ABSOLUTE AND UNCONDITIONAL CONVERGENCE IN NORMED LINEAR SPACES

By D. RUTOVITZ

Communicated by F. Smithies

Received 22 August 1961

In 1933 Orlicz proved various results concerning unconditional convergence in Banach spaces (4), which were noted by Banach ((1), p. 240) who remarked that absolute and unconditional convergence† are equivalent in finite dimensional Banach spaces, but that whether or not the two are non-equivalent in all infinite dimensional spaces was still an open question. MacPhail (3) gave a criterion for the equivalence of the two notions of convergence in a general Banach space and used it to prove non-equivalence in the spaces l_1 and L_1.‡ In 1950 Dvoretzky and Rogers demonstrated the non-equivalence of the two types of convergence in any infinite dimensional normed linear space, using an elegant and instructive geometrical approach (2). The result has also been proved by a different method by Grothendieck (5).

We propose here to give an independent proof, making use of a construction similar to that of MacPhail for the case of an l_1-space. In order to do this we have first to show that in a space of large dimension, a subspace can be found which has some of the properties of an l_1-space (Lemma 1, below). This corresponds to the Dvoretzky–Rogers construction of a subspace having 'quasi-orthogonality' properties but is much weaker; the remaining part of the proof of non-equivalence of the two modes of convergence is correspondingly more difficult.

Definitions and notation. If B is a normed linear space (over the real numbers,) $\|x\|$ will denote the norm of x. If S is a sequence $a_1, a_2, \ldots, a_n$ of elements of B, we write $L\{S\}$ for the subspace spanned by the a_i's,

$$|S| = \Sigma \|a_i\|,$$

and

$$|S|_* = \sup \|\sum_T v\| \quad \text{for} \quad T \subseteq S.$$

The MacPhail parameter of B, $\lambda(B)$, is defined by the equation

$$\lambda(B) = \inf |S|_*/|S|,$$

for all finite sequences S in B which contain a non-zero element; it is not difficult to show that $\lambda(B) > 0$ if and only if unconditional and absolute convergence are equivalent in B (see (3)). Let $\lambda_n = \sup\{\lambda(B); B \text{ of dimension } n\}$. We will prove that λ_n tends

† A series Σx_n in a normed linear space is said to be unconditionally convergent if Σx_{n_i} converges for every sequence of subscripts n_i, absolutely convergent if $\Sigma \|x_n\|$ is convergent.

‡ l_1 is the space of real or complex sequences (α_i) with norm $\Sigma|\alpha_i|$, L_1 is the space of Lebesgue measurable functions with norm $\int_0^\infty |f(x)|\,dx$.

https://doi.org/10.1017/S0305004100040597 Published online by Cambridge University Press

to zero as n increases. Since $\lambda(B_1) \leqslant \lambda(B_2)$ if $B_2 \subseteq B_1$ it will follow immediately that $\lambda(B) = 0$ whenever B is infinite dimensional.

Let S be a subset of a normed linear space; we say that a is norm-orthogonal to S if

$$\|a+s\| \geqslant \|a\|$$

for every s in $L\{S\}$, and write $a \perp S$ when this is the case.

We have found it convenient to use the same symbol for the positive integer m and the set of positive integers $k < m$, writing $k \in m$, $K \subseteq m$, etc.

LEMMA 1. *Let B be a Banach space of dimension 2^n, and let C be the unit sphere of B. Then there exists a sequence $S = \{e_i\}_{i \in n+1}$ of elements of C such that for $J \subseteq m \in n$*

$$\sum_{j \in J} e_j \perp L(e_i : m+1 \leqslant i \leqslant n+1).$$

(A set such as S will be termed semi-orthogonal; it is evident that if S is semi-orthogonal, so is any subset of S in which the initial ordering of the elements is retained; also, S is a linearly independent set.)

For e_1 choose any unit-normed vector in B. Let H_1 be a support hyperplane of C at e_1, and let $T_1 = H_1$.

