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Möbius disjointness along ergodic sequences for uniquely ergodic actions

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Abstract. We show that there is an irrational rotation $Tx = x + \alpha$ on the circle $\mathbb{T}$ and a continuous $\varphi : \mathbb{T} \to \mathbb{R}$ such that for each (continuous) uniquely ergodic flow $\mathcal{S} = (S_t)_{t\in\mathbb{R}}$ acting on a compact metric space Y, the automorphism $T_{\varphi,\mathcal{S}}$ acting on $(X \times Y, \mu \otimes \nu)$ by the formula $T_{\varphi,\mathcal{S}}(x, y) = (Tx, S_{\varphi(x)}(y))$, where μ stands for the Lebesgue measure on $\mathbb{T}$ and ν denotes the unique $\mathcal{S}$-invariant measure, has the property of asymptotically orthogonal powers. This gives a class of relatively weakly mixing extensions of irrational rotations for which Sarnak's conjecture on the Möbius disjointness holds for all uniquely ergodic models of $T_{\varphi,\mathcal{S}}$. Moreover, we obtain a class of 'random' ergodic sequences $(c_n) \subset \mathbb{Z}$ such that if $\boldsymbol{\mu}$ denotes the Möbius function, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\le N} g(S_{c_n} y)\boldsymbol{\mu}(n) = 0$$

for all (continuous) uniquely ergodic flows $\mathcal{S}$, all $g \in C(Y)$ and $y \in Y$.

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1. *Introduction*

In 2010, Sarnak [**40**] stated the following conjecture: *given a zero (topological) entropy homeomorphism S of a compact metric space Y, we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n\le N} g(S^n y)\boldsymbol{\mu}(n)=0 \tag{1}$$

for each $g\in C(Y)$ and $y\in Y$. Here $\boldsymbol{\mu}:\mathbb{N}\to\mathbb{C}$ denotes the Möbius function: $\boldsymbol{\mu}(1)=1$, $\boldsymbol{\mu}(p_1\cdots p_k)=(-1)^k$ for any distinct prime numbers $p_1,\dots,p_k$ and $\boldsymbol{\mu}(n)=0$ otherwise. Given a dynamical system (Y, S), we will say that (Y, S), or simply S, is *Möbius disjoint* when (1) holds for all $g\in C(Y)$ and $y\in Y$. Möbius disjointness has been proved in numerous cases of zero entropy dynamical systems, see e.g. [**3**, **11**, **16**–**18**, **24**, **25**, **39**].

One of the motivations of the present paper is to consider a 'randomized' version of Möbius disjointness in which we are given another compact metric space X, a homeomorphism $T:X\to X$ and a family $(S_x)_{x\in X}$ of homeomorphisms of Y (the map $(x, y)\mapsto S_x y$ is assumed to be continuous) and we aim at proving that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n\le N} g(S_x^{(n)} y)\boldsymbol{\mu}(n)=0 \tag{2}$$

for each $x\in X$, $g\in C(Y)$ and $y\in Y$. Here $S_x^{(n)}=S_{T^{n-1}x}\circ\cdots\circ S_{Tx}\circ S_x$ for $n\ge 0$ ($S_x^{(0)}=\mathrm{Id}$), so when X is the one-point space, clearly, (2) is equivalent to (1). An instance of such a 'randomized' version arises when we choose a continuous function $\varphi:X\to\mathbb{R}$ and consider $\mathcal{S}=(S_t)_{t\in\mathbb{R}}$ a continuous flow on Y. Now, (2) takes the form

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n\le N} g(S_{\varphi^{(n)}(x)} y)\boldsymbol{\mu}(n)=0 \tag{3}$$

for each $x\in X$, $g\in C(Y)$ and $y\in Y$ ($\varphi^{(n)}(x)=\sum_{j=0}^{n-1}\varphi(T^j x)$). We can now ask if we can find 'parameters' (T, φ) so that (3) holds for 'all' $\mathcal{S}$. Somewhat surprisingly, the answer to such a question will be positive, whenever by 'all' we mean the class of continuous uniquely ergodic flows, see Theorem A below.

To explain better our approach, suppose that S acting on Y is Möbius disjoint and, additionally, it is uniquely ergodic (as are most of the examples from the aforementioned papers). Assume that S' is another uniquely ergodic homeomorphism acting on Y'. If S and S' are topologically conjugate then S' is also Möbius disjoint. But both dynamical systems (Y, S) and (Y', S') have unique invariant measures, ν and ν', respectively, and it is natural to ask whether already a metric isomorphism of measure-theoretic dynamical systems $(Y, \mathcal{B}(Y), \nu, S)$ and $(Y', \mathcal{B}(Y'), \nu', S')$ implies that S' is Möbius disjoint ($\mathcal{B}(Y)$ and $\mathcal{B}(Y')$ stand for the σ-algebras of Borel sets on Y and Y', respectively). This ergodic approach to study convergence (1), in fact, allows us to ask under which measure-theoretic properties an ergodic automorphism W of a probability standard Borel space $(Z, \mathcal{D}, \rho)$ has the property that the Möbius disjointness holds for each homeomorphism S of a compact metric space Y, which is a uniquely ergodic model of $(Z, \mathcal{D}, \rho, W)$. The latter means that the measure-theoretic dynamical systems $(Z, \mathcal{D}, \rho, W)$ and $(Y, \mathcal{B}(Y), \nu, S)$, where ν is the unique S-invariant probability measure, are measure-theoretically isomorphic (we recall that by the Jewett–Krieger theorem, each ergodic W has a uniquely ergodic

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model). As noticed in [**11**], this question has the positive answer if the different prime powers of W are disjoint in Furstenberg's sense. A generalization of the disjointness of powers proposed in [**6**] (called asymptotically orthogonal power (AOP), see §3.1), allows one to cover the case when such powers are not disjoint and, in the extremal case, can even be isomorphic.

Each AOP automorphism has the property that all of its uniquely ergodic models are Möbius disjoint. For example, the Möbius disjointness is known to hold for each irrational rotation (e.g. [**14**]) but by [**6**] it follows that irrational rotations enjoy the AOP property. Hence, the Möbius disjointness holds in each ergodic model of an irrational rotation, e.g. in Sturmian models [**8**] or topologically mixing models [**29**]. Similarly, Möbius disjointness for zero entropy affine automorphisms has been proved in [**37**], while the main result of [**6**] asserts that all quasi-discrete spectrum automorphisms [**7**] enjoy the AOP property; for Möbius disjointness itself of zero entropy affine automorphisms (which are examples of automorphisms with a quasi-discrete spectrum), see [**37**]. It follows that Möbius disjointness holds in all uniquely ergodic models of zero entropy affine automorphisms. Recently, in [**19**], the AOP property has been proved for all unipotent diffeomorphisms of compact nil-manifolds. It can also be shown that some examples from [**27**, **42**] enjoy the AOP property. The classes of automorphisms we listed in this paragraph are examples of distal automorphisms [**21**], in fact, these are distal extensions of some totally ergodic rotations. From that point of view, in the present paper, we will deal with the notion complementary to distality. Namely, we will mainly consider (uniquely ergodic) relatively weakly mixing extensions [**22**] of irrational rotations, and the paper is aimed at a study of the AOP property for them. Once the AOP property is established, such relatively weakly mixing extensions of rotations will constitute a new class of dynamical systems for which the Möbius disjointness holds in all their uniquely ergodic models.

Let us describe more precisely the class of systems we intend to study. They are of the following form:

$$\begin{gathered} T_{\varphi,\mathcal{S}} : (X \times Y, \mathcal{B} \otimes \mathcal{C}, \mu \otimes \nu) \to (X \times Y, \mathcal{B} \otimes \mathcal{C}, \mu \otimes \nu), \\ T_{\varphi,\mathcal{S}}(x, y) = (Tx, S_{\varphi(x)}(y)), \end{gathered} \tag{4}$$

where $X = \mathbb{T} = [0, 1) \pmod 1$, $Tx = x + \alpha$ is an irrational rotation, $\varphi : X \to \mathbb{R}$ is measurable and $\mathcal{S} = (S_t)_{t\in\mathbb{R}}$ is an ergodic flow acting on a probability standard Borel space $(Y, \mathcal{C}, \nu)$. Under a weak assumption on φ, the skew products (4) are ergodic [**33**]. Furthermore, if additionally φ is continuous then $T_{\varphi,\mathcal{S}}$ is a homeomorphism which is uniquely ergodic for each uniquely ergodic flow $\mathcal{S}$. Under some further assumptions on φ and $\mathcal{S}$ such extensions are relatively weakly mixing extensions of T [**33**, **34**]. For example, these assumptions are satisfied when T_φ is ergodic and $\mathcal{S}$ is weakly mixing. Recall that T_φ is defined on $X \times \mathbb{R}$ by the formula $T_\varphi(x, t) = (Tx, \varphi(x) + t)$ and it preserves $\mu \otimes \lambda_{\mathbb{R}}$; we assume the ergodicity of T_φ with respect to this infinite measure. Furthermore, since the cocycles φ we use are recurrent, the entropy of $T_{\varphi,\mathcal{S}}$ will also be zero regardless of the entropy of $\mathcal{S}$ itself [**14**] (recall also that the AOP property implies zero entropy [**6**]). Finally, we note that given $g \in C(Y)$ if we set $F(x, y) = g(y)$ then

$$\frac{1}{N}\sum_{n\le N} F(T^n_{\varphi,\mathcal{S}}(x, y))\boldsymbol{\mu}(n) = \frac{1}{N}\sum_{n\le N} g(S^{(n)}_{\varphi(x)}y)\boldsymbol{\mu}(n) = \frac{1}{N}\sum_{n\le N} g(S_{\varphi^{(n)}(x)}y)\boldsymbol{\mu}(n),$$

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where $\varphi^{(n)}(x) = \varphi(x) + \varphi(Tx) + \cdots + \varphi(T^{n-1}x)$, which explains why the Möbius disjointness (1) for $T_{\varphi,\mathcal{S}}$ implies (3).

One of the main results of the paper is the following.

THEOREM A. *There is an irrational $\alpha \in \mathbb{T}$ and a measurable (even smooth) $\varphi : X \to \mathbb{R}$ such that $T_{\varphi,\mathcal{S}}$ has the AOP property for each ergodic flow $\mathcal{S}$. In particular, there is an irrational rotation T and a continuous $\varphi : \mathbb{T} \to \mathbb{R}$ such that 'randomized' Möbius disjointness (2) holds for all continuous, uniquely ergodic flows $\mathcal{S}$, i.e.*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\le N} g(S_{\varphi^{(n)}(x)}y)\boldsymbol{\mu}(n) = 0$$

for all $x \in X$, $g \in C(Y)$ and $y \in Y$.

In fact, we prove a stronger statement.

THEOREM A'. *For any function $\varphi : X \to \mathbb{R}$ of class C^∞ that is not a trigonometric polynomial, there exists α such that $T_{\varphi,\mathcal{S}}$ has the AOP property for each ergodic flow $\mathcal{S}$. The set of such α contains a G_δ-dense subset.*

For the proof, see Theorem 4.1, Corollary 4.17, Proposition 5.1 below.

Remark 1.1. Note that our method does not yield anything beyond a G_δ-dense subset of α's as it strongly relies on Theorem 5.4.

We also consider the affine case: $\varphi(x) = x - \frac{1}{2}$ and prove that $T_{\varphi,\mathcal{S}}$ has the AOP property for each ergodic flow $\mathcal{S}$ acting on $(Y, \mathcal{C}, \nu)$ whose spectrum on $L^2_0(Y, \mathcal{C}, \nu)$ is disjoint with $\mathbb{Q}$, see Theorem 7.10 below.

Assume now that $\varphi : X \to \mathbb{R}$ is continuous and the homeomorphisms $T_{\varphi,\mathcal{S}}$ enjoy the AOP property for each (continuous) uniquely ergodic $\mathcal{S}$. It follows from Theorem A that if we fix $x_0 \in X$ and set

$$c_n := \varphi^{(n)}(x_0) = \varphi(x_0) + \varphi(Tx_0) + \cdots + \varphi(T^{n-1}x_0), \tag{5}$$

then for each uniquely ergodic flow $\mathcal{S}$ on a compact metric space Y, $g \in C(Y)$ and $y \in Y$, we obtain

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\le N} g(S_{c_n}y)\boldsymbol{\mu}(n) = 0. \tag{6}$$

Thus, (c_n) is an example of a sequence along which the Möbius disjointness holds for each uniquely ergodic flow $\mathcal{S}$. By taking

$$a_n := [c_n], \quad n \ge 1, \tag{7}$$

and applying the standard suspension construction (see §2.7), we obtain

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\le N} b(R^{a_n}z)\boldsymbol{\mu}(n) = 0 \tag{8}$$

for each uniquely ergodic homeomorphism R of a compact metric space Z ($b \in C(Z)$, $z \in Z$). In fact, we prove the ***u*-disjointness**, i.e. we prove (8) in which $\boldsymbol{\mu}$ is replaced by a

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multiplicative function† or being more precise: for each multiplicative function $\boldsymbol{u}:\mathbb{N}\to\mathbb{C}$, $|\boldsymbol{u}|\le 1$, we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n\le N} b(R^{a_n}z)\boldsymbol{u}(n)=0 \tag{9}$$

for each uniquely ergodic homeomorphism R of a compact metric space Z, each $b\in C(Z)$, $\int_Z b\,d\rho=0$ (ρ stands for the unique R-invariant measure) and $z\in Z$.

Notice however that the existence of a sequence $(a_n)\subset\mathbb{Z}$ for which the subsequence version (8) of Möbius disjointness holds for each uniquely ergodic homeomorphisms R is not surprising. Indeed, each slowly increasing sequence (a_n) of integers will do the same because the sequence $(b(R^{a_n}z))_{n\ge 1}$ will be a bounded sequence which behaves like a constant sequence and (8) will follow from the fact that $(1/N)\sum_{n\le N}\boldsymbol{\mu}(n)\to 0$. On the other hand, the existence of (a_n) for which (9) holds for each multiplicative $\boldsymbol{u}:\mathbb{N}\to\mathbb{C}$, $|\boldsymbol{u}|\le 1$, and each uniquely ergodic R does not seem to be clear as there are multiplicative functions whose averages do not converge: $n\mapsto n^{it}=e^{it\log n}$ for $t\neq 0$, are examples of such. Moreover, whenever $\int_Z b\,d\rho=0$, the sequence $(b(R^{[a_n]}z))$ displays an orthogonality behaviour along different subsequences $(b(R^{[a_{pn}]}z)$ and $b(R^{[a_{qn}]}z)$ for different prime numbers p,q, namely, it will satisfy the assumptions of the Kátai–Bourgain–Sarnak–Ziegler (KBSZ) criterion (see Proposition 3.1 below). Finally, as proved recently in [**5**], the AOP property implies a property similar to (9) on a typical short interval. More precisely, it follows from [**5**] that the following holds.

COROLLARY B. *There is a Poincaré sequence $(a_n)_{n\in\mathbb{N}}\subset\mathbb{Z}$ such that for each multiplicative function $\boldsymbol{u}:\mathbb{N}\to\mathbb{C}$, $|\boldsymbol{u}|\le 1$, we have*

$$\frac{1}{M}\sum_{M\le m<2M}\left|\frac{1}{H}\sum_{m\le h<m+H} b(R^{a_h}z)\boldsymbol{u}(h)\right|\to 0,\quad H\to\infty,\ H/M\to 0 \tag{10}$$

for each uniquely ergodic homeomorphism R of a compact metric space Z, $b\in C(Z)\cap L^2_0(Z,\mathcal{D},\rho)$ and $z\in Z$.

We recall that a sequence (a_n) is called Poincaré if for each ergodic automorphism R of a probability standard Borel space $(Z,\mathcal{D},\rho)$ and each $C\in\mathcal{D}$, $\rho(C)>0$, we have $\rho(R^{-a_n}C\cap C)>0$ for infinitely many n. Each ergodic sequence is Poincaré.

We provide quite explicit sequences satisfying the above (see §7). For example, if α is irrational with bounded partial quotients and α, β, 1 are rationally independent then we can take

$$a_n=\left[n\beta+\frac{n(n-1)}{2}\alpha-\frac{n}{2}-\sum_{j=1}^{n-1}[\beta+j\alpha]\right],\quad n\ge 1.$$

Proceeding as in [**6**], it follows that for each degree $d>0$ polynomial $P\in\mathbb{R}[x]$ with the leading coefficient α_d irrational, we have

$$\frac{1}{M}\sum_{M\le m<2M}\left|\frac{1}{H}\sum_{m\le h<m+H} b(W^{a_h}z)e^{2\pi iP(a_h)}\boldsymbol{u}(h)\right|\to 0 \tag{11}$$

† The function $\boldsymbol{\mu}$ is multiplicative, that is, $\boldsymbol{\mu}(mn)=\boldsymbol{\mu}(n)\boldsymbol{\mu}(m)$ for each m, n coprime.

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when $H \to \infty$, $H/M \to 0$, for W a uniquely ergodic homeomorphism such that $(Z, \mathcal{D}, \rho, W)$ has no $e^{2\pi i k\alpha} \neq 1$ as its eigenvalue, $\boldsymbol{u}$ as above and arbitrary $b \in C(Z)$ and $z \in Z$†.

