq 0$ and $\dagger$ denoting either a box (possibly $!0$), a ∂ or a cocontraction.

$\mathsf{DiLL}_{\partial\varrho}$'s **erasing** reduction $\xrightarrow{\epsilon}$ is the context closure of the other cut-elimination steps and the canonical reductions (see Figures 11 and 5):

$$\epsilon := !0/? + \dagger/!0 + \dagger/?_0 + !_0/\ddagger + \mathtt{pebble} + \mathtt{n} + \mathtt{p} + \mathtt{z},$$

with $\dagger$ denoting either a box (possibly $!0$), ∂, $!_0$ or $!_2$, and $\ddagger$ denoting either a box, ϱ, $?_0$ or $?_2$.

If we now consider $\xi^{\triangle}$, with ξ defined as in Figure 9, we can see that the generated critical pair can now be closed under non-erasing reduction. Indeed we will show the general case to hold in Figure 13 while proving Proposition 8.

2.5. *Switching acyclicity*

Cut-elimination can be performed without any reference to any form of correctness. However we noticed in subsection 2.3 that certain cycles, intuitively corresponding to logical errors, give a deranged computational behaviour (even in Linear Logic and even for simply typable nets). Purely graph-theoretical conditions, called *correctness criteria*, have been presented in order to characterise the set of 'correct' nets. We here give the most known of such correctness criteria, *switching acyclicity*, presented originally by Danos and Regnier (1989) for the multiplicative fragment of Linear Logic, and then extended to DiLL in Ehrhard and Regnier (2006)[13].

First of all, one sets a subset of the symbols of the system to be **switching**. In DiLL the switching symbols are just $⅋$ and $?_2$, in $\mathsf{DiLL}_{\partial\varrho}$ ϱ is added as switching too. All the cells in a net with a switching symbol are called switching too. One then defines **switching paths** as intuitively graph-theoretical paths on a net which cannot pass through two premises of the same switching cell. Formally a switching path ϕ of a simple net α is a non-repeating sequence of ports $p_1 \cdots p_k$ in $\mathsf{p}_0(\alpha)$ such that:

— two consecutive ports p_i and p_{i+1} are either in the same cell or in the same wire, and $k \geqslant 2$ (i.e. the sequence defines an actual path);
— at most two ports from each cell appear in ϕ (which forces the path to alternate cells and wires and prevents it from recrossing or ending on a cell already traversed by it);
— no two ports in ϕ are the premises of the same switching cell (which corresponds to the actual switching condition).

[13] To be precise, Ehrhard and Regnier (2006) introduces switching acyclicity in the promotion-free fragment of DiLL, however its generalisation to boxes is trivial.

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A switching path in a net is a switching path in one of its summands. Notice in particular that boxes are non-switching cells, so paths may enter and exit any of their ports regardless of their contents (a sort of *black box* principle). A **switching cycle** is either a loop or a switching path whose starting and ending ports are on the same cell. Moreover switching paths are considered disjoint if they have not only no ports, but also no cells in common.

A net π is switching acyclic (AC) if it contains no switching cycles and for every box $!\rho$ with depth 0 in π, ρ is switching acyclic.

What follows are results renowned in literature, namely that switching acyclicity is preserved under reduction. As this is usually proved by showing that reduction does not create switching paths (in particular cycles), and we will need such more general result later in subsection 4.3, we state such a result more precisely (Lemma 4 below), while skipping on the details of the proof of Proposition 5.

We say that a relation R defined on nets with the same free ports **does not create switching paths** if whenever $\pi\ R\ \rho$, with $p, q \in \mathsf{fp}(\pi) = \mathsf{fp}(\rho)$ and with a switching path linking p and q in a summand of ρ, then there is also one linking the same ports in π.

Lemma 4. Both reduction $\to$ and equivalence $\sim$ do not create switching paths.

Proof. Let us define the following more refined property of a relation R, that implies that R does not create switching paths:

> let π, ρ be two nets sharing the same set of free ports. If $\pi\ R\ \rho$ and ϕ_i is a family of disjoint switching paths in a summand β of ρ linking some of its free ports, then there is a summand α of π and a family ϕ'_i of disjoint switching paths in α linking the same free ports as ϕ_i. (1)

First we will see that (1) is preserved under context closure. This is not the case in general for the property of not creating switching paths, which justifies the generalisation. Now suppose $\omega[\pi]\ \tilde{R}\ \omega[\rho]$ with $\pi\ R\ \rho$, and let ϕ_i be a family of disjoint switching paths in a summand β' of $\omega[\rho]$ linking some of its free ports. In case the depth of the hole in $\omega[\,]$ is greater than 0, or β' is a summand of $\omega[\,]$ not containing the hole, then the statement is trivial, since the switching paths linking the free ports of $\omega[\rho]$ do not depend on ρ. Otherwise, there is a summand β in ρ such that β' is a summand of $\omega[\beta]$. Then, any switching path ϕ_i in β', as it may pass through β several times, induces a possibly empty family ϕ_{ij} of k_i paths in β and a family ψ_{ij} of $k_i + 1$ paths in a unique summand of $\omega[\,]$ so that chaining together $\psi_{i1}, \phi_{i1}, \psi_{i2}, \ldots, \phi_{ik_i}, \psi_{ik_i+1}$ we obtain the path ϕ_i of β'. Moreover since the ϕ_i's are disjoint and all switching paths do not cross themselves, all the ϕ_{ij} for any pair of distinct i and j are also disjoint. We thus get by definition of (1) a summand α in π and disjoint paths ϕ'_{ij} in π, so that glueing back $\psi_{i1}, \phi'_{i1}, \ldots, \phi'_{ik_i}, \psi_{ik_i+1}$ we get the needed family of disjoint switching paths in $\omega[\alpha]$, which is a summand of $\omega[\pi]$.

To prove the lemma, one then needs just to check (1) for all the reduction and conversion rules, which is easily carried out by inspecting them.

□

Proposition 5. Reduction $\to$ and equivalence $\sim$ both preserve switching acyclicity.

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Notice that in $\mathsf{DiLL}_{\partial\varrho}$ the status of the new symbols (with ϱ switching and ∂ not) is in fact deducible from the corresponding translation into DiLL. It is indeed straightforward that π in $\mathsf{DiLL}_{\partial\varrho}$ is AC iff $\pi^{\triangledown}$ is, and the same the other way around, so that σ is AC iff $\sigma^{\triangle}$ is.

3. The conservation theorem of $\mathsf{DiLL}_{\partial\varrho}$

In this section we prove that non-erasing reduction is perpetual in $\mathsf{DiLL}_{\partial\varrho}$, or equivalently, that $\mathsf{WN}_{\neg\epsilon}$ implies $\mathsf{SN}_{\sim}$ (Theorem 19). The proof splits this implication into two parts: subsection 3.1 shows that $\mathsf{WN}_{\neg\epsilon}$ implies $\mathsf{SN}_{\neg\epsilon\sim}$ and subsection 3.2 proves that $\mathsf{SN}_{\neg\epsilon\sim}$ implies $\mathsf{SN}_{\sim}$.

The fact that $\mathsf{WN}_{\neg\epsilon}$ implies $\mathsf{SN}_{\neg\epsilon\sim}$ (Lemma 10) is achieved by applying Lemma 39: we give a measure on nets (Definition 6) which is increasing under the $\neg\epsilon$ reduction and stable under conversion (Proposition 7), and we prove that $\overset{\neg\epsilon}{\to}$ is locally confluent modulo $\sim$ (Proposition 8) and locally coherent with $\mapsto$ (Proposition 9). Let us underline that Propositions 8 and 9 (and hence conservation) need the switching acyclic hypothesis.

The fact that $\mathsf{SN}_{\neg\epsilon\sim}$ implies $\mathsf{SN}_{\sim}$ is obtained by showing that non-erasing steps can always be done before the erasing steps (Lemma 18) and that erasing reduction is $\mathsf{SN}_{\sim}$ (Proposition 12).

Throughout this section we take as system $\mathsf{DiLL}_{\partial\varrho}$, so we may omit to mention it.

3.1. *From non-erasing* WN *to Non-Erasing* $\mathsf{SN}_{\sim}$

The next definition uses the following notation from Section 2: $\sigma(c)$ denotes the symbol of the cell c, and in case c is a box, $\sigma(c)$ is a net (its contents) and $\#\sigma(c)$ is the number of summands of $\sigma(c)$, finally $\mathsf{c}_!(\pi)$ is the set of cells at any depth of π.

Definition 6. The **weight** $\mathrm{wg}(c)$ of a cell c at depth d in a net π is defined as follows

$$\mathrm{wg}(c) := \begin{cases} 1 & \text{if } c \text{ is a pebble,} \\ 1+d & \text{if } c \text{ is a cocontraction, coweakening, linear substitution} \\ & \text{or an empty box,} \\ (1+d)(2\#\sigma(c)-1) & \text{if } c \text{ is a non-empty box,} \\ 0 & \text{otherwise.} \end{cases}$$

The weight $\mathrm{wg}(\pi)$ of the net π is the sum of the weights of its cells: $\mathrm{wg}(\pi) = \sum_{c\in\mathsf{c}_!(\pi)} \mathrm{wg}(c)$.

Proposition 7. In $\mathsf{DiLL}_{\partial\varrho}$, if $\pi \overset{\neg\epsilon}{\to} \pi'$ then $\mathrm{wg}(\pi) < \mathrm{wg}(\pi')$, and if $\pi \sim \pi'$ then $\mathrm{wg}(\pi) = \mathrm{wg}(\pi')$.

Proof. First, one checks the property case by case for the non-erasing rules of Figure 11, of which we sketch some in the following. Consider contraction against a † cell c, for $\dagger = !_2$ or $!\pi$: the weight of the redex is the weight of c, which is always different from 0; the one of the contractum is twice the weight of c. Also, consider the reduction rule of linear substitution against box: the weight increases (independently from the contents of

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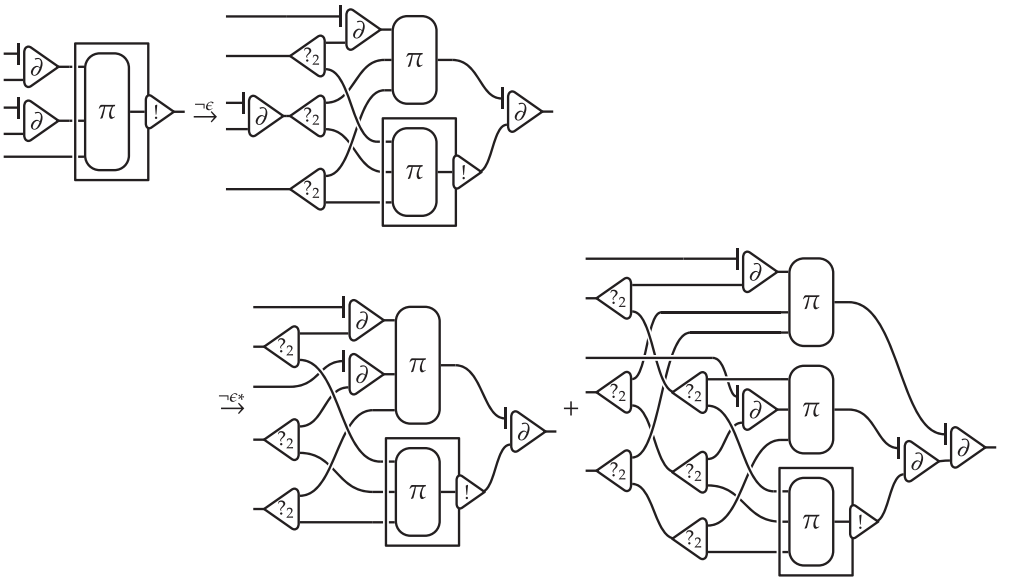


Fig. 12. Reduction of the critical pair with two ∂ cells on a box. The box has only one other auxiliary port for brevity. Starting with the other ∂ cell is completely symmetric, and an a-conversion equates the two results.

the box) since the contractum has at least one linear substitution more than the redex, as the contents of the box is not 0. As for conversion, consider the splitting of a sum in a box (Figure 5): the weight of the left-hand side net is $\mathrm{wg}(\pi) + \mathrm{wg}(\xi) + (2\#(\pi + \xi) - 1)$, which is equal to $\mathrm{wg}(\pi) + \mathrm{wg}(\xi) + 1 + (2\#\pi - 1) + (2\#\xi - 1)$, which is the weight of the net on the right-hand side. Notice again the importance of the condition $\pi, \xi \neq 0$ to yield the previous equality.

One generalises then to contexts noticing that $\neg\epsilon$ never decreases (resp. $\mapsto$ never changes) the number of summands in a sum, so $\mathrm{wg}(\omega[\alpha]) > \mathrm{wg}(\omega[\pi_\alpha])$ (resp. $\mathrm{wg}(\omega[\alpha]) = \mathrm{wg}(\omega[\pi_\alpha])$) whenever α, π_α is a $\neg\epsilon$ (resp. $\mapsto$) pair. □

Proposition 8. On switching acyclic nets of $\mathsf{DiLL}_{\partial\varrho}$, the reduction $\xrightarrow{\neg\epsilon}$ is locally confluent modulo $\sim$, i.e. $\xleftarrow{\neg\epsilon}\xrightarrow{\neg\epsilon} \subseteq \xrightarrow{\neg\epsilon*}\sim\xleftarrow{\neg\epsilon*}$.

Proof. There are not many more critical pairs to check than in Tranquilli (2009b,a). Apart from the confluence diagrams in common with DiLL (which can be seen to be using only non-erasing reductions if the critical pair is non erasing), the only non trivial diagrams are those of a box having two different auxiliary ports cut with two ∂ cells and of a box having the principal port cut with a ϱ cell and an auxiliary port cut with a ∂ cell.

For the former, Figure 12 shows the reduction starting with one of the two ∂ cells. The other ∂ cell leads to a symmetric form where the only difference is the order of the

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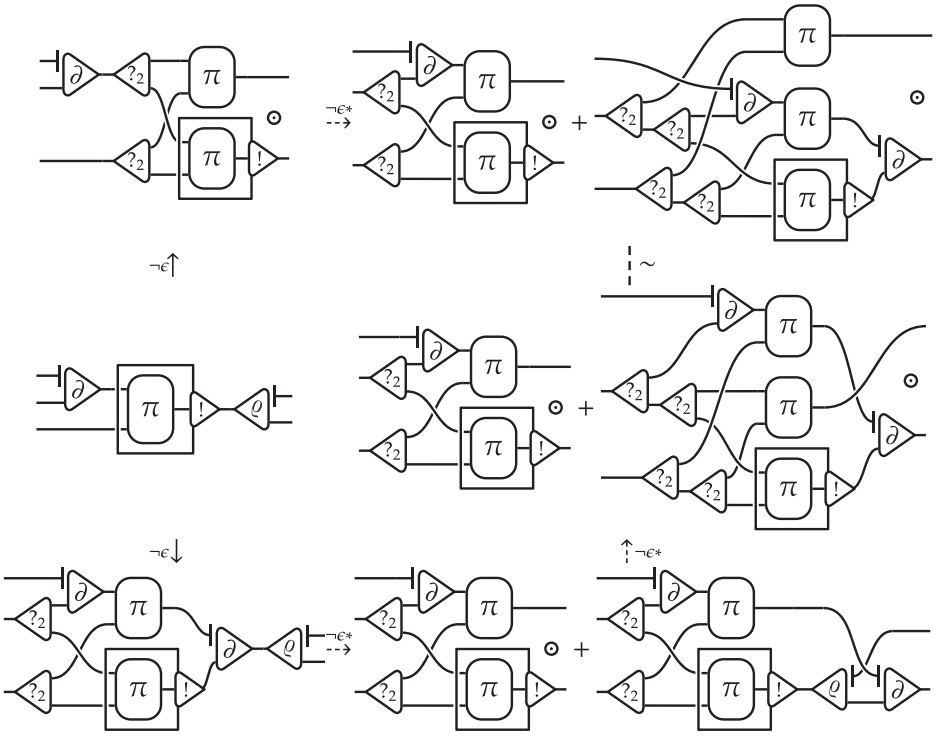


Fig. 13. Confluence diagram of the ∂ on box on ϱ critical pair.

rightmost ∂ cells and the shape of the contraction trees, so that an a-conversion equates the two results.

