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ON INTEGER SETS WITH THE SAME REPRESENTATION FUNCTIONS

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Abstract

Let $\mathbb{N}$ be the set of all nonnegative integers. For $S \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, let $R_S(n)$ denote the number of solutions of the equation $n = s_1 + s_2$, $s_1, s_2 \in S$ and $s_1 < s_2$. Let A be the set of all nonnegative integers which contain an even number of digits 1 in their binary representations and $B = \mathbb{N} \setminus A$. Put $A_l = A \cap [0, 2^l - 1]$ and $B_l = B \cap [0, 2^l - 1]$. We prove that if $C \cup D = [0, m] \setminus \{r\}$ with $0 < r < m$, $C \cap D = \emptyset$ and $0 \in C$, then $R_C(n) = R_D(n)$ for any nonnegative integer n if and only if there exists an integer $l \geq 1$ such that $m = 2^l$, $r = 2^{l-1}$, $C = A_{l-1} \cup (2^{l-1} + 1 + B_{l-1})$ and $D = B_{l-1} \cup (2^{l-1} + 1 + A_{l-1})$. Kiss and Sándor ['Partitions of the set of nonnegative integers with the same representation functions', *Discrete Math.* **340** (2017), 1154–1161] proved an analogous result when $C \cup D = [0, m]$, $0 \in C$ and $C \cap D = \{r\}$.

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1. Introduction

Let $\mathbb{N}$ be the set of all nonnegative integers. For $S \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, let $R_S(n)$ denote the number of solutions of the equation $n = s_1 + s_2$, $s_1, s_2 \in S$ and $s_1 < s_2$. Sárközy asked whether there exist two sets U and V of nonnegative integers with infinite symmetric difference and $R_U(n) = R_V(n)$ for all large enough integers n. Dombi [5] answered this problem affirmatively by using the Thue–Morse sequence. Later, Lev [7], Sándor [9] and Tang [10] gave simple proofs of this result. For related results, see [4, 12–15].

Let A be the set of all nonnegative integers which contain an even number of digits 1 in their binary representations and $B = \mathbb{N} \setminus A$. For any positive integer l, let $A_l = A \cap [0, 2^l - 1]$ and $B_l = B \cap [0, 2^l - 1]$. In 2017, Kiss and Sándor [6] proved that if C and D are two sets of nonnegative integers such that $C \cup D = [0, m]$, $C \cap D = \emptyset$ and $0 \in C$, then $R_C(n) = R_D(n)$ for every positive integer n if and only if there exists a positive integer l such that $C = A_l$ and $D = B_l$. In 2016, Tang [11] proved that for any given integer m, there do not exist subsets $C, D \subseteq \mathbb{N}$ with $\mathbb{N} = C \cup D$ and $C \cap D = \{km : k \in \mathbb{N}\}$ such that $R_C(n) = R_D(n)$ for all sufficiently large integers n.

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For any integer r and m, we define $r + m\mathbb{N} = \{r + ma : a \in \mathbb{N}\}$. Chen and Lev [3] proved that for any given positive integer l, there exist two subsets $C, D \subseteq \mathbb{N}$ satisfying $C \cup D = \mathbb{N}$ and $C \cap D = 2^{2l} - 1 + (2^{2l+1} - 1)\mathbb{N}$ such that $R_C(n) = R_D(n)$ for every positive integer n. They also posed the following two problems.

PROBLEM 1.1. Given that $R_C(n) = R_D(n)$ for every positive integer n, $C \cup D = \mathbb{N}$ and $C \cap D = r + m\mathbb{N}$ with integers $r \geq 0$ and $m \geq 2$, must there exist an integer $l \geq 1$ such that $r = 2^{2l} - 1$, $m = 2^{2l+1} - 1$?

PROBLEM 1.2. Given that $R_C(n) = R_D(n)$ for every positive integer n, $C \cup D = [0, m]$ and $C \cap D = \{r\}$ with integers $r \geq 0$ and $m \geq 2$, must there exist an integer $l \geq 1$ such that $r = 2^{2l} - 1$, $m = 2^{2l+1} - 2$, $C = A_{2l} \cup (2^{2l} - 1 + B_{2l})$ and $D = B_{2l} \cup (2^{2l} - 1 + A_{2l})$?

In 2017, Kiss and Sándor solved Problem 1.2 affirmatively. Later, Li and Tang [8] and Chen *et al.* [2] solved Problem 1.1 under the assumption $0 \leq r < m$. Recently, Chen and Chen [1] solved Problem 1.1 affirmatively.

In this paper, we consider two sets C and D satisfying $C \cup D = [0, m] \setminus \{r\}$ and $C \cap D = \emptyset$. We prove the following result.

