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Integrals with a large parameter: a double complex integral with four nearly coincident saddle-points

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Abstract. The method of steepest descents for finding the asymptotic expansion of contour integrals of the form $\int g(z)\exp(Nf(z))\,dz$ where N is a real parameter tending to $+\infty$ is familiar. As is well known, the principal contributions to the asymptotic expansion come from certain critical points; the most important are saddle-points where $df/dz = 0$. The original contour is deformed into an equivalent contour consisting of paths of steepest descent through certain saddle-points, the *relevant* saddle-points. The determination of these is a global problem which can be solved explicitly only in simple cases. The function $f(z)$ may also depend on parameters. The position of the saddle-points depends on the parameters and at a certain set of values of the parameters it may happen that two or more saddle-points coincide. The ordinary expansion is then non-uniform, but appropriate uniform expansions have been shown to exist in earlier work.

Much less is known about double integrals of the form

$$\iint g(x,y)\exp(Nf(x,y))\,dx\,dy$$

where $g(x,y)$ and $f(x,y)$ are analytic functions and where the integration is taken over a two-dimensional surface in the four-dimensional space of the two complex variables x and y. It is not known even in principle whether the surface of integration can be deformed into equivalent surfaces of steepest descent through saddle-points where $\partial f/\partial x = \partial f/\partial y = 0$, nor is it known whether the principal contributions come from saddle-points.

The present paper is concerned with double integrals of the special form

$$I(N,\alpha,\beta,\gamma) = \iint G(x,y,\alpha,\beta,\gamma)\exp\{iN(\tfrac{1}{3}x^3+\tfrac{1}{3}y^3+\alpha x+\beta y+\gamma xy)\}\,dx\,dy,$$

where the complex parameters α, β and γ are small. There are then 4 nearly coincident saddle-points near $(x,y) = (0,0)$. A complete asymptotic expansion for such integrals is obtained in terms of the function

$$\mathscr{A}(X,Y,Z) = \iint \exp\{i(\tfrac{1}{3}\xi^3+\tfrac{1}{3}\eta^3+X\xi+Y\eta+Z\xi\eta)\}\,d\xi\,d\eta$$

and its first derivatives with respect to X, Y and Z. Note that $\mathscr{A}(\ )$ and its derivatives are functions of three variables whereas $I(N,\alpha,\beta,\gamma)$ is a function of four variables. It is shown that the difficulties arising from the near coincidence of the saddle-points can be overcome by a generalization of the method that was used for single integrals, while the

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required deformation of surfaces is achieved only by making use of special properties of the integrand, together with the condition that α, β and γ are small.

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1. *Introduction.* The method of steepest descents is a familiar method for finding the asymptotic expansion of contour integrals of the form $\int g(z) \exp(Nf(z))\, dz$, where $g(z)$ and $f(z)$ are analytic functions of the complex variable z, and N is a real parameter tending to $+\infty$. To simplify the exposition let us suppose that $g(z)$ and $f(z)$ are analytic in the whole of the complex plane and that the contour of integration begins and ends at ∞. The principal contributions to the asymptotic expansion then come from saddle-points, i.e. from points where $df/dz = 0$. Let all the paths of steepest descent from all the saddle-points be drawn (actually paths of descent would be sufficient but paths of steepest descent make the construction unique, cf. Ursell (10)). Let the original contour be deformed so as to be equivalent to the sum of paths of steepest descent through saddle-points; it is assumed that such a deformation is always possible. The saddle-points lying on the new path are the *relevant* saddle-points, and these are the only saddle-points that can contribute to the asymptotic expansion. In practice it is not easy to determine the relevant saddle-points unless $f(z)$ is of very simple form (10).

In many applications the functions $g(\ )$ and $f(\ )$ also involve parameters

$$(\alpha_1, \alpha_2, \ldots, \alpha_l), = \boldsymbol{\alpha}, \text{ say.}$$

When $\boldsymbol{\alpha}$ varies the position of the saddle-points also varies, and it may happen that two or more saddle-points coincide at a certain value of $\boldsymbol{\alpha}$, at $\boldsymbol{\alpha} = \mathbf{0}$, say. The asymptotic expansions given by the ordinary method of steepest descents (which involve exponentials) are then non-uniform, but it is possible to obtain uniform expansions which involve suitable transcendental functions; thus the case of a single parameter and two nearly coincident saddle-points involves Airy functions. The appropriate form for l parameters and m nearly coincident saddle-points was first given by Bleistein (3) but his proof based on induction was incomplete. A complete proof based on different ideas was given by Ursell (11) and is applicable to any number of parameters and nearly coincident saddle-points. (References to this paper will be identified by the symbol I.) The ideas underlying this proof will be used in the present paper and are summarized at the end of this Introduction.

Much less is known about the asymptotic expansion of double integrals of the form

$$\iint g(x, y) \exp\{Nf(x, y)\}\, dx\, dy, \tag{1.1}$$

(see appendix 1), where x and y are complex variables and the integration extends over a two-dimensional surface (in the four-dimensional space of two complex variables) such that the double integral is convergent. (See however Budden (5), appendix B.) In past work the surface of integration has usually been taken to be the real (x, y) plane, and $f(x, y)$ has been taken to be either real or pure imaginary. The former case can be treated by generalizing the method of Laplace to two dimensions, the latter by generalizing the method of stationary phase. (See, for example, Bleistein and

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Handelsman (4) §§ 8·2 and 8·4.) In both cases the principal contributions are found to come from saddle-points, i.e. from points where $\partial f/\partial x = \partial f/\partial y = 0$.

Suppose now that we wish to extend the ideas of the method of steepest descents to two-dimensional integrals, not necessarily over the real plane. To simplify the exposition let it be assumed that $g(x, y)$ and $f(x, y)$ are analytic for all x and y, and that the surface of integration has all its boundary points at ∞. Proceeding by analogy with the one-dimensional case we would then find all the saddle-points where $\partial f/\partial x = \partial f/\partial y = 0$, and through each of these we would construct two-dimensional surfaces of descent or of steepest descent. Curves and surfaces of steepest descent are defined in appendix 5 at the end of the paper; surfaces of steepest descent are generated by curves of steepest descent. It will be shown, however, that curves of steepest descent do not in general remain curves of steepest descent under an analytic transformation of complex variables although evidently they remain curves of descent. Similarly surfaces of steepest descent do not remain surfaces of steepest descent. There is thus little advantage in the use of surfaces of steepest descent.

When two-dimensional surfaces of descent through saddle-points have been constructed there still remains the problem of combining these (in four-dimensional space) with suitable two-dimensional surfaces at ∞ so as to obtain a surface equivalent to the original surface of integration, and thus to identify the relevant saddle-points. Not much is known about the solution of this problem in the general case.

In the absence of a general theory it is of interest to consider special cases which can be treated by special arguments. In the present paper the integral

$$I(1, N, \alpha, \beta, \gamma) = \iint_{S(1)} G(x, y, \alpha, \beta, \gamma) \exp\{iN(\tfrac{1}{3}x^3 + \tfrac{1}{3}y^3 + \alpha x + \beta y + \gamma xy)\}\, dx\, dy \quad (1{\cdot}2)$$

$$= \iint G(x, y, \alpha, \beta, \gamma) \exp\{iN\phi(x, y, \alpha, \beta, \gamma)\}\, dx\, dy, \quad \text{say,} \quad (1{\cdot}3)$$

will be treated. When all the variables and parameters are real, the phase function ϕ is one of the canonical forms of catastrophe theory, the hyperbolic umbilic catastrophe. By a complex change of variables and parameters this can be transformed into the elliptic umbilic catastrophe. The integrals involving these catastrophes have interesting optical applications, see e.g. Berry (1), Nye (9), and Berry, Nye & Wright (2). There are also important applications in chemistry (Connor (6)).