For the latter, Figure 13 shows the confluence diagram. □

Proposition 9. On switching acyclic nets of $\mathsf{DiLL}_{\partial\varrho}$, the reduction $\xrightarrow{\neg\epsilon}$ is locally coherent with $\mapsto$, i.e. $\mapsto\xrightarrow{\neg\epsilon} \subseteq \xrightarrow{\neg\epsilon*}\sim\xleftarrow{\neg\epsilon*}$.

Proof. As usual, one must check all coherence diagrams. Most of them are obvious, or simple variations on the patterns already known for DiLL (see Tranquilli (2009a)). We show an interesting one in Figure 14, a ∂ cell on an s-convertible box. This is the only diagram that needs the 'mixed' associativity rule between cocontraction and linear substitution. Recall that $\pi_0, \pi_1 \neq 0$ is required by the s-conversion, so that no step is forbidden by the non-erasing discipline. □

Propositions 8 and 9 need the switching acyclicity hypothesis, in order to forbid the critical pairs where the boxes are cut between themselves. For example, Figure 15a gives a counterexample to local confluence of $\xrightarrow{\neg\epsilon}$ (as well as of $\xrightarrow[\sim]{\mathsf{cut}}$) in presence of switching cycles; analogously Figure 15b gives a counterexample to local coherence of $\xrightarrow{\neg\epsilon}$ with $\mapsto$. In fact the latter is also a counterexample to the conservation property in presence of switching cycles.

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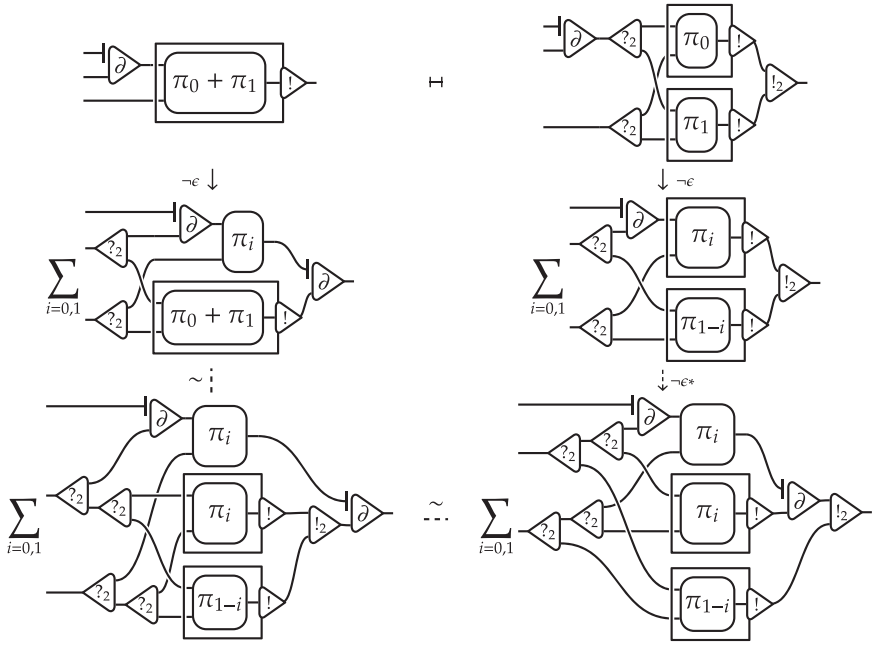
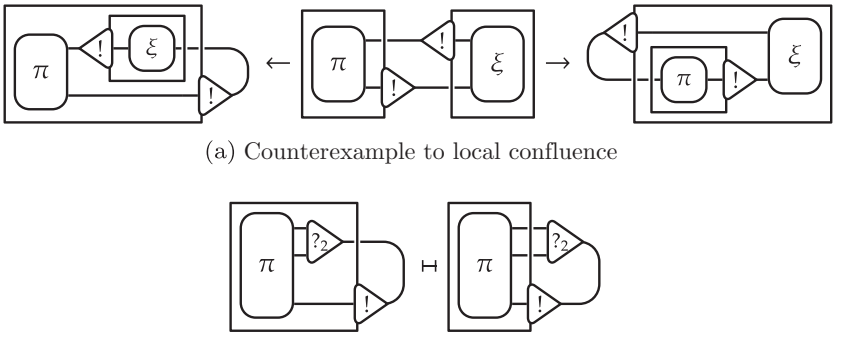


Fig. 14. Coherence diagram of a ∂ cell on an $\mathbf{s}$-conversion.



(a) Counterexample to local confluence

(b) Counterexample to local coherence

Fig. 15. Counterexamples to local confluence and coherence of $\neg\epsilon$ with $\sim$ in presence of switching cycles. The latter is also a counterexample to the conservation theorem.

Lemma 10 (conservation for non-erasing reduction). Given a net π in $\mathsf{DiLL}_{\partial\varrho}$, if π is switching acyclic and $\pi \in \mathsf{WN}_{\neg\epsilon}$ then $\pi \in \mathsf{SN}_{\neg\epsilon\sim}$.

Proof. Immediate consequence of Lemma 39 and the Propositions 7, 8 and 9. □

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3.2. *From non-erasing* $\mathsf{SN}_\sim$ *to* $\mathsf{SN}_\sim$ *in* $\mathsf{DiLL}_{\partial\varrho}$

We prove that non-erasing steps can always be anticipated with respect to the erasing ones (Lemma 18) and that erasing reduction is $\mathsf{SN}_\sim$ (Proposition 12). In this way we are able to prove conservation for $\mathsf{DiLL}_{\partial\varrho}$ (Theorem 19).

We recall again that, given a cell c, $\sigma(c)$ (resp. $\mathsf{deg}(c)$) denotes its symbol (resp. the number of its ports); namely if c is a box, $\#\,\sigma(c)$ is the number of the summands of the contents $\sigma(c)$ of c; finally, $\mathsf{c}_0(\pi)$ is the set of cells at depth 0 in π.

Definition 11. We define the **erasing weight** of a cell c and, mutually, the one of a net π by induction on the depth of the net:

$$\mathsf{e}(c) := \begin{cases} 2 & \text{if } \sigma(c) \in \{\, !_2, ?_2, \partial, \varrho \,\}, \\ 1 & \text{if } \sigma(c) = !_0 \text{ or } c \text{ is a pebble}, \\ 2\,\mathsf{deg}(c)\big(2\,\#(\sigma(c)) - 1\big) + 2\,\mathsf{e}(\sigma(c)) & \text{if } c \text{ is a non-empty box}, \\ \mathsf{deg}(c) & \text{if } c \text{ is an empty box}, \\ 0 & \text{otherwise;} \end{cases}$$

$$\mathsf{e}(\pi) := \sum_{c \in \mathsf{c}_0(\pi)} \mathsf{e}(c).$$

Proposition 12. In $\mathsf{DiLL}_{\partial\varrho}$, if $\pi \xrightarrow{\epsilon} \pi'$ then $\mathsf{e}(\pi) > \mathsf{e}(\pi')$, and if $\pi \sim \pi'$ then $\mathsf{e}(\pi) = \mathsf{e}(\pi')$. In particular, erasing reduction is $\mathsf{SN}_\sim$ in $\mathsf{DiLL}_{\partial\varrho}$.

Proof. As usual, one first checks the property on the erasing and conversion rules. We treat in detail only the more interesting cases. The definition of the reductions we consider here can be found in Figure 5.

For total pull reduction the weight clearly decreases since the box loses one auxiliary port. For cocontraction against an empty box, the latter having n ports, the weight of the redex is $2+n$ while that of the contractum is $n+1$. Notice that, in this step, it is crucial that cocontraction's weight be 2 and not 1. For weakening against linear substitutions (or coweakening against linear query), the contractum is the empty sum of nets, hence its weight is 0, while the redex weights 2.

As for the push conversion, using the names of Figure 5, the right-hand side simple net weights $2n(2\,\#(\pi)-1)+2(2\,\#(\pi)+\mathsf{e}(\pi))$, with n the number of ports of the box, while the left-hand side weights $2(n+1)(2\,\#(\pi)-1)+2+2\,\mathsf{e}(\pi) = 2n(2\,\#(\pi)-1)+4\,\#(\pi)+2\,\mathsf{e}(\pi)$, which is the same. Consider the sum conversion: at the left-hand side we have $2n(2\,\#(\pi)+2\,\#(\xi)-1)+2(\mathsf{e}(\pi)+\mathsf{e}(\xi))$, with n the number of ports of the box, and at the right one $2n(2\,\#(\pi)-1)+2\,\mathsf{e}(\pi)+2n(2\,\#(\xi)-1)+2\,\mathsf{e}(\xi)+2n = 2n(2\,\#(\pi)+2\,\#(\xi)-1)+2(\mathsf{e}(\pi)+\mathsf{e}(\xi))$.

One generalises the result to contexts arguing symmetrically with respect to the proof of Proposition 7: since ϵ-reduction never increases (resp. $\mapsto$ never changes) the number of summands in a sum, we have $\mathsf{e}(\omega[\alpha]) < \mathsf{e}(\omega[\pi_\alpha])$ (resp. $\mathsf{e}(\omega[\alpha]) = \mathsf{e}(\omega[\pi_\alpha])$) whenever (α, π_α) is an ϵ (resp. $\mapsto$) pair. □

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The next goal is to prove that non-erasing steps can always be anticipated with respect to the erasing ones (Lemma 18). In fact, we do not need commutation of the two, but merely the fact that we can bring at least a non-erasing step up front when there is one. Lemma 17 solves the case when one erasing step is close to a non-erasing one, but more effort is needed when some canonical conversions space out the two reductions, e.g. $\overset{\epsilon}{\to}\sim\overset{\neg\epsilon}{\to}$. We need either to postpone erasing steps past conversions, or anticipate non-erasing ones. But here comes a fine point: in Lemmas 14 and 15 we show that erasing steps can in fact be postponed past conversions (or generate a first non-erasing step) but for one case – when an $\mathtt{s}$-conversion merges boxes, one of which had its weakening just pulled out by a pull reduction. Fortunately, this same conversion allows non-erasing steps after it to be anticipated (by Lemma 16, which is not true for all conversions in general). We thus

— distinguish between the two directions of the bang sum conversion, denoted $\overset{\mathtt{s}}{\to}$ for splitting boxes (left to right direction of $\overset{\mathtt{s}}{\mapsto}$ as presented in Figure 5), and the corresponding merging expansion $\overset{\mathtt{s}}{\leftarrow}$;
— prove a series of lemmas that allow to rearrange reductions and conversions in order to bring non-erasing steps up-front, including a Church–Rosser modulo property of $\overset{\mathtt{s}}{\to}$ with respect to $\overset{\mathtt{ap}}{\sim}$ (Lemma 13).

Another fine point regards the presence of partial pull reductions, which can however be avoided as explained in the following proofs.

Lemma 13. $\overset{\mathtt{s}}{\to}$ is Church–Rosser modulo $\overset{\mathtt{ap}}{\sim}$, which amounts to stating $\overset{\mathtt{aps}}{\sim} = \overset{\mathtt{s}*}{\to}\overset{\mathtt{ap}}{\sim}\overset{\mathtt{s}*}{\leftarrow}$.

Proof. We apply Huet's lemma (Lemma 38), so we need the following three ingredients.

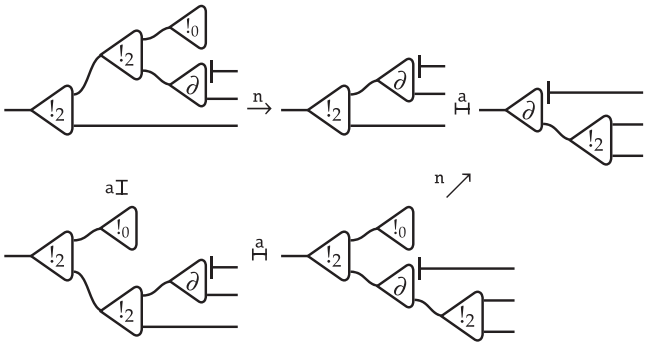
Normalisation modulo. Define the weight of a non-zero box as the number of its summands minus one and the weight of a net as the sum of the weight of its boxes (at any depth). This weight strictly decreases under $\overset{\mathtt{s}}{\to}$ and is invariant under $\mathtt{ap}$-conversions.

Local coherence. We can in fact show a stronger property, stating that $\overset{\mathtt{ap}}{\mapsto}\overset{\mathtt{s}}{\to} \subseteq \overset{\mathtt{s}}{\to}\overset{\mathtt{ap}}{\sim}$. The only diagram one has to check is when a $\mathtt{p}$-conversion (in one of the two directions) fires on a box involved then by an $\mathtt{s}$-reduction. After firing the $\mathtt{s}$-redex first, the diagram easily closes by carrying out the $\mathtt{p}$-conversion on both split boxes and then shuffling contractions in the right shape via $\mathtt{a}$-conversions.

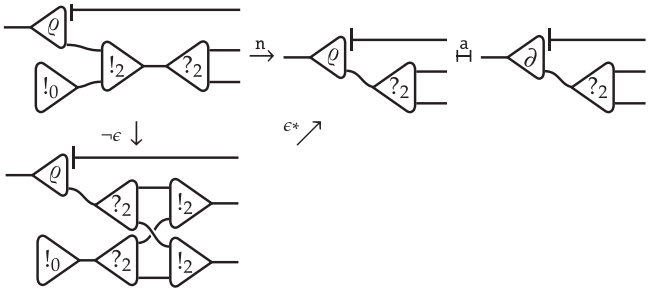
Local confluence modulo. The critical pair is when two different $\mathtt{s}$ reductions fire on the same box. We are in a situation where we split the box contents into two different sums $\pi_1 + \pi_2 = \sigma_1 + \sigma_2$. We can close the diagram by splitting the boxes further in order to have the same box contents on the two sides, and then rearranging contractions and cocontractions to the same shape by $\mathtt{a}$-conversions. □

Lemma 14. We have that $\overset{\epsilon}{\to}\overset{\mathtt{ap}}{\mapsto} \subseteq \overset{\neg\epsilon}{\to}\overset{*}{\sim}\to \cup \overset{\epsilon*}{\sim}\overset{\epsilon}{\to}$. Moreover, the right-hand side can be made to not contain any partial $\mathtt{p}$-reductions (see Figure 5).

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(a) n-reduction enabling an a-conversion of the same type.



(b) n-reduction enabling an a-conversion of opposing type.

Fig. 16. Examples of the two interesting cases in the proof of Lemma 14.

Proof. We need to inspect all cases of overlapping rules. Most of the ϵ-reductions cannot create nor modify a conversion instance, so the two can always be swapped. If a partial p-reduction is involved, we can, after permuting, turn it into a sequence $\overset{s}{\mapsto}\overset{p}{\to}\overset{n}{\to}$ with a total p-reduction.