THEOREM 1.3. *Let C and D be two sets of nonnegative integers such that $C \cup D = [0, m] \setminus \{r\}$ with $0 < r < m$, $C \cap D = \emptyset$ and $0 \in C$. Then for any nonnegative integer n, $R_C(n) = R_D(n)$ if and only if there exists an integer $l \geq 1$ such that $m = 2^l$, $r = 2^{l-1}$, $C = A_{l-1} \cup (2^{l-1} + 1 + B_{l-1})$ and $D = B_{l-1} \cup (2^{l-1} + 1 + A_{l-1})$.*

2. Preliminary lemmas

LEMMA 2.1. *For positive integers $0 < r_1 < r_2 < \cdots < r_s \leq m$, there exist at most one pair of sets (C, D) such that $C \cup D = [0, m] \setminus \{r_1, r_2, \ldots, r_s\}$, $0 \in C$, $C \cap D = \emptyset$ and $R_C(k) = R_D(k)$ for all $k \leq m$.*

PROOF. Suppose that there exist two pairs of sets $(C_1, D_1), (C_2, D_2)$ satisfying the conditions of Lemma 2.1. Let v be the minimal nonnegative integer such that $\chi_{C_1}(v) \neq \chi_{C_2}(v)$. Then, $C_1 \cap [0, v-1] = C_2 \cap [0, v-1]$ and $R_{C_1}(v) \neq R_{C_2}(v)$ since $0 \in C_1, C_2$. By $R_{C_1}(v) = R_{D_1}(v)$ and $R_{C_2}(v) = R_{D_2}(v)$, it suffices to prove that $R_{D_1}(v) = R_{D_2}(v)$ for a contradiction.

Since $0 \in C$ and $C \cap D = \emptyset$, it follows that $0 \notin D$, and so

$$R_{D_1}(v) = |\{(d, d') : d < d' < v, d, d' \in D_1, d + d' = v\}|.$$

Let t be an integer with $0 \leq t \leq s$ such that $r_t < v < r_{t+1}$. Here we define $r_0 = 0$, $r_{s+1} = +\infty$. Since

$$[0, v-1] \setminus \{r_1, r_2, \ldots, r_t\} = (C_1 \cap [0, v-1]) \cup (D_1 \cap [0, v-1]),$$

it follows that

$$\begin{aligned} D_1 \cap [0, v-1] &= ([0, v-1] \setminus \{r_1, r_2, \ldots, r_t\}) \setminus (C_1 \cap [0, v-1]) \\ &= [0, v-1] \setminus ((C_1 \cap [0, v-1]) \cup \{r_1, r_2, \ldots, r_s\}). \end{aligned}$$

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Similarly,

$$D_2 \cap [0, v-1] = [0, v-1] \setminus ((C_2 \cap [0, v-1]) \cup \{r_1, r_2, \ldots, r_t\}).$$

Hence, $D_1 \cap [0, v-1] = D_2 \cap [0, v-1]$ and so $R_{D_1}(v) = R_{D_2}(v)$. □

REMARK 2.2. If $\{r_1, r_2, \ldots, r_s\} = \emptyset$, then Lemma 2.1 also holds.

LEMMA 2.3. *Suppose thatC, D are two distinct sets with the properties $C \cup D = [0, m] \setminus \{r_1, \ldots, r_s\}$, $C \cap D = \emptyset$ and $R_C(n) = R_D(n)$ for any positive integer n. If $C' = m - C$, $D' = m - D$, then $C' \cup D' = [0, m] \setminus \{m - r_1, \ldots, m - r_s\}$, $C' \cap D' = \emptyset$, and $R_{C'}(n) = R_{D'}(n)$ for all nonnegative integers n.*

PROOF. For $k = 0, 1, \ldots, 2m$,

$$\begin{aligned} R_{C'}(k) &= |\{(c, c') : c < c', c, c' \in C', c + c' = k\}| \\ &= |\{(c, c') : c < c', m - c, m - c' \in C, c + c' = k\}| \\ &= |\{(m - c, m - c') : c < c', m - c, m - c' \in C, 2m - (c + c') = 2m - k\}| \\ &= R_C(2m - k). \end{aligned}$$

Similarly, $R_{D'}(k) = R_D(2m - k)$. Since $R_C(2m - k) = R_D(2m - k)$, it follows that $R_{C'}(k) = R_{D'}(k)$. □

LEMMA 2.4 (See [6, Claim 3]). *If for some positive integer M, the integers $M - 1$, $M - 2, M - 4, M - 8, \ldots, M - 2^{\lceil \log_2 M \rceil - 1}$ are all contained in the set A, then $\lceil \log_2 M \rceil$ is odd and $M = 2^{\lceil \log_2 M \rceil} - 1$.*

LEMMA 2.5 (See [6, Claim 4]). *If for some positive integer M, the integers $M - 1$, $M - 2, M - 4, M - 8, \ldots, M - 2^{\lceil \log_2 M \rceil - 1}$ are all contained in the set B, then $\lceil \log_2 M \rceil$ is even and $M = 2^{\lceil \log_2 M \rceil} - 1$.*

LEMMA 2.6. *If for some even positive integer M, the integers $M - 2, M - 4, \ldots, M - 2^{\lceil \log_2 M \rceil - 1}$ are all contained in the set A, then $M = 2^{\lceil \log_2 M \rceil} - 2$.*