It will be assumed that N is positive and that $G(x, y, \alpha, \beta, \gamma)$ is an analytic function of $x, y, \alpha, \beta, \gamma$ for all x and y, and for all α, β and γ lying in some neighbourhood

$$|\alpha| + |\beta| + |\gamma| \leqslant \rho$$

of the origin of the parameter space (α, β, γ). The integration is then over a two-dimensional surface, $S(1)$ say, with x and y each varying from $-\infty$ to $+\infty$ when α, β and γ are real; or equivalently from $\infty \exp(\frac{5}{6}\pi i)$ to $\infty \exp(\frac{1}{6}\pi i)$ to ensure the absolute convergence of the integral when α, β and γ are complex. (Surfaces of integration are discussed in detail in Sections 4 and 5 below.) It is readily seen that there are 4 saddle-points, obtained by solving the equations $\partial\phi/\partial x = \partial\phi/\partial y = 0$. (See appendix 2.) When α, β and γ are small these saddle-points are all nearly coincident at

$$(x, y) = (0, 0),$$

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and this is the case with which the present paper will be concerned. It will be shown that a complete asymptotic expansion can be obtained in terms of a certain function of three complex variables; the leading terms were given correctly but without proof by Connor (6).

The proof is a generalization to double integrals of the proof given in I for one-dimensional integrals, and it has therefore been thought unnecessary to give it in full detail. As in I, the argument begins with the construction of a Bleistein sequence which makes possible an integration by parts at every step. The main theorem then asserts a certain inequality which is proved if a certain (stronger) set of boundedness conditions can be proved. Such boundedness conditions are easily proved when the saddle-points are well separated; but here the saddle-points are nearly coincident. Two additional complex variables σ and τ (splitting parameters) are therefore introduced; when these are bounded away from 0 they cause the saddle-points to be well separated, and for these we can prove the boundedness conditions. The boundedness conditions for $\sigma = \tau = 0$ (nearly coincident saddle points) are then obtained from the maximum-modulus theorem for the complex variables σ and τ. This part of the proof is similar to I. The deformation of the surfaces of integration is treated in a less general manner, and uses the special property that $\phi(x, y, \sigma, \tau, 0)$ is the product of a function of x and a function of y.

2. *The form of the asymptotic expansion.* Throughout the present work it will be assumed that the function $G(x, y, \alpha, \beta, \gamma)$ is analytic for all x and y, and that the (two-dimensional) surface of integration extends to infinity, and has no part of its boundary at a finite distance from the origin $x = y = 0$. It is, however, expected that much of the present work can be adapted to finite surfaces of integration; compare the discussion in I, Section 6 for one-dimensional integrals with finite limits of integration.

The following is an adaptation of I, Section 3. Generalizing Bleistein ((3), p. 540), and noting that

$$\frac{\partial \phi}{\partial x} = x^2 + \alpha + \gamma y, \quad \frac{\partial \phi}{\partial y} = y^2 + \beta + \gamma x, \tag{2·1}$$

let us introduce the function sequences

$$G_k(x, y, \alpha, \beta, \gamma), \quad H_k(x, y, \alpha, \beta, \gamma), \quad K_k(x, y, \alpha, \beta, \gamma)$$
$$E_k(\alpha, \beta, \gamma), \quad A_k(\alpha, \beta, \gamma), \quad B_k(\alpha, \beta, \gamma), \quad C_k(\alpha, \beta, \gamma)$$

defined by the following equations:

$$\left.\begin{aligned} G(x, y, \alpha, \beta, \gamma) \equiv G_0(x, y, \alpha, \beta, \gamma) &= E_0 + A_0 x + B_0 y + C_0 xy \\ &\quad + (x^2 + \alpha + \gamma y) H_0 + (y^2 + \beta + \gamma x) K_0, \\ \frac{\partial H_0}{\partial x} + \frac{\partial K_0}{\partial y} \equiv G_1(\,) &= E_1 + A_1 x + B_1 y + C_1 xy \\ &\quad + (x^2 + \alpha + \gamma y) H_1 + (y^2 + \beta + \gamma x) K_1, \\ &\cdots\cdots\cdots\cdots \\ \frac{\partial H_k}{\partial x} + \frac{\partial K_k}{\partial y} \equiv G_{k+1}(\,) &= E_{k+1} + A_{k+1} x + B_{k+1} y + C_{k+1} xy \\ &\quad + (x^2 + \alpha + \gamma y) H_{k+1} + (y^2 + \beta + \gamma x) K_{k+1}. \end{aligned}\right\} \tag{2·2}$$

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It is shown in appendix 3 that such function sequences exist. Evidently the functions H_k and K_k are not uniquely defined for it is obvious that the equations would remain valid if H_k and K_k were replaced by $H_k+(y^2+\beta+\gamma x)\,F_k$ and $K_k-(x^2+\alpha+\gamma y)\,F_k$ where F_k is an arbitrary analytic function. It can however be seen (and it will follow from the form of the asymptotic expansion) that the coefficients E_k, A_k, B_k, C_k must be unique.

Stokes's theorem states that

$$\oint u\,dx+v\,dy=\iint\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)dx\,dy, \tag{2·3}$$

where the integration is taken over a finite area, and the line integral is taken along the boundary of the area. Taking $u=-K\exp(iN\phi)$ and $v=H\exp(iN\phi)$, then, noting that $\partial\phi/\partial x=x^2+\alpha+\gamma y$, and $\partial\phi/\partial y=y^2+\beta+\gamma x$, we obtain the result

$$\begin{aligned}\iint\exp(iN\phi)\{(x^2+\alpha+\gamma y)\,H+(y^2+\beta+\gamma x)\,K\}\,dx\,dy\\ =\frac{i}{N}\iint\exp(iN\phi)\left(\frac{\partial H}{\partial x}+\frac{\partial K}{\partial y}\right)dx\,dy-\frac{i}{N}\oint\exp(iN\phi)\,(H\,dy-K\,dx),\end{aligned} \tag{2·4}$$

which represents an integration by parts. By repeated integration by parts it is found that

$$\begin{aligned}&\iint G_0(x,y,\alpha,\beta,\gamma)\exp(iN\phi)\,dx\,dy-\left(E_0+\frac{i}{N}E_1+\ldots+\left(\frac{i}{N}\right)^n E_n\right)\iint\exp(iN\phi)\,dx\,dy\\ &-\left(A_0+\frac{i}{N}A_1+\ldots+\left(\frac{i}{N}\right)^n A_n\right)\iint x\exp(iN\phi)\,dx\,dy\\ &-\left(B_0+\frac{i}{N}B_1+\ldots+\left(\frac{i}{N}\right)^n B_n\right)\iint y\exp(iN\phi)\,dx\,dy\\ &-\left(C_0+\frac{i}{N}C_1+\ldots+\left(\frac{i}{N}\right)^n C_n\right)\iint xy\exp(iN\phi)\,dx\,dy\\ &+\int\exp(iN\phi)\left\{\left(\frac{i}{N}H_0(x,y)+\left(\frac{i}{N}\right)^2 H_1(x,y)+\ldots+\left(\frac{i}{N}\right)^{n+1}H_n(x,y)\right)dy\right.\\ &\left.-\left(\frac{i}{N}K_0(x,y)+\left(\frac{i}{N}\right)^2 K_1(x,y)+\ldots+\left(\frac{i}{N}\right)^{n+1}K_n(x,y)\right)dx\right\}\\ &\qquad=\left(\frac{i}{N}\right)^{n+1}\iint G_{n+1}(x,y,\alpha,\beta,\gamma)\exp(iN\phi)\,dx\,dy.\end{aligned} \tag{2·5}$$

In our applications it would be sufficient to take the integrations over any finite surface near the point $x=y=0$ since the factor $\exp(iN\phi)$ ensures that the contribution from the more distant parts is exponentially negligible. Similarly the line integral over the boundary is exponentially negligible and henceforth it will be formally omitted. We note that the integral (2·5) is of the same form as the original integral (1·2).