The interesting cases are the following.

— The erasing reduction is an n one on a contraction, creating a $\overset{x}{\mapsto}$ conversion for $x \in \{a, p\}$, with a on contractions/linear queries. Similarly, the n reduction may be on a cocontraction and enabling a cocontraction/linear substitutions $\overset{a}{\mapsto}$-conversion (Figure 16a). In this case, an additional $\overset{a}{\mapsto}$ conversion is needed for moving the (co)contraction blocking the $\overset{x}{\mapsto}$ conversion out of the way. $\overset{a}{\mapsto}$ can be performed since lax typing assures that the (co)contraction has the same !/? type as the neutral redex. We thus transform $\overset{n}{\to}\overset{x}{\mapsto}$ into $\overset{a}{\mapsto}\overset{x}{\mapsto}\overset{n}{\to}$.

— As above, but the $\overset{x}{\mapsto}$ instance is enabled by firing an n-reduction of the opposing type, i.e. there is either a neutral reduction of a cocontraction in case of contraction a-conversion (Figure 16b) or any p-conversions, or else a neutral reduction of a contraction in case of cocontraction $\overset{a}{\mapsto}$-conversion. This case can be closed by first performing a $\neg\epsilon$-reduction involving the previously n-reduced (co)contraction and then closing the diagram with other steps. It will in general be the case that we transform $\overset{n}{\to}\overset{x}{\mapsto}$ into $\overset{\neg\epsilon}{\to}\overset{\epsilon^*}{\to}\overset{x}{\mapsto}$. □

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Lemma 15. We have $\overset{\epsilon}{\to}\overset{\mathtt{s}^*}{\to} \subseteq \overset{\epsilon^*}{\rightsquigarrow}\overset{\epsilon}{\to}$. Moreover, the right-hand side can be made to not contain any partial p-reductions (see Figure 5).

Proof. We can show that $\overset{\epsilon}{\to}\overset{\mathtt{s}}{\to} \subseteq \overset{\epsilon^*}{\rightsquigarrow}\overset{\epsilon}{\to}$ and then conclude by iteration.

The only overlapping case is when the s-reduction is used to split the result of firing a pull redex. The diagram is closed by at most two s-reductions (one of these may come from the original pull redex in case it is partial, the other one is the original s-reduction) followed by at most two total p-reductions and at most two n ones. In any case, the new p-reductions can be total. □

We may notice that the overlapping case of $\overset{\mathtt{p}}{\to}\overset{\mathtt{s}}{\leftarrow}$, i.e. merging two boxes of which one has had a weakening just pulled out cannot be closed as done in the statement of the above lemma, which is the cause leading to the distinction of the two directions of $\overset{\mathtt{s}}{\mapsto}$ in these proofs.

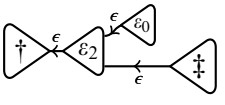
Lemma 16. We have $\overset{\mathtt{s}^*}{\leftarrow}\overset{\neg\epsilon}{\to} \subseteq \overset{\neg\epsilon}{\to}\overset{*}{\rightsquigarrow}$.

Proof. We can show that $\overset{\mathtt{s}}{\leftarrow}\overset{\neg\epsilon}{\to} \subseteq \overset{\neg\epsilon}{\to}\overset{*}{\rightsquigarrow}$ and then proceed by iteration. The critical cases are when a non-erasing reduction fires on a box $!(\pi + \xi)$ resulting from a merging of two distinct boxes (consider the right-to-left direction of the s conversion depicted in Figure 5). The possible kinds of the $\neg\epsilon$ redex are: $!(\pi + \xi)/\varrho$, $!(\pi + \xi)/?_2$, $\partial/!(\pi + \xi)$, $!_2/!(\pi + \xi)$, and $!\rho/!(\pi + \xi)$. All diagrams easily close by a $\rightsquigarrow$ reduction starting with a non-erasing step. □

Lemma 17. If we forbid partial p-reductions, we have $\overset{\epsilon}{\to}\overset{\neg\epsilon}{\to} \subseteq \overset{\neg\epsilon}{\to}\overset{*}{\to}$.

Proof. For all erasing reductions but the n one, the result is achieved by noticing that the erasing step (by hypothesis different from partial p) cannot have created the non-erasing redex fired immediately afterwards. Most of the times, the two reductions are orthogonal and therefore commute easily. The only exception is when the erasing step is on an auxiliary port of a box (i.e. a step of the type $\dagger/!0$ with $\dagger \in \{!\pi, !_2, !_0\}$) and the non-erasing one is on the box itself. All cases for the non-erasing step are then easily handled. For partial pull reductions the statement is false, so we explicitly left them out.

The n reduction is peculiar as it can 'create' the wire where the following non-erasing reduction takes place. By lax typing however, one of the cells taking place in the non-erasing step can also interact with the (co)contraction of the n-reduction. The situation is the following



where ε is either ? or ! and $\dagger$ can interact both with the (co)contraction (marked with symbol ε_2) and with $\ddagger$. Then

— if $\dagger$ is duplicated by ε_2 (i.e. $\dagger$ is a box, cocontraction or contraction): one copy can then reduce by a non-erasing step with $\ddagger$, the other will be erased by ε_0 and n-reductions will close the diagram;

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— if † is ∂ or ϱ, the reduction against ε_2 generates a sum: in one summand we can continue with the non-erasing † on ‡, then finish by erasing the other summand by ε_0 on ∂ or ϱ and the weakening-coweakening pair in the summand left;
— if † is a box and ε_2 a cocontraction, then we can make ε_2 enter † first, then fire the † on ‡ redex, and then make all the residuals of ε_0 $\mathtt{n}$-reduce with all the residuals of ε_2, possibly making them enter the residuals of †. □

Lemma 18. In $\mathsf{DiLL}_{\partial\varrho}$, we have $\overset{\epsilon^*}{\rightsquigarrow}\overset{\neg\epsilon}{\rightsquigarrow} \subseteq \overset{\neg\epsilon}{\rightsquigarrow}\overset{*}{\rightsquigarrow}$.

Proof. Consider a reduction $\pi \overset{\epsilon^*}{\rightsquigarrow} \xi_1 \overset{\neg\epsilon}{\to\!\!\sim} \xi$ where all partial $\mathtt{p}$-reductions have been converted to $\overset{\mathtt{s}}{\to}\overset{\mathtt{p}}{\to}\overset{\mathtt{n}}{\to}$ with a total $\mathtt{p}$. We proceed by induction on $\mathsf{e}(\pi) - \mathsf{e}(\xi_1)$, which is non-negative by Proposition 12. The base case is when there are no actual erasing steps and is trivial.

Suppose then that there is at least an erasing step. We can single out the last one and do the following chain of successive decompositions of the original reduction:

$$\begin{array}{c} \pi \overset{\epsilon^*}{\rightsquigarrow} \overset{\epsilon}{\to} \sim \xi_1 \overset{\neg\epsilon}{\to} \sim \xi \\ \vdots \quad \text{by Lemma 13} \\ \pi \overset{\epsilon^*}{\rightsquigarrow} \overset{\epsilon}{\to} \overset{\mathtt{s}^*}{\to} \overset{\mathtt{ap}^*}{\mapsto} \xi_2 \overset{\mathtt{s}^*}{\leftarrow} \xi_1 \overset{\neg\epsilon}{\to} \sim \xi \\ \vdots \quad \text{by Lemmas 15 and 16} \\ \pi \overset{\epsilon^*}{\rightsquigarrow} \overset{\epsilon}{\to} \overset{\mathtt{ap}^*}{\mapsto} \xi_2 \overset{\neg\epsilon}{\to} \overset{*}{\rightsquigarrow} \xi. \end{array}$$

Notice that from $\xi_2 \overset{\mathtt{s}^*}{\leftarrow} \xi_1$, we can deduce via Proposition 12 that $\mathsf{e}(\xi_2) = \mathsf{e}(\xi_1)$.

Now, we can repeatedly apply Lemma 14 on the segment $\overset{\epsilon^*}{\rightsquigarrow}\overset{\epsilon}{\to}\overset{\mathtt{ap}^*}{\mapsto}$ as long as it turns each subsequent $\overset{\epsilon}{\to}\overset{\mathtt{ap}}{\mapsto}$ into $\overset{\epsilon^*}{\rightsquigarrow}\overset{\epsilon}{\to}$. We have two possible outcomes.

— This conversion exhausts all $\mathtt{ap}$-conversions. We thus continue with

$$\begin{array}{c} \pi \overset{\epsilon^*}{\rightsquigarrow} \pi_1 \overset{\epsilon}{\to} \xi_2 \overset{\neg\epsilon}{\to} \overset{*}{\rightsquigarrow} \xi \\ \vdots \quad \text{by Lemma 17} \\ \pi \overset{\epsilon^*}{\rightsquigarrow} \pi_1 \overset{\neg\epsilon}{\to} \overset{*}{\rightsquigarrow} \xi \\ \vdots \quad \text{by inductive hypothesis} \\ \pi \overset{\neg\epsilon}{\rightsquigarrow} \overset{*}{\rightsquigarrow} \xi \end{array}$$

where we have concluded by inductive hypothesis as from $\pi_1 \overset{\epsilon}{\to} \xi_2$ we deduce, using Proposition 12, that $\mathsf{e}(\pi) - \mathsf{e}(\pi_1) < \mathsf{e}(\pi) - \mathsf{e}(\xi_2) = \mathsf{e}(\pi) - \mathsf{e}(\xi_1)$.

— We stumble on a case where Lemma 14 turns a $\overset{\epsilon}{\to}\overset{\mathtt{ap}}{\mapsto}$ into $\overset{\neg\epsilon}{\to}\overset{*}{\rightsquigarrow}$. We are therefore in a case where

$$\begin{array}{c} \pi \overset{\epsilon^*}{\rightsquigarrow} \pi_2 \overset{\epsilon}{\to} \overset{\mathtt{ap}}{\mapsto} \overset{\mathtt{ap}^*}{\mapsto} \xi_2 \overset{\neg\epsilon}{\to} \overset{*}{\rightsquigarrow} \xi \\ \vdots \quad \text{by hypothesis} \\ \pi \overset{\epsilon^*}{\rightsquigarrow} \pi_2 \overset{\neg\epsilon}{\to} \overset{*}{\rightsquigarrow} \xi \\ \vdots \quad \text{by inductive hypothesis} \\ \pi \overset{\neg\epsilon}{\rightsquigarrow} \overset{*}{\rightsquigarrow} \xi \end{array}$$

where from $\pi_2 \overset{\epsilon}{\to\!\!\sim} \overset{\mathtt{ap}}{\xi_2}$ we argue $\mathsf{e}(\pi) - \mathsf{e}(\pi_2) < \mathsf{e}(\pi) - \mathsf{e}(\xi_1)$ as for $\mathsf{e}(\pi) - \mathsf{e}(\pi_1)$ in the previous case. □

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Theorem 19 (conservation for $\mathsf{DiLL}_{\partial\varrho}$). Given a net π in $\mathsf{DiLL}_{\partial\varrho}$, if π is switching acyclic and $\pi \in \mathsf{WN}_{\neg\epsilon}$, then $\pi \in \mathsf{SN}_{\sim}$.

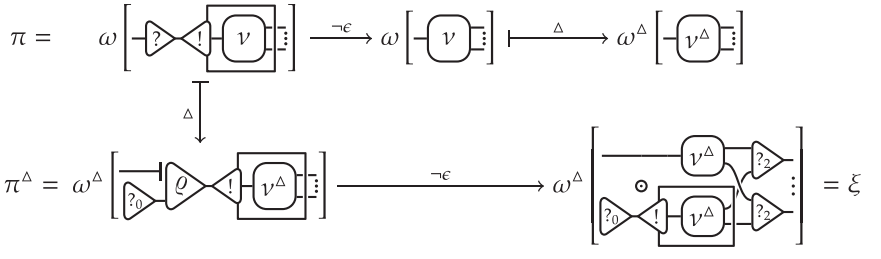
Proof.

Let $\pi \in \mathsf{WN}_{\neg\epsilon}$, which by Lemma 10 means that $\pi \in \mathsf{SN}_{\sim}$ for non-erasing reduction. Suppose we have an infinite chain R of reductions interleaved with $\sim$-conversions starting from π. If we are able to build by coinduction an infinite chain of non-erasing reductions starting from π we can conclude by contradiction.

By Proposition 12 R cannot be solely made of erasing steps, so it must necessarily start with a segment of the shape $\pi \overset{\epsilon*}{\rightsquigarrow}\overset{\neg\epsilon}{\rightsquigarrow} \pi'$, with an infinite reduction starting from π'. Now Lemma 18 ensures that we have $\pi \overset{\neg\epsilon}{\rightsquigarrow} \pi'' \overset{*}{\rightsquigarrow} \pi'$, so that π'' also has an infinite reduction from which we can extract an infinite non-erasing reduction by coinductive hypothesis (which can be used as we put a non-erasing step before it). □

4. Transferring conservation from $\mathsf{DiLL}_{\partial\varrho}$ to DiLL

In this section we achieve the conservation theorem for DiLL (Theorem 32). Let π be a switching acyclic DiLL net, we will prove that $\pi \in \mathsf{WN}_{\neg\epsilon}$ implies $\pi \in \mathsf{SN}_{\sim}$. The idea is to pass through $\mathsf{DiLL}_{\partial\varrho}$ using the translation $\pi^{\triangle}$ given in Figure 10: by the conservation of $\mathsf{DiLL}_{\partial\varrho}$ (Theorem 19) we have $\pi^{\triangle} \in \mathsf{WN}_{\neg\epsilon} \implies \pi^{\triangle} \in \mathsf{SN}_{\sim}$, so what remains to prove is $\pi \in \mathsf{WN}_{\neg\epsilon} \implies \pi^{\triangle} \in \mathsf{WN}_{\neg\epsilon}$ and $\pi^{\triangle} \in \mathsf{SN}_{\sim} \implies \pi \in \mathsf{SN}_{\sim}$. The latter is an immediate consequence of $\mathsf{DiLL}_{\partial\varrho}$ simulating DiLL $\twoheadrightarrow$ (Lemma 20). On the contrary, $\pi \in \mathsf{WN}_{\neg\epsilon} \implies \pi^{\triangle} \in \mathsf{WN}_{\neg\epsilon}$ is quite tough and the rest of this section is devoted to achieve it (Lemma 31). The problem is that both DiLL and $\mathsf{DiLL}_{\partial\varrho}$ need erasing steps to simulate the reduction of the other one, hence a non-erasing normalising chain for π cannot be translated directly to a non-erasing normalisation of $\pi^{\triangle}$. The core of the problem is DiLL's dereliction on box rule:



By induction hypothesis we can suppose that the translation $\omega^{\triangle}[\nu^{\triangle}]$ of the contractum of π is $\mathsf{WN}_{\neg\epsilon}$, but what about $\pi^{\triangle}$ and its contractum ξ? If we are to use a normalising chain of $\omega^{\triangle}[\nu^{\triangle}]$ to produce one for ξ we have to do 'more work', as we must account for a boxed copy of $\nu^{\triangle}$ and all the reductions it might spawn. We will tackle this in Lemma 30 by defining a relation that holds for $\omega^{\triangle}[\nu^{\triangle}]$ and ξ but is also sufficiently general to be preserved under non-erasing reduction and sufficiently precise to allow transferring a normalising chain of reductions. This relation is the composition $\twoheadrightarrow\rightsquigarrow$ of a relation $\twoheadrightarrow$ called *unfolding* (subsection 4.3) and a rewriting $\rightsquigarrow$ (subsection 4.2). Yet another remark: to make Lemma 30 work, we require the context $\omega[\,]$ to be linear, as otherwise the above

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mentioned relation gets too complicated to handle. Nevertheless this is enough to get $\pi \in \mathsf{WN}_{\neg\epsilon} \Longrightarrow \pi^{\triangle} \in \mathsf{WN}_{\neg\epsilon}$ thanks to a standardisation property that has its own interest (Theorem 23, shown in subsection 4.1).