PROOF. If for some even positive integer M, the integers $M - 2, M - 4, \ldots, M - 2^{\lceil \log_2 M \rceil - 1}$ are all contained in the set A, then $\frac{1}{2}M - 1, \frac{1}{2}M - 2, \ldots, \frac{1}{2}M - 2^{\lceil \log_2 M/2 \rceil - 1}$ are all contained in the set A. By Lemma 2.4, it follows that $\frac{1}{2}M = 2^{\lceil \log_2 M/2 \rceil} - 1$, and so $M = 2^{\lceil \log_2 M \rceil} - 2$. □

LEMMA 2.7 (See [6, Theorem 3]). *Let C and D be sets of nonnegative integers such that $C \cup D = [0, m]$ and $C \cap D = \emptyset$, $0 \in C$. Then, $R_C(n) = R_D(n)$ for every positive integer n if and only if there exists a nonnegative integer l such that $C = A_l$ and $D = B_l$.*

LEMMA 2.8 (See [6, Corollary 1]). *If $C = A \cap [0, m]$ and $D = B \cap [0, m]$, where m is a positive integer not of the form $2^l - 1$, then there exists a positive integer n with $m < n < 2m$ such that $R_C(n) \neq R_D(n)$.*

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3. Proofs

PROOF OF THEOREM 1.3. (*Sufficiency*). Suppose that there exists a positive integer $l \geq 1$ such that $C = A_{l-1} \cup (2^{l-1} + 1 + B_{l-1})$ and $D = B_{l-1} \cup (2^{l-1} + 1 + A_{l-1})$. Now we prove that $R_C(n) = R_D(n)$ for all nonnegative integers n. Clearly,

$$\begin{aligned} R_C(n) &= |\{(c,c') : c < c', c, c' \in A_{l-1}, c + c' = n\}| \\ &\quad + |\{(c,c') : c \in A_{l-1}, c' \in 2^{l-1} + 1 + B_{l-1}, c + c' = n\}| \\ &\quad + |\{(c,c') : c < c', c, c' \in 2^{l-1} + 1 + B_{l-1}, c + c' = n\}| \\ &= R_{A_{l-1}}(n) + |\{(c,c') : c \in A_{l-1}, c' \in B_{l-1}, c + c' = n - (2^{l-1} + 1)\}| \\ &\quad + R_{B_{l-1}}(n - 2(2^{l-1} + 1)). \end{aligned}$$

Similarly,

$$\begin{aligned} R_D(n) &= R_{B_{l-1}}(n) + |\{(d,d') : d \in B_{l-1}, d' \in A_{l-1}, d + d' = n - (2^{l-1} + 1)\}| \\ &\quad + R_{A_{l-1}}(n - 2(2^{l-1} + 1)). \end{aligned}$$

By Lemma 2.1, for any positive integer m, $R_{A_{l-1}}(m) = R_{B_{l-1}}(m)$. Hence, $R_C(n) = R_D(n)$.

(*Necessity*). Suppose that $C \cup D = [0, m] \setminus \{r\}$, $C \cap D = \emptyset$ and, for any positive integer n, $R_C(n) = R_D(n)$. Let $C' = m - C$, $D' = m - D$. Then by Lemma 2.3, we have $C' \cup D' = [0, m] \setminus \{m - r\}$, $C' \cap D' = \emptyset$ and $R_{C'}(n) = R_{D'}(n)$ for any positive integer n. Hence, we can assume that $r \leq m/2$ without loss of generality.

Let $p_C(x) = \sum_{i=0}^{m} \chi_C(i) x^i$, $p_D(x) = \sum_{i=0}^{m} \chi_D(i) x^i$. Since $C \cup D = [0, m] \setminus \{r\}$, $C \cap D = \emptyset$, it follows that

$$p_C(x) + p_D(x) = \frac{1 - x^{m+1}}{1 - x} - x^r. \tag{3.1}$$

Since $R_C(n) = R_D(n)$ for any positive integer n,

$$\sum_{n=0}^{\infty} R_C(n) x^n = \sum_{n=0}^{\infty} R_D(n) x^n.$$

Hence,

$$\begin{aligned} \frac{1}{2} p_C(x)^2 - \frac{1}{2} p_C(x^2) &= \frac{1}{2} \Big(\sum_{i \in C} x^i \Big)^2 - \frac{1}{2} \sum_{i \in C} x^{2i} = \sum_{\substack{i_1, i_2 \in C \\ i_1 < i_2}} x^{i_1} \cdot x^{i_2} = \sum_{\substack{i_1, i_2 \in C \\ i_1 < i_2}} x^{i_1 + i_2} \\ &= \sum_{n=0}^{\infty} R_C(n) x^n = \sum_{n=0}^{\infty} R_D(n) x^n = \frac{1}{2} p_D(x)^2 - \frac{1}{2} p_D(x^2), \end{aligned}$$

