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NOTES

CONSTANT-ELASTICITY-OF-SUBSTITUTION PRODUCTION FUNCTION

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We consider endogenous changes of inputs from labor to capital in the production of intermediate goods, i.e., a form of mechanization. We derive complementary relationships between capital accumulation and mechanization by assuming a Cobb–Douglas production function for the production of final goods from intermediate goods. A constant-elasticity-of-substitution production function in which the elasticity of substitution exceeds unity can be endogenously derived as the envelope of Cobb–Douglas production functions when the efficiency of inputs is assumed in a specific form. The difficulty of mechanization represents the elasticity of substitution.

Keywords: Mechanization, Micro Foundation for a CES Production Function

1. INTRODUCTION

Many analyses have shown, both theoretically and through growth regressions, that the accumulation of physical capital per labor unit is a crucial factor in explaining economic growth. However, strong positive relationships exist not only between gross domestic product (GDP) per labor unit and the amount of capital per labor unit, but also between GDP per labor unit and the range of the use of capital. In developed countries, the capital/labor ratio is high, and capital, e.g., machines, is extensively used to produce goods rather than labor. Furthermore, labor has been replaced by capital in some production stages; i.e., a high degree of mechanization exists in developed countries. For example, consider the production of clothes, which involves many stages such as cutting and sewing. One can differentiate

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between those stages in which labor is the main input and those in which capital is the main input by closely examining many of these stages. In some stages, a switch from labor to capital has occurred, because the introduction of machines has reduced labor costs. On the other hand, in developing countries, the capital/labor ratio is low, and labor is widely used to produce goods. Moreover, implementing mechanization seems difficult to do in developing countries.

In this paper, we consider the range of capital and labor use by assuming that intermediate goods can be produced by either labor or capital. We show that, even when we assume a standard Cobb–Douglas production function for the production of final goods from intermediate goods, the production function in equilibrium can imply a constant-elasticity-of-substitution (CES) production function. Our production function is dynamically derived as the envelope of the Cobb–Douglas production functions through mechanization such that a labor input is changed into a capital input in the production of intermediate goods.[1] Therefore, mechanization makes our long-run production function less concave than a Cobb–Douglas production function. In a special case, we can imply the CES production function in which the elasticity of substitution between labor and capital exceeds unity.

Arrow, Chenery, Minhas, and Solow (1961) mathematically derived a CES production function. In some studies, a CES production function has been endogenously derived from an economic model. For example, Acemoglu and Zilibotti (2001) developed a model in which technologies imported from the north by the south are relatively skill-complementary, while unskilled workers requiring more labor-complementary technologies are used in the south. They obtained a CES production function incorporating both skilled and unskilled labor in which the elasticity of substitution between inputs was two. Jones (2005) tried to derive a standard production function that exhibits steady-state growth from micro foundations. A global production function could be formed from the convex hull of the optimal local production functions, each of which applied to a single technique. He found that the standard production function could be derived by assuming a Pareto distribution of ideas.

In this paper, we make the following two important points. First, we partly contribute to the micro foundations of the CES production function by developing endogenous mechanization, although a CES production function is assumed exogenously in most studies. Second, we relate the difficulty of mechanization to economic growth in which said mechanization does not involve the growth of total factor productivity (TFP).

2. THE MODEL

In a closed economy, the final goods are produced from a continuum of intermediate goods, the total number of which is set to unity. This final output is used for consumption and for capital accumulation. The production function for the final

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goods is assumed to be of the following Cobb–Douglas function:

$$\ln Y_t = \int_0^1 \ln y_t(i)di, \tag{1}$$

where Y_t is the output of the final goods at time t and $y_t(i)$ is the input of intermediate good i at time t.

Each intermediate good can be produced by using one of two technologies. One uses labor and the other uses capital, such as machines, including robots. We represent the production function for intermediate good i by

$$y_t(i) = \eta(i)l_t(i) \quad \text{or} \quad \theta(i)x_t(i), \qquad \theta(i), \eta(i) > 0, \tag{2}$$

where $l_t(i)$ and $x_t(i)$ are the labor and capital inputs used in the production of intermediate good i at time t, respectively. The parameters $\eta(i)$ and $\theta(i)$, respectively, represent the efficiency of labor and capital in producing intermediate goods i.