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The preceding calculation suggests the following theorem.

THEOREM 1. (Cf. I, Theorem 2.) *There are positive numbers N_0 and ρ, such that, for all real $N > N_0$ and for all complex α, β and γ satisfying $|\alpha| + |\beta| + |\gamma| \leqslant \rho$, we have*

$$I(1, N, \alpha, \beta, \gamma) = \iint_{S(1)} G(x, y, \alpha, \beta, \gamma) \exp\{iN(\tfrac{1}{3}x^3 + \tfrac{1}{3}y^3 + \alpha x + \beta y + \gamma xy)\}\, dx\, dy \tag{2·6}$$

$$= J_{00}(1, N, \alpha, \beta, \gamma) \sum_0^n \left(\frac{i}{N}\right)^k E_k(\alpha, \beta, \gamma)$$

$$+ J_{10}(1, N, \alpha, \beta, \gamma) \sum_0^n \left(\frac{i}{N}\right)^k A_k(\alpha, \beta, \gamma)$$

$$+ J_{01}(1, N, \alpha, \beta, \gamma) \sum_0^n \left(\frac{i}{N}\right)^k B_k(\alpha, \beta, \gamma)$$

$$+ J_{11}(1, N, \alpha, \beta, \gamma) \sum_0^n \left(\frac{i}{N}\right)^k C_k(\alpha, \beta, \gamma)$$

$$+ \mathscr{R}_{n+1}(N, \alpha, \beta, \gamma), \tag{2·7}$$

where E, A_k, B_k, C_k are defined by (2·2) *above, where the limits of integration of $S(1)$ in* (2·6) *are at $\infty \exp(\frac{5}{6}\pi i)$ and $\infty \exp(\frac{1}{6}\pi i)$ for both x and y, and where*

$$|\mathscr{R}_{n+1}(N, \alpha, \beta, \gamma)| \leqslant \frac{M_{n+1}}{N^{n+1}} (|J_{00}(1, N, \alpha, \beta, \gamma)| + |J_{10}(1, N, \alpha, \beta, \gamma)| + |J_{01}(1, N, \alpha, \beta, \gamma)| + |J_{11}(1, N, \alpha, \beta, \gamma)|) \tag{2·8}$$

for some constant M_{n+1} depending on $G_0(x, y, \alpha, \beta, \gamma)$. Here by definition

$$J_{pq}(j, N, \alpha, \beta, \gamma) = \iint_{S(j)} x^p y^q \exp(iN\phi)\, dx\, dy$$

$$= \frac{1}{N^{\frac{1}{3}(p+q+2)}} \mathscr{A}_{pq}(j, N^{\frac{2}{3}}\alpha, N^{\frac{2}{3}}\beta, N^{\frac{1}{3}}\gamma), \tag{2·9}$$

where

$$\mathscr{A}_{pq}(j, X, Y, Z) = \iint_{S(j)} \xi^p \eta^q \exp\{i(\tfrac{1}{3}\xi^3 + \tfrac{1}{3}\eta^3 + X\xi + Y\eta + Z\xi\eta)\}\, d\xi\, d\eta. \tag{2·10}$$

(The surfaces $S(1)$, $S(2)$, $S(3)$, $S(4)$ will be defined later; see equation (3·4)). Theorem 1 will follow from a stronger result, stated here as

THEOREM 2. (Cf. I, Theorem 3.) *With the same assumptions and the same notation as in Theorem 1 we have*:

$$\iint_{S(1)} G(\,) \exp(iN\phi)\, dx\, dy$$

$$= E(N, \alpha, \beta, \gamma) J_{00}(1, N, \alpha, \beta, \gamma) + A(N, \alpha, \beta, \gamma) J_{10}(1, N, \alpha, \beta, \gamma)$$

$$+ B(N, \alpha, \beta, \gamma) J_{01}(1, N, \alpha, \beta, \gamma) + C(N, \alpha, \beta, \gamma) J_{11}(1, N, \alpha, \beta, \gamma), \tag{2·11}$$

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where the functions E, A, B, C, defined by (3·7) below satisfy the inequalities

$$\left.\begin{aligned}
&\left|E(N,\alpha,\beta,\gamma)-\sum_0^n\left(\frac{i}{N}\right)^k E_k(\alpha,\beta,\gamma)\right| \leqslant \frac{M_{n+1}}{N^{n+1}},\\
&\left|A(\quad)-\sum_0^n\left(\frac{i}{N}\right)^k A_k(\quad)\right| \leqslant \frac{M_{n+1}}{N^{n+1}},\\
&\left|B(\quad)-\sum_0^\infty\left(\frac{i}{N}\right)^k B_k(\quad)\right| \leqslant \frac{M_{n+1}}{N^{n+1}},\\
&\left|C(\quad)-\sum_0^\infty\left(\frac{i}{N}\right)^k C_k(\quad)\right| \leqslant \frac{M_{n+1}}{N^{n+1}}.
\end{aligned}\right\} \tag{2·12}$$

The proof of Theorem 2 will be given in Section 6 below.

3. *Definitions and notations.* As has already been stated, only infinite limits of integration will be considered in the present work, but there can be little doubt that the arguments can be modified (as in I, § 6) to take account of finite surfaces of integration. It will be assumed that $G(x,y,\alpha,\beta,\gamma)$ is suitably bounded at ∞ as a function of x and y, so that all the integrals under consideration are convergent. It will be convenient to consider not only the integral (2·6) over $S(1)$ but all the integrals

$$\iint G(x,y,\alpha,\beta,\gamma)\exp\{iN(\tfrac{1}{3}x^3+\tfrac{1}{3}y^3+\alpha x+\beta y+\gamma xy)\}\,dx\,dy, \tag{3·1}$$

where the limits of integration are two of the three valleys $\infty\exp(\frac{1}{6}\pi i)$, $\infty\exp(\frac{5}{6}\pi i)$, $\infty\exp(\frac{9}{6}\pi i)$ in the x-plane, and $\infty\exp(\frac{1}{6}\pi i)$, $\infty\exp(\frac{5}{6}\pi i)$, $\infty\exp(\frac{9}{6}\pi i)$ in the y-plane (Cf. I, §4.) Thus there are 36 possible integrals. We shall use the notation $I(p\xrightarrow[x]{} q, r\xrightarrow[y]{} s)$ to denote the integral (3·1) when the limits of integration are

$$\infty\exp\tfrac{1}{6}(p\pi i) \quad\text{and}\quad \infty\exp\tfrac{1}{6}(q\pi i) \text{ in the } x\text{-plane,}$$

and

$$\infty\exp\tfrac{1}{6}(r\pi i) \quad\text{and}\quad \infty\exp\tfrac{1}{6}(s\pi i) \text{ in the } y\text{-plane.}$$