To illustrate (10), consider R being the rotation on $Z = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$, $Ri = i + 1$, $b(i) = (-1)^i$ and $z = 0$. The validity of (10) for these parameters yields the following.

COROLLARY C. *There is a Poincaré sequence $(a_n) \subset \mathbb{Z}$ such that for each multiplicative function $\boldsymbol{u} : \mathbb{N} \to \mathbb{C}$, $|\boldsymbol{u}| \leq 1$, we have*

$$\frac{1}{M} \sum_{M \leq m < 2M} \left| \frac{1}{H} \sum_{m \leq h < m+H} (-1)^{a_h} \boldsymbol{u}(h) \right| \to 0, \quad H \to \infty, \ H/M \to 0. \tag{12}$$

Note that no assumption on the convergence of the averages of $\boldsymbol{u}$ is made. (If we assume that the averages of $\boldsymbol{u}$ are going to zero and $\boldsymbol{u}$ is real valued, then (12) holds for any constant sequence (a_n) [**38**].)

2. *Basic notions: cocycle, Mackey action, Rokhlin extension, joining*

2.1. *Mackey action associated to a cocycle.*

Let $(X, \mathcal{B}, \mu)$ be a probability standard Borel space. Let $\mathrm{Aut}(X, \mathcal{B}, \mu)$ denote the group of (measure-preserving) automorphisms of $(X, \mathcal{B}, \mu)$. Assume that $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ is ergodic. Let G be a locally compact second countable (lcsc) Abelian group and let $\varphi : X \to G$ be measurable (we say that φ is a *cocycle*). By T_φ we denote the corresponding *group extension*:

$$T_\varphi : (X \times G, \mathcal{B}(X \times G), \mu \otimes \lambda_G) \to (X \times G, \mathcal{B}(X \times G), \mu \otimes \lambda_G),$$
$$T_\varphi(x, g) = (Tx, \varphi(x) + g);$$

here λ_G stands for a Haar measure of G; hence, T_φ preserves the measure $\mu \otimes \lambda_G$ which is infinite if G is not compact. Note that $(T_\varphi)^k(x, g) = (T^k x, \varphi^{(k)}(x) + g)$, where

$$\varphi^{(k)}(x) = \begin{cases} \varphi(x) + \varphi(Tx) + \cdots + \varphi(T^{k-1}x) & \text{if } k \geq 1, \\ 0 & \text{if } k = 0, \\ -(\varphi(T^k x) + \cdots + f(T^{-1}x)) & \text{if } k < 0. \end{cases}$$

Let $\tau = (\tau_g)_{g \in G}$ be the natural G-action on $(X \times G, \mu \otimes \lambda_G)$:

$$\tau_g(x, g') = (x, g + g') \quad \text{for } (x, g') \in X \times G. \tag{13}$$

Then τ preserves the measure $\mu \otimes \lambda_G$ and for each $g \in G$, we have

$$T_\varphi \circ \tau_g = \tau_g \circ T_\varphi. \tag{14}$$

We say that φ is *ergodic* if T_φ is ergodic. In general, φ is not ergodic. For example, if

$$\varphi(x) = \theta(Tx) - \theta(x)$$

† We only need to show that the transformation R' which is behind the sequence $e^{2\pi i P(n)}$, $n \geq 0$, is disjoint from W as then $R := W \times R'$ is also uniquely ergodic and we can use Corollary B. As a matter of fact the transformation R' is a several step affine extension of the irrational rotation by $\alpha := d!\alpha_d$. Since W is disjoint with this rotation, it is also disjoint with its compact group extension (which does not add new eigenvalues) [**21**].

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for a measurable $\theta : X \to G$, i.e. when φ is a *coboundary*, then clearly T_φ is not ergodic as every set

$$\{\tau_g(x, \theta(x)) : x \in X\}, \quad g \in G, \tag{15}$$

is T_φ-invariant. Fix a probability measure λ equivalent to λ_G and note that now T_φ and τ become non-singular actions on the probability (standard Borel) space $(X \times G, \mu \otimes \lambda)$. Let $\mathcal{J}_\varphi = \mathcal{J}(T_\varphi)$ denote the σ-algebra of T_φ-invariant sets and $(C_\varphi, \mathcal{J}_\varphi, \kappa_\varphi)$ denote the corresponding probability (standard Borel) quotient space, called the *space of ergodic components of* T_φ. Since (14) holds, τ also acts on the space of ergodic components. This measurable and non-singular† G-action is called the *Mackey action* of T_φ (or of φ) and is denoted by $\mathcal{W}(\varphi)$ or $\mathcal{W}(\varphi, T)$ or even $\mathcal{W}(\varphi, T, \mu)$ if not expliciting the parameters may lead to a confusion. The Mackey action is always ergodic. Note that the Mackey action of an ergodic cocycle is trivial (the action on the one point space), while it is the action of G on itself (by translations) when φ is a coboundary, cf. (15).

2.2. *Essential values of a cocycle.* Let $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ be ergodic and $\varphi : X \to G$ a cocycle with values in an lcsc Abelian group G. Following [**41**], an element $g \in G$ is called an *essential value* of φ if for every $C \in \mathcal{B}$, $\mu(C) > 0$, and every open $V \subset G$, $g \in V$, there exists $N \in \mathbb{Z}$ such that $\mu(C \cap T^{-N}C \cap [\varphi^{(N)} \in V]) > 0$. By $E(\varphi)$ we denote the set of all essential values. In fact, it is a closed subgroup of G.

PROPOSITION 2.1. [**41**] *T_φ is ergodic if and only if $E(\varphi) = G$.*

We will later need the following fact from [**41**].

PROPOSITION 2.2. [**41**] *Let $K \subset G$ be compact and $K \cap E(\varphi) = \emptyset$. Then there exists $B \in \mathcal{B}$, $\mu(B) > 0$, such that for each integer $m \geq 1$, we have*

$$\mu(B \cap T^{-m}B \cap [\varphi^{(m)} \in K]) = 0.$$

A cocycle $\varphi : X \to G$ is called *regular* if there exist a closed subgroup $H \subset G$, a cocycle $\psi : X \to H$ and $f : X \to G$ measurable such that $\varphi = \psi + f \circ T - f$ and ψ is ergodic as a cocycle taking values in H (in fact, H must be equal to $E(\varphi)$). In particular, when $H = \{0\}$, i.e. when φ is a *coboundary*, then φ is regular. Moreover, each cocycle $\varphi : X \to G$ for which $E(\varphi)$ is cocompact is regular.

A method to compute essential values is contained in the following.

PROPOSITION 2.3. [**35**] *Assume that $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ is ergodic and rigid, that is, $T^{q_n} \to \mathrm{Id}$ (in $L^2(X, \mathcal{B}, \mu)$) for some increasing sequence $(q_n) \subset \mathbb{N}$. Let $\varphi : X \to G$ be a cocycle and suppose that the sequence $(\varphi^{(q_n)})_*(\mu)$ of probability measures on G weakly converges to a probability measure P (on G). Then each point g in the topological support of P belongs to $E(\varphi)$.*

† Given a G-action $\mathcal{W} = (W_g)_{g \in G}$ on a probability standard Borel space $(Z, \mathcal{D}, \rho)$, we say that it is *measurable* if the map $(z, g) \mapsto W_g z$ is measurable and it is *non-singular* if for each $g \in G$, the measure $\rho \circ W_g = (W_g)_*(\rho)$ given by $(W_g)_*(\rho)(A) := \rho(W_g^{-1}A)$ is equivalent to ρ. Implicitly, all actions under consideration are measurable.

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We will need to apply the above result when $\varphi, \psi : X \to G$ are given and we want to obtain some essential values for the cocycle $\varphi + \psi$ out of $E(\varphi)$ and $E(\psi)$. The lemma below will be applied when the assumptions of Proposition 2.3 are satisfied and $\varphi_n = \varphi^{(q_n)}$, $\psi_n = \psi^{(q_n)}$.

LEMMA 2.4. *Let $(X, \mathcal{B}, \mu)$ be a probability space, G an lcsc Abelian group with an invariant metric d. Assume that $\varphi_n, \psi_n : X \to G$ are measurable and taking values in a compact set $C \subset G$. Moreover, assume that ψ_n takes values in a finite set $F \subset C$, $n \geq 1$. Assume that $(\varphi_n)_*(\mu) \to \nu$, $(\psi_n)_*(\mu) \to \kappa$ and $(\varphi_n + \psi_n)_*(\mu) \to \rho$. Then for each $h_0 \in \operatorname{supp} \kappa$ there exists $g_0 \in \operatorname{supp} \nu$ such that $h_0 + g_0 \in \operatorname{supp} \rho$. Moreover, for each $g_0 \in \operatorname{supp} \nu$ there exists $h_0 \in \operatorname{supp} \kappa$ such that $h_0 + g_0 \in \operatorname{supp} \rho$.*

Proof. (a) Let $h_0 \in \operatorname{supp} \kappa$. Given $n \geq 1$, let $A_n := \{x \in X : \psi_n(x) = h_0\}$. By taking subsets of A_n if necessary, we can assume that there exists $c > 0$ such that $\mu(A_n) = c$ for all $n \geq 1$. Again, by passing to a subsequence if necessary, we can assume that

$$(\varphi_n|_{A_n})_*(\mu) \to \nu',$$

where ν' is concentrated on C and $\nu'(C) = c$. Take any $g_0 \in \operatorname{supp} \nu'$ (note that $\operatorname{supp} \nu' \subset \operatorname{supp} \nu$). Fix $\varepsilon > 0$. Then (by the invariance of d under translations)

$$B\left(g_0, \frac{\varepsilon}{2}\right) + B\left(h_0, \frac{\varepsilon}{2}\right) \subset B(g_0 + h_0, \varepsilon).$$

Since $g_0 \in \operatorname{supp} \nu'$, there exists $c' > 0$ such that

$$\liminf_{n \to \inf} (\varphi_n|_{A_n})_* \mu(B(g_0, \varepsilon/2)) \geq c',$$

so for n large enough there exists $A_n' \subset A_n$ such that $\mu(A_n') \geq c'/2$ and $\varphi_n(A_n') \subset B(g_0, \varepsilon/2)$. But $\psi_n(A_n') = \{h_0\}$, whence $(\varphi_n + \psi_n)(A_n') \subset B(g_0 + h_0, \varepsilon)$.

(b) Suppose now that $g_0 \in \operatorname{supp} \nu$. Fix $\varepsilon > 0$. Then there exist $c > 0$ and $A_n \subset X$ such that $\mu(A_n) = c$ and $\varphi_n(A_n) \subset B(g_0, \varepsilon)$ for each $n \geq 1$. Now, we have a partition

$$A_n = A_{n, f_1} \cup \cdots \cup A_{n, f_k},$$

where f_j is the (unique) value of ψ_n on A_{n, f_j}, $j = 1, \ldots, k$. Then, for some $1 \leq m \leq k$, we have $\mu(A_{n, f_m}) \geq c/k$ (passing to a subsequence if necessary). Now, the sets A_{n, f_m}, $n \geq 1$ 'realize' $h_0 := f_m$. Repeat the proof of (a) to find $A_n' \subset A_{n, f_m}$ so that for $g_0' \in \operatorname{supp} \nu''$ (with $(\varphi_n|_{A_{n, f_m}})_*(\mu) \to \nu''$), we have $g_0' + h_0 \in \operatorname{supp} \rho$. But $d(g_0, g_0') < \varepsilon$ whence $g_0 + h_0 \in \operatorname{supp} \rho$ (since the support is closed). □

2.3. *The group of L^∞-eigenvalues.* If $\mathcal{W} = (W_g)_{g \in G}$ is a non-singular G-action on a probability standard Borel space $(Z, \mathcal{D}, \rho)$ then a character $\chi \in \widehat{G}$ is called an L^∞-*eigenvalue* of $\mathcal{W}$ if for some $0 \neq f \in L^\infty(Z, \mathcal{D}, \rho)$ and each $g \in G$,

$$f \circ W_g = \chi(g) \cdot f \quad \rho\text{-almost everywhere.}$$

We will denote this group by $e(\mathcal{W})$†. When $\mathcal{W}$ is ergodic, we can assume that $|f| = 1$ and such an action is called *weakly mixing* if $e(\mathcal{W})$ consists of the trivial character solely.

† In contrast with the finite measure-preserving case, in the non-singular case, it may happen that $e(\mathcal{W})$ is uncountable. It is however always a Borel subgroup of $\widehat{G}$.

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It follows that for automorphisms ($\mathbb{Z}$-actions), say for T_φ considered above, the group $e(T_\varphi)$ consists of $c \in \mathbb{S}^1$ for which $F \circ T_\varphi = cF$ for some $F \in L^\infty(X \times G, \mu \otimes \lambda)$. If φ is ergodic then $c \in e(T_\varphi)$ if and only if for some $\chi \in \widehat{G}$,

$$\chi \circ \varphi = c \cdot \xi \circ T/\xi$$

for some measurable $\xi : X \to \mathbb{S}^1$, e.g. [**33**] (note that $F(x, g) = \overline{\xi(x)}\chi(g)$ is the corresponding eigenfunction).

Recall that for an arbitrary cocycle $\varphi : X \to G$ (over an ergodic $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$) the Mackey action $\mathcal{W}(\varphi) = (W_g)_{g \in G}$ is ergodic. Moreover, we have (e.g. [**33**])

$$\begin{aligned} e(\mathcal{W}(\varphi)) &= \Lambda(\varphi) \\ &:= \{\chi \in \widehat{G} : \chi \circ \varphi = \xi \circ T/\xi \text{ for a measurable } \xi : X \to \mathbb{S}^1\}. \end{aligned} \tag{16}$$

Note that when φ is ergodic then $\Lambda(\varphi) = \{\mathbb{1}\}$.

LEMMA 2.5. *If $\chi \in \widehat{G}$ is an L^∞-eigenvalue of the Mackey action $\mathcal{W}(\varphi, T)$, then $\chi \in E(\varphi)^\perp$, i.e. $\chi(g) = 1$ for each $g \in E(\varphi)$.*

Proof. Since $\overline{\chi(E(\varphi))} \subset E(\chi \circ \varphi)$ (e.g. [**35**]; the equality holds if φ is additionally regular), and $E(\chi \circ \varphi) = \{1\}$ since $\chi \circ \varphi$ is a coboundary, the result follows. □

2.4. *Rokhlin extensions.* Assume that $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ is ergodic and let $\varphi : X \to G$ be a cocycle. Assume, moreover, that $G \ni g \mapsto S_g \in \mathrm{Aut}(Y, \mathcal{C}, \nu)$ is a (measurable) G-representation, which we denote by $\mathcal{S}$†. We will always assume that $\mathcal{S}$ is ergodic. The G-action $\mathcal{S}$ induces a unitary G-representation (Koopman representation, we will use the same notation $\mathcal{S}$ to denote this representation) on $L^2(Y, \mathcal{C}, \nu)$ given by $f \mapsto f \circ S_g$. We denote by $\sigma_\mathcal{S}$ the maximal spectral type of $\mathcal{S}$ on the subspace $L^2_0(X, \mathcal{B}, \mu)$ of $L^2(X, \mathcal{B}, \mu)$ of zero mean functions. Then, the automorphism $T_{\varphi,\mathcal{S}} \in \mathrm{Aut}(X \times Y, \mathcal{B} \otimes \mathcal{C}, \mu \otimes \nu)$ given by

$$T_{\varphi,\mathcal{S}}(x, y) = (Tx, S_{\varphi(x)}(y)) \quad \text{for } (x, y) \in X \times Y$$

is called a *Rokhlin extension* of T‡. Note that for each $k \in \mathbb{Z}$, we have $(T_\varphi)^k(x, y) = (T^k x, S_{\varphi^{(k)}(x)}(y))$, hence

$$(T_{\varphi,\mathcal{S}})^k = (T^k)_{\varphi^{(k)},\mathcal{S}}. \tag{17}$$

PROPOSITION 2.6. [**30**, **34**] *Assume that $\mathcal{S}$ is ergodic. Then $T_{\varphi,\mathcal{S}}$ is ergodic if and only if $\sigma_\mathcal{S}(\Lambda_\varphi) = 0$.*

Remark 2.7. It follows that if φ is ergodic then $T_{\varphi,\mathcal{S}}$ is ergodic whenever $\mathcal{S}$ is ergodic.

We will also need the following.

PROPOSITION 2.8. [**34**] *Assume that $T_{\varphi,\mathcal{S}}$ is ergodic. Then $c \in e(T_{\varphi,\mathcal{S}})$ if and only if for some $\chi \in e(\mathcal{S})$, we have $\chi \circ \varphi = c \cdot \xi \circ T/\xi$ for some measurable $\xi : X \to \mathbb{S}^1$.*

† Measurability can also be expressed by the fact that for each $h \in L^2(Y, \mathcal{C}, \nu)$, the map $G \ni g \mapsto h \circ S_g$ is continuous (equivalently, it is weakly continuous).

‡ The map $X \ni x \mapsto S_x := S_{\varphi(x)} \in \mathrm{Aut}(X \times Y, \mu \otimes \nu)$ is a particular case of the so-called Rokhlin cocycle, e.g. [**15**, **30**, **33**, **34**].