If (e_i) has been defined for $i \in k$, and for each $J \subseteq k$ a support plane H_J of C at the point where $\sum_{i \in J} e_i$ cuts the unit sphere has been chosen, let T_k be the intersection of hyperplanes parallel to the H_J's for $J \subseteq k$. Since there are 2^k distinct subsets J of k, the dimension of T_k is at least $2^n - 2^k + 1$; provided $k \leqslant n$ we choose e_{k+1} to be any element of $C \cap T_k$; if $k < n$ we now add to the H_J's already chosen further support planes of C at each of the points

$$\sum_{i \in J} e_i, \quad J \subseteq k+1, \quad J \nsubseteq k.$$

Evidently,

$$L\{e_j : k+1 \leqslant j \leqslant n+1\} \subseteq T_k,$$

and since $\|v+s\| \geqslant \|v\|$ if $v \in C$ and s is a member of a support of C at v, we have, for $J \subseteq k$,

$$\sum_{i \in J} e_i \perp L\{e_i : k+1 \leqslant i \leqslant n+1\}.$$

Definition of certain functions of Rademacher type. Let

$$\rho_{1,p}(t) = \begin{cases} 1 & (2p < t \leqslant 2p+1), \\ -1 & (2p+1 < t \leqslant 2p+2), \\ 0 & \text{otherwise} \end{cases}$$

and let

$$\rho_{k,p}(t) = \rho_{1,p}(t/2^{k-1}).$$

We consider $\rho_{k,p}$ as being restricted to an interval $0 \leqslant t \leqslant 2^m$, and suppose that p and k are integers such that $0 \leqslant p < 2^{m-k}$, $k \in m$. These functions are, as it were, 'pieces' of Rademacher functions.

Notice that $\rho_{k,p}(t)$ is orthogonal to $\rho_{k',p'}(t)$ as an element of $L^2(0, 2^m)$† unless $k = k'$ and $p = p'$. Further, if P_k is a subset of $0, 1, \ldots, 2^{m-k} - 1$ and we define

$$\rho_k(t; P_k) = \rho(t; P_k) = \sum_{p \in P_k} \rho_{k,p}(t)$$

† I.e. the space of Lebesgue measurable square summable functions on the given interval.

https://doi.org/10.1017/S0305004100040597 Published online by Cambridge University Press

(setting $\rho(t; P_k) = 0$ if P_k is empty) we have

$$|\rho(t; P_k)| \leqslant 1$$

and $\rho(t; P_k)$ is orthogonal to $\rho(t; P_{k'})$ if $k \neq k'$.

Let $S = (a_i)_{i \in 2^m} \subset B$; by the *associated Rademacher sequence*, $R(S)$, we mean the sequence whose elements are

$$v_{k,p} = \sum_{j \in 2^m} \rho_{k,p}(j)\, a_j,$$

where $0 \leqslant p < 2^{m-k}$ and $k \in m$. For example, if $m = 2$,

$$\begin{aligned} v_{1,0} &= a_1 - a_2, \\ v_{1,1} &= a_3 - a_4, \\ v_{2,0} &= a_1 + a_2 - (a_3 + a_4). \end{aligned}$$

In general $v_{k,p}$ is a linear combination of 2^k consecutive terms of S, the first half of the terms bearing the coefficient $+1$, the second half -1; as p changes (for fixed k) we pick up the 2^{m-k} consecutive blocks of terms of length 2^k which make up the sequence S.

Let R be a subset of $R(S)$; then

$$R = \{v_{k,p};\ p \in P_k,\ k \in m\},$$

where for each k, P_k is a (possibly empty) subset of the integers $0 \leqslant p < 2^{m-k}$.