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and then, substituting for $p_D(x)$ from (3.1),

$$\begin{aligned} p_C(x)^2 - p_C(x^2) &= \left(\frac{1-x^{m+1}}{1-x} - p_C(x) - x^r\right)^2 - \left(\frac{1-x^{2m+2}}{1-x^2} - p_C(x^2) - x^{2r}\right) \\ &= \left(\frac{1-x^{m+1}}{1-x}\right)^2 + p_C(x)^2 + x^{2r} - 2\frac{1-x^{m+1}}{1-x}\cdot p_C(x) \\ &\quad - 2\frac{1-x^{m+1}}{1-x}\cdot x^r + 2p_C(x)x^r - \frac{1-x^{2m+2}}{1-x^2} + p_C(x^2) + x^{2r}. \end{aligned}$$

Therefore,

$$\begin{aligned} 2p_C(x^2) &= \frac{1-x^{2m+2}}{1-x^2} + 2p_C(x)\frac{1-x^{m+1}}{1-x} - \left(\frac{1-x^{m+1}}{1-x}\right)^2 \\ &\quad + 2x^r\frac{1-x^{m+1}}{1-x} - 2p_C(x)x^r - 2x^{2r}. \end{aligned} \tag{3.2}$$

We first check the necessity for $1 \le r \le 5$.

For $r = 1$, if there is a suitable decomposition $C \cup D = [0, m] \setminus \{1\}$ and $m \ge 4$, then $0 \in C$ and $2, 3, 4 \in D$. Let $E = m - C$, $F = m - D$. Then,

$$\begin{aligned} &(E \cap [0, m-2]) \cap (F \cap [0, m-2]) = \emptyset, \\ &(E \cap [0, m-2]) \cup (F \cap [0, m-2]) = [0, m-2]. \end{aligned}$$

By Lemma 2.3, $R_E(n) = R_F(n)$ for all integers $n \le m - 2$. Further, by Lemma 2.1, $F \cap [0, m-2] = A \cap [0, m-2]$ or $B \cap [0, m-2]$, but this is a contradiction, because $F \cap [0, m-2]$ contains three consecutive integers $m-4$, $m-3$ and $m-2$. Hence, the only solution is $m = 2$.

For $r = 2, 3, 4, 5$, the proof is similar. Here we give the details in the case where $r = 4$. Since $0 \in C$, we have $1 \in D$, otherwise $R_C(1) \ne R_D(1)$. Similarly, we can obtain $2 \in D$, $3 \in C$, $5 \in D$, $6 \in C$, $7 \in C$, $8 \in D$, $9 \in D$, $10 \in C$, $11 \in D$, $12 \in C$, $13 \in D$, and we cannot put 14 in both C and D. Hence if $m \ge 14$, there is no C and D. For $m < 13$, we can check only $m = 8$ holds and $C = \{0, 3, 6, 7\}$, $D = \{1, 2, 5, 8\}$.

Now we assume $r \ge 6$. First we shall prove that r is even. Suppose that r is odd. Let k be an even integer such that $r < k < 2r$. Then the coefficient of x^k in (3.2) is

$$2\chi_C\left(\frac{k}{2}\right) = 1 + 2\sum_{i \le k}\chi_C(i) - (k+1) + 2 - 2\chi_C(k-r), \tag{3.3}$$

and the coefficient of x^{k-1} in (3.2) is

$$0 = 0 + 2\sum_{i \le k-1}\chi_C(i) - k + 2 - 2\chi_C(k-1-r). \tag{3.4}$$

Calculating ((3.3) and (3.4)) $\times\frac{1}{2}$, we have

$$\chi_C\left(\frac{k}{2}\right) = \chi_C(k) - \chi_C(k-r) + \chi_C(k-1-r). \tag{3.5}$$

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By Lemma 2.1,

$$C \cap [0, r-1] = A \cap [0, r-1], \tag{3.6}$$

$$D \cap [0, r-1] = B \cap [0, r-1]. \tag{3.7}$$

Since $k-1-r$ is even, $k-r < r$. By (3.6) and the definition of A,

$$\chi_C(k-r) + \chi_C(k-1-r) = 1.$$

If $\chi_C(k-r) = 0, \chi_C(k-1-r) = 1$, then by (3.5), $\chi_C(k) = 0$ and $\chi_C(k/2) = 1$. If $\chi_C(k-r) = 1, \chi_C(k-1-r) = 0$, then by (3.5), $\chi_C(k) = 1$ and $\chi_C(k/2) = 0$. Hence, $\chi_C(k-1-r) = \chi_C(k/2)$.

Let $k = 2r - 2^{i+1} + 2$, where $i \geq 1$. Then $\frac{1}{2}k = r + 1 - 2^i$, $k - 1 - r = r + 1 - 2^{i+1}$. Hence, $\chi_C(r+1-2^i) = \chi_C(r+1-2^{i+1})$. Therefore,

$$\chi_C(r+1-2) = \chi_C(r+1-4) = \cdots = \chi_C(r+1-2^{\lceil \log_2(r+1)\rceil - 1}),$$

and so

$$\chi_A(r+1-2) = \chi_A(r+1-4) = \cdots = \chi_A(r+1-2^{\lceil \log_2(r+1)\rceil - 1}).$$

By Lemma 2.6, it follows that $r+1 = 2^u - 2$ and $\chi_A(r+1) = \chi_A(r)$.