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Remark 2.9. Assume that $G = \mathbb{Z}$ and let $Y = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ with $S(i) = i + 1 \bmod 2$ for $i = 0, 1$. Then the only non-trivial eigenvalue of S is -1 (more precisely, it is the character $\chi(n) = (-1)^n$) and it follows that (assuming that $T_{\varphi,\mathcal{S}}$ is ergodic) c is an eigenvalue of $T_{\varphi,\mathcal{S}}$ different from an eigenvalue of T if and only if

$$(-1)^{\varphi(\cdot)} = c \cdot \xi \circ T/\xi$$

for some measurable $\xi : X \to \mathbb{S}^1$. Similarly, if $Sy = y + \alpha$ is an irrational rotation on $\mathbb{T} = [0, 1)$ then $c \in e(T_{\varphi,\mathcal{S}})\backslash e(T)$ for an ergodic $T_{\varphi,\mathcal{S}}$ if and only if for some $k \neq 0$, we have

$$e^{2\pi i k\varphi(\cdot)\alpha} = c \cdot \xi \circ T/\xi$$

for a measurable $\xi : X \to \mathbb{S}^1$.

Clearly, if $\mathcal{S}$ is weakly mixing, then the ergodicity of $T_{\varphi,\mathcal{S}}$ implies $e(T_{\varphi,\mathcal{S}}) = e(T)$.

PROPOSITION 2.10. [**30**] *If T is uniquely ergodic, $\varphi : X \to G$ is continuous and T_φ is ergodic then $T_{\varphi,\mathcal{S}}$ is uniquely ergodic whenever $\mathcal{S}$ is uniquely ergodic.*

2.5. *Joinings.* Let $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ and $R \in \mathrm{Aut}(Z, \mathcal{D}, \rho)$ be ergodic. By a *joining* of T and R we mean any $T \times R$-invariant measure κ on $(X \times Z, \mathcal{B} \otimes \mathcal{D})$ whose projections on the coordinates are μ and ρ, respectively. The set of joinings between T and R is denoted by $J(T, R)$. By $J^e(T, R)$ we denote the subset of *ergodic joinings*, i.e. the subset of those $\kappa \in J(T, R)$ for which the automorphism $T \times R \in \mathrm{Aut}(X \times Z, \mathcal{B} \otimes \mathcal{D}, \kappa)$ is ergodic. Following [**21**], T and R are called *disjoint* if $J(T, R) = \{\mu \otimes \rho\}$.

An obvious example of a joining is just the product measure $\mu \otimes \rho$. Assume now that T and R are factors of $\overline{T} \in \mathrm{Aut}(\overline{X}, \overline{\mathcal{B}}, \overline{\mu})$ and $\overline{R} \in \mathrm{Aut}(\overline{Z}, \overline{\mathcal{D}}, \overline{\rho})$, respectively. By abuse of notation, we can assume that $\mathcal{B} \subset \overline{\mathcal{B}}$ and $\mathcal{D} \subset \overline{\mathcal{D}}$. Assume that $\kappa \in J(T, R)$. Then the measure $\widehat{\kappa}$ determined by

$$\int_{\overline{X}\times\overline{Z}} F \otimes G \, d\widehat{\kappa} = \int_{X\times Z} \mathbb{E}(F|\mathcal{B})\mathbb{E}(G|\mathcal{D}) \, d\kappa$$

is a joining of $\overline{T}$ and $\overline{R}$, i.e. $\widehat{\kappa} \in J(\overline{T}, \overline{R})$, called the *relatively independent extension* of κ [**23**]. Even if κ is ergodic, the relatively independent extension of it need not be ergodic. Note that

$$\int_{\overline{X}\times\overline{Z}} F \otimes G \, d\widehat{\kappa} = 0 \quad \text{whenever } \mathbb{E}(F|\mathcal{B}) = 0. \tag{18}$$

If $U \in \mathrm{Aut}(X, \mathcal{B}, \mu)$, $U \circ T = T \circ U$, i.e. if U belongs to the *centralizer* $C(T)$ of T, then U determines a self-joining $\mu_U \in J(T, T)$ (in fact, $\mu_U \in J^e(T, T)$) given by the formula $\mu_U(A \times B) = \mu(A \cap U^{-1}B)$ for $A, B \in \mathcal{B}$. Consider, as above, an ergodic extension $\overline{T} \in \mathrm{Aut}(\overline{X}, \overline{\mathcal{B}}, \overline{\mu})$ of T. If the relatively independent extension of μ_{Id} is ergodic, i.e. if $\widehat{\mu_{\mathrm{Id}}} \in J^e(\overline{T}, \overline{T})$ then we say that the extension

$$(\overline{X}, \overline{B}, \overline{\mu}, \overline{T}) \to (X, \mathcal{B}, \mu, T)$$

is *relatively weakly mixing*. We also say that $\overline{T}$ is relatively weakly mixing over T.

In what follows, we will study automorphisms of the form $T_{\varphi,\mathcal{S}}$ (as in §2.4). They are clearly extensions of T. There is a simple criterion for the relative weak mixing in this situation.

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PROPOSITION 2.11. [**33**, **34**] *$T_{\varphi,\mathcal{S}}$ is relatively weakly mixing over T if and only if $T_{\varphi,\mathcal{S}}$ is ergodic and $\mathcal{S} \subset \mathrm{Aut}(Y, \mathcal{C}, \nu)$ is weakly mixing. Moreover, if T is disjoint from all weakly mixing automorphisms, φ is ergodic and $\mathcal{S}$ is mildly mixing† then $T_{\varphi,\mathcal{S}}$ is disjoint from all weakly mixing automorphisms.*

2.6. *Joinings between Rokhlin extensions.* Let G be an lcsc Abelian group. Following [**32**], we will consider *algebraic couplings* of G which are subgroups $\mathcal{H} \subset G \times G$ whose projections on both coordinates are dense. Assume that $\mathcal{S} = (S_g)_{g\in G}$ is an ergodic G-action in $\mathrm{Aut}(Y, \mathcal{C}, \nu)$. By $\mathcal{S} \otimes \mathcal{S}$ we denote the product $G \times G$-action: $\mathcal{S} \otimes \mathcal{S} = (S_g \times S_{g'})_{(g,g')\in G\times G}$. It is considered with product measure $\nu \otimes \nu$‡. By $(\mathcal{S} \otimes \mathcal{S})|_{\mathcal{H}}$ we denote the restriction of the product action to $\mathcal{H}$. Let

$$\mathcal{M}(Y \times Y; \mathcal{H}) := \{\rho \in \mathcal{P}(Y \times Y) : \rho \text{ is } (\mathcal{S} \otimes \mathcal{S})|_{\mathcal{H}} - \text{invariant and both projections of } \rho \text{ are } \nu\}.$$

Then $\mathcal{M}(Y \times Y; \mathcal{H})$ is a simplex whose set of extremal points is the set $\mathcal{M}^e(Y \times Y; \mathcal{H})$ of ergodic measures in $\mathcal{M}(Y \times Y; \mathcal{H})$. Denote by $\sigma_{\mathcal{S}}$ the maximal spectral type of $\mathcal{S}$ on $L^2_0(Y, \mathcal{C}, \nu)$.

PROPOSITION 2.12. (Cf. [**32**, Proposition 8]) *Assume that $\mathcal{S}$ is ergodic and $\mathcal{H} \subset G \times G$ is an algebraic coupling of G. Consider $((\mathcal{S} \otimes \mathcal{S})|_{\mathcal{H}}, \nu \otimes \nu)$. The following conditions are equivalent:*

(1) *$(\mathcal{S} \otimes \mathcal{S})|_{\mathcal{H}}$ is ergodic;*
(2) *$(\sigma_{\mathcal{S}} \otimes \sigma_{\mathcal{S}} + \sigma_{\mathcal{S}} \otimes \delta_{\mathbb{1}} + \delta_{\mathbb{1}} \otimes \sigma_{\mathcal{S}})(\mathcal{H}^{\perp}) = 0$;*
(3) *if $\chi, \eta \in \widehat{G}$ are eigenvalues for $\mathcal{S}$ then*

$$(\chi, \eta) \in \mathcal{H}^{\perp} \Leftrightarrow \chi = \mathbb{1} = \eta.$$

Proof. Let $p : \widehat{G} \times \widehat{G} \to \widehat{\mathcal{H}}$ be the dual map corresponding to the embedding of $\mathcal{H}$ into $G \times G$. The kernel of this homomorphism is

$$\ker p = \mathcal{H}^{\perp} = \{\xi \in \widehat{G \times G} : \xi(\mathcal{H}) = \{1\}\}.$$

The maximal spectral type of $(\mathcal{S} \otimes \mathcal{S})|_{\mathcal{H}}$ on $L^2_0(Y \times Y, \nu \otimes \nu)$ is equal to

$$p_*(\sigma_{\mathcal{S}} \otimes \sigma_{\mathcal{S}} + \sigma_{\mathcal{S}} \otimes \delta_{\mathbb{1}} + \delta_{\mathbb{1}} \otimes \sigma_{\mathcal{S}}).$$

Hence we can obtain the trivial character in the image of p only for an eigenvalue for $\mathcal{S} \otimes \mathcal{S}$, belonging to $\mathcal{H}^{\perp}$ (and such are only products of eigenvalues for $\mathcal{S}$). □

COROLLARY 2.13. *If, additionally, $\mathcal{H}$ is cocompact and*

$$(\sigma_{\mathcal{S}} \otimes \sigma_{\mathcal{S}} + \sigma_{\mathcal{S}} \otimes \delta_{\mathbb{1}} + \delta_{\mathbb{1}} \otimes \sigma_{\mathcal{S}})(\mathcal{H}^{\perp}) = 0$$

then $\mathcal{M}(Y \times Y; \mathcal{H}) = \{\nu \otimes \nu\}$.

† Mild mixing means that for no set $A \in \mathcal{B}$, $0 < \mu(A) < 1$, we have $\liminf_{n\to\infty} \mu(T^{-n}A \triangle A) = 0$.
‡ In fact, by ergodicity, $\mathcal{M}(Y \times Y; G \times G) = \{\nu \otimes \nu\}$ [**32**].

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Proof. Let $\rho \in \mathcal{M}^e(Y \times Y; \mathcal{H})$. Then also $(S_g \times S_{g'})_*(\rho) \in \mathcal{M}^e(Y \times Y; \mathcal{H})$ for each $g, g' \in G$. Define

$$\widetilde{\rho} := \int_{(G\times G)/\mathcal{H}} (S_g \times S_{g'})_* \rho \, d((g, g')\mathcal{H}). \tag{19}$$

Then $\widetilde{\rho}$ is an $\mathcal{S} \otimes \mathcal{S}$-invariant measure, whence $\widetilde{\rho} = \nu \otimes \nu$. But by our assumption and Proposition 2.12, it follows that

$$((\mathcal{S} \otimes \mathcal{S})|_{\mathcal{H}}, \nu \otimes \nu) \text{ is ergodic.}$$

It follows that in the integral representation (19) of $\tilde{\rho} = \nu \otimes \nu$ all the measures are $\mathcal{H}$-invariant and ergodic and therefore they must be almost everywhere equal to $\nu \otimes \nu$ (since $\nu \otimes \nu \in \mathcal{M}^e(Y \times Y; \mathcal{H})$). Finally, if $(S_g \times S_{g'})_*(\rho) = \nu \otimes \nu$ for some $(g, g') \in G \times G$ then $\rho = \nu \otimes \nu$ and the result follows. □

Assume now that $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ is ergodic and let $\varphi, \psi : X \to G$ be cocycles. Fix $\lambda \in J_2^e(T)$ and consider

$$\varphi \times \psi : (X \times X, \lambda) \to G \times G.$$

To underline that we have dependence on λ, we will also write $(\varphi \times \psi)_\lambda$. Denote by $J(T_{\varphi,\mathcal{S}}, T_{\psi,\mathcal{S}}; \lambda)$ the subset of $J(T_{\varphi,\mathcal{S}}, T_{\psi,\mathcal{S}})$ consisting of these joinings whose restriction to $X \times X$ equals λ. Let

$$\mathcal{H}_\lambda := E((\varphi \times \psi)_\lambda). \tag{20}$$

Proceeding now as in [**32**], we obtain the following result.

THEOREM 2.14. *Under the above assumptions, suppose that $\varphi, \psi : X \to G$ are ergodic and let $(\varphi \times \psi)_\lambda$ be regular. Then there exists an affine isomorphism*

$$\Lambda_\lambda : J(T_{\varphi,\mathcal{S}}, T_{\psi,\mathcal{S}}; \lambda) \to \mathcal{M}(Y \times Y; \mathcal{H}_\lambda).$$

Directly from Proposition 2.12, Corollary 2.13 and Theorem 2.14, we obtain the following.

COROLLARY 2.15. *If additionally $\mathcal{H}_\lambda$ is cocompact and*

$$(\sigma_{\mathcal{S}} \otimes \sigma_{\mathcal{S}} + \sigma_{\mathcal{S}} \otimes \delta_{\mathbb{1}} + \delta_{\mathbb{1}} \otimes \sigma_{\mathcal{S}})(\mathcal{H}^\perp) = 0$$

then $J(T_{\varphi,\mathcal{S}}, T_{\psi,\mathcal{S}}; \lambda) = \{\lambda \otimes \nu \otimes \nu\}$.

2.7. *Ergodic sequences and suspensions.* Let G be an lcsc Abelian group.

Definition 2.1. (See e.g. [**9**]) A sequence $(a_n) \subset G$ is called *ergodic* if for each ergodic G-action $\mathcal{S} = (S_g)_{g\in G} \subset \mathrm{Aut}(Y, \mathcal{C}, \nu)$, we have

$$\frac{1}{N} \sum_{n\le N} f \circ S_{a_n} \to \int_Y f \, d\nu \quad \text{in } L^2(Y, \mathcal{C}, \nu)$$

for each $f \in L^2(Y, \mathcal{C}, \nu)$.

To obtain an ergodic sequence, one can use the following result†.

† For the ergodicity of sequences $([n^c])$ $(0 < c < 1)$, see [**9**], and for the case of a slowly increasing sequence, see [**10**].

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PROPOSITION 2.16. (See e.g. [**31**]) *If T is uniquely ergodic, $\varphi : X \to G$ is continuous and $\Lambda(\varphi) = \{\mathbb{1}\}$, i.e. the Mackey action $\mathcal{W}(\varphi)$ is weakly mixing, then for each $x \in X$, the sequence $(\varphi^{(n)}(x))$ is ergodic. In particular, the result holds if φ is ergodic.*

Proof. Since φ is continuous, so is $\chi \circ \varphi$ for each $\chi \in \widehat{G}$. Moreover, since $\Lambda(\varphi) = \{1\}$, for each $1 \neq \chi \in \widehat{G}$ the (compact) group extension $T_{\chi\circ\varphi}$ defined on $(X \times \mathbb{S}^1, \mu \otimes \lambda_{\mathbb{S}^1})$ is uniquely ergodic [**22**]. It follows that for $G(x, z) = z$ and every (x_0, z_0), we have

$$\frac{1}{N}\sum_{n\le N} G((T_{\chi\circ\varphi})^n(x_0, z_0)) \to \int G\, d(\mu \otimes \lambda_{\mathbb{S}^1}) = 0.$$

Therefore, whenever $\mathbb{1} \neq \chi \in \widehat{G}$, for all $x \in X$, we have

$$\frac{1}{N}\sum_{n\le N} \chi(\varphi^{(n)}(x)) \to 0. \tag{21}$$

Assume that $f \in L^2_0(Y, \mathcal{C}, \nu)$. By the spectral theorem, for each $x \in X$, we have

$$\left\| \frac{1}{N}\sum_{n\le N} f \circ S_{\varphi^{(n)}(x)} \right\|_{L^2(Y,\mathcal{C},\nu)} = \left\| \frac{1}{N}\sum_{n\le N} \chi(\varphi^{(n)}(x)) \right\|_{L^2(\widehat{G},\sigma_f)}, \tag{22}$$

where σ_f stands for the *spectral measure* of f (with respect to the Koopman representation of $\mathcal{S}$), i.e. the only measure on $\widehat{G}$ determined by

$$\widehat{\sigma_f}(g) := \int_{\widehat{G}} \chi(g)\, d\sigma_f(\chi) = \int_Y f \circ S_g \cdot \overline{f}\, d\nu, \quad g \in G.$$

Since $f \in L^2_0(Y, \mathcal{C}, \nu)$ and $\mathcal{S}$ is ergodic, σ_f has no atom at the trivial character $\mathbb{1}$, and the result follows from (21). □

Definition 2.2. [**22**, p. 72] A sequence $(a_n) \subset G$ is a *Poincaré sequence* if for each ergodic G-action $\mathcal{S} = (S_g)_{g\in G} \subset \mathrm{Aut}(Y, \mathcal{C}, \nu)$ and $A \subset Y$ of positive measure there exists $n \geq 1$ such that $\nu(S_{a_n}A \cap A)$ is positive.

Remark 2.17. Each ergodic sequence is a Poincaré sequence.

In what follows, we will be particularly interested in ergodic sequences for flows ($\mathbb{R}$-actions) and automorphisms ($\mathbb{Z}$-actions). If $(k_n) \subset \mathbb{Z}$ is an ergodic sequence for automorphisms then clearly it will be an ergodic sequence for all flows whose time 1 automorphism is ergodic.