(The notation $(p\xrightarrow[x]{} q,\ r\xrightarrow[y]{} s)$ will be used for the corresponding surfaces of integration.) Evidently we have

$$I(9\to 1,\ 5\to 9) = I(1\to 9,\ 9\to 5) = -I(1\to 9,\ 5\to 9) = -I(9\to 1,\ 9\to 5) \tag{3·2}$$

and similar equations; and also

$$I(1\to 9,\ 5\to 9) = I(5\to 9,\ 5\to 9) - I(5\to 1,\ 5\to 9) \tag{3·3}$$

and similar equations. Thus each of the 36 integrals can be expressed as a linear combination of any four independent integrals, say of the four integrals

$$\left.\begin{aligned}
&I(5\xrightarrow[x]{} 1,\ 5\xrightarrow[y]{} 1) \quad \text{over} \quad S(1),\\
&I(5\xrightarrow[x]{} 9,\ 5\xrightarrow[y]{} 1) \quad \text{over} \quad S(2),\\
&I(5\xrightarrow[x]{} 1,\ 5\xrightarrow[y]{} 9) \quad \text{over} \quad S(3),\\
&I(5\xrightarrow[x]{} 9,\ 5\xrightarrow[y]{} 9) \quad \text{over} \quad S(4),
\end{aligned}\right\} \tag{3·4}$$

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where the symbols $S(1)$, $S(2)$, $S(3)$, $S(4)$ denote the corresponding surfaces of integration. We shall write

$$\iint_{S(j)} G(\,)\exp(iN\phi)\,dx\,dy = I(j,N,\alpha,\beta,\gamma) \quad (j=1,2,3,4), \tag{3·5}$$

and similarly

$$\iint_{S(j)} x^p y^q \exp(iN\phi)\,dx\,dy = J_{pq}(j,N,\alpha,\beta,\gamma) \quad (j=1,2,3,4), \tag{3·6}$$

Sometimes abbreviations, such as $I(j)$, $J_{pq}(j)$, will be used for these integrals.

We now define the functions

$$E(N,\alpha,\beta,\gamma),\ A(N,\alpha,\beta,\gamma),\ B(N,\alpha,\beta,\gamma) \quad \text{and} \quad C(N,\alpha,\beta,\gamma)$$

as the solutions of the four equations

$$I(j) = EJ_{00}(j) + AJ_{10}(j) + BJ_{01}(j) + CJ_{11}(j) \quad (j=1,2,3,4); \tag{3·7}$$

cf. Theorem 2 above, and I, equation (4·3). Then by linear combination the corresponding equation is satisfied for all the 36 possible surfaces of integration. Solving the equations (3·7) we find that

$$E(N,\alpha,\beta,\gamma) = \Delta_{00}/\Delta, \tag{3·8}$$

where

$$\Delta_{00}(N,\alpha,\beta,\gamma) = \begin{vmatrix} I(1), J_{10}(1), J_{01}(1), J_{11}(1) \\ I(2), J_{10}(2), J_{01}(2), J_{11}(2) \\ I(3), J_{10}(3), J_{01}(3), J_{11}(3) \\ I(4), J_{10}(4), J_{01}(4), J_{11}(4) \end{vmatrix} \tag{3·9}$$

$$= [I(1), J_{10}(2), J_{01}(3), J_{11}(4)], \tag{3·10}$$

(in an abbreviated notation identifying determinants by their diagonal elements), and

$$\Delta(N,\alpha,\beta,\gamma) = [J_{00}(1), J_{10}(2), J_{01}(3), J_{11}(4)]. \tag{3·11}$$

Similarly

$$A = \Delta_{10}/\Delta,\ B = \Delta_{01}/\Delta,\ C = \Delta_{11}/\Delta, \tag{3·12}$$

where

$$\Delta_{10}(N,\alpha,\beta,\gamma) = [J_{00}(1), I(2), J_{01}(3), J_{11}(4)], \tag{3·13}$$

$$\Delta_{01}(N,\alpha,\beta,\gamma) = [J_{00}(1), J_{10}(2), I(3), J_{11}(4)], \tag{3·14}$$

$$\Delta_{11}(N,\alpha,\beta,\gamma) = [J_{00}(1), J_{10}(2), J_{01}(3), I(4)]. \tag{3·15}$$

It will be seen later (see appendix 4), that Δ can be found explicitly,

$$\Delta(N,\alpha,\beta,\gamma) = \Delta(N,0,0,0)\exp(iN\gamma^3), \tag{3·16}$$

where

$$\Delta(N,0,0,0) = (2\pi/N)^4 \neq 0. \tag{3·17}$$

We note that the determinants Δ_{00}, Δ_{10}, Δ_{01}, Δ_{11}, are analytic functions of α, β and γ in any domain in which the elements $I(j)$ are analytic. (Clearly all the elements $J_{pq}(j)$ are analytic.) To estimate these determinants it will be convenient to introduce two

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additional complex variables σ and τ. (Cf. I, equation 4·9.) We define a modified function

$$\begin{aligned}\phi^*(x, y, \alpha, \beta, \gamma, \sigma, \tau) &= \phi(x, y, \alpha+\sigma, \beta+\tau, \gamma)\\ &= \tfrac{1}{3}x^3+\tfrac{1}{3}y^3+(\alpha+\sigma)x+(\beta+\tau)y+\gamma xy\\ &= \phi(x, y, \alpha, \beta, \gamma)+\sigma x+\tau y, \end{aligned} \tag{3·18}$$

and another modified function

$$\phi^{**}(x, y, \sigma, \tau) = \phi(x, y, 0, 0, 0, \sigma, \tau) = \phi(x, y, \sigma, \tau, 0). \tag{3·19}$$

Similarly we define integrals

$$I^*(j, N, \alpha, \beta, \gamma, \sigma, \tau) = \iint_{S(j)} G(x, y, \alpha, \beta, \gamma) \exp(iN\phi^*)\, dx\, dy \tag{3·20}$$

and

$$I^{**}(j, N, \sigma, \tau) = \iint_{S(j)} G(x, y, 0, 0, 0) \exp(iN\phi^{**})\, dx\, dy, \tag{3·21}$$

and J^*_{pq}, J^{**}_{pq}; also determinants like Δ^*_{00} and Δ^{**}_{00} which involve elements I^*, J^*_{pq} and I^{**}, J^{**}_{pq} respectively instead of I, J_{pq}. All these functions are evidently analytic functions of σ and τ for all values of σ and τ.

4. *Global deformation of surfaces of integration for J^{**}_{pq} and I^{**}.* Throughout this and the following section it will be assumed that σ and τ are bounded away from 0,

$$|\sigma| = |\tau| = r, \text{ say;}$$

it will also be assumed that α, β and γ are sufficiently small, $|\alpha|+|\beta|+|\gamma| \leqslant \rho_1(r)$, say, to ensure that the four saddle-points remain well separated; cf. I, lemma 1. It will be seen that estimates obtained for well-separated saddle-points can nevertheless, as in I, be applied to nearly coincident saddle-points.

We shall in the next section obtain estimates for the integrals

$$J^*_{pq}(j, N, \alpha, \beta, \gamma, \sigma, \tau) = J_{pq}(j, N, \alpha+\sigma, \beta+\tau, \gamma) \quad \text{and} \quad I^*(j, N, \alpha, \beta, \gamma, \sigma, \tau)$$

by integrating over appropriate surfaces of integration equivalent to $S(j)$. These surfaces need to be defined with some care; in the present section we shall consider surfaces (but not estimates) for J^{**}_{pq} and I^{**}.