Next we shall prove $r \in B$. Suppose that $r \in A$. Since $0 \in C$, $r \notin C$, by (3.6), it follows that

$$R_C(r) = R_{C\cap[0,r-1]}(r) = R_{A\cap[0,r-1]}(r) = R_{A\cap[0,r]}(r) - 1.$$

Similarly, by (3.7), it follows that

$$R_D(r) = R_{D\cap[0,r-1]}(r) = R_{B\cap[0,r-1]}(r) = R_{B\cap[0,r]}(r).$$

Noting that

$$R_{A\cap[0,r]}(r) = R_A(r) = R_B(r) = R_{B\cap[0,r]}(r),$$

we have $R_C(r) \neq R_D(r)$, which is a contradiction. Hence, $r \in B$ and $r+1 \in B$, and so $r \notin A$.

By (3.6) and $r \notin C$, we have $C \cap [0, r] = A \cap [0, r]$. Hence,

$$R_C(r+1) = R_{C\cap[0,r]}(r+1) + \chi_C(r+1) = R_{A\cap[0,r]}(r+1) + \chi_C(r+1).$$

By (3.7) and $1, r \in B, r \notin D$,

$$R_D(r+1) = R_{B\cap[0,r]}(r+1) - 1.$$

Noting that $r, r+1 \in B$,

$$R_{A\cap[0,r]}(r+1) = R_{A\cap[0,r+1]}(r+1), \quad R_{B\cap[0,r]}(r+1) = R_{B\cap[0,r+1]}(r+1).$$

Since

$$R_{A\cap[0,r+1]}(r+1) = R_A(r+1) = R_B(r+1) = R_{B\cap[0,r+1]}(r+1),$$

it follows that $R_C(r+1) \neq R_D(r+1)$, which is a contradiction. Therefore, r is not odd.

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Suppose that r is even. Take an even integer k with $r \le k \le 2r$. Then, the coefficient of x^{k-2} in (3.2) is

$$2\chi_C\left(\frac{k-2}{2}\right) = 1 + 2\sum_{i\le k-2}\chi_C(i) - (k-1) + 2 - 2\chi_C(k-2-r), \tag{3.8}$$

and the coefficient of x^{k-1} in (3.2) is

$$0 = 0 + 2\sum_{i\le k-1}\chi_C(i) - k + 2 - 2\chi_C(k-1-r). \tag{3.9}$$

Calculating ((3.8) and (3.9)) $\times\frac{1}{2}$, we obtain

$$\chi_C\left(\frac{k-2}{2}\right) = 1 - \chi_C(k-1) - \chi_C(k-2-r) + \chi_C(k-1-r).$$

By Lemma 2.3, $k-1-r < r$ and $k-2-r$ is even. By (3.6) and the definition of A,

$$\chi_C(k-2-r) + \chi_C(k-1-r) = 1.$$

If $\chi_C(k-2-r) = 0$ and $\chi_C(k-1-r) = 1$, then $\chi_C(k-1) = 1, \chi_C((k-2)/2) = 1$. If $\chi_C(k-2-r) = 1$, $\chi_C(k-1-r) = 0$, then $\chi_C(k-1) = 0, \chi_C((k-2)/2) = 0$. From these two cases, $\chi_C(k-1-r) = \chi_C((k-2)/2)$. Take

$$k = 2r - 2^{i+1}, \quad i = 0, 1, \ldots, \lceil \log_2(r-1)\rceil - 2.$$

Then $\frac{1}{2}(k-2) = r-1-2^i$ giving $k-1-r = r-1-2^{i+1}$ and so $\chi_C(r-1-2^i) = \chi_C(r-1-2^{i+1})$. Hence,

$$\chi_C(r-1-1) = \chi_C(r-1-2) = \cdots = \chi_C(r-1-2^{\lceil \log_2(r-1)\rceil - 1}).$$

By Lemma 2.5, $r-1 = 2^{\lceil \log_2(r-1)\rceil} - 1$. Write $l = \lceil \log_2(r-1)\rceil + 1$. Then $r = 2^{l-1}$. Also $[0, 2r] \subseteq [0, m]$ because $r \le m/2$. By Lemma 2.1 and the sufficiency of Theorem 1.3,

$$C \cap [0, 2r] = A_{l-1} \cup (2^{l-1} + 1 + B_{l-1}), \quad D \cap [0, 2r] = B_{l-1} \cup (2^{l-1} + 1 + A_{l-1}).$$