Lemma 20 ($\mathsf{DiLL}_{\partial\varrho}$ simulates DiLL). If in DiLL $\pi \rightsquigarrow \xi$ then $\pi^{\triangle} \overset{+}{\rightsquigarrow} \xi^{\triangle}$ in $\mathsf{DiLL}_{\partial\varrho}$. In particular, if $\pi^{\triangle}$ is $\mathsf{SN}_{\sim}$, then so is π in DiLL.

Proof. The proof is an exercise in reducing nets. For equivalence relations, it is immediate that $\pi \mapsto \xi$ implies $\pi^{\triangle} \mapsto \xi^{\triangle}$. For non-erasing reductions we refer to the figures in the proof of Lemma 31, where for each non-erasing redex $\lambda \to \mu$ involving a dereliction or codereliction in DiLL, both $\mu^{\triangle}$ and the first step in reducing $\lambda^{\triangle}$ is shown. It is straightforward to see that reducing further all such immediate contracta with erasing steps gives $\mu^{\triangle}$ in all the cases. As for erasing steps, the only relevant cases are the reduction to 0 with (co)weakening on (co)dereliction, for which the result is immediate. □

4.1. *The standardisation theorem in* DiLL

What we will prove in the following is the equivalent of the standardisation theorem in λ-calculus (Barendregt 1981, Theorem 11.4.7). As the standardisation we need in Lemma 31 is for non-erasing reduction in DiLL, we state the theorem to encompass also this case. We recall that the depth of a step $\omega[\pi]\ \tilde{R}\ \omega[\xi]$ with $\tilde{R}$ the context closure of R and $\pi\ R\ \xi$ is the number of nested boxes containing the hole in $\omega[\ ]$. Let us denote by $\to_0$ (resp. $\to_{>0}$) a reduction happening at depth 0 (resp. greater than 0), and similarly for $\mapsto$, $\sim$ and the combination $\rightsquigarrow$. For the sake of clarity we will first prove a postponement property for non-erasing reduction, then we generalise to full $\rightsquigarrow$, and finally we arrive to standardisation of both. Notice that none of the results in this subsection needs the switching acyclicity hypothesis on any of the nets involved.

Lemma 21. $\overset{\neg\epsilon}{\to}_{>0}$ can be postponed with respect to $\overset{\neg\epsilon}{\to}_0$, i.e. $\overset{\neg\epsilon^*}{\to} \subseteq \overset{\neg\epsilon^*}{\to}_0 \overset{\neg\epsilon^*}{\to}_{>0}$.

Proof. We aim at applying Lemma 41, taking $\overset{1}{\to}$ and $\overset{2}{\to}$ to be the expansions $\overset{\neg\epsilon}{\leftarrow}_0$ and $\overset{\neg\epsilon}{\leftarrow}_{>0}$ respectively. We thus need an increasing measure $|\pi|$, and to obtain it we tweak the system: we add pebbles as in $\mathsf{DiLL}_{\partial\varrho}$, we redefine all non-erasing reductions to throw in a pebble in each addend of the contractum and finally we take $|\pi|$ to be the number of pebbles in π at any depth. Clearly postponement in this system entails the same in DiLL, as the latter can be recovered by just forgetting pebbles. The measure increases on all steps as no erasing is done. It remains to show that

$$\overset{\neg\epsilon}{\to}_{>0} \overset{\neg\epsilon}{\to}_0 \subseteq \overset{\neg\epsilon}{\to}_0 \overset{\neg\epsilon^*}{\to},$$

considering also pebbles. In all cases, the deeper step cannot have created the 0-depth redex, so the latter step can always be performed first. Then the deeper one can be performed on all the residuals of its redex, whether they have been duplicated additively, or by a reduction on the box containing it which may have also lowered its depth. □

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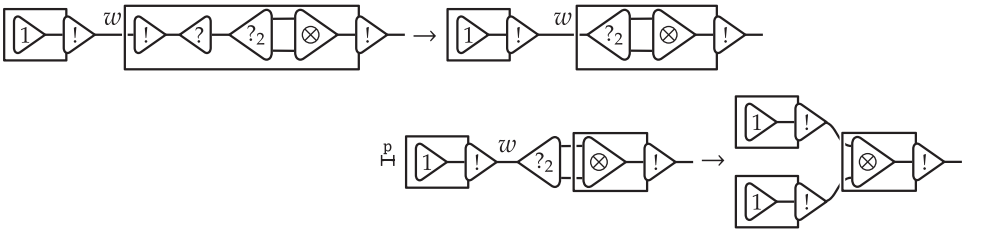


Fig. 17. A counter-example living in MELL to the standardisation property in the presence of push conversion. The non-standard chain of reductions and conversions does not have a standardised version.

Recall from subsection 2.2 that $\overset{\mathtt{can}}{\to}$ (resp. $\sim$) can denote either $\overset{\mathtt{n}}{\to}$ (resp. $\overset{\mathtt{a}}{\sim}$) or the npz-reduction (resp. the union of the a, p and s conversions). The following Lemma 22 and Theorem 23 hold however only for $\overset{\mathtt{n}}{\to}$ and $\overset{\mathtt{a}}{\sim}$, so for $\to = \overset{\mathtt{cut}}{\longrightarrow} \cup \overset{\mathtt{n}}{\to}$ and $\rightsquigarrow = \overset{\mathtt{a}}{\sim}\to\overset{\mathtt{a}}{\sim}$. A counter-example dealing with $\overset{\mathtt{aps}}{\sim}$ and $\overset{\mathtt{npz}}{\longrightarrow}$ is pictured in Figure 17 and discussed at the end of the subsection.

Lemma 22. Take $\sim$ and $\overset{\mathtt{can}}{\to}$ to be $\overset{\mathtt{a}}{\sim}$ and $\overset{\mathtt{n}}{\to}$ respectively. Then $\to_{>0} \cup \mapsto_{>0}$ can be postponed with respect to $\rightsquigarrow_0$, i.e. $\overset{*}{\rightsquigarrow} = (\rightsquigarrow_0 \cup \to_{>0} \cup \mapsto_{>0})^* \subseteq \overset{*}{\rightsquigarrow}_0 (\to_{>0} \cup \mapsto_{>0})^* = \overset{*}{\rightsquigarrow}_0 \overset{*}{\to}_{>0}$.

Proof. We adapt the proof of Lemma 21 to this case. However, we need to tweak the system further to prevent erasing steps from deleting the contribution of pebbles to the measure. Let us therefore introduce special nets $\underline{n}$ for $n \in \mathbb{N}$, considered to have any number and type of free ports. We redefine $|\pi|$ to be the total number of pebbles plus the sum of n for any $\underline{n}$ in π at any depth. We define how $\underline{n}$ interacts with other nets by setting $\omega[\underline{n}] = \underline{n + |\omega|}$ for $\omega[\ ]$ linear, i.e. it erases the rest of the context just like 0, 'saving' however its measure:

$$|\omega[\underline{n}]| = |\omega| + n, \tag{2}$$

To prevent 0-boxes from erasing part of the measure, we consider them as being $!\underline{0}$.

Now we redefine all reductions to introduce a pebble as in the proof of Lemma 21, but for $?_0/!\pi$, $?_0/!$ and $!_0/?$. Among the latter, we redefine the first adding $|\pi| + 1$ pebbles in the contractum; we change the other two to yield $\underline{1}$. Finally, in order to close the needed diagram, we add two dummy reductions:

(i) one creating a pebble from nothing, and
(ii) the other taking $\underline{n}$ to $\underline{n+1}$.

The measure $|\pi|$ now increases strictly on all reduction steps (and is invariant under $\overset{\mathtt{a}}{\sim}$). The delicate cases are the truly erasing ones: opening a box containing only $\sum \underline{n_i}$ (including $!0$ which has become $!\underline{0}$) is increasing due to (2); a box entering a $!(\sum \underline{n_i})$ is settled for the same reason, as the new contents of the box are indeed a linear context given by the entering box and applied to the old contents; $?_0/!\pi$, $?_0/!$ and $!_0/?$ again restore the erased part of the measure. Moreover, normal DiLL can be promptly recovered by forgetting both pebbles and $\underline{n}$ nets, and the redefined reductions map to normal ones,

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possibly loosing the new steps. In particular, proving postponement in the tweaked system entails the same property in DiLL.

Now in order to use Lemma 41, we show that

$$(\rightarrow_{>0} \cup \mapsto_{>0}) \rightsquigarrow_0 \subseteq \rightsquigarrow_0 \overset{*}{\rightsquigarrow}.$$

The reasoning goes as in Lemma 21, as a-conversions do not pose any serious problem. The only really different case is when the 0-depth step erases the location of the deeper one. In this case making the 0-depth one first gives necessarily a strictly lower number of pebbles or $\underline{n}$-net, so that dummy steps (i) and (ii) are needed to close the diagram again. □

Notice that postponement of $\rightsquigarrow_{>0}$ with respect to $\rightsquigarrow_0$ does not give that $\overset{*}{\rightsquigarrow} \subseteq \overset{*}{\rightsquigarrow_0} \overset{*}{\rightsquigarrow_{>0}}$, as for example we would not entail anything on a chain of the form $\rightarrow_0 \mapsto_{>0} \rightarrow_0$, which is in $\overset{*}{\rightsquigarrow}$ but not in $(\rightsquigarrow_0 \cup \rightsquigarrow_{>0})^*$. This justifies the peculiar statement of the above lemma.

Theorem 23 (Standardisation in DiLL). Take $\sim$ and $\overset{\mathtt{can}}{\rightarrow}$ to be exclusively $\overset{\mathtt{a}}{\sim}$ and $\overset{\mathtt{n}}{\rightarrow}$ respectively. Let π be a DiLL net. If $\pi \overset{*}{\rightsquigarrow} \pi'$ (resp. $\pi \overset{\neg\epsilon^*}{\rightarrow} \pi'$), then there is a chain (called **standard**) of reductions and conversions (resp. of non-erasing reductions) from π to π' such that their depth is monotone increasing.

Proof. The proof goes by induction on the depth of π'. We can transform the given chain into $\pi \overset{*}{\rightsquigarrow_0} \nu \overset{*}{\rightsquigarrow_{>0}} \pi'$ (resp. $\pi \overset{\neg\epsilon^*}{\rightarrow_0} \nu \overset{\neg\epsilon^*}{\rightarrow_{>0}} \pi'$) by applying Lemma 22 (resp. Lemma 21). Such a chain induces a bijection f from $\mathsf{b}_0(\nu)$ to $\mathsf{b}_0(\pi')$ (the set of the 0-depth boxes), so that for every $c \in \mathsf{b}_0(\nu)$ we have $\sigma(c) \overset{*}{\rightsquigarrow} \sigma(f(c))$ (the box contents), or resp. non-erasing steps. By inductive hypothesis there are standard chains taking all $\sigma(c)$ to $\sigma(f(c))$, and as these can be freely interleaved within ν, we end up with a standard chain from ν to π', which prepended with the 0-depth reductions from π to ν gives the desired standard chain of steps. □

Leaving out the general conversions and canonical steps is necessary, or else standardisation fails. As an example take a cut w between the principal port of a box $!\pi$ and an auxiliary port of a box $!\xi$ and suppose that a reduction happening in ξ creates a contraction cell over the port corresponding to w. Such a situation is depicted in Figure 17. By a conversion we can push this contraction out of the box, so changing w evolves into a contraction/box cut. Performing one reduction step on w yields a net that cannot be obtained by the starting one by a standard reduction, as the only possible 0-depth step brings irreversibly $!\pi$ inside $!\xi$. Similar examples can be done using bang sum conversion and the two other canonical reductions. Notice that this drawback is not specific to DiLL but is also present in MELL.

4.2. *Dead ends and their manipulation in* $\mathsf{DiLL}_{\partial\varrho}$

In this section we introduce and study the rewriting $\rightsquigarrow$ (Figure 18), one of the two main tools used in the proof of Lemma 31 (the other being the unfolding defined in subsection 4.3).

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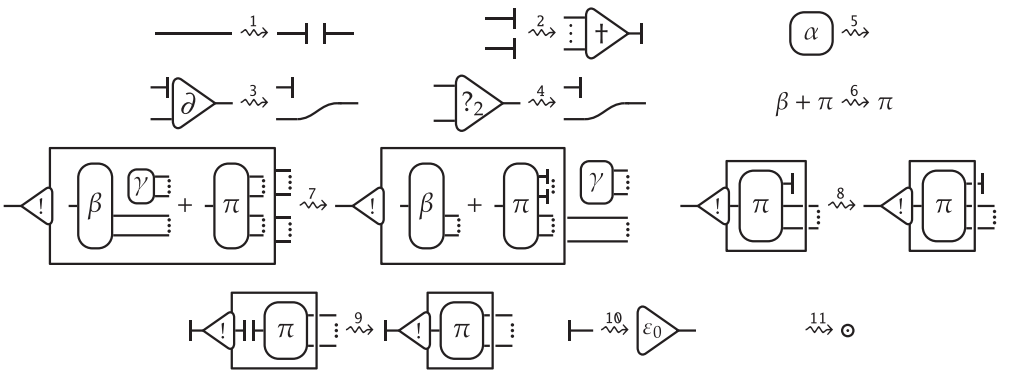


Fig. 18. Rewriting rules for dead ends. † is any non-box cell, α is any simple net without free ports, β any simple net, γ any simple net not containing any ∂ or ϱ cells at depth 0, ε_0 a weakening or coweakening. Rule 11 creates a pebble from the empty graph. Lax typing is omitted altogether as it is irrelevant for this rewriting.

We introduce in $\mathsf{DiLL}_{\partial\varrho}$ a new cell with one inactive port and no reduction rules, the **dead end** (graphically depicted by $\dashv$). As such cell is inert, it does not change anything of the computational properties of $\mathsf{DiLL}_{\partial\varrho}$, so all the results proved in Section 3 are still valid. In order to manipulate such cells we introduce however a rewriting relation denoted by $\rightsquigarrow$ which is the context closure of the rules in Figure 18. Notice that $\rightsquigarrow$ does not necessarily preserve correctness, due to $\overset{2}{\rightsquigarrow}$. In the following, we will prove the important properties of $\rightsquigarrow$.

Lemma 24. $\rightsquigarrow$ and $\overset{\neg\epsilon}{\rightarrow}$ commute, i.e. we have the following confluence diagram:

$$\begin{array}{ccc} \pi & \rightsquigarrow & \pi_1 \\ \downarrow{\scriptstyle \neg\epsilon^* \neg\epsilon^*} & & \downarrow \\ \pi_2 & \rightsquigarrow & \pi_3 \end{array}$$

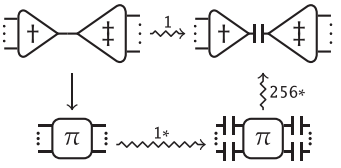
Proof. One proves that $\forall \pi, \pi_1, \pi_2, \exists \pi_3$:

$$\begin{array}{ccc} \pi & \rightsquigarrow & \pi_1 \\ \downarrow{\scriptstyle \neg\epsilon^* \neg\epsilon} & & \downarrow{=} \\ \pi_2 & \rightsquigarrow & \pi_3 \end{array}$$

so that the result follows by Hindley's Lemma (Lemma 40). We here list the critical pairs, organised by the numbers associated with the $\rightsquigarrow$ reduction rules given in Figure 18.