With equation (2), we can consider the number of intermediate goods for which capital is the input. We can see that number as the range of the use of capital in producing the final goods. We term an intermediate good a "capital intermediate good" when capital is the input; when labor is the input, we have a "labor intermediate good." We define the number of capital intermediate goods as a_t. An increase in a_t means that an input is changed from labor to capital in the production of some intermediate goods, i.e., mechanization.

We arrange the intermediate goods from those in which capital is used relatively more efficiently than labor to those in which capital is used relatively less efficiently than labor. The function $\Psi(i) \equiv \theta(i)/\eta(i)$, which indicates the relative efficiency of capital to labor, is assumed to have the following properties:

$$\Psi(i) \geq 0, \qquad \Psi'(i) < 0, \qquad i \in [0, 1], \tag{3}$$

where $\Psi(0) > 0$.

Given these properties, the production of capital intermediate goods starts with those in which a relatively high efficiency of capital to labor exists. Furthermore, mechanization becomes increasingly difficult.

The shape of the function $\Psi(i)$ influences the difficulty of implementing mechanization. For example, compare the production of agricultural goods and manufacturing goods. Dividing the production of agricultural goods into many monotonous stages is more difficult than doing so for manufacturing goods because the production of agricultural goods requires more delicate handwork than does the production of manufacturing goods. Therefore, we suggest that a relatively low efficiency of capital to labor exists in the production of agricultural goods. Moreover, the function $\Psi(i)$ is steeper for agricultural goods.[2]

While we suppose that the economy is centrally planned, to explain our model simply, we first consider the maximization problem for production of the final

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goods separately from the utility maximization problem:

$$\max_{\{a_t, x_t(i)|i\in[0,a_t], l_t(j)|j\in(a_t,1]\}} \ln Y_t = \int_0^1 \ln y_t(i)di$$
$$= \int_0^{a_t} \ln \theta(i)x_t(i)di + \int_{a_t}^1 \ln \eta(j)l_t(j)dj, \tag{4}$$
$$\text{s.t.} \qquad K_t = \int_0^{a_t} x_t(i)di \qquad \text{and} \qquad L_t = \int_{a_t}^1 l_t(j)dj,$$

where K_t and L_t are the amount of capital and the population of the economy, respectively.[3]

We consider both the type of input that should be used to produce each intermediate good and the quantities of inputs. We obtain the following first-order conditions for the quantities of inputs:

$$x_t(i) = x_t \equiv \frac{1}{\lambda_{1t}}, \qquad i \in [0, a_t],$$
$$l_t(j) = l_t \equiv \frac{1}{\lambda_{2t}}, \qquad j \in (a_t, 1], \tag{5}$$

where λ_{1t} and λ_{2t} are the Lagrangian multipliers.

The number of capital intermediate goods must satisfy the following first-order condition:

$$\ln \theta(a_t)x_t - \ln \eta(a_t)l_t = 0. \tag{6}$$

With equations (5) and (6) and the resource constraints, we obtain the following relationship between the number of capital intermediate goods and the ratio of input quantities of labor to capital:

$$\Psi(a_t) = \frac{l_t}{x_t} = \frac{L_t/(1-a_t)}{K_t/a_t}. \tag{7}$$

If function $\Psi(i)$ took a lower value, a decrease in the number of capital intermediate goods would occur; i.e., labor, rather than capital, would be widely used.

In addition, using equation (7), we can represent technology, as measured by the number of capital intermediate goods, by the capital/labor ratio of the economy, denoted by k_t, as follows:

$$a_t = a(k_t), \quad a'(k_t) > 0, \qquad a(0) = 0, \qquad \lim_{k_t\to\infty} a(k_t) = 1. \tag{8}$$

An economy that has accumulated little capital per labor unit has few capital intermediate goods; i.e., labor is used extensively. In an economy with a high capital per labor unit, the number of capital intermediate goods is high and capital is used extensively.