To see some relation in the opposite direction consider the following construction. Assume that $R \in \mathrm{Aut}(Z, \mathcal{D}, \rho)$ is ergodic. Consider $\varphi : Z \to \mathbb{R}$, $\varphi(z) = 1$ and let λ be a probability measure equivalent to $\lambda_{\mathbb{R}}$. If $F : Z \times \mathbb{R} \to \mathbb{C}$ is measurable and $F \circ R_\varphi = F$ $\mu \otimes \lambda$-almost everywhere then F is entirely determined by $F|_{Z\times[0,1)}$; indeed $F(z, s) := F(R^{-n}z, s - n)$, where $n \in \mathbb{Z}$ is unique so that $0 \leq s - n < 1$. It easily follows that the Mackey action of R_φ is the *suspension flow* [**13**] $\widetilde{R} = (\widetilde{R}_t)_{t\in\mathbb{R}}$ of R, which acts on the space $\widetilde{Z} := Z \times [0, 1]/\sim$, where $(z, 1) \sim (Rz, 0)$, by the formula

$$\widetilde{R}_t(z, s) = (R^{[t+s]}z, \{s + t\}). \tag{23}$$

This flow preserves the product measure denoted by $\widetilde{\mu}$.

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Remark 2.18. Note that if $(a_n) \subset \mathbb{R}$ is ergodic then (a_n) is uniformly distributed modulo 1. Indeed, a sequence (a_n) is ergodic then $(1/N)\sum_{n\le N} e^{2\pi i\theta a_n} \to 0$ for each $\theta \in \mathbb{R}\setminus\{0\}$ (cf. the proof of Proposition 2.16). Hence $(\{a_n\})$ is uniformly distributed on $\mathbb{T}$ by the Weyl criterion.

Remark 2.19. Assume that $(a_n) \subset \mathbb{R}$ is an ergodic sequence. Then $([a_n])$ is a Poincaré sequence for each ergodic automorphism R. Indeed, if we take a set $A \subset Z$ of positive measures then for some a_n (arbitrarily large)

$$\widetilde{\mu}(\widetilde{R}_{-a_n}(A \times [0, 1/2)) \cap (A \times [1/2, 1))) > 0.$$

It follows that $\mu(R^{-[a_n]}A \cap A) > 0$.

As shown in [**36**], it is however possible that $(a_n) \subset \mathbb{R}$ is an ergodic sequence for flows but $([a_n])$ is not an ergodic sequence for automorphisms.

3. *The AOP property and multiplicative functions*

3.1. *Automorphisms with the AOP property.* Let $\mathscr{P}$ denote the set of prime numbers. Following [**6**], $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ is said to have AOPs if for each $f, g \in L^2_0(X, \mathcal{B}, \mu)$, we have

$$\limsup_{r\ne s, r,s\in\mathscr{P}, r,s\to\infty} \sup_{\kappa\in J^e(T^r,T^s)} \left|\int_{X\times X} f \otimes g \, d\kappa\right| = 0. \tag{24}$$

The condition above means that, intuitively, the prime powers of an automorphism become more and more disjoint. Clearly, the AOP property is inherited by factors, therefore the AOP property of T implies its total ergodicity†. As shown in [**6**], all totally ergodic rotations enjoy the AOP property‡. For larger classes of AOP automorphisms, see [**6**, **19**].

3.2. *Orthogonality of AOP observables with multiplicative functions.* We will consider *arithmetic functions*, i.e. $\boldsymbol{u} : \mathbb{N} \to \mathbb{C}$, $\boldsymbol{u} = (\boldsymbol{u}(n))_{n\in\mathbb{N}}$. Recall that such a function is *multiplicative* if $\boldsymbol{u}(mn) = \boldsymbol{u}(m)\boldsymbol{u}(n)$ whenever $(m, n) = 1$. Many basic arithmetic functions as the Möbius function $\boldsymbol{\mu}$ or the Liouville $\boldsymbol{\lambda}$ function are multiplicative§. It is a general problem in number theory to show that a given sequence $(a_n) \subset \mathbb{C}$ is *orthogonal* to a multiplicative function $\boldsymbol{u}$, i.e.

$$\frac{1}{N}\sum_{n\le N} a_n \boldsymbol{u}(n) \to 0 \quad \text{when } N \to \infty. \tag{25}$$

The basic criterion to show such an orthogonality is the following version (see [**6**]) of the KBSZ criterion.

PROPOSITION 3.1. [**11**, **26**] *Assume that $(a_n) \subset \mathbb{C}$ is bounded and*

$$\limsup_{r\ne s, r,s\in\mathscr{P}, r,s\to\infty} \left(\limsup_{N\to\infty} \left|\frac{1}{N}\sum_{n\le N} a_{rn}\overline{a}_{sn}\right|\right) = 0.$$

Then (a_n) is orthogonal to any multiplicative function $\boldsymbol{u}$, $|\boldsymbol{u}| \le 1$.

† Ergodic rotations of finite non-trivial cyclic groups do not have the AOP property.

‡ Note that such rotations can have all non-zero powers isomorphic. However, when r, s are growing, the graphs of such isomorphisms which yield ergodic joinings between T^r and T^s are more and more 'mixing'.

§ Dirichlet characters are other examples of multiplicative functions. Also, for each $t \in \mathbb{R}$, the function $n \mapsto n^{it}$ is multiplicative.

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Given a homeomorphism T of a compact metric space X, $f \in C(X)$ and $x \in X$, we are interested in the orthogonality of the observable $(f(T^n x))_{n\geq 0}$ with $\boldsymbol{u}$. For example, Sarnak's conjecture (cf. (1)) claims that whenever T has topological entropy zero then all observables as above are orthogonal to the Möbius function.

To see a relationship of the KBSZ criterion with joinings, assume that T is a uniquely ergodic homeomorphism of a compact metric space X (the unique T-invariant measure is denoted by μ). Assume that (X, μ, T) is totally ergodic and let $x \in X$. Consider the sequence

$$\frac{1}{N}\sum_{n\leq N} \delta_{(T^r\times T^s)^n(x,x)}, \quad N \geq 1, \tag{26}$$

of probability measures on $X \times X$. Any accumulation point κ of such measures is $T^r \times T^s$-invariant, and by the unique ergodicity of T (and the total ergodicity assumption), $\kappa \in J(T^r, T^s)$. It follows that if we are able to show that T^r and T^s are disjoint for $r \neq s$ sufficiently large then

$$\frac{1}{N}\sum_{n\leq N} \delta_{(T^r\times T^s)^n(x,x)} \to \mu\otimes\mu,$$

whence

$$\begin{aligned}&\frac{1}{N}\sum_{n\leq N} f(T^{rn}x)\overline{f(T^{sn}x)}\\ &\quad=\int f\otimes\overline{f}\,d\Big(\frac{1}{N}\sum_{n\leq N}\delta_{(T^r\times T^s)^n(x,x)}\Big)\to\int f\otimes\overline{f}\,d(\mu\otimes\mu)=0\end{aligned}$$

whenever $f \in L^2_0(X, \mathcal{B}, \mu)$. It follows that the KBSZ criterion applies.

It turns out, however, that to apply Proposition 3.1 we do not need to obtain that the only accumulation point of the sequence (26) is a product measure. We just need to show that when $r \neq s$ are large, such accumulation points are close (in the weak topology) to the product measure. This is guaranteed if the AOP property holds (see [**6**] for details). In fact, the AOP property implies even more.

PROPOSITION 3.2. [**5**] *Assume that T is a uniquely ergodic homeomorphism of a compact metric space X such that (X, μ, T) enjoys the AOP property. Then for each multiplicative $\boldsymbol{u}$, $|\boldsymbol{u}| \leq 1$, each sequence (b_k) of natural numbers with $b_{k+1} - b_k \to \infty$ and each choice of $x_k \in X$, $k \geq 1$, and $f \in C(X)$, $\int f\,d\mu = 0$, we have*

$$\frac{1}{b_{K+1}}\sum_{k\leq K}\Big|\sum_{b_k\leq n<b_{k+1}} f(T^n x_k)\boldsymbol{u}(n)\Big|\to 0 \quad \textit{when } K\to\infty.$$

The above property is referred to as the *strong* $\boldsymbol{u}$-MOMO property (in other words AOP implies the strong MOMO property). As noticed already in [**6**], from the strong $\boldsymbol{u}$-MOMO property, we obtain the following:

$$\frac{1}{M}\sum_{M\leq m<2M}\Big|\frac{1}{H}\sum_{m\leq h<m+H} f(T^h x)\boldsymbol{u}(n)\Big|\to 0 \tag{27}$$

when $H \to \infty$, $H/M \to 0$, for each $x \in X$ and f, $\boldsymbol{u}$ as above.

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We can extend the notion of a strong $\boldsymbol{u}$-MOMO property to arbitrary lcsc Abelian groups by considering sequences of group elements. Fix a sequence $(a_n) \subset G$.

Definition 3.1. A uniquely ergodic G-action $\mathcal{S}$ of a compact metric space Y (the unique invariant measure is denoted by ν) is said to satisfy *the strong $\boldsymbol{u}$-MOMO property along* (a_n) if for each sequence (b_k) of natural numbers with $b_{k+1} - b_k \to \infty$ and each choice of $y_k \in Y$, $k \geq 1$, and $f \in C(Y)$, $\int f \, d\nu = 0$, we have

$$\frac{1}{b_{K+1}} \sum_{k \leq K} \left| \sum_{b_k \leq n < b_{k+1}} f(S_{a_n} y_k) \boldsymbol{u}(n) \right| \to 0 \quad \text{when } K \to \infty.$$

4. *Lifting the AOP property by Rokhlin cocycles*

4.1. *The main result.* Assume that $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ has the AOP property. Assume that G is an lcsc Abelian group and let $\varphi : X \to G$ be an ergodic cocycle. We are interested in a sufficient condition on T and φ to ensure that the Rokhlin extensions $T_{\varphi,\mathcal{S}}$ have the AOP property for all ergodic G-actions $\mathcal{S} = (S_g)_{g \in G}$.

THEOREM 4.1. *Assume that T has the AOP property. Assume, moreover, that for each $r \neq s$, $r, s \in \mathscr{P}$ and arbitrary $\eta \in J^e(T^r, T^s)$:*

$$\text{the group extension } (T_\varphi)^r \times T^s \text{ is ergodic (over } (T^r \times T^s, \eta)); \tag{28}$$

$$\text{the Mackey action } \mathcal{W}((\varphi^{(r)} \times \varphi^{(s)}, T^r \times T^s, \eta) \text{ is weakly mixing.} \tag{29}$$

Let $\mathcal{S} = (S_g)_{g \in G}$ be an ergodic G-action on $(Y, \mathcal{C}, \nu)$. Then $T_{\varphi,\mathcal{S}}$ has the AOP property.

4.2. *Remarks on total ergodicity.* Recall that the AOP property implies total ergodicity. We will show now that each of (28) and (29) also implies the total ergodicity of $T_{\varphi,\mathcal{S}}$ for each ergodic $\mathcal{S}$.

PROPOSITION 4.2. *Condition (28) implies the total ergodicity of $T_{\varphi,\mathcal{S}}$ for each ergodic $\mathcal{S}$.*

Proof. Suppose that for some ergodic $\mathcal{S}$ acting on $(Y, \mathcal{C}, \nu)$, $T_{\varphi,\mathcal{S}}$ is not totally ergodic. Then for some prime number r, $(T_{\varphi,\mathcal{S}})^r$ is not ergodic. Since $(T_{\varphi,\mathcal{S}})^r = (T^r)_{\varphi^{(r)},\mathcal{S}}$ and $(T_{\varphi,\mathcal{S}})^r$ is a factor of $(T^r \times T^s, \eta)_{\varphi^{(r)},\mathcal{S}}$, we obtain a contradiction to (28), cf. Remark 2.7. □

Note also that (28) implies the total ergodicity of $T_{\varphi,\mathcal{S}}$ itself. Indeed, the following lemma holds.

LEMMA 4.3. *Suppose that R is a non-singular automorphism of a probability standard Borel space $(Z, \mathcal{D}, \rho)$. Suppose that R^k is ergodic for all $k \in \mathscr{P}$. Then R is totally ergodic.*

Proof. Suppose that R^n is not ergodic for some $n > 1$ and let n be the smallest number with this property. Let A be non-trivial such that $R^n A = A$. Let $J \subset \{0, \ldots, n-1\}$ be a maximal subset such that $B := \bigcap_{j \in J} R^j A$ is of positive measure. Clearly, $R^n B = B$ and we claim that $\{R^i B\}_{i=0}^{n-1}$ is a Rokhlin tower. Indeed, all we need to show is that $\mu(B \cap R^i B) = 0$ for $i = 1, \ldots, n-1$. In other words, $J \neq J + i \bmod n$. However, if $J = J + i \bmod n$ then $R^i B = B$ which contradicts the ergodicity of R^i by the choice of n (recall that

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$1 \le i < n$). Let p be a prime dividing n. Then $B' := B \cup R^p B \cup \cdots \cup R^{n-p} B$ is a non-trivial R^p-invariant set which contradicts the ergodicity of R^p. □

PROPOSITION 4.4. *Condition (29) implies the total ergodicity of $T_{\varphi,\mathcal{S}}$ for each ergodic $\mathcal{S}$.*

Proof. Suppose that $e^{2\pi i/k}$ is an eigenvalue of $T_{\varphi,\mathcal{S}}$. Take any primes $r \neq s$ such that $r = s \bmod k$. Since φ is ergodic, so is $T_{\varphi,\mathcal{S}}$ and by Proposition 2.8, for some $\chi \in \widehat{G}$, we have $\chi \circ \varphi = e^{2\pi i/k} \cdot \xi \circ T/\xi$ for some measurable $\xi : X \to \mathbb{S}^1$. Hence

$$\chi \circ \varphi^{(r)} = e^{2\pi ir/k} \cdot \xi \circ T^r/\xi \quad \text{and} \quad \chi \circ \varphi^{(s)} = e^{2\pi is/k} \cdot \xi \circ T^s/\xi.$$

If we fix any $\eta \in J^e(T^r, T^s)$ then we obtain

$$\chi \circ \overline{\chi}(\varphi^{(r)}(x), \varphi^{(s)}(x')) = \xi \otimes \overline{\xi} \circ (T^r \times T^s)(x, x')/\xi \otimes \overline{\xi}(x, x')$$

for η-almost every $(x, x') \in X \times X$ which contradicts (29), cf. (16). □

4.3. *More on non-singular actions, cocycles and Mackey actions.* Throughout, we assume that G is an lcsc Abelian group.

LEMMA 4.5. *Assume that $\mathcal{W} = (W_g)_{g\in G}$ is a non-singular action of G on a probability standard Borel space $(C, \mathcal{J}, \kappa)$. Let $H \subset G$ be a closed subgroup and assume that the subaction $(W_h)_{h\in H}$ is ergodic. Let $\chi \in \widehat{H}$ be an L^∞-eigenvalue for $(W_h)_{h\in H}$. Then there is an L^∞-eigenvalue $\widetilde{\chi} \in \widehat{G}$ of $(W_g)_{g\in G}$ such that $\widetilde{\chi}|_H = \chi$.*

Proof. By assumption, there is a measurable function F, $|F| = 1$, such that $F \circ W_h = \chi(h) \cdot F$ for each $h \in H$. Then, for each $g \in G$, $F \circ W_g$ is also an eigenfunction corresponding to χ, so by ergodicity, $F \circ W_g = c_g \cdot F$ for a (unique) number c_g, $|c_g| = 1$. Now, $g \mapsto c_g = F \circ W_g/F$ is a measurable homomorphism, whence there exists $\widetilde{\chi} \in \widehat{G}$ such that $\widetilde{\chi}(g) = c_g$, $g \in G$. Since $c_h = \chi(h)$ for $h \in H$, the result follows. □

Immediately from Lemma 4.5 we obtain the following.

LEMMA 4.6. *Assume that $\mathcal{W} = (W_g)_{g\in G}$ is a weakly mixing non-singular G-action. Let $H \subset G$ be a closed subgroup such that the subaction $(W_h)_{h\in G}$ is ergodic. Then $(W_h)_{h\in H}$ is weakly mixing.*

We will also need the following classical result.

LEMMA 4.7. *Assume that $\mathcal{S} = (S_g)_{g\in G} \subset \mathrm{Aut}(Y, \mathcal{C}, \nu)$ is ergodic. Let $\mathcal{W} = (W_g)_{g\in G}$ be a non-singular, weakly mixing G-action on $(C, \mathcal{J}, \kappa)$. Then the G-action $\mathcal{S} \times \mathcal{W} := (S_g \times W_g)_{g\in G}$ on $(Y \times C, \mathcal{C} \otimes \mathcal{J}, \nu \otimes \kappa)$ is ergodic.*

Proof. We use Keane's criterion (see [**1**, Theorem 2.7.1]) for the ergodicity of the direct product of an ergodic finite measure-preserving action and an ergodic non-singular action. If by $\sigma_{\mathcal{S}}$ we denote the maximal spectral type of $\mathcal{S}$ on $L^2_0(Y, \mathcal{C}, \nu)$ then the product G-action $\mathcal{S} \times \mathcal{W}$ is ergodic if and only if $\sigma_{\mathcal{S}}(e(\mathcal{W})) = 0$. In our case $e(\mathcal{W}) = \{\mathbb{1}\}$ and since $\mathcal{S}$ is ergodic, $\sigma_{\mathcal{S}}$ has no atom at the trivial character. □

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Definition 4.1. [**15**] Given a non-singular G-action $\mathcal{W}$ on $(C, \mathcal{J}, \kappa)$ and a G-invariant σ-algebra $\mathcal{A} \subset \mathcal{J}$, the corresponding extension

$$\pi : (C, \mathcal{J}, \kappa, \mathcal{W}) \to (C/\mathcal{A}, \mathcal{A}, \kappa|_{\mathcal{A}}, \mathcal{W}|_{C/\mathcal{A}}) \tag{30}$$

is called *relatively finite measure-preserving* (rfmp for short) if the Radon–Nikodym derivative $d\kappa \circ W_g/d\kappa$ is $\mathcal{A}$-measurable for each $g \in G$.