It will be helpful to consider first the integrals

$$J_{pq}(j, N, \sigma, \tau, 0) = J^{**}_{pq}(j, N, \sigma, \tau),$$

before considering the integrals

$$I^*(j, N, 0, 0, 0, \sigma, \tau) = I^{**}(j, N, \sigma, \tau).$$

We observe that the integral $J^{**}_{pq}(1, N, \sigma, \tau)$

$$\begin{aligned}&= \left(\int_{\infty \exp(5\pi i/6)}^{\infty \exp(\pi i/6)} x^p \exp\{iN(\tfrac{1}{3}x^3+\sigma x)\}dx\right)\left(\int_{\infty \exp(5\pi i/6)}^{\infty \exp(\pi i/6)} y^q \exp\{iN(\tfrac{1}{3}y^3+\tau y)\}dy\right).\\ &= J_p(5 \xrightarrow[x]{} 1)\, J_q(5 \xrightarrow[y]{} 1) \text{ in an obvious notation,}\end{aligned} \tag{4·1}$$

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is the product of single integrals. The asymptotic expansion of J^{**}_{pq} is readily found (but is not needed) because each single integral is a derivative either of the Airy integral

$$J_0(5 \xrightarrow[x]{} 1) = \int_{\infty \exp(5\pi i/6)}^{\infty \exp(\pi i/6)} \exp\{iN(\tfrac{1}{3}x^3 + \sigma x)\}\, dx = 2\pi N^{-\frac{1}{3}} \operatorname{Ai}(N^{\frac{2}{3}}\sigma), \tag{4·2}$$

or of the corresponding integral $J_0(5 \xrightarrow[y]{} 1)$ involving τ; see e.g. Jeffreys (8) (§ 2·5). The relevant saddle-points for $J_0(5 \xrightarrow[x]{} 1)$ are known to be

$$\text{at} \quad x = i\sigma^{\frac{1}{2}} \qquad \text{when} \quad \tfrac{1}{6}\pi < \operatorname{ph}(i\sigma^{\frac{1}{2}}) < \tfrac{5}{6}\pi, \tag{4·3}$$

$$\text{at} \quad x = i\sigma^{\frac{1}{2}} \quad \text{and} \quad x = -i\sigma^{\frac{1}{2}} \qquad \text{when} \quad \tfrac{5}{6}\pi < \operatorname{ph}(i\sigma^{\frac{1}{2}}) < \tfrac{7}{6}\pi, \tag{4·4}$$

$$\text{at} \quad x = -i\sigma^{\frac{1}{2}} \qquad \text{when} \quad \tfrac{7}{6}\pi < \operatorname{ph}(i\sigma^{\frac{1}{2}}) < \tfrac{13}{6}\pi. \tag{4·5}$$

We now define a surface of integration $S^{**}(1)$ equivalent to $S(1)$ when the integrals $J_p(5 \xrightarrow[x]{} 1)$ and $J_q(5 \xrightarrow[y]{} 1)$ each involve only one saddle-point in their respective complex planes; to fix ideas, let us suppose that

$$\tfrac{1}{6}\pi < \operatorname{ph}(i\sigma^{\frac{1}{2}}) < \tfrac{5}{6}\pi \quad \text{and} \quad \tfrac{1}{6}\pi < \operatorname{ph}(i\tau^{\frac{1}{2}}) < \tfrac{5}{6}\pi. \tag{4·6}$$

Near the saddle-point $x = i\sigma^{\frac{1}{2}}$, let a new variable ξ_0 be introduced (as in the ordinary method of steepest descents) by the relation $i(\frac{1}{3}x^3 + \sigma x) = -\frac{2}{3}\sigma^{\frac{3}{2}} - \xi_0^2$, or equivalently by

$$\tfrac{1}{3}i(x - i\sigma^{\frac{1}{2}})^2(x + 2i\sigma^{\frac{1}{2}}) = -\xi_0^2; \tag{4·7}$$

let the square root be taken so that $\xi_0 \sim \sigma^{\frac{1}{4}}(x - i\sigma^{\frac{1}{2}})$ near $\xi_0 = 0$. It is known that, near the saddle-point, ξ_0 is an analytic function of $x - i\sigma^{\frac{1}{2}}$, and $x - i\sigma^{\frac{1}{2}}$ is an analytic function of ξ_0. This is a local mapping which can be extended to a global mapping along the path of steepest descent in the x-plane. For it can be shown by conformal mapping (as in Ursell (10), example 1) that x moves along the path of steepest descent from $\infty \exp(\frac{5}{6}\pi i)$ through $i\sigma^{\frac{1}{2}}$ to $\infty \exp(\frac{1}{6}\pi i)$ when ξ_0 moves along real values from $-\infty$ to $+\infty$. Similarly there is a regular transformation of the whole path of steepest descent in the y-plane to a variable η_0 such that

$$\tfrac{1}{3}i(y - i\tau^{\frac{1}{2}})^2(y + 2i\tau^{\frac{1}{2}}) = -\eta_0^2, \tag{4·8}$$

where η_0 moves from $-\infty$ to $+\infty$ along real values.

We can now define the surface $S^{**}(1)$ through the saddle-point $(i\sigma^{\frac{1}{2}}, i\tau^{\frac{1}{2}})$: it is the surface in the four-dimensional (x, y) space on which ξ_0 and η_0 are real; in this definition it is still assumed that $\frac{1}{6}\pi < \operatorname{ph}(i\sigma^{\frac{1}{2}}) < \frac{5}{6}\pi$ and $\frac{1}{6}\pi < \operatorname{ph}(i\tau^{\frac{1}{2}}) < \frac{5}{6}\pi$, so that only a single saddle-point is involved. (It is not difficult to see that $S^{**}(1)$ is a surface of steepest descent.) In a similar way we can define a wider class of surfaces of descent by taking ξ_0 and η_0 as complex-valued coordinates when x and y lie on paths of descent (not necessarily paths of steepest descent) in the x-plane and y-plane respectively; see Ursell (10), example 1.

Suppose next that τ is kept fixed while σ moves along the circle $|\sigma| = r$ into the sector $\frac{5}{6}\pi < \operatorname{ph}(i\sigma^{\frac{1}{2}}) < \pi$ where two saddle-points are relevant. To find the appropriate path in the x-plane the contour $(5 \xrightarrow[x]{} 1)$ is deformed (as is well known,

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see e.g. Jeffreys(8), ch. 2) into the sum of paths through saddle-points in the complex x-plane,

$$(5 \xrightarrow[x]{} 1) = (5 \xrightarrow[x]{} 9) + (9 \xrightarrow[x]{} 1), \tag{4·9}$$

while the path on the y-plane remains unchanged. We now observe that this deformation of curves can also be regarded as a deformation of surfaces in a three-dimensional subspace in which x occupies two dimensions and y (more precisely, η_0) occupies one dimension,

$$(5 \xrightarrow[x]{} 1,\ 5 \xrightarrow[y]{} 1) = (5 \xrightarrow[x]{} 9,\ 5 \xrightarrow[y]{} 1) + (9 \xrightarrow[x]{} 1,\ 5 \xrightarrow[y]{} 1); \tag{4·10}$$

thus for this deformation the full four-dimensional space is not needed. Consider in this way the first surface on the right-hand side of (3·4). This involves only the single saddle-point $(i\sigma^{\frac{1}{2}}, i\tau^{\frac{1}{2}})$, and it can therefore be deformed into a surface S^{**} with real coordinates ξ_0 and η_0 as in the previous example. Similarly the second surface on the right-hand side of (3·4), through the single saddle-point $(-i\sigma^{\frac{1}{2}}, i\tau^{\frac{1}{2}})$, can be deformed into a surface S^{**}.