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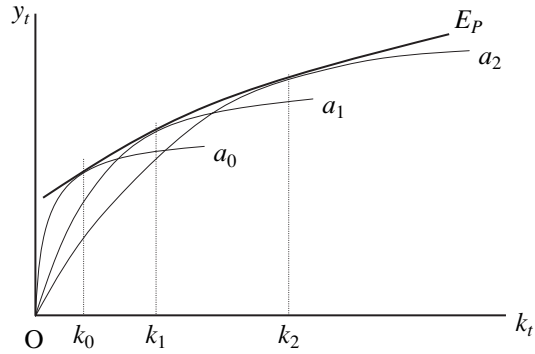


FIGURE 1. Envelope of Cobb–Douglas production function.

We describe the utility maximization problem of a central planner as follows:

$$\max_{\{c_t|t\in[0,\infty]\}} \int_0^\infty \frac{1}{1-\phi} c_t^{1-\phi} \exp(-\rho t) dt, \qquad \phi, \rho > 0, \tag{9}$$

$$\text{s.t.} \qquad \dot{k}_t = y_t - c_t - (n+\delta)k_t, \tag{10}$$

where c_t is consumption per labor unit, y_t is output per labor unit, ϕ^{-1} represents the elasticity of substitution between consumption at any two points in time, ρ is the time preference rate, n is the population growth rate, and δ is the depreciation rate on capital.

3. THE DYNAMICS AND TECHNOLOGICAL CHANGE

The equilibrium growth path of the economy must satisfy the following two differential equations relating to consumption per labor unit and capital per labor unit, respectively:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\phi}[f'(k_t) - \rho - n - \delta], \tag{11}$$

$$\frac{\dot{k}_t}{k_t} = \frac{y_t}{k_t} - \frac{c_t}{k_t} - (n+\delta), \tag{12}$$

where $f(k_t)$ is output per labor unit. Note that by using equations (4), (5), (6), and (7), the production function in equilibrium, which is illustrated as the envelope in Figure 1, can be written as

$$f(k_t) = \Omega[a(k_t)]\{a(k_t)^{a(k_t)}[1-a(k_t)]^{1-a(k_t)}\}^{-1} {k_t}^{a(k_t)}, \tag{13}$$

where $\ln \Omega[a(k_t)] \equiv \int_0^{a(k_t)} \ln \theta(i) di + \int_{a(k_t)}^1 \ln \eta(j) dj$.

Here, we explain the dynamics of our model. The accumulation of capital urges an increasing number of capital intermediate goods, i.e., mechanization occurs

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because capital rather than labor must be used more extensively to maximize the output of the final goods. This mechanization causes further capital accumulation because the marginal product of capital declines by less as a result of capital accumulation than it would if the number of capital intermediate goods were constant. Therefore, complementary relationships exist between capital accumulation and mechanization.

If mechanization is intrinsically easy to implement, those complementary relationships will be strong and they will continue permanently. The production function would asymptotically become the Ak technology at the limit. Along the envelope in Figure 1, the marginal product of capital does not go down to zero as in the Inada conditions, but instead it remains positive. Therefore, if it is greater than the sum of the population growth rate, the time preference rate, and the depreciation rate, the economy can attain endogenous growth.[4] However, those complementary relationships are weak when mechanization is intrinsically difficult to implement, and they cease as the capital per labor unit accumulates.

In Figure 1, the number of capital intermediate goods increases, as shown by a_0, a_1, and a_2, which correspond to the accumulation of capital per labor unit, k_0, k_1, and k_2, respectively. The Cobb–Douglas production functions for the numbers of capital intermediate goods, a_0, a_1, and a_2, shift in a northeasterly direction because the relative efficiency of capital to labor declines with mechanization. The increase in output per labor unit can be represented by the envelope E_P, which corresponds to the change in the amount of capital per labor unit. That is, the economy moves along a single production function noted in equation (13) that is derived by the maximization of equation (4). The TFP does not grow because this single production function does not shift. Zeira (1998) focused on that type of technological innovation that saves on labor but uses more capital in the context of a small and open economy. In his model, this change is adopted only in countries that have large shift parameters in their production functions; furthermore, this change increases differences in productivity levels between countries.