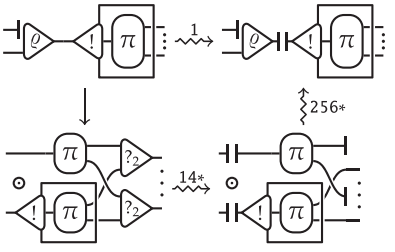
1. The $\overset{1}{\rightsquigarrow}$ rewriting may interfere with a reduction by interrupting the cut to be reduced. Let us first tackle the case in which neither of the two interacting cells is a box. Then

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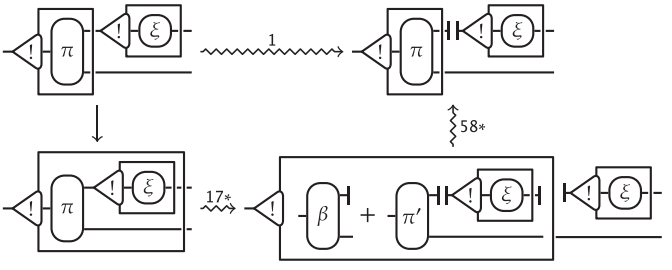


Here and in the following cases it is important that the reduction is non-erasing, so that in particular $\pi \neq 0$. This is to assure the $\neg\epsilon$-contractum is a non-empty graph, so that one can apply $\rightsquigarrow$. Step 6 may be necessary to delete addends added by the reduction.

Suppose now one of the two interacting cells is a box, so step 2 cannot be used to rebuild it. We have five cases, given by the other cell being ϱ, ∂, $?_2$, a box or a $!_2$.



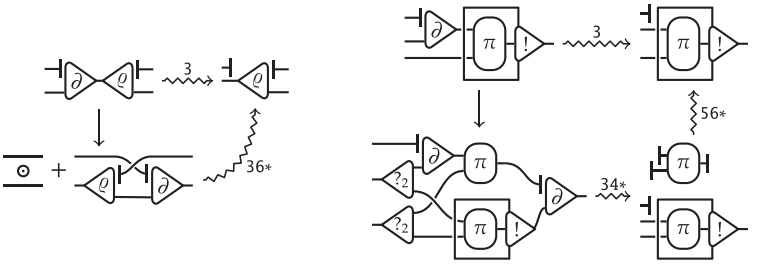
The cases for $?_2$ and ∂ are similar. As for box on box:



where $\pi = \beta + \pi'$ (as $\pi \neq 0$), and the number of wires is simplified. The case for $!_2$ is identical.

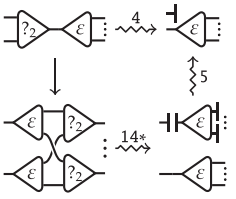
2. This step does not give rise to any critical pairs.

3. The critical pairs arise when the ∂ cell interacts with another cell. We show all but the case for a $?_2$-cell, as it is similar to the following ϱ one.



4. The case for a $?_2$ cell interacting with a ∂ cell is similar to the ∂/ϱ case above. Let ε be either $!_2$ or $!\pi$, then:

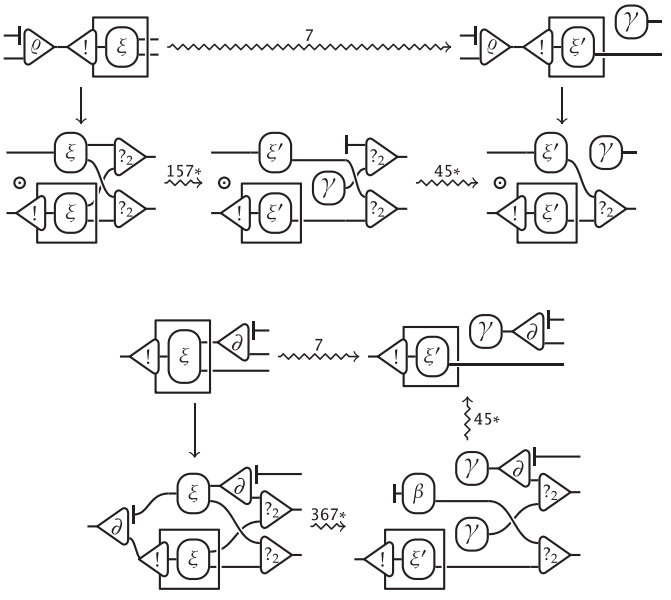
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5, 6. These steps pose no problem, as they interfere with a reduction only by completely deleting the location of the redex, which can be done also after firing it. In both cases, more 6-steps could be necessary to erase additional addends spawned by the reduction.

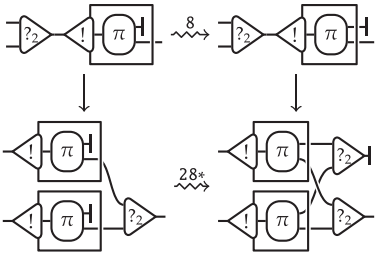
7. Let us use the notation of Figure 18. First suppose the reduction takes place inside γ. As no 0-depth ∂ or ϱ cell is present in it, this reduction cannot spawn new addends, so the diagram is easily closed. If it takes place in β, then we can close by extracting γ from one of the addends originated by the reduction, and then eliminating all other instances of γ by means of 5-steps (as 7 severs their wires to the auxiliary ports). No difficulty arises when the reduction is in π.

Finally, let us consider the case the reduction deals with one port of the box fired by the rewriting $\overset{7}{\rightsquigarrow}$. Let ξ be the contents of the box before the $\overset{7}{\rightsquigarrow}$ step, and ξ' afterwards with γ extracted and π's wires severed. Notice that $\xi \overset{15^*}{\rightsquigarrow} \xi'$ up to dead ends over some of the free ports (corresponding to γ): i.e. there is ξ'' differing from ξ' by having more free ports all directly connected to dead ends, such that $\xi \overset{15^*}{\rightsquigarrow} \xi''$. We show the cases for $\varrho/!\xi$ and $\partial/!\xi$ with the ∂ cell on a port corresponding to γ. The other cases for ∂ and $?_2$ are very similar to the $\varrho/!\xi$ one, while $!_2$ and $!\rho$ on $!\xi$ are immediate (possibly glueing the entering cell to γ). We make a simplification on the number of wires, giving one free port to γ and one to β.

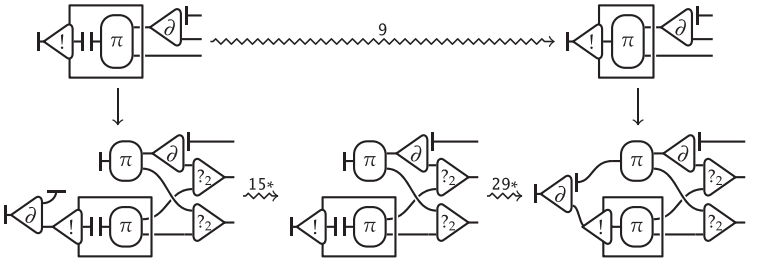


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8. Reductions happening in π (using the notation of Figure 18), or $!_2$ and $!\rho$ cells entering the box pose no problems. The other cases are similar to the one for $?_2$:



9. As above, but the only case of relevance is $\partial/!\pi$:

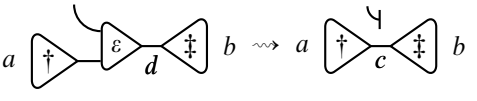


10, 11. These steps have no critical pairs. □

Lemma 25. If $\pi \rightsquigarrow \xi$ and π is normal for $\neg\epsilon$-reduction, then so is ξ.

Proof. One reasons by cases on which rule of $\rightsquigarrow$ was applied. In fact the only cases that could give a problem are the steps 3 or 4, replacing a ∂ or $?_2$ cell by a wire and a dead end, as the other steps cannot possibly introduce cuts.

Let c be a reducible cut in ξ that is not present in π. It must therefore be the wire resulting from one of the above mentioned $\rightsquigarrow$-reductions. The situation is thus the following:



with ε either ∂ or $?_2$, and with b neither a weakening nor a coweakening. b could be a 0 box only if $\dagger = ?_2$ and c is on the principal port. As c is reducible, and therefore not $\mathfrak{O}$-typed, so also d is not $\mathfrak{O}$-typed. It must be therefore the case that d is a reducible cut, so that π cannot be normal. □

Lemma 26. If π is $\mathsf{WN}_{\neg\epsilon}$ and $\pi \overset{*}{\rightsquigarrow} \xi$, then also ξ is $\mathsf{WN}_{\neg\epsilon}$.

Proof. Direct consequence of Lemmas 24 and 25. □

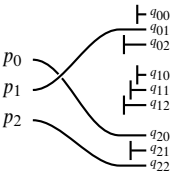
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4.3. *Unfolding*

Given any function $f : n \to k$ between finite ordinals[14], we define a net $\hat{f}$ which intuitively links n ports with k groups of n ports each, so that the ith port among the n ones gets connected to the ith port of the $f(i)$th group, while the uncovered ports are severed by dead ends. Formally, let $\hat{f}$ be the net with:

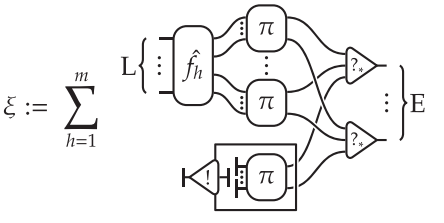
— free ports $\mathsf{fp}(\hat{f}) = n + kn$, denoted with $\{p_i\}_{i\in n} \cup \{q_{ji}\}_{j\in k, i\in n}$;
— cells $\mathsf{c}(\hat{f}) = \{\, c^{ji} \mid j \in k, i \in n, j \neq f(i) \,\}$, all dead ends;
— wires $\mathsf{w}(\hat{f}) = \{\, \{p_i, q_{f(i)i}\} \mid i \in n \,\} \cup \left\{\, \{c_0^{ji}, q_{ji}\} \mid c^{ji} \in \mathsf{c}(\hat{f}) \,\right\}$, where c_0^{ij} denotes the sole port of the dead end c^{ij}.

For example, take $f : 3 \to 3$ given by $f(0) = f(2) = 2$ and $f(1) = 0$. Then

$\hat{f} =$ 

where for the sake of presentation we have written the name of the ports in the figure. We define a relation on pairs of contexts and nets, called **unfolding** and denoted by $\omega[\pi] \twoheadrightarrow \omega[\xi]$. It holds when:

1. $\omega[\,]$ is a linear context, with its hole interface partitioned into two interfaces L and E which we will call **linear** and **exponential**;
2. π is a net, with interface conveniently identified with $L + E$;
3. $\omega[\pi]$ is switching acyclic;
4. all wires passing through E in $\omega[\pi]$, with direction from π outward, are laxly typed by either ? or $\mathfrak{V}$;
5. drawing L left and E right, ξ is

$\xi :=$ 

with $m \geqslant 1$, and $f_h : \# L \to k_h$ for each h, with $k_h \geqslant 1$, where $?_*$ stands for any tree of contractions with a suitable number of leaves (namely $k_h + 1$ for the hth addend);
6. for every switching path (see subsection 2.5) in $\omega[\,]$ linking two ports of the hole, we have that both of its ends are in L. Moreover if these ends are the ith and jth ports of L, then $f_h(i) = f_h(j)$ for all addends of ξ, i.e. they are linked to the same linear copy of π.

[14] We recall that all natural numbers may be identified with the set of strictly less numbers, i.e. $n = \{0, \ldots, n-1\}$.

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Notice that because of the last point the relation is not contextual, even when restricted to linear contexts. Notice also that the last condition entails that also $\omega[\xi]$ is switching acyclic: it could fail to be so if a switching cycle was formed by passing through two auxiliary ports of the new box, but this can't happen by definition.

To get a first glimpse of what the above conditions mean, we can foretell we will use unfolding on the result of a $?/!\pi$ reduction: initially L will be the wire put in place of the principal port of $!\pi$, while the wires from the former auxiliary ports will form E. The last condition reflects what remains of the correctness criterion of boxes after they are opened. The shape of ξ is an evolution of what can be found after replacing the dereliction in $?/!\pi$ with its translation in $\mathsf{DiLL}_{\partial!}$ and reducing the resulting cut, which always leaves a copy of the box behind.

Lemma 27. If $\omega[\pi] \twoheadrightarrow \omega[\xi]$, then there are $\omega'[\,]$, π', ξ' and v such that

$$\begin{array}{ccc} \omega[\pi] & \longrightarrow\!\!\!\!\!\rightarrow & \omega[\xi] \\ \| & & \searrow^{\neg\epsilon*} \\ \omega'[\pi'] & \twoheadrightarrow & \omega'[\xi'] \overset{*}{\rightsquigarrow} v \end{array} \qquad (*)$$

where $\omega'[\pi']$ is a decomposition of $\omega[\pi]$ such that no wire connecting π' or ξ' to $\omega'[\,]$ is a $\neg\epsilon$-reducible cut. Moreover, if $\omega[\,]$ is normal then so is $\omega'[\,]$.

Proof. We reason by induction on the size of $\omega[\,]$. The base of induction is when there is no cut glueing π or ξ to $\omega[\,]$. In this case the statement of the lemma is trivial, by setting $\omega'[\,] = \omega[\,]$, $\pi' = \pi$, $\xi' = \xi$ and $v = \omega[\xi]$. The inductive step is when there is a $\neg\epsilon$-reducible cut w connecting π or ξ to $\omega[\,]$. What we prove is that there is $\omega'[\,]$, π', ξ' and v satisfying diagram $(*)$ with $\omega'[\,]$ smaller than $\omega[\,]$ (in fact a subnet, so that the condition for a normal $\omega[\,]$ follows). Then, the inductive step is solved by

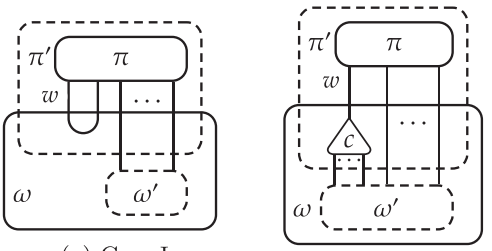
$$\begin{array}{ccccc} \omega[\pi] & \longrightarrow\!\!\!\!\!\rightarrow & \omega[\xi] & & \\ \| & \text{to be proved} & \searrow^{\neg\epsilon*} & & \\ \omega'[\pi'] & \longrightarrow\!\!\!\!\!\rightarrow & \omega'[\xi'] \overset{*}{\rightsquigarrow} v & & \\ \| & \text{IH} & \searrow^{\neg\epsilon*}\text{L24} & \searrow^{\neg\epsilon*} & \\ \omega''[\pi''] & \longrightarrow\!\!\!\!\!\rightarrow & \omega''[\xi''] \overset{*}{\rightsquigarrow} v' & \overset{*}{\rightsquigarrow} v'' & \end{array}$$

where $\omega''[\,]$, π'' and ξ'' are obtained by inductive hypothesis with no cuts along the glueing, and the rightmost square is by Lemma 24.

The proof splits in different cases, as depicted in Figure 19.