Next we shall prove $m = 2r$. First, we prove $m < 3\cdot 2^{l-1} = 2^{l-1} + 2^l$. Suppose that $m \ge 3\cdot 2^{l-1}$. First we shall prove the following structure of C and D:

$$C \cap [0, 3\cdot 2^{l-1}] = A_{l-1} \cup (2^{l-1} + 1 + B_{l-1}) \cup (2^l + 1 + B_{l-1}) := E,$$
$$D \cap [0, 3\cdot 2^{l-1}] = B_{l-1} \cup (2^{l-1} + 1 + A_{l-1}) \cup (2^l + 1 + A_{l-1}) := F.$$

If $n \le 2^l$, then clearly $R_E(n) = R_F(n)$. Now we assume that $2^l < n \le 3\cdot 2^{l-1}$. It follows that

$$\begin{aligned} R_E(n) &= |\{(c, c') : c \in A_{l-1}, c' \in 2^{l-1} + 1 + B_{l-1}, c + c' = n\}| \\ &\quad + |\{(c, c') : c \in A_{l-1}, c' \in 2^l + 1 + B_{l-1}, c + c' = n\}| \\ &\quad + |\{(c, c') : c < c', c, c' \in 2^{l-1} + 1 + B_{l-1}, c + c' = n\}| \\ &= |\{(c, c') : c \in A_{l-1}, c' \in B_{l-1}, c + c' = n - (2^{l-1} + 1)\}| \\ &\quad + |\{(c, c') : c \in A_{l-1}, c' \in B_{l-1}, c + c' = n - (2^l + 1)| + R_{B_{l-1}}(n - 2(2^{l-1} + 1)) \end{aligned}$$

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and

$$\begin{aligned}R_F(n) &= |\{(d,d') : d \in B_{l-1}, d' \in 2^{l-1}+1+A_{l-1}, d+d'=n\}|\\ &\quad + |\{(d,d') : d \in B_{l-1}, d' \in 2^{l}+1+A_{l-1}, d+d'=n\}|\\ &\quad + |\{(d,d') : d < d', d, d' \in 2^{l-1}+1+B_{l-1}, d+d'=n\}|\\ &= |\{(d,d') : d \in B_{l-1}, d' \in A_{l-1}, d+d'=n-(2^{l-1}+1)\}|\\ &\quad + |\{(d,d') : d \in B_{l-1}, d' \in A_{l-1}, d+d'=n-(2^{l}+1)| + R_{A_{l-1}}(n-2(2^{l-1}+1)).\end{aligned}$$

By Lemma 2.7, $R_{A_{l-1}}(m) = R_{B_{l-1}}(m)$ for any positive integers m and so $R_E(n) = R_F(n)$. By Lemma 2.1, $C \cap [0, 3\cdot 2^{l-1}] = E$ and $D \cap [0, 3\cdot 2^{l-1}] = F$.

Next we shall prove that $m \geq 3\cdot 2^{l-1}$ is impossible. If $m > 3\cdot 2^{l-1}$ then, since $0 \in C$,

$$\begin{aligned}&R_C(3\cdot 2^{l-1}+1)\\ &\quad = \chi_C(3\cdot 2^{l-1}+1) + |\{(c,c') : c \in A_{l-1}, c' \in 2^{l}+1+B_{l-1}, c+c' = 3\cdot 2^{l-1}+1\}|\\ &\qquad + |\{(c,c') : c < c', c, c' \in 2^{l-1}+1+B_{l-1}, c+c' = 3\cdot 2^{l-1}+1\}|\\ &\quad = \chi_C(3\cdot 2^{l-1}+1) + |\{(c,c') : c \in A_{l-1}, c' \in B_{l-1}, c+c' = 2^{l-1}\}| + R_{B_{l-1}}(2^{l-1}-1).\end{aligned}$$

Similarly,

$$R_D(3\cdot 2^{l-1}+1) = |\{(d,d') : d \in B_{l-1}, d' \in A_{l-1}, d+d' = 2^{l-1}\}| + R_{A_{l-1}}(2^{l-1}-1).$$

By Lemma 2.7, $R_{A_{l-1}}(2^{l-1}-1) = R_{B_{l-1}}(2^{l-1}-1)$ so $R_C(3\cdot 2^{l-1}+1) = R_D(3\cdot 2^{l-1}+1)$ if and only if $\chi_C(3\cdot 2^{l-1}+1) = 0$, that is, $3\cdot 2^{l-1}+1 \in D$.