There is, however, a discontinuous change in the definition of these coordinate systems as $i\sigma^{\frac{1}{2}}$ crosses the Stokes line ph $(i\sigma^{\frac{1}{2}}) = \frac{5}{6}\pi$. A second relevant saddle-point is then introduced at $-i\sigma^{\frac{1}{2}}$ but its contribution is exponentially negligible for ph $(i\sigma^{\frac{1}{2}})$ near $\frac{5}{6}\pi$, e.g. when $\frac{5}{6}\pi \leqslant$ ph $(i\sigma^{\frac{1}{2}}) \leqslant \frac{11}{12}\pi$. The path of steepest descent from $i\sigma^{\frac{1}{2}}$ in one direction then still terminates at $\infty \exp(\frac{5}{6}\pi i)$. In the opposite direction the path of steepest descent now terminates at $\infty \exp(\frac{9}{6}\pi i)$ but it can be shown by a straightforward application of the method of Ursell((10) example 1) that a path of descent can still be drawn from $i\sigma^{\frac{1}{2}}$ to terminate at $\infty \exp(\frac{1}{6}\pi i)$ and that this path can even be chosen to be a path of steepest descent locally near $i\sigma^{\frac{1}{2}}$. Paths satisfying the latter condition will be described as paths of *local steepest descent* and the variable ξ_0 along such a path is real near the saddle-point $i\sigma^{\frac{1}{2}}$. The variation of $i\phi^{**}$ along the corresponding two-dimensional surface of descent is real or nearly real so that the method of Laplace can be shown to remain applicable. Similarly the movement of τ across the Stokes line in the y-plane can be treated by a deformation in a different three-dimensional subspace. It is now evident how for the integrals J^{**}_{pq} the apparent difficulties associated with Stokes lines can be overcome by using the theory of a single complex variable. (An alternative treatment of Stokes lines was given in I p. 58. This is not so suitable here because a further local deformation will be described later, when integrals I^* are considered.) Consider next the double integral

$$I^{**}(1, N, \sigma, \tau) = \iint_{S(1)} G(x, y, 0, 0, 0) \exp\{iN(\tfrac{1}{3}x^3 + \tfrac{1}{3}y^3 + \sigma x + \tau y)\}\, dx\, dy, \tag{4·11}$$

which is not in general a product of single integrals. Suppose at first that $S(1)$ involves only the single saddle point $(i\sigma^{\frac{1}{2}}, i\tau^{\frac{1}{2}})$ in four dimensions. Let the same new variables ξ_0 and η_0 be introduced as in (4·7), (4·8) above, and let the surface $S(1)$ be deformed (if necessary) into the equivalent surface $S^{**}(1)$ on which ξ_0 and η_0 are real. In terms of these coordinates we can write

$$I^{**}(1, N, \sigma, \tau) = \exp\{iN\phi^{**}(i\sigma^{\frac{1}{2}}, i\tau^{\frac{1}{2}}, \sigma, \tau)\} \iint_{-\infty}^{\infty} G^{**}(\xi_0, \eta_0) \exp\{-N(\xi_0^2 + \eta_0^2)\}\, d\xi_0\, d\eta_0, \tag{4·12}$$

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and this can be expanded asymptotically by the well-known two-dimensional method of Laplace, ((4) § 8·2) but we shall not need the result. If more than one saddle-point is involved then we can deform the surface $S(1)$ by three-dimensional deformations into the sum of surfaces of steepest descents exactly as for the integrals J^{**}_{pq}; these deformations are determined by considering the paths of steepest descent for the factor $\exp\{iN(\frac{1}{3}x^3+\sigma x)\}$ in the x-plane and the factor $\exp\{iN(\frac{1}{3}y^3+\tau y)\}$ in the y-plane. (Near Stokes lines in these planes we can use paths of descent or paths of local steepest descent instead.)

5. *Local deformations and estimates for I^*.* We can now estimate the integral

$$I^*(1,N,\alpha,\beta,\gamma,\sigma,\tau) = \iint_{S(1)} G(x,y,\alpha,\beta,\gamma)\exp\{iN(\tfrac{1}{3}x^3+\tfrac{1}{3}y^3+(\alpha+\sigma)x + (\beta+\tau)y+\gamma xy)\}\,dx\,dy$$
$$= \iint G(x,y,\alpha,\beta,\gamma)\exp(iN\phi^*)\,dx\,dy \qquad (5{\cdot}1)$$

by means of the following lemma.

LEMMA 1. *Let $(x_j(\alpha+\sigma,\beta+\tau,\gamma), y_j(\alpha+\sigma,\beta+\tau,\gamma))$ be a saddle point of ϕ^*, i.e. a solution of the equations*

$$\frac{\partial\phi^*}{\partial x} = \frac{\partial\phi^*}{\partial y} = 0. \qquad (5{\cdot}2)$$

Given r, there are positive numbers ϵ, δ and ρ_2 with the following property. For any fixed σ and τ satisfying $|\sigma| = |\tau| = r$, and for any α, β and γ satisfying $|\alpha|+|\beta|+|\gamma| \leqslant \rho_2$, there is a 4-dimensional neighbourhood $|x-x_j^*|+|y-y_j^*| < \epsilon$ of the saddle point (x_j^*, y_j^*) in which there is an analytic transformation

$$\left.\begin{aligned}\xi &= \xi(x-x_j^*, y-y_j^*, \alpha+\sigma, \beta+\tau, \gamma),\\ \eta &= \eta(x-x_j^*, y-y_j^*, \alpha+\sigma, \beta+\tau, \gamma),\end{aligned}\right\} \qquad (5{\cdot}3)$$

and an inverse analytic transformation in a neighbourhood $|\xi|+|\eta| < \delta$, such that

$$i\phi^*(x,y) - i\phi^*(x_j^*, y_j^*) = -\xi^2-\eta^2.$$

When $\alpha=\beta=\gamma=0$, this transformation reduces to the transformation

$$\left.\begin{aligned}\xi &= \xi_0(x-x_j^*,\sigma)\\ \eta &= \eta_0(y-y_j^*,\tau)\end{aligned}\right\} \qquad (5{\cdot}4)$$

obtained in (4·7) and (4·8) above.