Remark 4.8. [**15**] If (30) is rfmp then

$$\frac{d(\kappa \circ W_g)}{d\kappa}(c) = \frac{d(\kappa|_{\mathcal{A}} \circ W_g)}{d(\kappa|_{\mathcal{A}})}(\pi(c))$$

for κ-almost every $c \in C$ and every $g \in G$.

Remark 4.9. Let $H \subset G$ be a closed subgroup. It follows by Remark 4.8 that if a G-extension is rfmp, then the corresponding H-extension is also rfmp.

Remark 4.10. [**15**] Let $\kappa = \int_{C/\mathcal{A}} \kappa_{\overline{c}}\, d\overline{\kappa}(\overline{c})$ (where $\overline{\kappa} = \kappa|_{\mathcal{A}}$). It follows by Remark 4.8 that extension (30) is rfmp if and only if $\kappa_{W_g(\pi(c))} \circ W_g = \kappa_{\pi(c)}$ for ν-almost every $c \in C$ and for all $g \in G$.

Note that if the extension (30) is rfmp and the action $\mathcal{W}|_{C/\mathcal{A}}$ preserves the measure $\kappa|_{\mathcal{A}}$, then $\mathcal{W}$ also preserves the measure κ.

LEMMA 4.11. See [**15**, Lemma 5.3] *Assume that $\mathcal{W} = (W_g)_{g\in G}$ is an ergodic, non-singular G-action on $(C, \mathcal{J}, \kappa)$. Let $(Z, \mathcal{D})$ be a standard Borel space. Assume that γ is a probability measure on $(C \times Z, \mathcal{J} \otimes \mathcal{D})$ whose projection on the C-coordinate is ν and which is $(W_g \times \mathrm{Id})_{g\in G}$-quasi-invariant. Assume, moreover, that the extension*

$$(C \times Z, \mathcal{J} \otimes \mathcal{D}, \gamma, \mathcal{W} \times \mathrm{Id}) \to (C, \mathcal{J}, \kappa, \mathcal{W})$$

is rfmp. Then $\gamma = \kappa \otimes \rho$ for a probability measure ρ on $(Z, \mathcal{D})$.

Proof. Writing $\gamma = \int_C \gamma_c\, d\kappa(c)$ and using Remark 4.10, we obtain that $\gamma_{W_g c} = \gamma_c$ for κ-almost every $c \in C$ (and all $g \in G$). Since the map $c \mapsto \gamma_c$ is measurable, the result follows by the ergodicity of $\mathcal{W}$. □

Let $(C, \mathcal{J})$ be a standard Borel space. Denote by $\mathcal{P}(C, \mathcal{J})$ the space of probability measures on $(C, \mathcal{J})$. Given a Borel G-action $\mathcal{W}$ on $(C, \mathcal{J})$ and a G-invariant σ-algebra $\mathcal{A} \subset \mathcal{J}$ with a quasi-invariant probability measure κ, we set

$$\mathcal{P}(\mathcal{W}, \mathcal{A}, \kappa) := \{\tilde{\kappa} \in \mathcal{P}(C, \mathcal{J}) : \tilde{\kappa} \text{ is quasi-invariant, } \tilde{\kappa}|_{\mathcal{A}} = \kappa \text{ and the extension } (C, \mathcal{J}, \tilde{\kappa}, \mathcal{W}) \to (C/\mathcal{A}, \mathcal{A}, \kappa, \mathcal{W}) \text{ is rfmp}\}.$$

LEMMA 4.12. *Assume that $\mathcal{S}^{(i)} = (S_g^{(i)})_{g\in G} \subset \mathrm{Aut}(Y_i, \mathcal{C}_i, \nu_i)$ is ergodic for $i = 1, 2$. Assume, moreover, that $\mathcal{W} = (W_{(g_1,g_2)})_{(g_1,g_2)\in G\times G}$ is a non-singular, weakly mixing $G \times G$-action on a probability standard Borel space $(\widetilde{C}, \widetilde{\mathcal{J}}, \widetilde{\kappa})$. If the G-subaction $\mathcal{W}|_{\{0\}\times G}$ is ergodic then*

$$\{\rho \in \mathcal{P}((\mathcal{S}^{(1)} \otimes \mathcal{S}^{(2)}) \times \mathcal{W}, \widetilde{\mathcal{J}}, \widetilde{\kappa}) : \rho|_{\mathcal{C}_2\otimes\widetilde{\mathcal{J}}} = \nu_2 \otimes \widetilde{\kappa},\ \rho|_{\mathcal{C}_1} = \nu_1\} = \{\nu_1 \otimes \nu_2 \otimes \widetilde{\kappa}\},$$

where $\mathcal{S}^{(1)} \otimes \mathcal{S}^{(2)}$ stands for the $G \times G$-action $(g_1, g_2) \mapsto S_{g_1}^{(1)} \times S_{g_2}^{(2)}$ on the space $(Y_1 \times Y_2, \mathcal{C}_1 \otimes \mathcal{C}_2, \nu_1 \otimes \nu_2)$.

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Proof. Take $\rho \in \mathcal{P}((\mathcal{S}^{(1)} \otimes \mathcal{S}^{(2)}) \times \mathcal{W}, \widetilde{\mathcal{J}}, \widetilde{\kappa})$ such that $\rho|_{\mathcal{C}_2 \otimes \widetilde{\mathcal{J}}} = \nu_2 \otimes \widetilde{\kappa}$, $\rho|_{\mathcal{C}_1} = \nu_1$. By Lemma 4.6, the subaction $\mathcal{W}|_{\{0\}\times G}$ is weakly mixing. Therefore, since $\mathcal{S}^{(2)}$ is ergodic, it follows by Lemma 4.7 that the (non-singular) G-action $\mathcal{S}^{(2)} \times (\mathcal{W}|_{\{0\}\times G})$ on $(Y_2 \times \widetilde{C}, \nu_2 \otimes \widetilde{\kappa}) = (Y_2 \times \widetilde{C}, \rho|_{\mathcal{C}_2 \otimes \widetilde{\mathcal{J}}})$ is ergodic. In view of Remark 4.9, the G-extension $(Y_1 \times Y_2 \times \widetilde{C}, \rho, \mathrm{Id} \otimes \mathcal{S}^{(2)} \times (\mathcal{W}|_{\{0\}\times G})) \to (\widetilde{C}, \widetilde{\kappa}, W|_{\{0\}\times G})$ is rfmp. By Lemma 4.11 (applied to $\mathcal{W}' := (\mathcal{S}^{(2)} \times (\mathcal{W}|_{\{0\}\times G}), \rho|_{\mathcal{C}_2 \otimes \widetilde{\mathcal{J}}})$ and $\gamma = \rho$), we obtain that $\rho = \nu' \otimes (\nu_2 \otimes \widetilde{\kappa})$. The result follows now from the equality $\rho|_{\mathcal{C}} = \nu_1$. □

Assume that $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ is ergodic. Let $\theta : X \to J$ be a cocycle with values in an lcsc Abelian group J. Let $\mathcal{W}(\theta) = (W(\theta)_j)_{j\in J}$ denote the associated Mackey J-action on the space $(C_\theta, \mathcal{J}_\theta, \kappa_\theta)$ of ergodic components of T_θ.

Let $\mathcal{R} = (R_j)_{j\in J} \subset \mathrm{Aut}(Z, \mathcal{D}, \rho)$ be an ergodic J-action. We will need the following.

LEMMA 4.13. ([**15**, Proposition 6.1, Remark 6.2], [**33**, Proposition 2.1]) *There exists an affine isomorphism $A : \mathcal{P}(T_{\theta,\mathcal{R}}, \mathcal{B}, \mu) \to \mathcal{P}(\mathcal{W}(\theta) \times \mathcal{R}, \mathcal{J}_\theta, \kappa_\theta)$ such that whenever $\mathcal{E} \subset \mathcal{D}$ is an $\mathcal{R}$-invariant sub-σ-algebra, $\eta \in \mathcal{P}(T_{\theta,\mathcal{R}}, \mathcal{B}, \mu)$ satisfies $\eta|_{\mathcal{B}\otimes\mathcal{E}} = \mu \otimes \nu_1$, where ν_1 is $\mathcal{R}$-invariant, then*

$$A(\eta)|_{\mathcal{J}_\theta \otimes \mathcal{E}} = \kappa_\theta \otimes \nu_1. \tag{31}$$

Remark 4.14. If θ above is ergodic then $\mu \otimes \rho$ is the only $T_{\theta,\mathcal{R}}$-invariant measure whose projections on X and Z are μ and ρ, respectively [**30**].

For $\varphi, \psi : X \to G$, we define $\varphi \times \psi : X \to G \times G$ by

$$(\varphi \times \psi)(x) := (\varphi(x), \psi(x)) \quad \text{for } x \in X.$$

LEMMA 4.15. *Let $\varphi, \psi : X \to G$ be ergodic cocycles. Then:*

(i) *the G-subactions $\mathcal{W}(\varphi \times \psi)|_{\{0\}\times G}$ and $\mathcal{W}(\varphi \times \psi)|_{G\times\{0\}}$ are ergodic†;*

(ii) *$\mathcal{W}(\varphi \times \psi)$ is weakly mixing if and only if the only character $(\chi, \eta) \in \widehat{G} \times \widehat{G}$ for which there exists a measurable $\xi : X \to \mathbb{S}^1$ satisfying*

$$\chi(\varphi(x))\eta(\psi(x)) = \xi(Tx)/\xi(x)$$

is the trivial character ($\chi = \eta = 1$);

(iii) *if $\mathcal{W}(\varphi \times \psi)$ is weakly mixing, so are the subactions $\mathcal{W}(\varphi \times \psi)|_{\{0\}\times G}$ and $\mathcal{W}(\varphi \times \psi)|_{G\times\{0\}}$.*

Proof. (i) (See [**15**]) If $F \in L^\infty(C_{\varphi\times\psi}, \kappa_{\varphi\times\psi})$ satisfies $F \circ T_{\varphi\times\psi} = F$ and $F(x, g_1, g_2 + g) = F(x, g_1, g_2)$ for each $g \in G$ then $F = F(x, g_1)$ and F is constant by the ergodicity of T_φ.

(ii) Follows directly from (16) applied to $\theta = \varphi \times \psi$.

(iii) This claim follows from (i) and Lemma 4.6. □

LEMMA 4.16. *Assume that $T, T' \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ are ergodic. Fix $\eta \in J^e(T, T')$. Let $\varphi : X \to G$ ($\varphi' : X \to G$) be an ergodic cocycle over T (over T') such that $(T \times T', \eta)_\varphi$‡ and*

† In what follows, see Lemma 4.16 and the proof of Theorem 4.1 below, we will consider the situation in which T is replaced by $(T^r \times T^s, \eta)$ (with $\eta \in J^e(T^r, T^s)$) and then we will consider the group extension $(T^r \times T^s)_{\varphi^{(r)}\times\varphi^{(s)}}$. Note that then the assumption (28) will imply the validity of (i) with T replaced by $(T^r \times T^s, \eta)$.

‡ φ is treated as if it was defined on $(X \times X, \eta)$: $\varphi(x, y) = \varphi(x)$.

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$(T \times T', \eta)_{\varphi'}$ are also ergodic. Assume, moreover, that the Mackey action $\mathcal{W}(\varphi \times \varphi') = \mathcal{W}(\varphi \times \varphi', T \times T', \eta)$ associated to $\varphi \times \varphi' : X \times X \to G \times G$ is weakly mixing. Finally, let $\mathcal{S} = (S_g)_{g \in G} \subset \mathrm{Aut}(Y, \mathcal{C}, \nu)$ be an ergodic G-action. Then the set

$$J(T_{\varphi,\mathcal{S}}, T'_{\varphi',\mathcal{S}}; \eta) := \{\eta' \in J(T_{\varphi,\mathcal{S}}, T'_{\varphi',\mathcal{S}}) : \eta'|_{\mathcal{B}\otimes\mathcal{B}} = \eta\}$$

is a singleton (consisting of the relatively independent extension of η).

Proof. Notice first that $J(T_{\varphi,\mathcal{S}}, T'_{\varphi',\mathcal{S}}; \eta)$ is a subset of $\mathcal{P}((T \times T')_{\varphi\times\varphi',\mathcal{S}\otimes\mathcal{S}}, \mathcal{B} \otimes \mathcal{B}, \eta)$. Note that the coordinate σ-algebras in $(Y \times Y, \mathcal{C} \otimes \mathcal{C})$ (which are $\mathcal{S} \otimes \mathcal{S}$-invariant) yield two $(T \times T')_{\varphi\times\varphi',\mathcal{S}\otimes\mathcal{S}}$-invariant σ-algebras $\mathcal{B} \otimes \mathcal{B} \otimes \mathcal{E}_i$, $i = 1, 2$. More precisely, $J(T_{\varphi,\mathcal{S}}, T'_{\varphi',\mathcal{S}}; \eta)$ can be identified (cf. Remark 4.14) with

$$\mathcal{P}_1 := \{\eta' \in \mathcal{P}((T \times T')_{\varphi\times\varphi',\mathcal{S}\otimes\mathcal{S}}, \mathcal{B} \otimes \mathcal{B}, \eta) : \eta'|_{\mathcal{B}\otimes\mathcal{B}\otimes\mathcal{E}_i} = \eta \otimes \nu,\ i = 1, 2\} \tag{32}$$

and it is enough to show that $\mathcal{P}_1$ is a singleton. But, in view of Lemma 4.13, the affine isomorphism

$$A : \mathcal{P}(T_{\varphi\times\varphi',\mathcal{S}\otimes\mathcal{S}}, \mathcal{B} \otimes \mathcal{B}, \eta) \to \mathcal{P}(\mathcal{W}(\varphi \times \varphi') \times (\mathcal{S} \otimes \mathcal{S}), \mathcal{J}_{\varphi\times\varphi'}, \kappa_{\varphi\times\varphi'})$$

satisfies $A(\eta')|_{\mathcal{J}_{\varphi\times\varphi'}\otimes\mathcal{E}_i} = \kappa_{\varphi\times\varphi'} \otimes \nu$, $i = 1, 2$, provided that $\eta' \in \mathcal{P}_1$. Moreover, it is not hard to see that the assumptions of Lemma 4.12 are fulfilled (cf. Lemma 4.15(i)). The result follows. □

4.4. *Proof of Theorem 4.1.* We want to prove the AOP property of $T_{\varphi,\mathcal{S}}$ and to obtain it, we need to show that (see (24))

$$\limsup_{r\neq s, r,s\in\mathscr{P}, r,s\to\infty} \sup_{\kappa\in J^e((T_{\varphi,\mathcal{S}})^r,(T_{\varphi,\mathcal{S}})^s)} \left| \int_{X\times Y\times X\times Y} f_1 \otimes g_1 \otimes f_2 \otimes g_2 \, d\kappa \right| = 0$$

if $f_i \otimes g_i \in L^2_0(X \times Y, \mu \otimes \nu)$, $i = 1, 2$. In fact, since T is assumed to enjoy the AOP property, we can assume that at least one of the functions g_i is in $L^2_0(Y, \mathcal{C}, \nu)$. But then the assumptions of Theorem 4.1 and Lemma 4.16 applied to T^r, T^s, $\varphi^{(r)}$ and $\varphi^{(s)}$ instead of T, T', φ and φ' will tell us that the only members in $J^e((T_{\varphi,\mathcal{S}})^r, (T_{\varphi,\mathcal{S}})^s)$ are the relatively independent extensions of ergodic joinings in $J(T^r, T^s)$. The result follows from (18). □

4.5. *A special case when T is a totally ergodic rotation.* Let us consider now the special case of Theorem 4.1 in which T is a totally ergodic rotation. Then T has the AOP property [**6**]. With no loss of generality, we can assume that X is a compact metric monothetic group, $Tx = x + \alpha$, where $\{n\alpha : n \in \mathbb{Z}\}$ is dense in X. The measure μ is then λ_X the Haar measure on X.