Since $1, 3\cdot 2^{l-1}+1, 2^l+1, 2^{l-1}+1 \in D$ and

$$3\cdot 2^{l-1}+2 = 1+(1+3\cdot 2^{l-1}) = (2^{l-1}+1)+(2^l+1),$$

it follows that

$$\begin{aligned}R_D(3\cdot 2^{l-1}+2) &= 1+1+|\{(d,d') : d \in B_{l-1}, d' \in 2^{l}+1+A_{l-1}, d+d' = 3\cdot 2^{l-1}+2\}|\\ &\quad + |\{(d,d') : d < d', d, d' \in 2^{l-1}+1+A_{l-1}, d+d' = 3\cdot 2^{l-1}+2\}|\\ &= 2 + |\{(d,d') : d \in B_{l-1}, d' \in A_{l-1}, d+d' = 2^{l-1}+1\}| + R_{A_{l-1}}(2^{l-1}).\end{aligned}$$

Similarly,

$$\begin{aligned}R_C(3\cdot 2^{l-1}+2) &= \chi_C(3\cdot 2^{l-1}+2) + |\{(c,c') : c \in A_{l-1}, c' \in B_{l-1}, c+c' = 2^{l-1}+1\}|\\ &\quad + R_{B_{l-1}}(2^{l-1}).\end{aligned}$$

Hence, $R_C(3\cdot 2^{l-1}+2) \neq R_D(3\cdot 2^{l-1}+2)$, which is a contradiction.

If $m = 3\cdot 2^{l-1}$, by $3\cdot 2^{l-1}+2 = (2^{l-1}+1)+(2^l+1)$, then

$$\begin{aligned}R_C(3\cdot 2^{l-1}+2) &= |\{(c,c') : c \in A_{l-1}, c' \in B_{l-1}, c < c', c+c' = 2^{l-1}+1|\} + R_{B_{l-1}}(2^{l-1}),\\ R_D(3\cdot 2^{l-1}+2) &= 1 + |\{(d,d') : d \in B_{l-1}, d' \in A_{l-1}, d < d', d+d' = 2^{l-1}+1|\\ &\quad + R_{A_{l-1}}(2^{l-1}).\end{aligned}$$

Hence, $R_C(3\cdot 2^{l-1}+2) \neq R_D(3\cdot 2^{l-1}+2)$, which is a contradiction.

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Therefore, $m < 3 \cdot 2^{l-1}$. Now we have

$$C = A_{l-1} \cup (2^{l-1} + 1 + B_{l-1}) \cup (2^l + 1 + (B_{l-1} \cap [0, m - (2^l + 1)])),$$
$$D = B_{l-1} \cup (2^{l-1} + 1 + A_{l-1}) \cup (2^l + 1 + (A_{l-1} \cap [0, m - (2^l + 1)])).$$

Next we prove $m = 2^l$. Suppose that $m > 2^l$.

If $m - (2^l + 1) \neq 2^k - 1$, where k is a nonnegative integer, then by Lemma 2.8, there exists an integer u satisfying $m - (2^l + 1) < u < 2[m - (2^l + 1)]$ such that

$$R_{A\cap[0,m-(2^l+1)]}(u) = R_{A_{l-1}\cap[0,m-(2^l+1)]}(u) \neq R_{B_{l-1}\cap[0,m-(2^l+1)]}(u) = R_{B\cap[0,m-(2^l+1)]}(u). \tag{3.10}$$

Since $m + (2^l + 1) < u + 2(2^l + 1) < 2m$, it follows that

$$\begin{aligned} R_C(2(2^l + 1) + u) &= R_{2^l+1+(B_{l-1}\cap[0,m-(2^l+1)])}(2(2^l + 1) + u) \\ &= R_{B_{l-1}\cap[0,m-(2^l+1)]}(u) = R_{B\cap[0,m-(2^l+1)]}(u). \end{aligned}$$

Similarly,

$$\begin{aligned} R_D(2(2^l + 1) + u) &= R_{2^l+1+(A_{l-1}\cap[0,m-(2^l+1)])}(2(2^l + 1) + u) \\ &= R_{A_{l-1}\cap[0,m-(2^l+1)]}(u) = R_{A\cap[0,m-(2^l+1)]}(u). \end{aligned}$$

Since $R_C(2(2^l + 1) + u) = R_D(2(2^l + 1) + u)$, it follows that $R_{A\cap[0,m-(2^l+1)]}(u) = R_{B\cap[0,m(2^l+1)]}(u)$, which is a contradiction with (3.10).

Now we suppose that $m - (2^l + 1) = 2^k - 1$. Noting that $m < 3 \cdot 2^{l-1}$, we have $k < l - 1$. Hence,

$$C = A_{l-1} \cup (2^{l-1} + 1 + B_{l-1}) \cup (2^l + 1 + B_k),$$
$$D = B_{l-1} \cup (2^{l-1} + 1 + A_{l-1}) \cup (2^l + 1 + A_k).$$

If $k = 0$, then $2^l + 1 \in D$. Now we divide into the following two cases.

Case 1: $2^{l-1} - 1 \in A_{l-1}$. That is, $2^l \in D$. Hence, $2^{l+1} + 1 = (2^l + 1) + 2^l$ and so $R_C(2^{l+1} + 1) = 0$, $R_D(2^{l+1} + 1) = 1$, which is a contradiction.

Case 2: $2^{l-1} - 1 \in B_{l-1}$. That is, $2^l \in C$. Hence, $2^{l-1} - 2 \in A_{l-1}$ and so $2^l - 1 \in D$. Since $2^{l+1} = (2^l + 1) + (2^l - 1)$, it follows that $R_C(2^{l+1}) = 0$, $R_D(2^{l+1}) = 1$, which is a contradiction.