Proof. Let σ and τ be fixed points on $|\sigma| = r$ and $|\tau| = r$ respectively. Near the saddle point there is evidently an expansion of the form

$$i\phi^*(x,y) - i\phi^*(x_j^*, y_j^*) = ix_j^*(x-x_j^*)^2 + \tfrac{1}{3}i(x-x_j^*)^3 + iy_j^*(y-y_j^*)^2 + \tfrac{1}{3}i(y-y_j^*)^3 + i\gamma(x-x_j^*)(y-y_j^*), \qquad (5{\cdot}5)$$

where the coefficients of $(x-x_j^*)^2$ and of $(y-y_j^*)^2$ are bounded away from 0 when $\alpha=\beta=\gamma=0$. (In this case we have $|x_j^*| = |y_j^*| = r^{\frac{1}{2}}$.) They remain bounded away

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from 0, when α, β and γ are small, $|\alpha|+|\beta|+|\gamma| \leqslant \rho_2(\sigma,\tau)$ say. When γ is small, the term $i\gamma(x-x_j^*)\,(y-y_j^*)$ can be removed by a linear transformation close to the identity transformation, and in terms of new variables x' and y' we then have

$$i\phi^*(x',y')-i\phi^*(x_j^*,y_j^*) = a_{20}(x'-x_j^*)^2+a_{02}(y'-y_j^*)^2 + a_{30}(x'-x_j^*)^3+a_{21}(x'-x_j^*)^2(y'-y_j^*)+a_{12}(x'-x_j^*)\,(y'-y_j^*)^2+a_{03}(y'-y_j^*)^3. \quad (5{\cdot}6)$$

We wish to show that

$$i\phi^*(x',y')-i\phi^*(x_j^*,y_j^*) = -\xi^2-\eta^2 \quad (5{\cdot}7)$$

Write

$$(-a_{20})^{\frac{1}{2}}(x'-x_j^*) = \xi(1+\mu(\xi,\eta)), \quad (5{\cdot}8)$$

$$(-a_{02})^{\frac{1}{2}}(y'-y_j^*) = \eta(1+\nu(\xi,\eta)), \quad (5{\cdot}9)$$

where we recall that a_{20} and a_{02} are bounded away from 0. Then (5·7) becomes

$$\xi^2(2\mu+\mu^2)+\eta^2(2\nu+\nu^2)+b_{30}\xi^3(1+\mu)^3+b_{21}\xi^2\eta(1+\mu)^2(1+\nu) + b_{12}\xi\eta^2(1+\mu)\,(1+\nu)^2+b_{03}\eta^3(1+\nu)^3 = 0, \quad (5{\cdot}10)$$

where $b_{30}, b_{21}, b_{12}, b_{03}$ are analytic functions of $\alpha+\sigma, \beta+\tau$ and γ. The equation (5·10) can be satisfied for many choices of μ and ν, for example when

$$2\mu+\mu^2+b_{30}\xi(1+\mu)^3+b_{21}\eta(1+\mu)^2(1+\nu) = 0 \quad (5{\cdot}11)$$

and

$$2\nu+\nu^2+b_{12}\xi(1+\mu)\,(1+\nu)^2+b_{03}\eta(1+\nu)^3 = 0. \quad (5{\cdot}12)$$

By a fundamental implicit-function theorem the system (5·11, 5·12) has a unique small solution (μ,ν) when ξ and η are small enough, and $|\alpha|+|\beta|+|\gamma| \leqslant \rho_2(\sigma,\tau)$. Then $x'-x_j^*$ and $y'-y_j^*$ are given in terms of ξ and η by (5·8) and (5·9), and by the same implicit-function theorem these can be solved for ξ and η when $x'-x_j^*$ and $y'-y_j^*$ are small enough, and therefore when $x-x_j^*$ and $y-y_j^*$ are small enough,

$$|x-x_j^*|+|y-y_j^*| < \epsilon(\sigma,\tau), \text{ say.}$$

Moreover, when $\alpha=\beta=\gamma=0$ we see that b_{12} and b_{21} both vanish, and that the equations are then equivalent to (4·7) and (4·8). We note that $\rho_2(\sigma,\tau)$ and $\epsilon(\sigma,\tau)$ can be replaced by bounds independent of σ and τ, for $\rho_2(\sigma,\tau)$ and $\epsilon(\sigma,\tau)$ can be taken to be continuous functions of σ and τ on $|\sigma| = |\tau| = r$, and therefore have strictly positive lower bounds. This concludes the proof of Lemma 1.

Consider now the integral

$$I^*(1,N,\alpha,\beta,\gamma,\sigma,\tau) = \iint_{S^{**}(1)} G(x,y,\alpha,\beta,\gamma)\exp(iN\phi^*)\,dx\,dy \quad (5{\cdot}13)$$

over the surface $S^{**}(1)$, and suppose that it contains only the single four-dimensional saddle-point $(i\sigma^{\frac{1}{2}}, i\tau^{\frac{1}{2}})$ of ϕ^{**}. When $|\alpha|+|\beta|+|\gamma|$ is small there is a saddle-point (x_1^*, y_1^*) of ϕ^* near the saddle-point $(i\sigma^{\frac{1}{2}}, i\tau^{\frac{1}{2}})$ of ϕ^{**}. Near the saddle-point (x_1^*, y_1^*) introduce the transformation of Lemma 1 to new variables ξ, η, and let the surface of integration $S^{**}(1)$ near $(i\sigma^{\frac{1}{2}}, i\tau^{\frac{1}{2}})$ be deformed into the surface $S^*(1)$ on which ξ and η are real. (The surface $S^*(1)$ tends to $S^{**}(1)$ when $|\alpha|+|\beta|+|\gamma|$ tends to 0.) Let δ be the parameter occurring in the statement of Lemma 1.

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Define

$$\left.\begin{aligned} S^*_{\text{int}}(1, \tfrac{1}{2}\delta) \quad &\text{as the part of } S^*(1) \quad \text{on which} \quad |\xi| + |\eta| \leqslant \tfrac{1}{2}\delta, \\ S^*_{\text{ext}}(1, \tfrac{1}{2}\delta) \quad &\text{as the part of } S^{**}(1) \quad \text{on which} \quad |\xi_0 + |\eta_0| \geqslant \tfrac{1}{2}\delta. \end{aligned}\right\} \tag{5·14}$$

The boundary curves of these surfaces are close to each other when $|\alpha| + |\beta| + |\gamma|$ is small; let them be joined by narrow two-dimensional strips, $S^{***}(1, \frac{1}{2}\delta)$, say. Then the deformed surface of integration is taken to be

$$S^*_{\text{int}}(1, \tfrac{1}{2}\delta) \cup S^{***}(1, \tfrac{1}{2}\delta) \cup S^*_{\text{ext}}(1, \tfrac{1}{2}\delta), \quad = S^*(1, \tfrac{1}{2}\delta), \quad \text{say.} \tag{5·15}$$

We can now proceed to estimate the integral I^*. It will first be shown, that, when $|\alpha| + |\beta| + |\gamma|$ is small enough, the contributions from S^{***} and S^{**}_{ext} to the asymptotic expansion are exponentially negligible. For example, on S^{**}_{ext} we have

$$\begin{aligned} \exp(iN\phi^*) &= \exp\{iN\phi^{**} + iN(\alpha x + \beta y + \gamma xy)\} \\ &= \exp\{-N(\xi_0^2 + \eta_0^2) + iN(\alpha x + \beta y + \gamma xy)\}. \end{aligned} \tag{5·16}$$

Since $|x| = O(|\xi_0|^{\frac{3}{2}})$ and $|y| = O(|\eta_0|^{\frac{3}{2}})$ it is easy to see that, for x and y along the paths of steepest descent, the ratios

$$\frac{|x|}{\xi_0^2 + \eta_0^2}, \quad \frac{|y|}{\xi_0^2 + \eta_0^2}, \quad \frac{|xy|}{\xi_0^2 + \eta_0^2}$$

all tend to 0 when $\xi_0^2 + \eta_0^2 \to \infty$, and it follows that there is a bound $m(\frac{1}{2}\delta)$ such that

$$\frac{|\alpha x| + |\beta y| + |\gamma xy|}{\xi_0^2 + \eta_0^2} \leqslant (|\alpha| + |\beta| + |\gamma|)\, m(\tfrac{1}{2}\delta) \tag{5·17}$$

on S^{**}_{ext}. Choose

$$|\alpha| + |\beta| + |\gamma| < \min\left(\frac{1}{2m(\frac{1}{2}\delta)}, \rho_1, \rho_2\right),$$

then

$$|\alpha x| + |\beta y| + |\gamma xy| < \tfrac{1}{2}(\xi_0^2 + \eta_0^2) \quad \text{on} \quad S^{**}_{\text{ext}}, \tag{5·18}$$

and

$$\begin{aligned} |\exp(iN\phi^*)| &< \exp(-N(\xi_0^2 + \eta_0^2)) \exp\{N(|\alpha x| + |\beta y| + |\gamma xy|)\}| \exp\{iN\phi^{**}(i\sigma^{\frac{1}{2}}, i\tau^{\frac{1}{2}})\}| \\ &< \exp(-\tfrac{1}{2}N(\xi_0^2 + \eta_0^2))| \exp\{iN\phi^{**}(i\sigma^{\frac{1}{2}}, i\tau^{\frac{1}{2}})\}|. \end{aligned} \tag{5·19}$$

