Student Problems

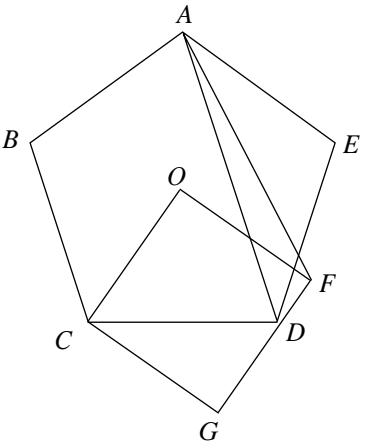
Students up to the age of 19 are invited to send solutions to either or both of the following problems to Stan Dolan, 126A Harpenden Road, St Albans, Herts., AL3 6BZ.

Two prizes will be awarded - a first prize of £25, and a second prize of £20 - to the senders of the most impressive solutions for either problem. It is, therefore, not necessary to submit solutions to both. Solutions should arrive by January 20th 2017. Please give your School year, the name and address of your School or College, and the name of a teacher through whom the award will be made. Please print your own name clearly! The names of all successful solvers will be published in the March 2017 edition.

Problem 2016.5 (Abdilkadir Altintaş)

Let $ABCDE$ be a regular pentagon with centre O and let $OCGF$ be a square, as shown.

Find $\angle DAF$.



Problem 2016.6 (Stan Dolan)

Let x, y, z and t be four non-zero real numbers such that

$$x - \frac{1}{y} = y - \frac{1}{z} = z - \frac{1}{t} = t - \frac{1}{x}.$$

Show that either $x = y = z = t$ or $xyzt = 1$.

https://doi.org/10.1017/mag.2016.130 Published online by Cambridge University Press

Solutions to 2016.3 and 2016.4

As usual after the Summer holidays, there were very few correct solutions. Problem 2016.3 was solved by Ruizhen Ma (Dulwich College). Problem 2016.4 was solved by Francis Zhu (Dulwich College).

Problem 2016.3 (Nick Lord)

Find all whole numbers n such that

$$\frac{a_1^n + a_2^n + \ \dots \ + a_n^n}{n} \geqslant a_1 a_2 \dots a_n$$

for all real numbers $a_1, a_2, \dots, a_n$ with $a_1 + a_2 + \ \dots \ + a_n \geqslant 0$.

Solution

The inequality is true for all even n and for $n = 1$ and 3.

If n is even, we can suppose all a_i are non-negative and then, as noted by Ruizhen, the result is the generalised AM-GM inequality.

If $n = 1$, the result is trivial.

If $n = 3$, then

$$\frac{a_1^3 + a_2^3 + a_3^3}{3} = a_1 a_2 a_3 + \tfrac{1}{6}(a_1 + a_2 + a_3)\left[(a_1 - a_2)^2 + (a_2 - a_3)^2 + (a_3 - a_1)^2\right]$$

$$\geqslant a_1 a_2 a_3.$$

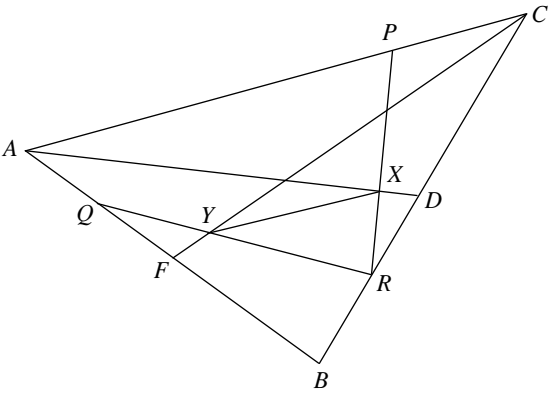
If $n \geqslant 5$ is odd, then the example

$$(a_1, a_2, \dots, a_n) = \left(-1, \frac{1}{2}, \frac{1}{2}, 0, \dots, 0\right)$$

shows that the inequality is not true.

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Problem 2016.4 (Geoff Strickland)



ABC is a triangle. Points *P*, *Q* and *R* lie on *CA*, *AB* and *BC*, respectively. So that

$$CP : PA = AQ : QB = BR : RC.$$

Points *F* and *D* are the midpoints of *AB* and *BC*, respectively. *AD* meets *PR* at *X* and *CF* meets *QR* at *Y*.

Prove that *XY* is parallel to *CA*.

Solution

Francis produced an imaginative but very long geometrical proof of this result. A much shorter proof can be given using vectors.

Let $CP : PA = \lambda : 1 - \lambda$ and take the origin *O* at the centroid of the triangle. Let *A*, *B* and *C* have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively, then

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = 0. \tag{1}$$

Then there are numbers v and t such that

$$\overrightarrow{OX} = v\mathbf{a} = t\big(\lambda\mathbf{a} + (1 - \lambda)\mathbf{c}\big) + (1 - t)\big(\lambda\mathbf{c} + (1 - \lambda)\mathbf{a}\big). \tag{2}$$

These equations can be used to find v as a function of λ. However, there is no need to do this since the same equations hold for $\overrightarrow{OY}$ but with $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ replaced by $\mathbf{c}$, $\mathbf{a}$ and $\mathbf{b}$, respectively.

Therefore $\overrightarrow{OY} = v\mathbf{c}$ and $\overrightarrow{XY} = v(\mathbf{c} - \mathbf{a})$. So *XY* is parallel to *CA*, as required.

The first prize of £25 is awarded to Francis Zhu. The second prize of £20 is awarded to Ruizhen Ma.

10.1017/mag.2016.130 STAN DOLAN

https://doi.org/10.1017/mag.2016.130 Published online by Cambridge University Press