We first repeat the argument from [**27**] giving rise to the full description of ergodic joinings between T^r and T^s with $(r, s) = 1$. For this aim, choose $a, b \in \mathbb{Z}$ so that $ar + bs = 1$. Fix $u \in X$ and consider

$$A_u := \{(x, y + u) \in X \times X : sx = ry\}.$$

Then the map $V_u(x, y + u) = ax + by$ settles a topological isomorphism of the action of $T^r \times T^s$ on A_u and of T on X with the latter action being uniquely ergodic. Hence, we described $J^e(T^r, T^s)$. Moreover,

$$X \times X \times G \times G = \bigcup_{u\in X} A_u \times G \times G$$

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is a decomposition of $X \times X \times G \times G$ into pairwise disjoint closed sets invariant under $(T^r \times T^s)_{\varphi^{(r)} \times \varphi^{(s)}}$. Moreover, $(T^r \times T^s)_{\varphi^{(r)} \times \varphi^{(s)}}|_{A_u \times G \times G}$ is topologically isomorphic to $T_{\varphi^{(r)}(r\cdot) \times \varphi^{(s)}(s\cdot + u)}$. Indeed, the isomorphism is given by $J = J_u : A_u \times G \times G \to X \times G \times G$, $J(x, y + u, g, h) = (ax + by, g, h)$. Finally, notice that if $\eta \in J^e(T^r, T^s)$ then $(T^r \times T^s, \eta)_{\varphi^{(r)}}$ has, via the new coordinates given by J_u, the form $T_{\varphi^{(r)}(r\cdot)}$. In view of Theorem 4.1, we hence obtain the following result.

COROLLARY 4.17. *Assume that $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ is a totally ergodic ergodic rotation on a compact metric monothetic group X. Let G be an lcsc Abelian group. Assume that $\varphi : X \to G$ is a cocycle such that:*

$$\varphi^{(r)}(r\,\cdot) \text{ is ergodic for each } r \in \mathscr{P}; \tag{33}$$

$$\text{the Mackey action } W(\varphi^{(r)}(r\,\cdot) \times \varphi^{(s)}(U(s\cdot))) \text{ is weakly mixing} \tag{34}$$

for each $r \neq s$, $r, s \in \mathscr{P}$ and arbitrary $U \in C(T)$†. Let $\mathcal{S} = (S_g)_{g \in G} \subset \mathrm{Aut}(Y, \mathcal{C}, \nu)$ be an ergodic G-action. Then $T_{\varphi, \mathcal{S}}$ has the AOP property.

5. *Smooth cocycles over a generic irrational rotation*

5.1. *Smooth Anzai skew products having the AOP property.* In this section our aim will be to prove the following.

PROPOSITION 5.1. *Assume that $f \in C^{1+\delta}(\mathbb{T})$, $\int_{\mathbb{T}} f \, d\lambda_{\mathbb{T}} = 0$, for some $\delta > 0$ and it is not a trigonometric polynomial. Then, there exists a dense G_δ set of α such that, for $Tx = x + \alpha$, we have*

$$f^{(r)}(r\cdot) \text{ is ergodic for each } r \in \mathscr{P}; \tag{35}$$

$$W(f^{(r)}(r\cdot) \times f^{(s)}(s\cdot + h)) \text{ is weakly mixing for } r < s,\ r, s \in \mathscr{P},\ h \in \mathbb{T}. \tag{36}$$

As a consequence of Proposition 5.1 and Corollary 4.17, we obtain the following.

COROLLARY 5.2. *Under the assumptions of Proposition 5.1, for each ergodic flow $\mathcal{S} = (S_t)_{t \in \mathbb{R}} \subset \mathrm{Aut}(Y, \mathcal{C}, \nu)$, the automorphism $T_{\varphi, \mathcal{S}}$ has the AOP property.*

Combining Corollary 5.2 with Proposition 2.11, we obtain the following.

COROLLARY 5.3. *There are relatively weakly mixing extensions $T_{\varphi, \mathcal{S}}$ of an irrational rotation T which have the AOP property and are disjoint from all weakly mixing transformations.*

5.2. *Proof of Proposition 5.1.* Let $f : \mathbb{R} \to \mathbb{R}$ (periodic of period 1) be in $L^2(\mathbb{T})$, $f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x}$. Assume that f has zero mean and its Fourier transform is absolutely summable. Let

$$\begin{aligned} f_m(x) &:= f(x) + f\left(x + \frac{1}{m}\right) + \cdots + f\left(x + \frac{m-1}{m}\right) \\ &= m \sum_{l=-\infty}^{\infty} \widehat{f}(lm) e^{2\pi i l m x} \quad \text{for } m \geq 1. \end{aligned}$$

Recall the following ergodicity criterion.

† We recall that the centralizer of an ergodic rotation on a compact metric group consists of all other rotations.

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THEOREM 5.4. [**2**, Theorem 5.1] *Suppose that there exist a sequence $(q_n) \subset \mathbb{N}$ and a constant $C > 0$ such that:*
- $q_n \sum_{l=-\infty}^{\infty} |\widehat{f}(lq_n)| \le C \| f_{q_n} \|_{L^2}$ *for* $n \ge 1$;
- $0 < \| f_{q_n} \|_{L^2} \to 0$.

Then there exists a dense G_δ set of irrational numbers α such that the corresponding group extension $T_f : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$, where $Tx = x + \alpha$, is ergodic.

We will now prove a modified version of Theorem 5.4 (the proof follows the lines of the proof of Theorem 5.1 in [**2**]).

THEOREM 5.5. *Under the assumptions of Theorem 5.4, there exists a dense G_δ set of irrational numbers α such that the group extensions $T_{f^{(r)}(r\cdot)}$, where $Tx = x + \alpha$, are ergodic for all $r \ge 1$.*

We will need the following lemma.

LEMMA 5.6. [**2**] *Given $C > 0$, there exist numbers K, L, M such that $0 < K < L < 1$, $0 < M < 1$ and for each $h \in L^4(\mathbb{T})$ if $\|h\|_{L^4} \le C\|h\|_{L^2}$ then*

$$\mu(\{x \in \mathbb{T} : K\|h\|_{L^2} \le |h(x)| \le L\|h\|_{L^2}\}) > M.$$

Recall also (see e.g. [**28**]) that given an infinite set $\{q_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$ and a positive real valued function $R = R(q_n)$ the set

$$\mathcal{A} = \{\alpha \in [0, 1) : \text{for infinitely many } n \text{ we have } |\alpha - p_n/q_n| < R(q_n), \text{ where } p_n/q_n \text{ are convergents of } \alpha\} \text{ is residual.} \tag{37}$$

Proof of Theorem 5.5. For $n \ge 1$, let

$$\widetilde{f}_n(x) := q_n \sum_{l=-\infty}^{\infty} \widehat{f}(lq_n) e^{2\pi i l x}.$$

Fix $r \ge 1$ and let $g(x) := f^{(r)}(rx)$. Then for all m,

$$g^{(m)}(x) = f^{(rm)}(rx). \tag{38}$$

Moreover, in view of the assumptions of the theorem:
- $\widetilde{f}_n(q_n x) = f_{q_n}(x)$, whence $\|\widetilde{f}_n\|_{L^2} = \|f_{q_n}\|_{L^2} \to 0$;
- $\|\widetilde{f}_n\|_{L^4} \le \|\widetilde{f}_n\|_{L^\infty} \le q_n \sum_{l=-\infty}^{\infty} |\widehat{f}(lq_n)| \le C\|f_{q_n}\|_{L^2} = C\|\widetilde{f}_n\|_{L^2}$.

Therefore, by Lemma 5.6, for all $n \ge 1$,

$$\mu(\{x \in \mathbb{T} : K\|\widetilde{f}_n\|_{L^2} \le |\widetilde{f}_n(x)| \le L\|\widetilde{f}_n\|_{L^2}\}) > M.$$

Let $\{D_R : R \ge 1\}$ be a family of disjoint, closed intervals of the form $D_R = [c_R, d_R]$, where $d_R/c_R = 100(L/K)$ and $0 < d_R \to 0$. Let $(D'_n)_{n\ge1} \subset \{D_R : R \ge 1\}$ be a sequence such that for all $R \ge 1$, we have $\#\mathcal{N}_R = \infty$, where

$$\mathcal{N}_R = \{n \in \mathbb{N} : D'_n = D_R\}.$$

Fix $n \in \mathbb{N}$. Choose $k_n, l_n \in \mathbb{N}$ so that

$$[k_n K\|\widetilde{f}_{l_n}\|_{L^2}, k_n L\|\widetilde{f}_{l_n}\|_{L^2}] \subset \widetilde{D}'_n \subsetneq D'_n,$$

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where $\widetilde{D}'_n$ is a strict closed subinterval of D'_n. This gives us two sequences $(k_n)_{n\geq 1}$ and $(l_n)_{n\geq 1}$, such that

$$\mu(\{x \in \mathbb{T} : |k_n \widetilde{f}_{l_n}(x)| \in \widetilde{D}'_n\}) \geq M.$$

We fix now $R \geq 1$. We claim that the set of numbers α such that there exists an infinite subset $\mathcal{N}_R^\alpha \subset \mathcal{N}_R$ satisfying the two conditions

(i) $\|k_n q_{l_n} \alpha\| \xrightarrow[n\in\mathcal{N}_R^\alpha]{} 0$,

(ii) for $n \in \mathcal{N}_R^\alpha$, we have

$$\mu(\{y \in I : |f^{(k_n q_{l_n})}(T^{ik_n q_{l_n}} y)| \in D'_n \text{ and } \operatorname{sgn}(f^{(k_n q_{l_n})}(T^{ik_n q_{l_n}} y)) \text{ is the same for } 0 \leq i < r\}) \geq M|I|$$

for every interval I such that $|I| = t/q_{l_n}$, $t = 1, \ldots, q_{l_n}$,

is a dense G_δ subset of $\mathbb{T}$. In view of (37), it suffices to show that the above conditions describe a speed of approximation of α by rational numbers. This is clearly the case for (i). We will now deal with (ii). For $0 \leq i < r$, we have

$$\begin{aligned}
|f^{(kq)}(T^{ikq} y) - k f_q(y)| &\leq \sum_{j=0}^{k-1} |f^{(q)}(y + ikq\alpha + jq\alpha) - f_q(y)| \\
&= \sum_{j=0}^{k-1} \left| \sum_{w=0}^{q-1} f(y + ikq\alpha + jq\alpha + w\alpha) - \sum_{w'=0}^{q-1} f\left(y + \frac{w'}{q}\right) \right| \\
&= \sum_{j=0}^{k-1} \left| \sum_{w=0}^{q-1} \left(f(y + ikq\alpha + jq\alpha + w\alpha) - f\left(y + w\frac{p}{q}\right)\right) \right| \\
&\leq \sum_{j=0}^{k-1} \sum_{w=0}^{q-1} \omega\left(f, ikq\alpha + jq\alpha + w\left(\alpha - \frac{p}{q}\right)\right),
\end{aligned}$$

where $p = p_{l_n}$, $q = q_{l_n}$ (recall that p_{l_n} and q_{l_n} are relatively prime) and $\omega(f, \cdot)$ is the modulus of continuity of f. With k and q fixed, the above quantity depends only on the distance between α and p/q.

We denote the obtained dense G_δ set of numbers α by Y_R and take $Y = \bigcap_{R=1}^\infty Y_R$. This set is again a dense G_δ. Pick $\alpha \in Y$ and fix $R \geq 1$. Let J be an interval such that $|J| = t/rq_{l_n}$ for some $t = 1, \ldots, q_{l_n}$. Then, for $I := rJ$, we have $|I| = t/q_{l_n}$, i.e. condition (ii) holds. Therefore

$$\begin{aligned}
&\mu(\{x \in J : |f^{(rk_n q_{l_n})}(rx)| \in rD'_n\}) \\
&\quad \geq \mu(\{x \in J : |f^{(k_n q_{l_n})}(T^{ik_n q_{l_n}} rx)| \in D'_n \text{ and } \operatorname{sgn}(f^{(k_n q_{l_n})}(T^{ik_n q_{l_n}} rx)) \text{ is the same for } 0 \leq i < r\}) \\
&\quad = \frac{1}{r} \cdot \mu(\{y \in I : |f^{(k_n q_{l_n})}(T^{ik_n q_{l_n}} y)| \in D'_n \text{ and } \operatorname{sgn}(f^{(k_n q_{l_n})}(T^{ik_n q_{l_n}} y)) \text{ is the same for } 0 \leq i < r\}) \\
&\quad \geq \frac{1}{r} \cdot M|I| = M|J|. \qquad (39)
\end{aligned}$$

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Suppose now that $E(g) = \lambda\mathbb{Z}$ for some $\lambda \in \mathbb{R}$. Let $R \geq 1$ be large enough, so that the set $K_R := r(D_R \cup (-D_R))$ is disjoint from $\lambda\mathbb{Z}$. By Proposition 2.2, there exists a Borel set B with $\mu(B) > 0$ and such that for all $m \geq 1$, we have

$$\mu(B \cap T^{-m}B \cap \{x \in \mathbb{T} : g^{(m)}(x) \in K_R\}) = 0.$$

Therefore, taking $m = k_n q_{l_n}$, in view of (38), we obtain

$$\mu(B \cap T^{-k_n q_{l_n}} B \cap \{x \in \mathbb{T} : f^{(rk_n q_{l_n})}(rx) \in K_R\}) = 0. \tag{40}$$

If now $n \in \mathcal{N}_R^\alpha$ then $\mu(B \triangle T^{k_n q_{l_n}} B) \to 0$ by condition (i) ($k_n q_{l_n}$ is a rigidity sequence for T). If x is a density point of B then for an interval J, containing x, with $|J| = t/rq_{l_n}$, for some $t = 1, \ldots, q_{l_n}$, we have $\mu(B \cap J) > (1 - M/2)|J|$. In view of (39), it follows that there exists a measurable subset $A_n \subset B$ with $\mu(A_n) \geq (M/2)\mu(B)$ such that

$$|f^{(rk_n q_{l_n})}(rx)| \in rD'_n \quad \text{for } x \in A_n.$$

Let $\widetilde{A}_n := A_n \cap T^{-k_n q_{l_n}} B$. Then, by condition (i), for n large enough, $\mu(\widetilde{A}_n) \geq (M/4)\mu(B)$. Hence

$$\mu(B \cap T^{-k_n q_{l_n}} B \cap \{x \in \mathbb{T} : f^{(rk_n q_{l_n})}(rx) \in K_R\}) \geq \mu(\widetilde{A}_n) \geq \frac{M}{4}\mu(B).$$

This contradicts (40) and completes the proof. □

Remark 5.7. It was shown in [**27**] that the assumptions of Theorem 5.4 are satisfied for each zero mean function $f \in C^{1+\delta}(\mathbb{T})$, $\delta > 0$, which is not a trigonometric polynomial.

THEOREM 5.8. [**27**, Corollary 2.5.6] *Assume that $f \in C^{1+\delta}(\mathbb{T})$ for some $\delta > 0$ and it is not a trigonometric polynomial. Then, for a dense G_δ set of $\alpha \in \mathbb{T}$, we have: for arbitrary $A, B \in \mathbb{R}$, $|A| + |B| > 0$, any relatively prime numbers $r, s \in \mathbb{N}$, any $|c| = 1$ and $h \in \mathbb{R}$, the cocycle*

$$ce^{2\pi i(Af^{(r)}(r\cdot) + Bf^{(s)}(s\cdot + h))}$$

considered over $Tx = x + \alpha$, is not a T-coboundary.

Proposition 5.1 follows immediately by Theorems 5.5, 5.8 and Lemma 4.15(ii).

6. *Applications—average orthogonality on short intervals*

6.1. *Universal sequences (a_n) for the strong MOMO property along (a_n).* Assume that T is a uniquely ergodic homeomorphism of a compact metric space X with the unique T-invariant measure μ (hence $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ is ergodic). Assume that G is an lcsc Abelian group and let $\varphi : X \to G$ be continuous. Assume, moreover, that $\mathcal{S} = (S_g)_{g \in G}$ is a continuous† uniquely ergodic G-action on a compact metric space Y (with the unique invariant measure ν). It follows that $T_{\varphi,\mathcal{S}}$ is a homeomorphism of $X \times Y$. According to Lemma 4.13, the simplex of invariant measures for $T_{\varphi,\mathcal{S}}$ is affinely isomorphic to $\mathcal{P}(\mathcal{W}(\varphi) \times \mathcal{S}, \mathcal{J}_\varphi, \kappa_\varphi)$. It follows that if $\mathcal{W}(\varphi)$ is trivial then $T_{\varphi,\mathcal{S}}$ will be uniquely ergodic. We hence proved the following.

† I.e. the map $(g, x) \to S_g x$ is continuous.

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PROPOSITION 6.1. (Cf. [**30**]) *If T, $\mathcal{S}$ are uniquely ergodic, $\varphi : X \to G$ is continuous and ergodic then $T_{\varphi,\mathcal{S}}$ is a uniquely ergodic homeomorphism of $X \times Y$.*

Using Proposition 3.2, we will now obtain the existence of universal sequences $(a_n) \subset G$ for which the strong MOMO property along (a_n) holds for all continuous uniquely ergodic G-actions. Such sequences will be ergodic by Proposition 2.16.

THEOREM 6.2. *Assume that T is uniquely ergodic, $\varphi : X \to G$ is continuous and ergodic, and the other assumptions in Theorem 4.1 are also satisfied. Let $x \in X$ and set $a_n := \varphi^{(n)}(x)$, $n \geq 0$. Then each continuous uniquely ergodic G-action $\mathcal{S} = (S_g)_{g\in G}$ on a compact metric space Y satisfies the strong MOMO property along (a_n).*

In particular, each sequence $(f(S_{a_n}x))$ (with $f \in C(Y) \cap L^2_0(Y, \mathcal{C}, \nu)$) is orthogonal to an arbitrary multiplicative function $\boldsymbol{u}$, $|\boldsymbol{u}| \leq 1$.