Case I. Suppose the cut w linking π or ξ to $\omega[\,]$ comes from a wire in $\omega[\,]$ connecting two ports of the hole (Figure 19a). By item 4.3 in the definition of $\omega[\pi] \twoheadrightarrow \omega[\xi]$, this wire must be on two ports in L which are connected both to the same copies of π in ξ. Define $\omega'[\,]$ as $\omega[\,]$ without this wire, and π' accordingly glueing the wire back. ξ' is immediately obtained from π' by following the choices made in ξ and just ignoring what was chosen for the missing ports of the interface. Then $\omega'[\xi']$ differs from $\omega[\xi]$ just by those copies of π in ξ that do not get the wire from $\omega[\,]$. The diagram is then closed using

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(a) Case I (b) Case II

Fig. 19. The different cases of the inductive step in the proof of Lemma 27. The wire w is a cut contained in the interface between π and $\omega[\,]$, which are drawn in solid lines. The dashed lines denote how we shrink the context to $\omega'[\,]$ (and consistently grow the net to π') in order to apply to it inductive hypothesis. Notice that in Case II what we draw as the principal port of the cell c may as well be an active auxiliary port of a box.

several steps 1 of $\rightsquigarrow$ interrupting the glued wire in where necessary. No $\neg\epsilon$-reduction of $\omega[\xi]$ is necessary (i.e. $v = \omega[\xi]$).

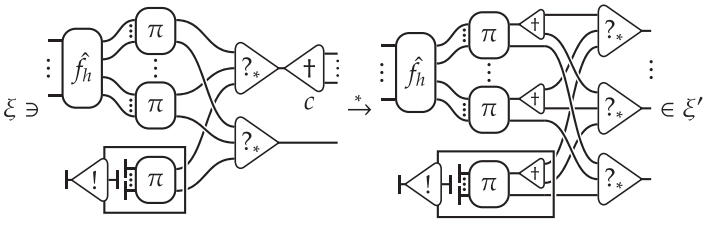
Case II. If the cut w is not from a wire in $\omega[\,]$ on the hole, then at least one of the cells (say c) involved in the cut is in $\omega[\,]$ (Figure 19b). We define $\omega'[\,]$ and π' by detaching c from $\omega[\,]$ and glueing it to π. We split then in two sub-cases, depending whether w links c with a port in the linear interface L or in the exponential interface E of $\omega[\,]$.

Case II.a. If w is on the linear interface of $\omega[\,]$, the linear interface L' of $\omega'[\,]$ has one less port (say p, the one on the cut w) and some new ones due to the other ports of c. ξ' is obtained by following the same choices used for ξ, assigning to the new ports the same linear copies of π that were assigned to p. The only difference between $\omega[\xi]$ and $\omega'[\xi']$ is that all the copies of π that in $\omega[\xi]$ were not connected to p (including the one in the box), now have a copy of c, with dead ends on the ports not involved in the cut w. Steps 1 and 5 of $\rightsquigarrow$ easily bring $\omega'[\xi']$ back to $\omega[\xi]$, again with no needed $\neg\epsilon$-reduction (i.e. $v = \omega[\xi]$). For showing $\omega'[\pi'] \twoheadrightarrow \omega'[\xi']$, now just the condition on switching paths needs to be checked (item 4.3 in the definition of $\twoheadrightarrow$). Take a switching path in $\omega'[\,]$ linking two ports on the hole, exactly one among the new ones otherwise the condition is trivially met: if both are, then they are assigned to the same copies of π' by definition; if none are, it follows from $\omega[\pi] \twoheadrightarrow \omega[\xi]$. Notice that as p was on the cut w, thus linked to an active port of c (be it a principal port or an auxiliary one of a box), this path induces one in $\omega[\,]$ which is still switching and linking p, and the condition follows.

Case II.b. Suppose finally that the cut w has a port in the exponential interface E. By the lax type condition in the definition of $\twoheadrightarrow$ (item 4.3), for the cut w to be reducible the cell c must be either a duplicable cell (i.e. a box or cocontraction) which we will denote by a generic symbol †, or a ∂ cell. In the first case, letting all the new ports of the hole

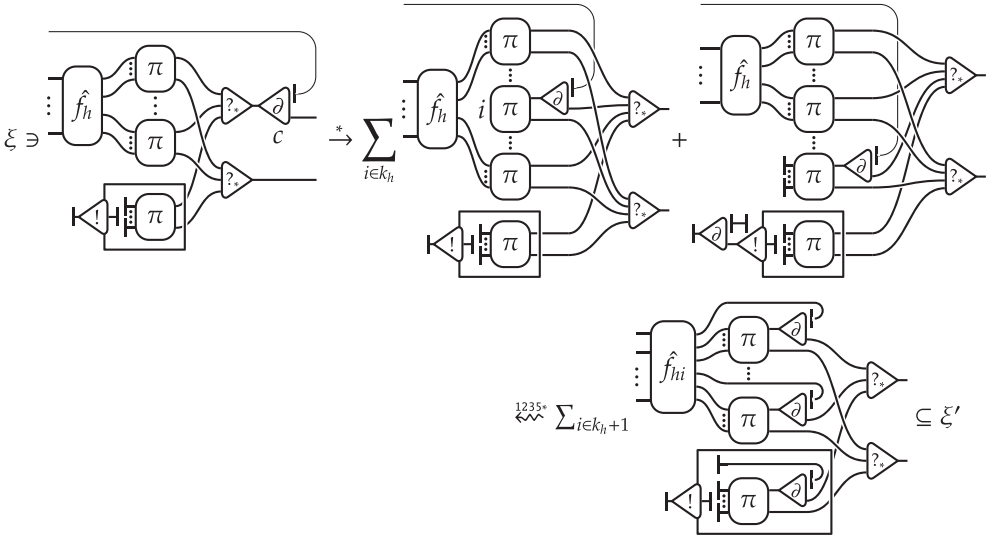
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of $\omega'[\,]$ (i.e. the auxiliary ones of c) be in the exponential interface and defining ξ' by following the same choices made for ξ, we immediately get (depicting a single addend of ξ and ξ')



The condition on switching paths is met as † is always non-switching: no paths link its auxiliary ports (if any) among themselves or to other ports of the new interfaces. Notice that we have $v = \omega[\epsilon']$ (no $\rightsquigarrow$ step is necessary).

In the latter case (i.e. c being a ∂ cell), we let one of the new ports be in the exponential interface of $\omega'[\,]$ and the other in the linear one and close the diagram in the following way (by appropriately defining ξ')



where if $f_h : n \to k_h$ then for $i \in k_h$ we set $f_{hi} : n \to k_h$ defined by $f_{hi}(0) = i$ and $f_{hi}(j+1) = f_h(j)$ (i.e. we complete f_h in all the possible ways for the new port), and finally $f_{hk_h} : n \to k_h + 1$ defined by $f_{hk_h}(0) = k_h$ and $f_{hk_h}(j+1) = f_h(j)$ (i.e. we further complete it by spawning a new linear copy of π). 3-steps take out the surplus of ∂ cells, letting 5-steps erase the resulting disconnected wires. A 1 and a 2 step are needed to recreate the ∂ cell linked to the box in the $k_h + 1$th addend.

Again, the last thing to do is to check the condition on switching paths (item 4.3 in the definition of $\twoheadrightarrow$). No path links the new ports with any other port on the interfaces, otherwise $\omega[\,]$ itself would not meet the condition. Moreover the two new ports are not linked between themselves as ∂ is non-switching. □

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Lemma 28. If $\omega[\pi] \twoheadrightarrow \omega[\xi] \overset{*}{\rightsquigarrow} v$ and $\omega[\pi]$ is normal for $\neg\epsilon$-reduction, then $v \in \mathsf{WN}_{\neg\epsilon}$.

Proof. Using Lemmas 27 and 24, we get

$$\begin{array}{ccccc} \omega[\pi] & \twoheadrightarrow & \omega[\xi] & \overset{*}{\longrightarrow} & v \\ \| & \text{L27} & & \overset{\neg\epsilon*}{\searrow}\ \text{L24} & \overset{\neg\epsilon*}{\searrow} \\ \omega'[\pi'] & \twoheadrightarrow & \omega'[\xi'] & \overset{*}{\rightsquigarrow} v' & \overset{*}{\rightsquigarrow} v'' \end{array}$$

As $\omega[\pi] = \omega'[\pi']$ is normal, then so are $\omega'[\,]$, π' and therefore ξ'. By Lemma 27 we deduce that $\omega'[\xi']$ is normal for $\neg\epsilon$-reduction, as that lemma explicitly forbids the plugging between ξ' and $\omega'[\,]$ from forming new cuts. By Lemma 25 also v'' is normal (in fact $\omega'[\xi'] \overset{*}{\rightsquigarrow} v''$). We conclude $v \in \mathsf{WN}_{\neg\epsilon}$. □

Lemma 29. If $\omega[\pi] \twoheadrightarrow \omega[\xi] \overset{*}{\rightsquigarrow} v$, $\omega[\,]$ is normal and $\omega[\pi] \overset{\neg\epsilon}{\to} \rho$, then there are $\omega'[\,]$ normal, π', ξ' and v' with

$$\begin{array}{ccccc} \omega[\pi] & \twoheadrightarrow & \omega[\xi] & \overset{*}{\rightsquigarrow} & v \\ {\scriptstyle\neg\epsilon}\downarrow & & & & \\ \rho & & & & \downarrow{\scriptstyle\neg\epsilon*} \\ \| & & & & \\ \omega'[\pi'] & \twoheadrightarrow & \omega'[\xi'] & \overset{*}{\rightsquigarrow} & v' \end{array}$$

Proof. First of all we apply Lemma 27, getting $\omega'[\pi''] = \omega[\pi]$ and ξ'', satisfying the diagram marked by L27 below and with $\omega'[\,]$ normal and such that no wire connecting π'' or ξ'' to $\omega'[\,]$ is a cut reducible by a $\neg\epsilon$-reduction. As $\omega'[\pi''] \overset{\neg\epsilon}{\to} \rho$ but no cut is reducible along the glueing or in $\omega'[\,]$, the reduction must happen in π''. We set $\pi' \overset{\neg\epsilon}{\leftarrow} \pi''$, by firing the same redex, so that clearly $\rho = \omega'[\pi']$. Moreover from $\pi'' \overset{\neg\epsilon}{\to} \pi'$ we get $\xi' \overset{\neg\epsilon*}{\leftarrow} \xi''$ with $\omega'[\pi'] \twoheadrightarrow \omega'[\xi']$, by firing the appropriate redexes in all copies made of π'' in its unfolding ξ'' . Summing up:

$$\begin{array}{ccccc} \omega[\pi] & \twoheadrightarrow & \omega[\xi] & \overset{*}{\rightsquigarrow} & v \\ \| & \text{L27} & & \overset{\neg\epsilon*}{\searrow}\ \text{L24} & \overset{\neg\epsilon*}{\searrow} \\ \omega'[\pi''] & \twoheadrightarrow & \omega'[\xi''] & \overset{*}{\rightsquigarrow} v'' & \overset{*}{\rightsquigarrow} \cdot \\ {\scriptstyle\neg\epsilon}\downarrow & & \downarrow{\scriptstyle\neg\epsilon*} & & \downarrow{\scriptstyle\neg\epsilon*} \\ \rho & & \text{L24} & & \\ \| & & & & \\ \omega'[\pi'] & \twoheadrightarrow & \omega'[\xi'] & \overset{*}{\rightsquigarrow} & v' \end{array}$$

where the whole diagram is closed using commutation (Lemma 24). The condition of normality of $\omega[\,]$ is needed to prevent the context from additively duplicating its hole, and thus ceasing to be a context. □

It is now possible to prove the following lemma.

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Lemma 30. If $\omega[\pi] \twoheadrightarrow \omega[\xi] \overset{*}{\rightsquigarrow} v$ and $\omega[\pi] \in \mathsf{WN}_{\neg\epsilon}$, then $v \in \mathsf{WN}_{\neg\epsilon}$.

Proof. First, we reduce to the case where $\omega[\ ]$ is $\neg\epsilon$-normal. By Lemma 10 (the conservation property for non-erasing reduction) $\omega[\pi] \in \mathsf{SN}_{\neg\epsilon\sim}$, and in particular $\omega[\ ] \in \mathsf{WN}_{\neg\epsilon}$. Reducing the context to normal form for $\neg\epsilon$-reduction we get $\omega[\pi] \overset{*}{\to} \sum_i \omega_i[\pi]$ and $\omega[\xi] \overset{*}{\to} \sum_i \omega_i[\xi]$ with $\omega_i[\ ]$ normal: as $\omega[\ ]$ is linear (item 4.3 of the definition of $\omega[\pi] \twoheadrightarrow \omega[\xi]$) the hole may be duplicated only additively. We see that for each i we have $\omega_i[\pi] \twoheadrightarrow \omega_i[\xi]$, as reduction does not create switching paths (Lemma 4), so item 4.3 and item 4.3 of the definition of $\omega_i[\pi] \twoheadrightarrow \omega_i[\xi]$ are met. The other items hold trivially. Moreover, $\omega_i[\pi]$ is $\mathsf{WN}_{\neg\epsilon}$, in fact $\omega_i[\pi]$ is $\mathsf{SN}_{\neg\epsilon\sim}$. Finally, by Lemma 24 applied to $\sum_i \omega_i[\xi] \overset{*}{\leftarrow} \omega[\xi] \overset{*}{\rightsquigarrow} v$ we get a net v' such that $\sum_i \omega_i[\xi] \overset{*}{\rightsquigarrow} v' \overset{*}{\leftarrow} v$. Notice that the rewriting $\sum_i \omega_i[\xi] \overset{*}{\rightsquigarrow} v'$ induces a partition $\sum_i v_i = v'$ such that for every i, $\omega_i[\xi] \overset{*}{\rightsquigarrow} v_i$. So we have reduced the proof to the case where $\omega[\ ]$ is normal.

Now, let us prove the lemma supposing $\omega[\ ]$ is $\neg\epsilon$-normal. We reason by induction on $\omega[\ ]$ with respect to $\neg\epsilon$-reduction, which is well-founded (i.e. SN) because of the hypothesis $\omega[\pi] \in \mathsf{WN}_{\neg\epsilon}$ and the conservation theorem for $\mathsf{DiLL}_{\partial!}$ (Theorem 19). The base case is in fact Lemma 28, while if $\omega[\pi] \to \rho$ we can apply Lemma 29 (in fact $\omega[\ ]$ is supposed to be $\neg\epsilon$-normal) and then use the inductive hypothesis, which holds for $\omega'[\pi']$ and get the result. □

4.4. *The main lemma*

We are finally ready to prove the main lemma enabling to transfer the conservation theorem from $\mathsf{DiLL}_{\partial!}$ to DiLL via the translation $\pi^{\triangle}$.

Lemma 31. Let π be a pure DiLL switching acyclic net. If $\pi \in \mathsf{WN}_{\neg\epsilon}$ then $\pi^{\triangle} \in \mathsf{WN}_{\neg\epsilon}$ in $\mathsf{DiLL}_{\partial!}$.

Proof. First, using Theorem 23, consider a standard normalising reduction chain R of π to normal form. Let us reason by induction on the lexicographically ordered pair (n, d), where n is the length of R and d is the depth of the first reduction step (i.e. the minimum depth of all steps in R).

