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Conservation law of turbulent dispersion

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A side remark in the old paper by Zeldovich and co-authors leads to the recent discovery of a universal conservation law of turbulent dispersion.

In a spatially smooth but temporally random flow, trajectories separate exponentially. When the observation time exceeds the flow correlation time, the statistics of separation are described by the theory of dynamical chaos with universal notions like Lyapunov exponents etc. However, this beautiful mathematics is not of much use for physicists trying to make any predictions for the future using the knowledge of the present state. In complicated situations the only hope to make quantitative predictions is to identify all-time conservation laws, dynamical or statistical. The interest in statistical conservation laws was enhanced when some were identified as responsible for the breakdown of scale invariance and an anomalous scaling in turbulence – those laws were of geometrical nature and related to multiparticle configurations (Falkovich et al. 2001). Until recently, lacking was an answer to the simple question: are there conservation laws, expressed solely via the single distance between two particles, $R(t)$?

A path to answer this question started (at least for the author of this review) from a side remark in the old paper by Zeldovich et al. (1984). There, with the reference to the older paper by Furstenberg (1963) it was claimed that for a spatially smooth statistically isotropic incompressible random flow in d dimensions $\langle [R(t)]^{-d} \rangle$ does not change at $t \to \infty$. That statement was quite natural since at large times all moments are expected to depend exponentially on time: $\langle [R(t)]^n \rangle \propto \exp[E(n)t]$. Negative moments of orders $n < -1$ must be dominated by the contribution of initial directions $\mathbf{R}(0)$, perpendicular to the eigendirection of the senior Lyapunov exponent. At $n < 1-d$, the main contribution comes from a small subset of directions in a solid angle $\propto \exp(d\lambda_d t)$ around the eigendirection of the most negative Lyapunov exponent λ_d. For such n, $\langle R^n \rangle \propto \exp[\lambda_d(d+n)t]$ and that $\langle R^{-d} \rangle$ does not change at the long time limit. Since $E(n)$ is a convex function, it cannot have other zeroes except $-d$ and 0 if it does not vanish identically between those values. That consideration (inspired by the discussion of the author with K. Khanin) was included into the 2001 review (Falkovich et al. 2001) together with the references to Zeldovich and Furstenberg.

It took another 13 years to come back to what seemed like a settled issue. It all started from an attempt by Anna Frishman and the author to build an analytic formalism that would bridge an Eulerian set of single-time velocity correlation functions (velocity snapshot) and a Lagrangian description of fluid trajectories evolving in time. The idea of the formalism is very simple: the time derivative of any order of the Lagrangian correlation function taken at zero time can be expressed

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via Eulerian correlation functions (Falkovich and Frishman 2013). In particular, if one manages to find a quantity with all time derivatives zero and to make sure of analyticity in time, then the conservation law is established.

To show how the formalism works, let us calculate the time derivatives of correlation functions involving R:

$$\mathscr{P}(R,t;R_0,0)=\left\langle \delta\left(\vec{\boldsymbol{R}}-\vec{\boldsymbol{R}}_0-\int_0^t \vec{\boldsymbol{u}}(t')dt'\right)\right\rangle . \tag{0.1}$$

Here, $\mathscr{P}(R,t;R_0,0)$ is the transition probability. The integral is over Lagrangian trajectories and averaging here and everywhere is over the statistics of the velocity difference $\vec{\boldsymbol{u}}$. To calculate time derivatives at $t=0$ of any mean,

$$\left\langle F\left(R(t)\right)\right\rangle =\int F(R)\mathscr{P}(R,t;R_0,0)dR, \tag{0.2}$$

we first differentiate the δ-function (and velocities), then set $t=0$ turning the δ-function into $\delta(R-R_0)$:

$$\left.\frac{\partial^n\langle F(R)\rangle}{\partial t^n}\right|_{t=0}=\left\langle\frac{d^nF(R)}{dt^n}\right\rangle_{t=0}=\left\langle\frac{d^n}{dt^n}\right\rangle F(R_0), \tag{0.3}$$

where $d/dt\equiv\partial_t+\vec{\boldsymbol{u}}\cdot\nabla_R$ is implied. Here and below, unless stated otherwise, averages without an explicit specification of the time are taken at $t=0$.

When the velocity statistics is stationary,

$$\left\langle\frac{d^2}{dt^2}\right\rangle=\langle u^i\nabla_i u^j\rangle\nabla_j\,,\quad \left\langle\frac{d^3}{dt^3}\right\rangle=\langle u^i\nabla_i u^j\nabla_j u^k\rangle\nabla_k, \tag{0.4}$$

$$\left\langle\frac{d^4}{dt^4}\right\rangle=\langle u^i\nabla_i u^j\nabla_j u^k\nabla_k u^l\rangle\nabla_l-\nabla_i\left\langle\frac{\partial u^i}{\partial t}\frac{\partial u^j}{\partial t}\right\rangle\nabla_j. \tag{0.5}$$

Velocity time derivatives (acceleration, etc.) appear only starting from $n=4$. Therefore, in a general case the velocity snapshot contains enough information to describe evolution up to t^4 terms.