6.2. *From universal sequences for flows to universal sequences for automorphisms.* So far the only cocycles satisfying the assumptions of Theorem 4.1 (and the assumptions of Theorem 6.2) are smooth cocycles over some irrational rotations. We will now show how to use them to obtain integer-valued sequences universal for the strong MOMO property for uniquely ergodic homeomorphisms.

Remark 6.3. Note that if R is a uniquely ergodic homeomorphism of a compact metric space Z, then the suspension $\widetilde{R}$ is a continuous flow on the compact metric space $\widetilde{Z}$ and the flow is also uniquely ergodic†.

Let T be a uniquely ergodic homeomorphism of a compact metric space X and let $\varphi : X \to \mathbb{R}$ be continuous and ergodic. Fix $x \in X$, set $a_n = \varphi^{(n)}(x)$. Assume that we have the strong MOMO property along (a_n) for the suspension flow $\widetilde{R}$. If we fix $f \in C(X)$, $\int f\, d\mu = 0$, and some $\delta > 0$ then we can find $F \in C(\widetilde{Z})$, $\int F\, d(\mu \otimes \lambda_{[0,1]}) = 0$, $\|F\|_\infty = \|f\|_\infty$, and

$$F(z, s) = f(z) \quad \text{whenever } \delta \leq s \leq 1 - \delta. \tag{41}$$

Indeed, it is enough to set

$$F(z, s) = \begin{cases} \dfrac{s}{\delta} f(z) & \text{if } 0 \leq s \leq \delta, \\ f(z) & \text{if } \delta \leq s \leq 1 - \delta, \\ \dfrac{1}{\delta}(1 - s) f(z) & \text{if } 1 - \delta \leq s \leq 1. \end{cases}$$

Choose $(z_k) \subset Z$ arbitrarily. Now, we have

$$\frac{1}{b_{K+1}} \sum_{k\leq K} \left| \sum_{b_k \leq n < b_{k+1}} F(\widetilde{R}_{a_n}(z_k, 0))\boldsymbol{u}(n) \right| \to 0.$$

That is

$$\frac{1}{b_{K+1}} \sum_{k\leq K} \left| \sum_{b_k \leq n < b_{k+1}} F(R^{[a_n]}(z_k, \{a_n\}))\boldsymbol{u}(n) \right| \to 0. \tag{42}$$

† The metric on $Z \times [0, 1]/\sim$ is the quotient metric of the natural product metric on $Z \times [0, 1]$. With this metric, the map $((x, s), t) \mapsto (R^{[t]}x, \{s + t\})$ becomes continuous.

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However, the sequence (a_n) is uniformly distributed mod 1 (see Remark 2.18), hence

$$\frac{1}{b_{K+1}}|\{0 \le n < b_{K+1} : \{a_n\} \in [0, \delta] \cup [1-\delta, 1)\}| < 3\delta$$

if K is large enough. Hence, taking into account additionally (41) and (42), we obtain that

$$\frac{1}{b_{K+1}} \sum_{k \le K} \left| \sum_{b_k \le n < b_{k+1}} f(R^{[a_n]} z_k) \boldsymbol{u}(n) \right| \to 0. \tag{43}$$

We have proved the following.

COROLLARY 6.4. *Assume that T is uniquely ergodic, $\varphi : X \to \mathbb{R}$ is continuous and ergodic, and the other assumptions in Theorem 4.1 are also satisfied. Let $x \in X$ and set $a_n = \varphi^{(n)}(x)$, $n \ge 0$. Then each uniquely ergodic homeomorphism R of a compact metric space Z satisfies the strong MOMO property along* $([a_n])$.

In particular, each sequence $(f(R^{[a_n]}z))$ (with $f \in C(Z) \cap L^2_0(Z, \mathcal{D}, \rho)$) is orthogonal to an arbitrary multiplicative function $\boldsymbol{u}$, $|\boldsymbol{u}| \le 1$.

7. *Affine cocycles over irrational rotations. AOP and the strong MOMO property*

We continue to study the AOP property for automorphisms of the form $T_{\varphi,\mathcal{S}}$, where T is an irrational rotation. From now on, we assume that $\varphi(x) = x - \frac{1}{2} = \{x\} - \frac{1}{2}$. This map is considered as a cocycle over an irrational rotation by α. (Our analysis is true for a general (zero mean) affine cocycle $\varphi(x) = mx + c$ with $0 \ne m \in \mathbb{Z}$ but for simplicity of notation we will only consider the case $m = 1$.)

The cocycle $\varphi : X \to \mathbb{R}$ is not continuous, so the first task will be to show how to bypass this inconvenience. In fact, φ is a continuous cocycle by a slight extension of the base—in §7.1 we will show that studying φ over this extension still yields the results we are interested in.

7.1. *Lifting generic points in the Cartesian square.*

Assume that T and $\widehat{T}$ are uniquely ergodic homeomorphisms of compact metric spaces X and $\widehat{X}$, with the unique invariant measures μ and $\widehat{\mu}$, respectively. Assume, moreover, that $\pi : \widehat{X} \to X$ is continuous and $T \circ \pi = \pi \circ \widehat{T}$, in particular, (X, T) is a topological factor of $(\widehat{X}, \widehat{T})$.

PROPOSITION 7.1. *Assume that $(\widehat{X}, \widehat{\mu}, \widehat{T})$ and (X, μ, T) are measure-theoretically isomorphic. Assume that T is measure-theoretically coalescent†. Finally, assume that each pair $(x, y) \in X \times X$ is generic for an ergodic $T \times T$-invariant measure. Then each pair $(\widehat{x}, \widehat{y}) \in \widehat{X} \times \widehat{X}$ is generic for some $\widehat{T} \times \widehat{T}$-invariant measure. Moreover, if $(\widehat{x}, \widehat{y}) \in \widehat{X} \times \widehat{X}$ is generic for $\widehat{\rho}$ then $(\widehat{X} \times \widehat{X}, \widehat{\rho}, \widehat{T} \times \widehat{T})$ is isomorphic to $(X \times X, \rho, T \times T)$, where $\rho = (\pi \times \pi)_*(\widehat{\rho})$.*

Proof. Take $(\widehat{x}, \widehat{y}) \in \widehat{X} \times \widehat{X}$ and let $(x, y) := (\pi(\widehat{x}), \pi(\widehat{y}))$. Assume that

$$\frac{1}{N_k} \sum_{n \le N_k} \delta_{(\widehat{T} \times \widehat{T})^n (\widehat{x}, \widehat{y})} \to \widehat{\rho} \quad \text{when } k \to \infty. \tag{44}$$

† T is called coalescent if each measure-preserving map W on (X, μ) commuting with T is invertible. Equivalently, for each factor $\mathcal{A} \subset \mathcal{B}$ if $T|_{\mathcal{A}}$ is isomorphic to T then $\mathcal{A} = \mathcal{B}$.

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Since (x, y) is generic, for some measure ρ, we have

$$\frac{1}{N}\sum_{n\le N}\delta_{(T\times T)^n(x,y)}\to\rho \quad \text{when } N\to\infty.$$

By the continuity of π, for $f, g\in C(X)$, we obtain

$$\begin{aligned}\int_{X\times X} f\otimes g\, d((\pi\times\pi)_*(\widehat{\rho})) &= \int_{\widehat{X}\times\widehat{X}}(f\circ\pi)\otimes(g\circ\pi)\, d\widehat{\rho}\\ &= \lim_{k\to\infty}\frac{1}{N_k}\sum_{n\le N_k} f\circ\pi(\widehat{T}^n\widehat{x})\cdot g\circ\pi(\widehat{T}^n\widehat{y})\\ &= \lim_{k\to\infty}\frac{1}{N_k}\sum_{n\le N_k} f(T^n x)\cdot g(T^n y)=\int_{X\times X} f\otimes g\, d\rho.\end{aligned}$$

Hence $\rho=(\pi\times\pi)_*(\widehat{\rho})$. By coalescence, $\pi^{-1}(\mathcal{B}(X))=\mathcal{B}(\widehat{X})$ (modulo $\widehat{\mu}$), so there exists a T-invariant set $\widehat{X}_0\subset\widehat{X}$ such that $\widehat{\mu}(\widehat{X}_0)=1$ and $\pi|_{\widehat{X}_0}$ is 1-1. It follows that

$$\widehat{\rho}(\widehat{X}_0\times\widehat{X}_0)=\widehat{\rho}((\widehat{X}\times\widehat{X}_0)\cap(\widehat{X}_0\times\widehat{X}))=1$$

since $\widehat{\rho}(\widehat{X}\times\widehat{X}_0)=\widehat{\rho}(\widehat{X}_0\times\widehat{X})=\widehat{\mu}(\widehat{X}_0)=1$ ($(\widehat{X},\widehat{T})$ is uniquely ergodic, so the projections of $\widehat{\rho}$ on both coordinates are equal to $\widehat{\mu}$). Since π is 1-1 on $\widehat{X}_0$, $\pi\times\pi$ is 1-1 on $\widehat{X}_0\times\widehat{X}_0$. Moreover, for Borel sets $\widehat{A}, \widehat{B}\subset\widehat{X}$, we have

$$\begin{aligned}\widehat{\rho}(\widehat{A}\times\widehat{B}) &= \widehat{\rho}((\widehat{A}\cap\widehat{X}_0)\times(\widehat{B}\cap\widehat{X}_0))\\ &= \widehat{\rho}(\pi^{-1}(\pi(\widehat{A}\cap\widehat{X}_0))\times\pi^{-1}(\pi(\widehat{B}\cap\widehat{X}_0)))=\rho(\pi(\widehat{A}\cap\widehat{X}_0)\times\pi(\widehat{B}\cap\widehat{X}_0)).\end{aligned}$$

As $\widehat{X}_0$ depends only on $\widehat{\mu}$, it follows that $\widehat{\rho}$ does not depend on the choice of (N_k) in (44), i.e. $(\widehat{x},\widehat{y})$ is generic. Moreover, $(X\times X,\rho,T\times T)$ is isomorphic to $(\widehat{X}\times\widehat{X},\widehat{\rho},\widehat{T}\times\widehat{T})$. In particular, $\widehat{\rho}$ is ergodic. □

Remark 7.2. Proposition 7.1 remains true for Cartesian products of two different systems (the proof is the same). More precisely, we consider uniquely ergodic extensions $\widehat{T}_i$ of T_i, $i=1, 2$, where both T_1 and T_2 are coalescent and isomorphic to $\widehat{T}_1$ and $\widehat{T}_2$, respectively. In particular, we can apply this to powers T^r, T^s knowing that each measure-preserving map commuting with T^k commutes with T, for each $k\in\mathbb{Z}\setminus\{0\}$.

Coming back to our affine situation, we consider $X=\mathbb{T}$, $Tx=x+\alpha$, an irrational rotation. Then we make $\varphi:\mathbb{T}\to\mathbb{R}$, $\varphi(x)=x-\frac{1}{2}$ continuous by considering φ in the coordinates given by a Sturmian system $\widehat{X}\subset\{0, 1\}^{\mathbb{Z}}$ considered with the shift $\widehat{T}$. More precisely, $\widehat{X}$ is given by the closure of the names of 0 for the partition $[0, 1/2)$, $[1/2, 1)$ of the circle, see [**8**] for details. Now, the Sturmian system $(\widehat{X},\widehat{T})$ is uniquely ergodic (with the unique invariant measure $\widehat{\mu}$). Set $\widehat{\varphi}(\widehat{x}):=\varphi(\pi(\widehat{x}))$ which is a continuous cocycle on $\widehat{X}$. Note that:

- the map $(\widehat{x}, y)\mapsto(\pi(\widehat{x}), y)$ settles a natural factor map from $\widehat{T}_{\widehat{\varphi},\mathcal{S}}$ to $T_{\varphi,\mathcal{S}}$, for each $\mathcal{S}$;
- $\widehat{T}_{\widehat{\varphi},\mathcal{S}}$ is uniquely ergodic whenever $\mathcal{S}$ is uniquely ergodic (in view of Proposition 2.10), and the map $(\widehat{x}, y)\mapsto(\pi(\widehat{x}), y)$ becomes a measure-theoretic isomorphism.

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Notice that

$$\widehat{\varphi}^{(n)}(\widehat{x}) = \varphi^{(n)}(\pi(\widehat{x})). \tag{45}$$

Using Proposition 7.1, we can also see that if (x, x) is a generic point for an ergodic $T^r \times T^s$-invariant measure ρ then so is $(\widehat{x}, \widehat{x})$ for $\widehat{T}^r \times \widehat{T}^s$ ($\pi(\widehat{x}) = x$). Moreover, if $((x, y), (x, y))$ is generic for $(T_{\varphi,\mathcal{S}})^r \times (T_{\varphi,\mathcal{S}})^s$ for a unique measure $\widetilde{\rho}$ satisfying $\widetilde{\rho}|_{X\times X} = \rho$, also $((\widehat{x}, y), (\widehat{x}, y))$ is generic for $(\widehat{T}_{\widehat{\varphi},\mathcal{S}})^r \times (\widehat{T}_{\widehat{\varphi},\mathcal{S}})^s$.

7.2. *Ergodicity of* $\varphi^{(r)}(r\cdot)$. We assume that $\varphi(x) = \{x\} - \frac{1}{2}$. Then for each $r \geq 1$, we have

$$\begin{aligned}\varphi^{(r)}(rx) &= \sum_{j=0}^{r-1}\left(\{rx + j\alpha\} - \frac{1}{2}\right)\\ &= \sum_{j=0}^{r-1}\left(rx + j\alpha - [rx - j\alpha] - \frac{1}{2}\right) = r^2x - \sum_{j=0}^{r-1}[rx + j\alpha] + \frac{r(r-1)}{2}\alpha.\end{aligned}$$

It follows that

$$\varphi^{(r)}(rx) = \Theta_r(x) - \Psi_r(x), \tag{46}$$

where

$$\Theta_r x = r^2x - \frac{r^2}{2}, \quad \Psi_r(x) = \sum_{j=1}^{r-1}[rx + j\alpha] - \frac{r(r-1)}{2}(\alpha + 1). \tag{47}$$

Note that the first cocycle is real valued, while the second one takes values in $(r(r-1)/2)\alpha + \mathbb{Z}$. Each of them is of bounded variation and has zero mean.

Remark 7.3. Our main idea is now to consider $\Theta_r^{(q_n)}$ and $\Psi_r^{(q_n)}$, $n \geq 1$, where (q_n) is the sequence of denominators of α. Then, by the Denjoy–Koksma inequality, the distributions of the above random variables are contained in a bounded subset of $\mathbb{R}$. It is well known, e.g. [**32**, **35**], that any limit point of $(\Theta_r^{(q_n)})_*$ is an absolutely continuous measure. The values of $\Psi_r^{(q_n)}$, by passing to a subsequence if necessary, are taken in a fixed finite set. In other words, we are in the situation of Lemma 2.4.

Consider the set $\{\{j\alpha\} : j = 0, 1, \ldots, r-1\}$. Then, order this set so that

$$1 - \{j_0\alpha\} < 1 - \{j_2\alpha\} < \cdots < 1 - \{j_{r-1}\alpha\}.$$

It follows that Ψ_r is a step cocycle and the discontinuities of Ψ_r are the points

$$\frac{k - \{j_0\alpha\}}{r} < \frac{k - \{j_1\alpha\}}{r} < \cdots < \frac{k - \{j_{r-1}\alpha\}}{r}, \quad k = 1, \ldots, r.$$

At each of the above points the jump is equal to 1 or to $-(r-1)$.

Remark 7.4. Assume that α has bounded partial quotients. The crucial observation is that in this case the points $(k - \{j_\ell\alpha\})/r$, $k = 1, \ldots, r$, $\ell = 0, 1, \ldots, r-1$ are badly approximated by α, e.g. [**12**, **20**]. In other words, the intervals whose endpoints are the consecutive discontinuities of the cocycle $\Psi_r^{(q_n)}$ are all of length 'comparable' with $1/q_n$. It easily follows that if κ is a limit point of the set $\{(\Psi_r^{(q_n)})_* : n \geq 1\}$, then it has an atom at a (even non-zero) member of the set F (see Remark 7.3).

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By Remarks 7.3 and 7.4, we obtain that the topological support of a limit point of $\{(\Theta_r^{(q_n)} - \Psi_r^{(q_n)})_* : n \geq 1\}$ contains a translation of an uncountable subset of $\mathbb{R}$. In this way, we have proved the following.

PROPOSITION 7.5. *If α has bounded partial quotients, then $\varphi^{(r)}(r\cdot)$ is ergodic for each $r \geq 1$.*

It follows immediately that all cocycles of the form $\varphi^{(s)}(s \cdot + c)$ are also ergodic.

7.3. *Regularity of $(\varphi^{(r)}(r\cdot), \varphi^{(s)}(s\cdot))$.* Our aim will now be to study the cocycles

$$(\varphi^{(r)}(r\cdot), \varphi^{(s)}(s\cdot)) = (\Theta_r, \Theta_s) - (\Psi_r, \Psi_s) \tag{48}$$

taking values in $\mathbb{R}^2$, $r, s \in \mathbb{N}$. We will constantly assume that α has bounded partial quotients.