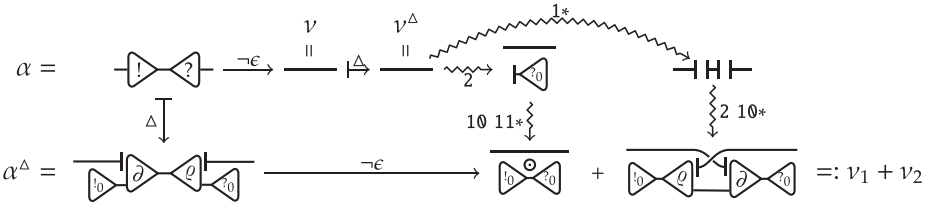
If $n = 0$, i.e. π is normal for $\overset{\neg\epsilon}{\to}$, then so is $\pi^{\triangle}$. If both $n > 0$ and $d > 0$ then let $v_1, \dots, v_k$ be all the contents of the boxes at depth 0 in π. As the reduction is standard, π (and therefore $\pi^{\triangle}$) is normal at depth 0 and all R can be partitioned into standard chains R_i reducing each v_i to normal form. Now the inductive hypothesis kicks in: if the partition is trivial (i.e. the reduction happens all in one v_i) then R_i has the same length but lesser depth in v_i; if it is not trivial then each R_i has strictly less reductions. In any case $v_i^{\triangle}$ are $\mathsf{WN}_{\neg\epsilon}$, and following such normalising reductions in $\pi^{\triangle}$ also leads to a normal form.

Finally, suppose $n > 0$ and $d = 0$, so that $\pi = \omega[\alpha] + \rho \overset{\neg\epsilon}{\to} \omega[v] + \rho$ with α and v a redex/contractum pair, $\omega[\ ]$ a linear context, and by inductive hypothesis $(\omega[v] + \rho)^{\triangle} = \omega^{\triangle}[v^{\triangle}] + \rho^{\triangle} \in \mathsf{WN}_{\neg\epsilon}$. For every reduction which does not touch derelictions or coderelictions, the inductive step is immediate, as the translation in $\mathsf{DiLL}_{\partial!}$ leaves the reduction unchanged apart from possibly differing on pebbles. In that case we have that

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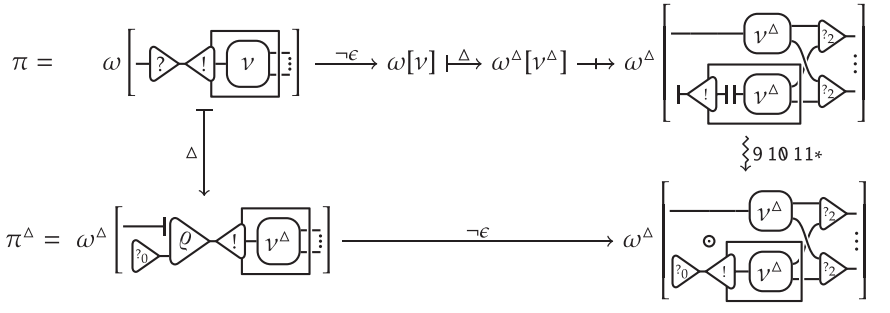
$\pi^\Delta \overset{\neg\epsilon}{\to} \overset{11*}{\rightsquigarrow} (\omega[\nu] + \rho)^\Delta$, so that Lemma 26 yields what needed. Let us see all the remaining cases. We will drop ρ, as closing by sum is trivial.

Dereliction on codereliction:



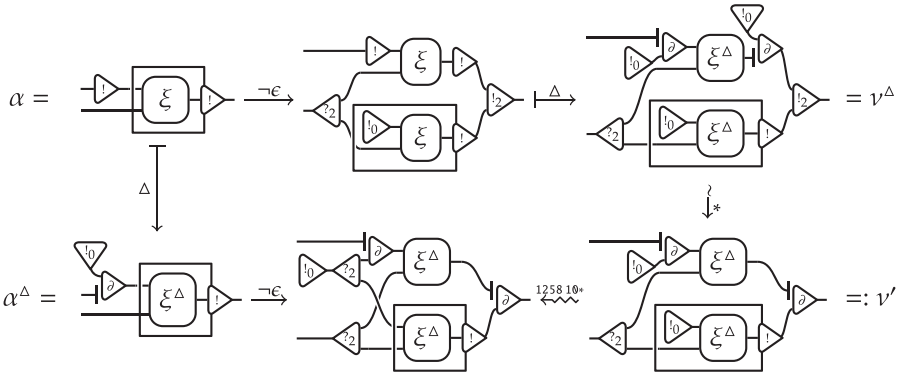
As $\omega^\Delta[\nu^\Delta]$ is $\mathsf{WN}_{\neg\epsilon}$ and $\omega^\Delta[\nu^\Delta] \overset{*}{\rightsquigarrow} \omega^\Delta[\nu_i]$ for both i, it follows from Lemma 26 that both $\omega^\Delta[\nu_i] \in \mathsf{WN}_{\neg\epsilon}$, so $\pi^\Delta \overset{\neg\epsilon}{\to} \omega^\Delta[\nu_1] + \omega^\Delta[\nu_2]$ is also $\mathsf{WN}_{\neg\epsilon}$.

(Co)dereliction on co(co)ntraction:



Notice that $\omega^\Delta[\nu^\Delta]$ satisfies the requirements of $\twoheadrightarrow$ as π and π^Δ are switching acyclic. Lemma 30 applied to $\omega^\Delta[\nu^\Delta] \in \mathsf{WN}_{\neg\epsilon}$ directly gives the result.

Codereliction on box:



By inductive hypothesis, $\omega^\Delta[\nu^\Delta]$ is $\mathsf{WN}_{\neg\epsilon}$, so by Theorem 19 we have $\omega^\Delta[\nu^\Delta] \in \mathsf{SN}_\sim$. This implies that $\omega^\Delta[\nu']$, obtained by an associative conversion and a neutral reduction, is also $\mathsf{SN}_\sim$ and $\mathsf{WN}_{\neg\epsilon}$. Lemma 26 therefore gives the desired result. □

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Theorem 32 (Conservation for DiLL). Given a net π in DiLL, if π is switching acyclic and $\pi \in \mathsf{WN}_{\neg\epsilon}$, then $\pi \in \mathsf{SN}_{\rightsquigarrow}$.

Proof. The proof is now immediate. Take a DiLL switching acyclic net π. Then

$$\begin{aligned} \pi \in \mathsf{WN}_{\neg\epsilon} &\implies \pi^{\triangle} \in \mathsf{WN}_{\neg\epsilon} && \text{by Lemma 31;} \\ &\implies \pi^{\triangle} \in \mathsf{SN}_{\rightsquigarrow} && \text{by Theorem 19, Conservation for } \mathsf{DiLL}_{\partial!}; \\ &\implies \pi \in \mathsf{SN}_{\rightsquigarrow} && \text{by Lemma 20.} \end{aligned}$$

□

5. An application: $\mathsf{SN}_{\rightsquigarrow}$ of simply typed DiLL and $\mathsf{DiLL}_{\partial!}$

We introduce simply typed DiLL and $\mathsf{DiLL}_{\partial!}$: both systems can be proved to be $\mathsf{WN}_{\neg\epsilon}$ by adapting the technique from Pagani (2009) (based on a decreasing measure) or the one from Gimenez (2011) (based on reducibility candidates), therefore conservation yields $\mathsf{SN}_{\rightsquigarrow}$. We define simple typing as a type assignment characterising a subclass of laxly typed nets. Indeed we here move from the assumption that the syntactic objects we are dealing with are inherently laxly typed as we defined them at the beginning of this paper. However, we will assure that the class of simply typable nets is closed under $\twoheadrightarrow$ and that in fact lax types can be forgotten altogether (Proposition 33), so that the simply typed system *is* the one expected. Analogously one may apply the conservation theorem also on second order DiLL, which is not presented here.

Simple types of Linear Logic are given by the following grammar, where X ranges over a fixed set of propositional variables:

$$A, B ::= X \mid X^{\perp} \mid 1 \mid \bot \mid A \otimes B \mid A \parr B \mid !A \mid ?A.$$

Linear negation is an involution ($A^{\perp\perp} = A$) defined via the De Morgan laws: $(X)^{\perp} := X^{\perp}$, $(A \otimes B)^{\perp} := A^{\perp} \parr B^{\perp}$, $(!A)^{\perp} := ?A^{\perp}$. There is a function U (a sort of forgetful map) from simple types to lax ones, given by $U(X) = U(X^{\perp}) = \Omega$, $U(c) = c$, $U(A \square B) = \square$ and $U(\diamond A) = \diamond$ for c, $\square$ and $\diamond$ one of the two units 1 and $\bot$, the two connectives $\otimes$ and $\parr$ and the two exponential modalities ! and ? respectively.

A differential net π is **simply typable** whenever there is a function ℓ from the oriented wires of π to the set of simple types respecting the conditions of Figure 20, such that opposite directions of a wire are associated with dual types and such that the underlying lax typing ℓ' enjoys $\ell'(d) = U(\ell(d))$. So the lax typing can be recovered from the simple one, and indeed the rules for simple typing of Figure 20 assure that also the ones for lax typing (Figures 2 and 10) are satisfied by $U \circ \ell$. Notice that in particular this means that a simply typed net has no $\mathfrak{V}$-typed wires. Notice also that both the translations $(\,)^{\triangle}$ and $(\,)^{\triangledown}$ from DiLL to $\mathsf{DiLL}_{\partial!}$ and back preserve simple typability.

Proposition 33. In DiLL and $\mathsf{DiLL}_{\partial!}$ if $\pi \to \rho$ (resp. $\pi \mapsto \rho$) and π is simply typable then so is ρ.

Proof. Easy case inspection. In fact given any rule $\alpha \to \nu$ or $\alpha \mapsto \nu$ and a simple typing ℓ of α, it is straightforward to define a typing ℓ' of ν preserving the ℓ types on terminal wires. Plugging in the context then gives the result. □

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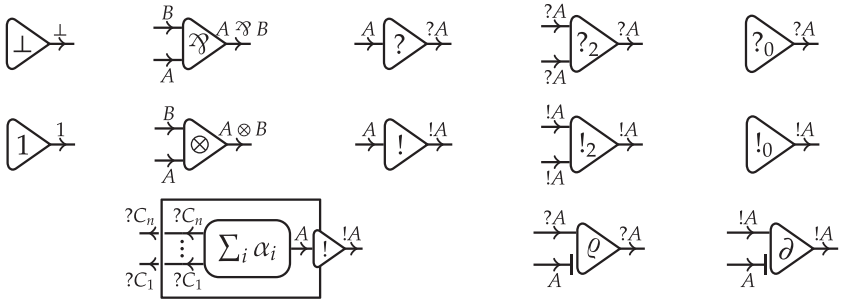


Fig. 20. Simple typing for DiLL and $\mathsf{DiLL}_{\partial\varrho}$.

The above property means also that $\mathfrak{V}$ never arises in a reduction starting from a simply typed net so that no cut gets blocked by lax typing (in particular banning clashes). As the only *raison d'être* of lax typing is therefore missing, we can altogether forget it.

Proposition 34. In DiLL simply typed and switching acyclic differential nets are $\mathsf{WN}_{\neg\epsilon}$.

Proof (Sketch). We adapt to non-erasing reduction and somewhat simplify (at the cost of using a stronger theorem along the way) the proof of WN given in Pagani (2009) for DiLL.

Let the grade of a wire in π be the size of its type, i.e. the number of connectives in the formula, which is invariant under linear negation. We can then use the finite developments theorem from Tranquilli (2009a) to obtain that, given a fixed grade, reducing only non-erasing cuts with that grade is necessarily strongly normalising. Indeed there are two types of 'new'cuts blocked by the definition of developments: on one side there are clashes, that are of no concern here, on the other are cuts whose type is necessarily 'simpler' (i.e. of strictly less grade) than the one fired. So the reduction reducing cuts of a fixed grade is a development (i.e. it is a reduction of the system DiLL° following the terminology of Tranquilli (2009a)) and is therefore terminating.

A round-by-round normalising strategy is now easy to define: one reduces non-erasing cuts in decreasing order of grade, in any order for cuts of equal grade. The maximum grade of the net then decreases strictly at each round involving the cuts with that grade. □

Theorem 35. In DiLL simply typed and switching acyclic differential nets are $\mathsf{SN}_{\rightsquigarrow}$.

Proof. Consequence of Proposition 34 and Theorem 32. □

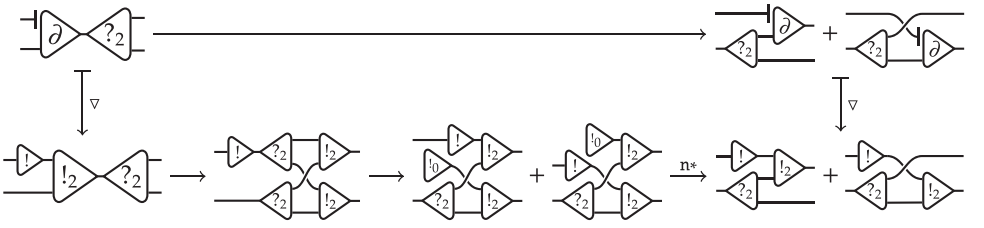
We can achieve $\mathsf{SN}_{\rightsquigarrow}$ of simply typed $\mathsf{DiLL}_{\partial\varrho}$ following two different ways: either we prove $\mathsf{WN}_{\neg\epsilon}$ using again a variant of Pagani (2009) and then we apply $\mathsf{DiLL}_{\partial\varrho}$ conservation, or we use the translation $(\ )^{\nabla}$ for simulating $\mathsf{DiLL}_{\partial\varrho}$ by means of DiLL (which is needed anyway to obtain the finite developments theorem for $\mathsf{DiLL}_{\partial\varrho}$). Let us follow this last

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direction, the former presenting no new interesting point. The following lemma is the $\mathsf{DiLL}_{\partial\varrho}$ counterpart of Lemma 20.

Lemma 36 (DiLL simulates $\mathsf{DiLL}_{\partial\varrho}$). If in $\mathsf{DiLL}_{\partial\varrho}$ $\pi \rightsquigarrow \rho$ then $\pi^{\triangledown} \overset{+}{\rightsquigarrow} \rho^{\triangledown}$ in DiLL. In particular, if $\pi^{\triangledown}$ is $\mathsf{SN}_{\sim}$, then so is π in $\mathsf{DiLL}_{\partial\varrho}$.

Proof. The only interesting cases are the ones involving one ϱ or ∂ cell. We leave all but one as an exercise for the reader. Let us detail the linear substitution on contraction case:



□

Theorem 37. In $\mathsf{DiLL}_{\partial\varrho}$, simply typed and switching acyclic differential nets are $\mathsf{SN}_{\sim}$.

Proof. Consequence of Lemma 36 and Theorem 35, recalling that $\pi^{\triangledown}$ is simply typed and switching acyclic whenever π so is. □

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Appendix A. Compendium of Rewriting Theory

The aim of this appendix is making the reader acquainted with the notations and notions of rewriting and rewriting modulo that we employ in this paper. We refer to Terese (2003) for a more in-depth presentation with proofs. Let us note however that Lemmas 39 and 41 are new results, as far as we know.

Let $(A, \to)$ be an abstract reduction system. The relations $\overset{=}{\to}$, $\overset{+}{\to}$, $\overset{*}{\to}$, $\leftarrow$ are respectively the reflexive, transitive, reflexive-transitive closures and the transpose of $\to$. The composition of two relations R and T is denoted by juxtaposition, so that $s\ RT\ t$ iff $\exists u$ with $s\ R\ u$ and $u\ T\ s$. A term t is **normal** if there is no u with $t \to u$; t is **weakly normalisable** ($\mathsf{WN}_{\to}$) whenever there is u normal such that $t \overset{*}{\to} u$; t is **strongly $\to$-normalisable** ($\mathsf{SN}_{\to}$) whenever there is no infinite sequence $(t_i)_{i\in\mathbb{N}}$ such that $t_0 = t$ and $t_i \to t_{i+1}$. We say that $\to$ is weak (resp. strongly) normalisable if so is every term of A. We may avoid specifying $\to$ in $\mathsf{WN}_{\to}$ and $\mathsf{SN}_{\to}$ whenever it is clear which relation we refer to.