To produce the final goods, a Cobb–Douglas production function is endowed at each moment in time. However, to maximize its output, a Cobb–Douglas production function that has a greater number of capital intermediate goods is chosen as the capital per labor unit accumulates. This implies that in the long run a less concave production function than a Cobb–Douglas production function appears as a result of mechanization.

Next, we show that the envelope E_P includes part of a CES production function.[5] We specify the efficiency of capital relative to labor as follows[6]:

$$\Psi(i) = \left(\frac{b}{1-b}\right)^{1+\gamma}\left(\frac{1-i}{i}\right)^{\gamma}, \qquad \gamma > 0, 1 > b > 0. \tag{14}$$

This assumption implies that the relative efficiency of capital to labor evaluated at a_t falls log-linearly over the ratio of the number of labor intermediate goods to the number of capital intermediate goods. Moreover, this ratio crucially

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influences the shape of our production function. Given assumption (14), we have $(Y_t/K_t)/(Y_t/L_t)$ and $(\partial Y_t/\partial K_t)/(\partial Y_t/\partial L_t)$ are proportional to $\{(1-a_t)/a_t\}^{1+\gamma}$ and $\{(1-a_t)/a_t\}^{\gamma}$, respectively. That is, the ratio of the number of labor intermediate goods to the number of capital intermediate goods plays a crucial role in determining not only the ratio of the average product of capital to the average product of labor, but also the ratio of the marginal product of capital to the marginal product of labor.

Given assumption (14), and using equation (13), we explain the growth rate of output per labor unit as

$$\frac{\dot{y}_t}{y_t} = a(k_t)\frac{\dot{k}_t}{k_t} = \frac{k_t^{1/(1+\gamma)}}{(1-b)/b + k_t^{1/(1+\gamma)}}\frac{\dot{k}_t}{k_t}. \tag{15}$$

By integrating this equation, we have the following CES production function:

$$y_t = C\{bk_t^{1/(1+\gamma)} + (1-b)\}^{1+\gamma}, \tag{16}$$

where $C = e^{-\gamma}$ because of the equality between equations (13) and (16).[7]

We can see that the elasticity of substitution between labor and capital of our production function becomes $\gamma^{-1} + 1$. While the elasticity of substitution is unity at each moment in time, mechanization makes possible taking a higher value in the long run. A decrease in γ that implies a decrease in the difficulty of mechanization raises the elasticity of substitution. In the case that $\gamma \to 0+0$, i.e., the efficiency of capital and labor intermediate goods would not change as a result of mechanization, our production function would become the Ak technology.

The results of estimating the elasticity of substitution could be a controversial issue. However, Sato (1975, Chapter 10), for example, explained that for the long run, elasticity of substitution between labor and capital could possibly exceed unity. Moreover, Duffy and Papageorgiou (2000) estimated a CES production function and found that the elasticity of substitution exceeded unity. Therefore, our model provides a micro foundation for these findings.

NOTES

1. As is explained in Section 3, our technological change is essentially the same as that of Zeira (1998).

2. Galor and Mountford (2004) argue that international trade has been a major cause of divergences in economic growth rates because, according to comparative advantages, countries specializing in the production of manufacturing goods have had sustained economic growth, whereas those countries specializing in the production of agricultural goods have stagnated.

3. Even if we considered the decentralized economy, we would obtain the identical equilibrium growth path noted in equations (11) and (12) because our model does not involve any market failure.

4. Jones and Manuelli (1990) investigated the relationship between economic growth and the convex technology in which the marginal product of capital remains positive.

5. Nakamura (2006) tried to derive a CES production function from fixed coefficients production functions by assuming that the efficiency of capital and labor inputs is influenced by the range of capital uses.

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6. The normalization such that $b + (1 - b) = 1$ helps us to consider the distribution parameter of a CES production function.

7. Given assumption (14), our production function noted in equation (13) becomes as $f(k_t) = \{[a(k_t)/b]^{a(k_t)}\{[1 - a(k_t)]/(1 - b)\}^{1-a(k_t)}\}^{-(1+\gamma)}e^{-\gamma}{k_t}^{a(k_t)}$. We assume that $\theta(i) = b^{1+\gamma}(1 - i)^{\gamma}$ and $\eta(i) = (1 - b)^{1+\gamma}i^{\gamma}$, implying assumption (14).

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