Observe that $$\rho_{k,p}(j) = \int_{j-1}^{j} \rho_{k,p}(x)\, dx.$$

LEMMA 2. *If $S = (a_i)_{i \in 2^m}$ is a sequence of 2^m linearly independent elements of C (the unit sphere of B) then*

$$|R(S)|_* \leqslant m^{\frac{1}{2}} 2^m.$$

Let R be a non-empty subset of $R(S)$. Then

$$\begin{aligned} \Big\|\sum_R v\Big\| &= \Big\| \sum_{k \in m} \sum_{p \in P_k} \sum_{j \in 2^m} \rho_{k,p}(j)\, a_j \Big\| \\ &\leqslant \sum_{j \in 2^m} \Big| \sum_{k \in m} \rho(j; P_k) \Big| \\ &= \sum_{j \in 2^m} \left| \int_{j-1}^{j} \sum_{k \in m} \rho(t; P_k)\, dt \right| \\ &\leqslant \int_0^{2^m} \Big| \sum_{k \in m} \rho(t; P_k) \Big|\, dt \\ &\leqslant \left\{ \int_0^{2^m} dt \right\}^{\frac{1}{2}} \left\{ \int_0^{2^m} \Big| \sum_{k \in m} \rho(t; P_k) \Big|^2 dt \right\}^{\frac{1}{2}} \\ &= 2^{\frac{1}{2}m} \left\{ \int_0^{\frac{1}{2}m} \sum_{k \in m} |\rho(t; P_k)|^2\, dt \right\}^{\frac{1}{2}} \end{aligned}$$

(using the orthogonality of the $\rho(t; P_k)$'s)

$$\leqslant m^{\frac{1}{2}} 2^m.$$

https://doi.org/10.1017/S0305004100040597 Published online by Cambridge University Press

Essentially the same lemma is given by MacPhail, but we require this slightly stronger form for the 'pieces' of Rademacher functions, $\rho(t; P_k)$.

LEMMA 3. *If* $S = (e_i)$, $i \in 2^m$, *is semi-orthogonal,* $\|e_i\| = 1$, $v = \Sigma e_i$, *and* $\|v\| \geqslant \theta 2^m$, $0 < \theta < 1$, then

$$|R(S)| \geqslant \tfrac{1}{8}\theta^2 m 2^m.$$

The proof will be by induction on m, though we shall have to consider the first two cases separately.

For $m = 1$, $v = e_1 + e_2$, $R(S) = (e_1 - e_2)$,

$$|R(S)| = \|e_1 - e_2\| \geqslant 1 > \tfrac{1}{8}\theta^2 1.2.$$

For $m = 2$, $v = e_1 + e_2 + e_3 + e_4$,

$$\begin{aligned} |R(S)| &= \|e_1 + e_2 - e_3 - e_4\| + \|e_1 - e_2\| + \|e_3 - e_4\| \\ &\geqslant 3 > \tfrac{1}{8}\theta^2 2.4. \end{aligned}$$

Suppose $m \geqslant 3$, and assume the result established for $m-1$; let

$$v_1 = \sum_{j \in 2^{m-1}} e_j, \quad v_2 = v - v_1,$$

$$\|v_1\| = 2^m\phi, \quad \|v_2\| = 2^m\psi;$$

since $\|v\| \leqslant \|v_1\| + \|v_2\|$, we have $\phi + \psi \geqslant \theta$

and *a fortiori* $\phi^2 + \psi^2 \geqslant \tfrac{1}{2}\theta^2$.

Then $|R(S)| = \|v_1 - v_2\| + |R(S_1)| + |R(S_2)|$,

where $S_1 = (e_i), \quad i \in 2^{m-1}$

and $S_2 = (e_i), \quad 2^{m-1} < i \leqslant 2^m$;

since S_1 and S_2 are semi-orthogonal along with S, we have by the inductive hypothesis

$$|R(S_1)| \geqslant \tfrac{1}{8}(2\phi)^2 (m-1)\, 2^{m-1},$$

$$|R(S_2)| \geqslant \tfrac{1}{8}(2\psi)^2 (m-1)\, 2^{m-1};$$

thus $|R(S)| \geqslant (\phi + \tfrac{1}{4}(\phi^2 + \psi^2)(m-1))\, 2^m$.