Let $\mapsto$ be a symmetric relation on A generating an equivalence $\sim$, i.e. $\sim = \mapsto^*$ (possibly $\mapsto$ can be $\sim$ itself). We denote by $\rightsquigarrow$ the concatenation $\sim\!\to\!\sim$. We speak of **weak/strong normalisation modulo** $\sim$ ($\mathsf{WN}_{\to\sim}$ and $\mathsf{SN}_{\to\sim}$ for short) intending the corresponding notion of weak/strong normalisation of $\rightsquigarrow$. We say that (see Figure 21) $\to$ is **confluent**

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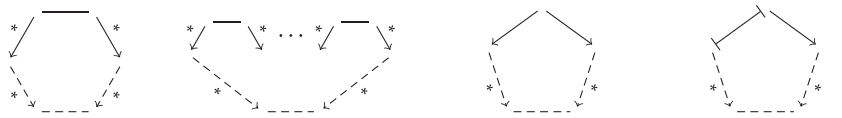


Fig. 21. Various confluence patterns. A straight line indicates the equivalence $\sim$. In local coherence $\mapsto$ generates $\sim$, i.e. $\sim = \mapsto^*$.

modulo $\sim$ if $\overset{*}{\leftarrow}\sim\overset{*}{\rightarrow} \subseteq \overset{*}{\rightarrow}\sim\overset{*}{\leftarrow}$; **Church–Rosser modulo** $\sim$ (or $\mathsf{CR}\sim$) if $(\rightarrow \cup \leftarrow \cup \sim)^* \subseteq \overset{*}{\rightarrow}\sim\overset{*}{\leftarrow}$; **locally confluent modulo** $\sim$ if $\leftarrow\rightarrow \subseteq \overset{*}{\rightarrow}\sim\overset{*}{\leftarrow}$; **locally coherent with** $\mapsto$ if $\mapsto\rightarrow \subseteq \overset{*}{\rightarrow}\sim\overset{*}{\leftarrow}$.

Clearly by setting $\sim$ to be the identity on terms one recovers the usual terminology of (local) confluence and Church–Rosser. Notice however that being Church–Rosser modulo $\sim$ is strictly stronger than plain confluence in absence of WN (Terese 2003, Remarks 14.3.6, Exercise 14.3.7).

We will use the following important generalisation of Newman's lemma.

Lemma 38 (Huet (1980)). Let $(A, \rightarrow, \mapsto)$ be an abstract reduction system, $\mapsto$ being a symmetric relation generating the equivalence $\sim$. If $\rightarrow$ is $\mathsf{SN}_\sim$, locally coherent with $\mapsto$ and locally confluent modulo $\sim$, then it is $\mathsf{CR}\sim$.

The following lemma extends to this setting item (iii) of Theorem 1.2.3 at page 18 in Terese (2003). Recalling the discussion in the introduction, this lemma upgrades the Nederpelt's/Gandy's method to reduction modulo.

Lemma 39 (normalisation modulo by increasing measure). Let $(A, \rightarrow, \mapsto)$ be an abstract reduction system, $\mapsto$ being a symmetric conversion step generating an equivalence $\sim$. Suppose

1. $\rightarrow$ is locally confluent modulo $\sim$,
2. $\rightarrow$ is locally coherent with $\mapsto$,
3. there is a measure $|\cdot| : A \rightarrow \mathbb{N}$ that is strictly increasing under $\rightarrow$ and invariant under $\sim$.

Then $t \in \mathsf{WN}$ implies $t \in \mathsf{SN}_\sim$.

Proof. First notice that 2 and 3 entail

$$\text{every term equivalent to a normal form is a normal form.} \tag{3}$$

In fact suppose $t_0 \mapsto t$ for a normal form t_0. If t is not a normal form, there is a term s s.t. $t \rightarrow s$. By local coherence there is $s' \sim t_0$ s.t. $s \overset{*}{\rightarrow} s'$. By 3 we have $|t_0| = |t| < |s| \leqslant |s'| = |t_0|$, which is absurd. Therefore both $\mapsto$ and $\sim = \mapsto^*$ preserve being a normal form. In particular, all normal forms are also $\mathsf{SN}_\sim$, which need not be in general.

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We will show that if $t \xrightarrow{*} t_0$ with t_0 normal then $t \in \mathsf{SN}_\sim$, reasoning by induction on $|t_0| - |t|$, which is positive by 3. If it is 0, then necessarily $t = t_0$ and by (3) $t \in \mathsf{SN}_\sim$. Suppose therefore that it is strictly positive, which implies $t \to t_1 \xrightarrow{*} t_0$ with $t_1 \in \mathsf{SN}_\sim$ by inductive hypothesis.

First, we see that if $t \mapsto s$ then there is a normalising chain $s \xrightarrow{*} s_0$ with $t_0 \sim s_0$. The situation is indeed the following.

$$\begin{array}{ccccc} t & \longrightarrow & t_1 & \xrightarrow{*} & t_0 \\ \rotatebox{90}{$\vdash$} & \text{(i)} & \downarrow^{*} & \text{(ii)} & \wr \\ s & \xrightarrow{*} & s_1 & \xrightarrow{*} & s_0 \end{array}$$

(i) is local coherence modulo, from item 2, while (ii) is entailed by $\mathsf{CR}\sim$ on the $\rightrightarrows$-descendants of t_1, which is in turn given by items 1 and 2 and applying Lemma 38 as $t_1 \in \mathsf{SN}_\sim$. Notice that as $|s_0| - |s| = |t_0| - |t|$ (item 3), we can iterate the above and get the same if starting from $t \sim s$.

Now take any s such that $t \rightrightarrows s$, i.e. $t \sim t' \to s' \sim s$. By the above we get $t' \to t'_1 \xrightarrow{*} t'_0 \sim t_0$ with $|t'_0| - |t'| = |t_0| - |t|$ (with t'_1 existing as the latter is strictly positive). Now we can repeat the above confluence diagram with t' and s' in place of t and s, and using local confluence modulo (item 1) rather than local coherence, to get $s' \xrightarrow{*} s'_0 \sim t_0$. As $|s'_0| - |s'| < |t'_0| - |t'|$, inductive hypothesis gives $s' \in \mathsf{SN}_\sim$ which in turn entails that $s \in \mathsf{SN}_\sim$ also. As all $\rightrightarrows$ steps from t brings inside $\mathsf{SN}_\sim$, then necessarily $t \in \mathsf{SN}_\sim$. □

Let $(A, \xrightarrow{1}, \xrightarrow{2})$ be an abstract reduction system. The reduction $\xrightarrow{12}$ is the union of reductions $\xrightarrow{1}$ and $\xrightarrow{2}$. The following is in Hindley's PhD Thesis, it can be also found in (Terese 2003, Exercise 1.3.6).

Lemma 40 (Hindley (1964)). Let $(A, \xrightarrow{1}, \xrightarrow{2})$ be an abstract reduction system such that $\forall u, t, v \in A, u \xleftarrow{1} t \xrightarrow{2} v$ implies $\exists z \in A, u \xrightarrow{2*} z \xleftarrow{1=} v$. Then $\xrightarrow{1}$ and $\xrightarrow{2}$ commute, i.e. $\xleftarrow{1*}\xrightarrow{2*} \subseteq \xrightarrow{2*}\xleftarrow{1*}$.

Lemma 41. Let $(A, \xrightarrow{1}, \xrightarrow{2})$ be an abstract reduction system such that there is a measure $|\cdot| : A \to \mathbb{N}$ with $|x| > |y|$ (resp. $|x| \geqslant |y|$) whenever $x \xrightarrow{1} y$ (resp. $x \xrightarrow{2} y$), and such that $\xrightarrow{1}\xrightarrow{2} \subseteq \xrightarrow{12*}\xrightarrow{1}$. Then $\xrightarrow{1}$ can be postponed with respect to $\xrightarrow{2}$, i.e. $\xrightarrow{12*} \subseteq \xrightarrow{2*}\xrightarrow{1*}$.

Proof. Take $x \xrightarrow{12*} y$, and let us reason by induction on $|x| - |y|$, which by hypothesis is a positive integer. Suppose in the reduction chain there is at least a 1-reduction, or else the result is trivial. We then have $x \xrightarrow{12*}\xrightarrow{1}\xrightarrow{2*} y$: we can turn it into $x \xrightarrow{12*} y' \xrightarrow{1} y$ iterating the hypothesis $\xrightarrow{1}\xrightarrow{2} \subseteq \xrightarrow{12*}\xrightarrow{1}$ (formally reasoning by a secondary induction on the length of the final 2-reduction). Now $|x| - |y'| < |x| - |y|$, so by inductive hypothesis we get $x \xrightarrow{2*}\xrightarrow{1*} y' \xrightarrow{1} y$ which concludes the proof. □

Notice that the above lemma may be employed dually, i.e. if $|.|$ increases strictly (resp. does not decrease) for $\xrightarrow{1}$ (resp. for $\xrightarrow{2}$) and $\xrightarrow{2}\xrightarrow{1} \subseteq \xrightarrow{1}\xrightarrow{12*}$ then $\xrightarrow{2}$ can be delayed with respect to $\xrightarrow{1}$: it suffices to use the lemma on the expansions $\xleftarrow{1}$ and $\xleftarrow{2}$, transposing all the equations on the relations.

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Barendregt, H. (1981). *The Lambda Calculus, Its Syntax and Semantics*. Number 103 in Studies in Logic and the Foundations of Mathematics, 1st edition, North-Holland.

Bonelli, E.A. (2001). *Substitutions explicites et réécriture de termes*. Thèse de doctorat, Université Paris XI.

Danos, V. (1990). *La Logique Linéaire appliquée à l'étude de divers processus de normalisation (principalement du λ-calcul)*. Thèse de doctorat, Université Paris VII.

Danos, V. and Regnier, L. (1989). The structure of multiplicatives. *Archive for Mathematical Logic* **28**(3) 181–203.

de Falco, M. (2009). An explicit framework for interaction nets. In: Treinen, R. (ed.) RTA. *Springer Lecture Notes in Computer Science* **5595** 209–223.

Di Cosmo, R. and Guerrini, S. (1999). Strong normalization of proof nets modulo structural congruences. In: Narendran, P. and Rusinowitch, M. (eds.) Proceedings Rewriting Techniques and Applications, 10th International Conference, RTA-99, Trento, Italy, July 2–4. *Springer Lecture Notes in Computer Science* **1631** 75–89.

Di Cosmo, R. and Kesner, D. (1997). Strong normalization of explicit substitutions via cut elimination in proof nets (extended abstract). In: *Proceedings, 12th Annual IEEE Symposium*

https://doi.org/10.1017/S0960129515000456 Published online by Cambridge University Press

on Logic in Computer Science, Warsaw, Poland, June 29–July 2, 1997, IEEE Computer Society 35–46.

Di Cosmo, R., Kesner, D. and Polonovski, E. (2003). Proof nets and explicit substitutions. *Mathematical Structures in Computer Science* **13** (3) 409–450.

Ehrhard, T. (2005). Finiteness spaces. *Mathematical Structures in Computer Science* **15** (4) 615–646.

Ehrhard, T. and Laurent, O. (2007). Interpreting a finitary pi-calculus in differential interaction nets. In: CONCUR 2007 - Concurrency Theory, Proceedings of the 18th International Conference, CONCUR 2007, Lisbon, Portugal, September 3–8, 2007, *Springer Lecture Notes in Computer Science* **4703** 333–348.

Ehrhard, T. and Regnier, L. (2003). The differential lambda-calculus. *Theoretical Computer Science* **309** (1) 1–41.

Ehrhard, T. and Regnier, L. (2006). Differential interaction nets. *Theoretical Computer Science* **364** (2) 166–195.

Fiore, M.P. (2007). Differential structure in models of multiplicative biadditive intuitionistic linear logic. In: Ronchi Della Rocca, S. (eds.) TLCA. *Springer Lecture Notes in Computer Science* **4583** 163–177.

Gandy, R.O. (1980). Proofs of strong normalization. In: Seldin, J.P. and Hindley, J.R. (eds.) *To H.B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism*, Academic Press Limited 457–477.

Gimenez, S. (2011). Realizability proof for normalization of full differential linear logic. In: Luke Ong, C.-H. (eds.) Proceedings of the Typed Lambda Calculi and Applications - 10th International Conference, TLCA 2011, Novi Sad, Serbia, June 1–3. *Springer Lecture Notes in Computer Science* **6690** 107–122.

Girard, J.-Y. (1972). *Interprétation Fonctionnelle et Élimination des Coupures de l'Arithmétique d'Ordre Supérieur*. Thèse de doctorat d'etat, Université Paris VII.

Girard, J.-Y. (1987a). Linear Logic. *Theoretical Computer Science* **50** 1–102.

Girard, J.-Y. (1987b). *Proof Theory and Logical Complexity*. Studies in Proof-theory. Bibliopolis, Napoli.

Girard, J.-Y., Lafont, Y. and Taylor, P. (1989). *Proofs and Types*, Cambridge Tracts in Theoretical Computer Science volume 7, Cambridge University Press.

Hindley, J.R. (1964). *The Church-Rosser Property and a Result in Combinatory Logic*, Ph.D. thesis, University of Newcastle-upon-Tyne.

Huet, G.P. (1980). Confluent reductions: Abstract properties and applications to term rewriting systems. *Journal of the ACM* **27** (4) 797–821.

Lafont, Y. (1990). Interaction nets. In: *POPL '90: Proceedings of the 17th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*, New York, NY, USA. ACM 95–108.

Nederpelt, R.P. (1973). *Strong Normalization for a Typed Lambda Calculus with Lambda Structured Types*, Ph.D. thesis, Technische Hogeschool Eindhoven.

Pagani, M. (2009). The cut-elimination theorem for differential nets with promotion. In: Curien, P.-L. (eds.) TLCA. *Springer Lecture Notes in Computer Science* **5608** 219–233.

Pagani, M. and Tortora de Falco, L. (2010). Strong normalization property for second order linear logic. *Theoretical Computer Science* **411** (2) 410–444.

Sørensen, M.H. (1997). Strong normalization from weak normalization in typed λ-calculi. *Information and Computation* **133** (1) 35–71.

Tait, W.W. (1967). Intensional interpretation of functionals of finite type. *Journal of Symbolic Logic* **32** (2) 187–199.

https://doi.org/10.1017/S0960129515000456 Published online by Cambridge University Press

Terese. (2003). *Term Rewriting Systems*, Cambridge Tracts in Theoretical Computer Science volume 55, Cambridge University Press.

Tortora de Falco, L. (2003). Additives of linear logic and normalization- part I: a (restricted) Church-Rosser property. *Theoretical Computer Science* **294** (3) 489–524.

Tranquilli, P. (2009a). Confluence of pure differential nets with promotion. In: Grädel, E. and Kahle, R. (eds.) CSL. *Springer Lecture Notes in Computer Science* **5771** 500–514.

Tranquilli, P. (2009b). *Nets Between Determinism and Nondeterminism* Ph.D. thesis, Università Roma Tre/Université Paris Diderot (Paris 7).

Tranquilli, P. (2011). Intuitionistic differential nets and lambda-calculus. *Theoretical Computer Science* **412** (20) 1979–1997.

Vaux, L. (2007). *λ-calcul différentiel et logique classique : interactions calculatoires*. Thèse de doctorat, Université de la Méditerranée.

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