Note that $s^2x - s^2/2 = (s^2/r^2)(r^2x - r^2/2)$, so $(\Theta_r, \Theta_s)(\mathbb{T}) \subset P_{r,s}$, where

$$P_{r,s} := \left\{\left(t, \frac{s^2}{r^2}t\right) : t \in \mathbb{R}\right\}$$

is a closed subgroup of $\mathbb{R}^2$. By examining the sequence $((\Theta_r, \Theta_s)^{(q_n)})_*$, $n \geq 1$, it is not hard to see that each limit point of such distributions is an absolutely continuous measure on the line $P_{r,s}$. It easily follows that $E((\Theta_r, \Theta_s)) = P_{r,s}$. In order to study (Ψ_r, Ψ_s), we need first the following observation.

LEMMA 7.6. *Assume that $(r, s) = 1$. Then the non-zero discontinuity points*

$$\frac{k - \{j_\ell\alpha\}}{r}, \quad \frac{k' - \{j'_\ell\alpha\}}{s}, \quad \ell = 0, 1, \ldots, r-1,$$

for $\Psi_r^{(q_n)}$ and $\Psi_s^{(q_n)}$, respectively, are pairwise different.

Proof. Suppose that $(m - \{j\alpha\})/r = (u - \{k\alpha\})/s$ with $m, u \in \mathbb{Z}$. It follows that

$$s(j\alpha - [j\alpha]) - r(k\alpha - [k\alpha]) = sm - ru,$$

whence $sj\alpha - rk\alpha \in \mathbb{Z}$. If $j \neq 0$ or $k \neq 0$, this is possible only if $sj = rk$. But $(r, s) = 1$ implies $s \mid k$ and $r \mid j$ and we seek solution for $0 \leq j < r$ and $0 \leq k < s$, a contradiction. A similar argument works in the case $j = k = 0$ and either $m \neq r$ or $u \neq s$. □

It now again will follow from the analysis of limit points of the distributions $((\Psi_r, \Psi_s)^{(q_n)})_*$, $n \geq 1$, that there is a non-zero point in the topological support of a limit of such distributions. Taking into account the form of closed subgroups of $\mathbb{R}^2$, Lemma 2.4 and setting

$$\mathcal{H}_{r,s} := E((\varphi^{(r)}(r\cdot), \varphi^{(s)}(s\cdot))), \tag{49}$$

we obtain the following.

PROPOSITION 7.7. *Assume that α has bounded partial quotients and $(r, s) = 1$. Then there exists $0 \neq d \in \mathbb{R}$ such that*

$$\mathcal{H}_{r,s} = P_{r,s} + (d\mathbb{Z}, 0).$$

In particular, $(\varphi^{(r)}(r\cdot), \varphi^{(s)}(s\cdot))$ is regular.

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Proof. We have already shown that for some $d' \neq 0$, the group $P_{r,s} + (d'\mathbb{Z}, 0)$ is contained in the group of essential values of the cocycle under consideration. Hence, we only need to notice that $(\varphi^{(r)}(r\cdot), \varphi^{(s)}(s\cdot))$ is not ergodic. Indeed, let $\widetilde{\chi}(u, v) = e^{2\pi i(2s^2u - 2r^2v)}$. It is not hard to see that

$$\mathbb{1} \neq \widetilde{\chi} \in \Lambda_{(\varphi^{(r)}(r\cdot), \varphi^{(s)}(s\cdot))}. \tag{50}$$

The result follows. □

PROPOSITION 7.8. *Under the above assumptions on α, r and s, we have*

$$\mathcal{H}_{r,s}^{\perp} \subset \mathbb{Q} \times \mathbb{Q}.$$

Proof. In view of Lemma 2.5, it follows that (cf. (50) in the proof of Proposition 7.7)

$$\widetilde{\chi} \in \mathcal{H}_{r,s}^{\perp}. \tag{51}$$

Now,

$$\mathcal{H}_{r,s}^{\perp} = (P_{r,s} + (d\mathbb{Z}, 0))^{\perp} \subset P_{r,s}^{\perp} = \{e^{2\pi i(u, -(r^2/s^2)u)(\cdot,\cdot)} : u \in \mathbb{R}\}.$$

By (51), $\widetilde{\chi}(d, 0) = 1$, whence $e^{2\pi i 2s^2 d} = 1$ and hence $d \in \mathbb{Q}$. Now, $(d, 0) \in \mathcal{H}_{r,s}$ and if

$$e^{2\pi i(du - (r^2/s^2)u\cdot 0)} = 1$$

then $ud \in \mathbb{Z}$ and it follows that $u \in \mathbb{Q}$. Hence $(u, -(r^2/s^2)u) \in \mathbb{Q} \times \mathbb{Q}$ and the result follows. □

7.4. *Regularity of $(\varphi^{(r)}(r\cdot), \varphi^{(s)}(s\cdot + c))$.*

PROPOSITION 7.9. *Assume that α has bounded partial quotients and $(r, s) = 1$. Then:*

(i) *if $c \notin \mathbb{Q}\alpha + \mathbb{Q}$ then the cocycle $(\varphi^{(r)}(r\cdot), \varphi^{(s)}(s\cdot + c))$ is ergodic;*

(ii) *if $c \in \mathbb{Q}\alpha + \mathbb{Q}$ then the cocycle $(\varphi^{(r)}(r\cdot), \varphi^{(s)}(s\cdot + c))$ is regular, its group of essential values $\mathcal{H}_{r,s,c}$ is cocompact of the form $P_{r,s} + (d\mathbb{Z}, 0)$ and $\mathcal{H}_{r,s,c}^{\perp} \subset \mathbb{Q} \times \mathbb{Q}$.*

Proof. Let us prove (i) first. For this aim, notice that

$$\varphi^{(s)}(sx + c) = \Theta_s(x) - (\Psi_s(x + c) - sc). \tag{52}$$

As before, we would like to study the limit points of distributions

$$((\Theta_r, \Theta_s)^{(q_n)} - (\Psi_r, \Psi_s(\cdot + c/s))^{(q_n)} - (0, q_n sc))_*, \quad n \geq 1.$$

However, we can also consider distributions on the circle, namely

$$(e^{2\pi i((a,b)\cdot((\Theta_r,\Theta_s)^{(q_n)} - (\Psi_r, \Psi(\cdot + c/s))^{(q_n)} + (0, q_n sc)))})_*,$$

with $a, b \in \mathbb{Z}\backslash\{0\}$. Then

$$e^{2\pi i((a,b)\cdot((\Theta_r,\Theta_s)^{(q_n)} - (\Psi_r, \Psi_s(\cdot + c))^{(q_n)} + (0, q_n sc)))} = e^{2\pi i((a,b)\cdot((\Theta_r,\Theta_s)^{(q_n)} + (0, q_n sc)))}.$$

Now, since α has bounded partial quotients and $bsc \notin \mathbb{Q}\alpha + \mathbb{Q}$, the sequence $(e^{2\pi i q_n bsc})$, $n \geq 1$, has infinitely many accumulation points. If $d \in \mathbb{T}$ is any of these accumulation points, this means that there is a line $P_{r,s} + (dm, 0)$ with some $m \neq \mathbb{Z}\backslash\{0\}$ meeting $\mathcal{H}_{r,s,c}$ (cf. the proof of Lemma 3 in [**32**]). Since we have infinitely many d at our disposal, $\mathcal{H}_{r,s,c} = \mathbb{R} \times \mathbb{R}$.

The proof of (ii) is much the same as the proof in case $c = 0$. □

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7.5. *AOP property for Rokhlin extensions given by affine cocycles.* Using Corollary 2.15, Proposition 7.9 and the fact that the group of eigenvalues of $\mathcal{S}\otimes\mathcal{S}$ is the Cartesian square of the group of eigenvalues of $\mathcal{S}$, we obtain the following result.

THEOREM 7.10. *Assume that $Tx = x+\alpha$ with α irrational of bounded partial quotients. Let $\varphi(x) = x - \frac{1}{2}$. Let $\mathcal{S}$ be any ergodic flow acting on a probability standard Borel space $(Y, \mathcal{C}, \nu)$. If the group of eigenvalues of $\mathcal{S}$ on $L^2_0(Y, \mathcal{C}, \nu)$ does not meet $\mathbb{Q}$ then $T_{\varphi,\mathcal{S}}$ has the AOP property.*

By replacing the base rotation T by its Sturmian model $\widetilde{T}$, we can view $\varphi(x) = x - \frac{1}{2}$ as a continuous function (see §7.1). Then for each ergodic $\mathcal{S} = (S_t)$ which has no non-trivial rational eigenvalues, we obtain $T_{\varphi,\mathcal{S}}$ uniquely ergodic, enjoying the strong MOMO property.

COROLLARY 7.11. *For each $(y_k)\subset Y$, $f\in C(Y)$ and each sequence $(b_k)\subset\mathbb{N}$ with $b_{k+1}-b_k\to\infty$, we have*

$$\frac{1}{b_{K+1}}\sum_{k\le K}\left|\sum_{b_k\le n<b_{k+1}} f(S_{\varphi^{(n)}(0)}y_k)\boldsymbol{u}(n)\right|\to 0 \quad \textit{when } K\to\infty,$$

for each multiplicative $\boldsymbol{u}:\mathbb{N}\to\mathbb{C}$, $|\boldsymbol{u}|\le 1$. Here,

$$\varphi^{(n)}(0) = \frac{n(n-1)}{2}\alpha - \frac{n}{2} - \sum_{j=1}^{n-1}[j\alpha].$$

Assume that R is an ergodic automorphism of $(Z, \mathcal{D}, \rho)$. Consider its suspension $\tilde{R} = (\tilde{R}_t)_{t\in\mathbb{R}}$, and change time in it:

$$\overline{R}_t := \widetilde{R}_{\beta t}.$$

The (additive) group of eigenvalues of $\overline{R}$ is the multiplication by β of the group of eigenvalues of the suspension (for the suspension the group of eigenvalues is the group for R meant as a subgroup of $[0, 1)$ and then we add $\mathbb{Z}$ to obtain an additive subgroup of $\mathbb{R}$). We can choose β so that the group of eigenvalues of $\overline{R}$ is hence disjoint from $\mathbb{Q}$ (mod the common element 0).

By considering suspensions (and their change of time as above), we obtain the following.

COROLLARY 7.12. *If R is an ergodic automorphism of $(Z, \mathcal{D}, \rho)$ and $\beta\in\mathbb{R}$ is such that $\beta e(R)\subset\mathbb{Q}^c\cup\{0\}$ then for each $(z_k)\subset Z$, $f\in C(Z)$ and each sequence $(b_k)\subset\mathbb{N}$ with $b_{k+1}-b_k\to\infty$, we have*

$$\frac{1}{b_{K+1}}\sum_{k\le K}\left|\sum_{b_k\le n<b_{k+1}} f(R^{[\beta\varphi^{(n)}(0)]}y_k)\boldsymbol{u}(n)\right|\to 0 \quad \textit{when } K\to\infty,$$

for each multiplicative $\boldsymbol{u}:\mathbb{N}\to\mathbb{C}$, $|\boldsymbol{u}|\le 1$.

Finally, by taking suspensions over the rotation by 1 on $\mathbb{Z}/\ell\mathbb{Z}$, we obtain the following.

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COROLLARY 7.13. *Let $\beta \in \mathbb{Q}^c$ be such that $\beta e(R) \subset \mathbb{Q}^c \cup \{0\}$. Then, for each sequence $(b_k) \subset \mathbb{N}$ with $b_{k+1} - b_k \to \infty$, we have*

$$\frac{1}{b_{K+1}} \sum_{k \le K} \left| \sum_{b_k \le n < b_{k+1}} e^{2\pi i [\beta \varphi^{(n)}(0)]/\ell} \boldsymbol{u}(n) \right| \to 0 \quad \text{when } K \to \infty,$$

for each multiplicative $\boldsymbol{u} : \mathbb{N} \to \mathbb{C}$, $|\boldsymbol{u}| \le 1$.

7.6. *The strong MOMO property for the affine case.* In §7.1 we have considered $T_{\varphi,\mathcal{S}}$ as a uniquely ergodic automorphism (whenever $\mathcal{S}$ is a uniquely ergodic flow). However, we have proved the AOP property of $T_{\varphi,\mathcal{S}}$ only for those flows $\mathcal{S}$ that have no non-trivial rational spectrum.

Consider a sequence of the form $(x, y_k) \subset X \times Y$. We have

$$\frac{1}{b_{K+1}} \sum_{k \le K} \left(\sum_{b_k \le n < b_{k+1}} \delta_{(T_{\varphi,\mathcal{S}})^{rn} \times (T_{\varphi,\mathcal{S}})^{sn}((x,y_k),(x,y_k))} \right) \to \widetilde{\rho},$$

when $K \to \infty$. It follows from the above that

$$\frac{1}{b_{K+1}} \sum_{k \le K} \left(\sum_{b_k \le n < b_{k+1}} \delta_{(T^{rn} \times T^{sn})(x,x)} \right) \to \rho,$$

that is, $\widetilde{\rho}$ is a lift of ρ. But (x, x) is a generic point for an ergodic measure (formally, we should have considered $\widehat{T}$ and $\widehat{\varphi}$, see Proposition 7.1), that is, $\rho \in J^e(T^r, T^s)$, so if we know that it has a unique extension to a joining of $(T_{\varphi,\mathcal{S}})^r$ and $(T_{\varphi,\mathcal{S}})^s$, the joining must be the relatively independent extension of ρ (in other words, up to a permutation of coordinates, $\widetilde{\rho} = \rho \otimes \nu \otimes \nu$).

LEMMA 7.14. *If $\beta \notin \mathbb{Q}\alpha + \mathbb{Q}$ then (β, β) is generic for an ergodic measure $\rho \in J(T^r, T^s)$, for which: for an arbitrary uniquely ergodic $\mathcal{S}$ acting on $(Y, \mathcal{C}, \nu)$ the only $\widetilde{\rho} \in J(T^r_{\varphi,\mathcal{S}}, T^s_{\varphi,\mathcal{S}})$, $\widetilde{\rho}|_{X \times X} = \rho$ is the measure $\rho \otimes \nu \otimes \nu$.*

Proof. Recall that the ergodic components of $T^r \times T^s$ are of the form $I_c := \{(x, y + c) : sx = ry\}$. If $(\beta, \beta) \in I_c$ then $s\beta = r(\beta - c)$ and $c = ((r - s)/r)\beta$, and therefore $c \notin \mathbb{Q}\alpha + \mathbb{Q}$. The claim follows from Proposition 7.9 and Corollary 2.13. □

It follows that we can now prove the counterparts of Corollaries 7.11–7.13 for the sequence

$$[\varphi^{(n)}(\beta)] = \left[n\beta + \frac{n(n-1)}{2}\alpha - \frac{n}{2} - \sum_{j=0}^{n-1} [\beta + j\alpha] \right], \quad n \ge 1,$$

but with the absolute values replaced by parentheses. We have however, the following result.

COROLLARY 7.15. *Let $\boldsymbol{u} : \mathbb{N} \to \mathbb{C}$ be a multiplicative function satisfying $|\boldsymbol{u}| \le 1$ and $\ell \ge 2$. Then*

$$\frac{1}{M} \sum_{M \le m < 2M} \left| \frac{1}{H} \sum_{m \le h < m+H} e^{2\pi i [\varphi^{(h)}(\beta)]/\ell} \boldsymbol{u}(h) \right| \to 0$$

when $H \to \infty$, $H/M \to 0$.

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Proof. Consider $Ry := y + 1$ on $\mathbb{Z}/\ell\mathbb{Z}$ and $\chi(y) := e^{2\pi i y/\ell}$. Then, by the above, for each $(b_k) \subset \mathbb{N}$, $b_{k+1} - b_k \to \infty$, and each choice of $(y_k) \subset \mathbb{Z}/\ell\mathbb{Z}$, we have

$$\frac{1}{b_{k+1}} \sum_{k \le K} \left(\sum_{b_k \le n < b_{k+1}} \chi(R^{[\varphi^{(n)}(\beta)]} y_k) \boldsymbol{u}(n) \right) \to 0$$

when $K \to \infty$, that is,

$$\frac{1}{b_{k+1}} \sum_{k \le K} \chi(y_k) \left(\sum_{b_k \le n < b_{k+1}} e^{2\pi i [\varphi^{(n)}(\beta)]/\ell} \boldsymbol{u}(n) \right) \to 0$$

when $K \to \infty$. If $\ell \ge 3$, by choosing y_k so that $\chi(y_k)(\sum_{b_k \le n < b_{k+1}} e^{2\pi i [\varphi^{(n)}(\beta)]/\ell} \boldsymbol{u}(n))$ belongs to the cone $\{0\} \cup \{z \in \mathbb{C} : \arg(z) \in [-\pi/3, \pi/3]\}$ (or, when $\ell = 2$, obtaining $(\chi(y_k))$ and arbitrary sequence of ± 1), we obtain

$$\frac{1}{b_{k+1}} \sum_{k \le K} \left| \sum_{b_k \le n < b_{k+1}} e^{2\pi i [\varphi^{(n)}(\beta)]/\ell} \boldsymbol{u}(n) \right| \to 0$$

when $K \to \infty$ (cf. [**4**]). Since (b_k) was arbitrary, as in [**6**], we obtain our assertion. □

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