Now if $\phi \geqslant \tfrac{1}{8}\theta$ we have, since $\theta^2 + \psi^2 \geqslant \tfrac{1}{2}\theta^2$,

$$\begin{aligned} |R(S)| &\geqslant (\tfrac{1}{8}\theta + \tfrac{1}{8}\theta^2(m-1))\, 2^m \\ &\geqslant \tfrac{1}{8}\theta^2 m 2^m; \end{aligned}$$

and if $\phi \leqslant \tfrac{1}{8}\theta$ then $\psi \geqslant \tfrac{7}{8}\theta$ so that again

$$\begin{aligned} |R(S)| &\geqslant \frac{49(m-1)}{4.64.m}\theta^2 m 2^m > \frac{48.2}{4.64.3}\theta^2 m 2^m \\ &= \tfrac{1}{8}\theta^2 m 2^m. \end{aligned}$$

THEOREM. $\lambda_n \to 0$ *as* $n \to \infty$.

If this is not the case, there exists $\epsilon > 0$ such that $\lambda_n > \epsilon$ for arbitrarily large n; in particular for any m there exists a Banach space B with $\lambda(B) > \epsilon$, which contains a semi-orthonormal† sequence $S = (e_i)$, $i \in p$ such that $p > 2^{2m}/\epsilon$.

† I.e. S is semi-orthogonal, and its elements have norm 1.

https://doi.org/10.1017/S0305004100040597 Published online by Cambridge University Press

Since $\lambda(B) > \epsilon$ there exists a subset S_1 of S for which

$$\|\Sigma\{e_i;\ e_i \in S_1\}\| \geqslant \epsilon\Sigma\|e_i\| > 2^{2m}.$$

It follows in particular that S_1 has at least 2^{2m} elements. Now any integer $k > 2^{2m}$ can be expressed as a sum of integers of the form 2^m or 2^{m+1}; for if

$$k = 2^m j + r, \quad 0 \leqslant r < 2^m, \quad j \geqslant 2^m > r,$$

then

$$k = (j-r)\,2^m + r(2^m+1).$$

In consequence, we may split S_1 into disjoint subsets containing either 2^m or 2^{m+1} elements; at least one of these, S_2, say, must satisfy

$$\|\sum_{S_2} v\| \geqslant \epsilon 2^m$$

(because $\|\sum_{S_1} v\| \geqslant \epsilon \sum_{S_1} \|v\|$). If S_2 contains 2^m elements, let $T = S_2$; if S_2 contains $2^m + 1$ elements let T consist of any 2^m of these. In either case, taking $\epsilon < 2$,

$$\|\sum_{T} v\| \geqslant \epsilon 2^m - 1 \geqslant \tfrac{1}{2}\epsilon 2^m = \theta 2^m, \quad \text{say},$$

for all sufficiently large m.

Arrange the elements of T in the same relative order as in S; they will be semi-orthonormal, and we have

$$\|\sum_{T} v\| \geqslant \theta 2^m;$$

hence

$$|R(T)| \geqslant \tfrac{1}{8}\theta^2 m 2^m \quad \text{(Lemma 2)}$$

and

$$|R(T)|_* \leqslant m^{\frac{1}{2}} 2^m \quad \text{(Lemma 3)}.$$

Combining these two results we have

$$\lambda(B) \leqslant \frac{|R(T)|_*}{|R(T)|} \leqslant \frac{32}{m^{\frac{1}{2}}\epsilon^2} < \epsilon$$

for all sufficiently large m. This contradiction proves the theorem.

REFERENCES

(1) Banach, S. *Théorie des opérations linéaires* (Warsaw, 1932).

(2) Dvoretzky, A. and Rogers, C. A. Absolute and unconditional convergence in normed linear spaces. *Proc. Nat. Acad. Sci. U.S.A.* 36 (1950), 192–197.

(3) MacPhail, M. S. Absolute and unconditional convergence. *Bull. American Math. Soc.* 53 (1947), 121–123.

(4) Orliz, W. Über unbedingte Konvergenz in Funktionenräumen (II). *Studia Math.* 4 (1933), 41–47.

(5) Grothendieck, A. Produits tensoriels topologiques et espaces nucléaires. *Mem. American Math. Soc.* 16 (1955).

The University
Manchester

https://doi.org/10.1017/S0305004100040597 Published online by Cambridge University Press