Let us apply the formalism to a spatially smooth random flow with isotropic but not necessarily stationary statistics. In this case, the relative velocity is proportional to the distance, $\vec{\boldsymbol{u}}=\hat{\sigma}(t)\vec{\boldsymbol{R}}(t)$, where $\hat{\sigma}(t)$ is a random traceless matrix with isotropic statistics. Such is the case, for example, in the viscous interval of turbulence, where the separation between the particles is much smaller than the viscous scale, for elastic turbulence and many other cases related to dynamical chaos. Let us show that all the time derivatives at $t=0$ are zero for $F(R)=R^{-d}$. This is a consequence of $\left\langle d^nR^{-d}/dt^n\right\rangle=0$ when taken at $R=R_0$ without additionally setting $t=0$ in $\hat{\sigma}(t)$. To show this we note that the first time derivative at $R=R_0$ is zero by virtue of isotropy:

$$\left\langle dR^{-d}/dt\right\rangle=\langle\sigma_{ij}(t)\rangle\,R_0^iR_0^jR_0^{-d-2}=0. \tag{0.6}$$

We can now use induction and express the $n+1$ time derivative at $R=R_0$:

$$\left\langle\frac{d^{n+1}}{dt^{n+1}}R^{-d}\right\rangle=\partial_t\left\langle\frac{d^n}{dt^n}R^{-d}\right\rangle+\nabla_i\left\langle u^i\frac{d^n}{dt^n}R^{-d}\right\rangle.$$

On the right side, the first term is zero by the induction hypothesis, and the second term is zero because of $\left\langle u^i d^nR^{-d}/dt^n\right\rangle\propto R_0^iR_0^{-d}$ at $R=R_0$. Our simple argument shows that all time derivatives are zero. Since all time dependencies are analytic we

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conclude that

$$\left\langle [R(t)]^{-d} \right\rangle = R_0^{-d}, \tag{0.7}$$

for all t even for a time-dependent statistics. The universality of this statistical conservation law follows from the simple dynamical statement: in every realization of an incompressible spatially smooth flow, the integral of $[R(t)]^{-d}$ over the directions of the initial vector, $\vec{\boldsymbol{R}}_0$, is constant in time. That statement can actually be found by taking a closer look at the argument (not only result) of Zeldovich et al. (1984). For isotropic flows this integral can be interpreted as an average over the angular degrees of freedom, which gives (0.7). Indeed, for each velocity realization, the volume of an infinitesimal d-dimensional hyper-spherical sector, with radius R_0 and differential solid angle $d\Omega_0$, at $t = 0$, is equal to its transformation under the flow at time t:

$$R_0^d d\Omega_0 = R^d(t) d\Omega_t. \tag{0.8}$$

It is due to the linearity of the transformation (i.e. spatial smoothness of the flow) that a spherical sector is transformed to another spherical sector. The conservation law directly follows: for a pair of particles in the flow, starting with the separation vector $\mathbf{R_0}$

$$\frac{\left\langle R(t)^{-d} \right\rangle}{R_0^{-d}} = \int \frac{\left\langle R(t)^{-d} \right\rangle}{R_0^{-d}} \frac{d\Omega_0}{S_d} = \int \frac{d\Omega_t}{S_d} = 1, \tag{0.9}$$

with S_d the volume of the $d-1$-dimensional unit sphere and $d\Omega_0$ parameterizing the direction of $\mathbf{R_0}$. The first equality is a consequence of the assumption that the flow is statistically isotropic, the average over velocity realizations for a scalar quantity thus being independent of the direction of $\mathbf{R_0}$. In the second equality the average and the integration are interchanged and (0.8) is used. Therefore, careful reading of Zeldovich paper reveals all the ingredients needed for establishing the all time conservation. On the contrary, no matter how many times I have read the Furstenberg paper, no traces of even asymptotic conservation were found there. Apparently, Furstenberg formalism inspired the approach of Zeldovich et al. (1984). Yet it seems that the statement on asymptotic conservation and dynamical measure argument must be credited to Zeldovich et al. (1984), while all time conservation to Falkovich and Frishman (2013).

Understanding that (0.7) holds for all times, even in the case of decaying turbulence, opens the door for its experimental verification which is ongoing (Anna Frishman, private communication).

Does analogous conservation exist for non-smooth velocity as in the inertial interval of turbulence? Assuming velocity fast-fluctuating (so-called Kazantsev–Kraichnan model) one can show that all time derivatives are zero for $\langle R^{2-\zeta_2-d} \rangle$, where ζ_2 is the scaling exponent of the second velocity structure function (2 for smooth flow and 2/3 for Kolmogorov phenomenology). And yet this quantity is not a conservation law! The reason is that the time dependence is non-analytic in the Kazantsev–Kraichnan model (Falkovich and Frishman 2013). For non-smooth in space non-white in time velocity, one can show that at most three first time derivatives can be made zero (for $\langle R^{2-\zeta_2-d} \rangle$ which is thus $\propto t^3$ i.e. is a short-time analog of the long time Richardson law). No function of $R(t)$ is conserved in the inertial interval of turbulence. A special case of $2d$ vorticity cascade is close to smooth yet with logarithmic corrections, which makes for approximate conservation of $\langle R^{-d} \rangle$. Lagrangian conservation laws must involve both velocity and separation between trajectories (Falkovich and Frishman 2013). In the non-smooth case typical timescales depend on the distances, so the

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integrals of motion must depend on the initial distance – they are actually zero modes described by Falkovich et al. (2001).

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