Therefore, we can assume that $k > 0$. Let $C' = 2^l + 2^k - C$, $D' = 2^l + 2^k - D$.

Case 1: $2 \nmid l - 1$, $2 \nmid k$. It follows that

$$C' = A_k \cup (2^k + A_{l-1}) \cup (2^{l-1} + 2^k + 1 + B_{l-1}),$$
$$D' = B_k \cup (2^k + B_{l-1}) \cup (2^{l-1} + 2^k + 1 + A_{l-1}).$$

Case 2: $2 \nmid l - 1$, $2 \mid k$. Then,

$$C' = B_k \cup (2^k + A_{l-1}) \cup (2^{l-1} + 2^k + 1 + B_{l-1}),$$
$$D' = A_k \cup (2^k + B_{l-1}) \cup (2^{l-1} + 2^k + 1 + A_{l-1}).$$

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Case 3: $2 \mid l-1, 2 \nmid k$. Then,

$$C' = A_k \cup (2^k + B_{l-1}) \cup (2^{l-1} + 2^k + 1 + A_{l-1}),$$
$$D' = B_k \cup (2^k + A_{l-1}) \cup (2^{l-1} + 2^k + 1 + B_{l-1}).$$

Case 4: $2 \mid l-1, 2 \mid k$. Then,

$$C' = B_k \cup (2^k + B_{l-1}) \cup (2^{l-1} + 2^k + 1 + A_{l-1}),$$
$$D' = A_k \cup (2^k + A_{l-1}) \cup (2^{l-1} + 2^k + 1 + B_{l-1}).$$

Hence, $C' \cup D' = [0, 2^l + 2^k] \setminus \{2^{l-1} + 2^k\}$, $C' \cap D' = \emptyset$. By Lemma 2.3, it follows that $R_{C'}(n) = R_{D'}(n)$ for all positive integers n. By Lemma 2.1, it suffices to consider the following two cases.

Case I: $C' \cap [0, 2^{l-1} + 2^k - 1] = A \cap [0, 2^{l-1} + 2^k - 1]$ and $D' \cap [0, 2^{l-1} + 2^k - 1] = B \cap [0, 2^{l-1} + 2^k - 1]$.

Case II: $C' \cap [0, 2^{l-1} + 2^k - 1] = B \cap [0, 2^{l-1} + 2^k - 1]$ and $D' \cap [0, 2^{l-1} + 2^k - 1] = A \cap [0, 2^{l-1} + 2^k - 1]$.

If Case 1 and Case I hold, then $C' \cap [0, 2^k - 1] = A \cap [0, 2^k - 1] = A_k$. By the property of A, we have $C' \cap [2^k, 2^{k+1} - 1] = 2^k + B_k$. Then, by Case 1, we have $C' \cap [2^k, 2^{k+1} - 1] = 2^k + A_k$, which is a contradiction.

If Case 1 and Case II hold, then $C' \cap [0, 2^k - 1] = B \cap [0, 2^k - 1] = B_k$. By Case 1, we have $C' \cap [0, 2^k - 1] = A_k$, which is a contradiction.

If Case 2 and Case I hold, then $C' \cap [0, 2^k - 1] = A_k$. Then, by Case 2, we have $C' \cap [0, 2^k - 1] = B_k$, which is a contradiction.

If Case 2 and Case II hold, then $k < l - 1$ and so $2^k + 1 < 2^{l-1} - 1$. By the property of A, it follows that $2^k + 1 \in A_{l-1}$ and then $2^k + (2^k + 1) \in 2^k + A_{l-1} \subseteq C'$. Hence, $2^k + (2^k + 1) \in C'$. By Case II, $2^k + (2^k + 1) \in B$. But, $2^k + (2^k + 1) = 2^{k+1} + 1 \in A$, which is a contradiction.

If Case 3 and Case I hold, then $2^k + (2^k + 1) \in 2^k + A_{l-1} \subseteq D'$ because we have $2^k + 1 \in A_{l-1}$. By Case I, $2^k + (2^k + 1) \in B$. However, by the property of A, $2^k + (2^k + 1) = 2^{k+1} + 1 \in A$, which is a contradiction.

If Case 3 and Case II hold, then $C' \cap [0, 2^k - 1] = B \cap [0, 2^k - 1] = B_k$. By Case 3, $C' \cap [0, 2^k - 1] = A_k$, which is a contradiction.

If Case 4 and Case I hold, then $C' \cap [0, 2^k - 1] = A_k$. By Case 4, $C' \cap [0, 2^k - 1] = B_k$, which is a contradiction.

If Case 4 and Case II hold, then $C' \cap [0, 2^k - 1] = B \cap [0, 2^k - 1] = B_k$. By the property of B, $C' \cap [2^k, 2^{k+1} - 1] = 2^k + A_k$. By Case 4, $C' \cap [2^k, 2^{k+1} - 1] = 2^k + B_k$, which is a contradiction.

Therefore, $m = 2r$. □

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