By the application of a similar argument we see that, on S^{**}_{ext},

$$|\exp(iN\phi^*)| < \exp(-\tfrac{1}{4}N(\xi_0^2 + \eta_0^2))| \exp\{iN\phi^*(x_1^*, y_1^*)\}|, \tag{5·20}$$

when $|\alpha| + |\beta| + |\gamma|$ is small enough, $|\alpha| + |\beta| + |\gamma| < \rho_3$, say. (We can take ρ_3 independent of σ and τ, as in Lemma 1.) Thus the contribution from S^{**}_{ext} is exponentially negligible, and similarly the contribution from S^{***} can be shown to be exponentially negligible. It follows that the dominant contribution comes from S^{**}_{int}, and this will now be found from the two-dimensional method of Laplace.

LEMMA 2. *With the same assumptions on $\alpha, \beta, \gamma, \sigma, \tau$ as in Lemma 1, suppose that $I^*(1, N, \alpha, \beta, \gamma, \sigma, \tau)$ involves only the single well-separated saddle-point (x_1^*, y_1^*), and that $|\alpha| + |\beta| + |\gamma| \leqslant \rho_3$. Then, as $N \to +\infty$,*

$$I^*(1, N, \alpha, \beta, \gamma, \sigma, \tau) \sim \frac{2\pi i}{N} \frac{G(x_1^*, y_1^*, \alpha, \beta, \gamma)}{(4x_1^* y_1^* - \gamma^2)^{\frac{1}{2}}} \exp\{iN\phi^*(x_1^*, y_1^*)\}. \tag{5·21}$$

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Proof. As has just been shown, it is sufficient to take the integration over $S^*_{\text{int}}(1, \frac{1}{2}\delta)$. On this surface change to variables ξ and η. Thus, over S^*_{int},

$$\iint G(\,)\exp(iN\phi^*)\,dx\,dy = \exp\{iN\phi^*(x_1^*, y_1^*)\}\iint G(\,)\exp\{-N(\xi^2+\eta^2)\}\frac{\partial(x,y)}{\partial(\xi,\eta)}\,d\xi\,d\eta, \tag{5·22}$$

which can be evaluated by the two-dimensional method of Laplace. The value of the Jacobian determinant at the saddle-point is readily found from equation (5·5), and the result (5·21) is thus obtained. We note that proof and result remain applicable when $i\sigma^{\frac{1}{2}}$ or $i\tau^{\frac{1}{2}}$ lies near a Stokes line, provided that paths of local steepest descent are used to define ξ_0 and η_0. (See Section 4 above.)

This concludes the proof of Lemma 2. We shall actually need only the weaker result that, for $N > N_0$,

$$|I^*| \leqslant \frac{M}{N}|\exp\{iN\phi^*(x_1^*, y_1^*)\}|, \quad \text{where } M \text{ depends on } G(\,). \tag{5·23}$$

6. *Bounds for the determinants Δ^*_{pq} and Δ_{pq}.* The intermediate result at which we are aiming is

THEOREM 3. (Cf. I, Theorem 4.) *There is a constant M and a constant ρ, such that, for all real $N > N_0$ and for all α, β and γ satisfying $|\alpha|+|\beta|+|\gamma| \leqslant \rho$ we have*

$$|E(N,\alpha,\beta,\gamma)| \leqslant M, |A(N,\alpha,\beta,\gamma)| \leqslant M,$$
$$|B(N,\alpha,\beta,\gamma)| \leqslant M, |C(N,\alpha,\beta,\gamma)| \leqslant M,$$

where E, A, B and C are defined by (3·7).

This is evidently a weak form of Theorem 2. To obtain these bounds it is necessary to obtain bounds for Δ_{pq}, but the bounds (5·23) which we have just obtained are only applicable when the saddle-points are well separated. We therefore, in Lemma 4, obtain bounds for Δ^*_{pq}, and note that these bounds must also apply to Δ_{pq}, by the maximum–modulus theorem for the two complex variables σ and τ.

We shall first consider the simpler determinants Δ^{**}_{pq}. It will then be seen that the same arguments can be applied with little modification to the determinants Δ^*_{pq}. For example, let us take

$$\Delta^{**}_{00} = [I^{**}(1,\sigma,\tau),\ J^{**}_{10}(2,\sigma,\tau).\ J^{**}_{01}(3,\sigma,\tau),\ J^{**}_{11}(4,\sigma,\tau)] \tag{6·1}$$

(cf. 3·9) where the integrals also depend on N, and where the surfaces of integration in successive rows are

$$\left.\begin{aligned} S^{**}(1) &= (5 \xrightarrow[x]{} 1,\ 5 \xrightarrow[y]{} 1),\\ S^{**}(2) &= (5 \xrightarrow[x]{} 9,\ 5 \xrightarrow[y]{} 1),\\ S^{**}(3) &= (5 \xrightarrow[x]{} 1,\ 5 \xrightarrow[y]{} 9),\\ S^{**}(4) &= (5 \xrightarrow[x]{} 9,\ 5 \xrightarrow[y]{} 9) \end{aligned}\right\} \tag{6·2}$$

respectively, defined in Section 4 above. As before, the complex parameters σ and τ lie on $|\sigma| = r$ and $|\tau| = r$ respectively. Suppose that initially

$$\tfrac{1}{6}\pi < \text{ph}\,(i\sigma^{\frac{1}{2}}) < \tfrac{1}{2}\pi \quad \text{and} \quad \tfrac{1}{6}\pi < \text{ph}\,(i\tau^{\frac{1}{2}}) < \tfrac{1}{2}\pi, \tag{6·3}$$

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then it is not difficult to see from (4·3)–(4·5) that each of the four surfaces $S^{**}(j)$ involves exactly one saddle-point, and that each row involves a different saddle-point, (x_j^{**}, y_j^{**}), say, where (x_j^{**}, y_j^{**}) is one of the four saddle-points $(\pm i\sigma^{\frac{1}{2}}, \pm i\tau^{\frac{1}{2}})$. Thus an estimate of each element of the determinant (6·1) can be found by use of Lemma 2. The usual expansion of the determinant contains 24 terms, where each term is the product of four integrals. In this way we find that, subject to (6·3),

$$|\Delta_{00}^{**}| \leqslant M\left(\frac{2\pi}{N}\right)^4 |\exp\{iN\Sigma\phi^{**}(x_j^{**}, y_j^{**})\}|, \tag{6·4}$$

where M is a constant depending on the function G. Suppose next that σ moves along $|\sigma| = r$ so that

$$\tfrac{1}{2}\pi < \mathrm{ph}\,(i\sigma^{\frac{1}{2}}) < \tfrac{5}{6}\pi, \quad \text{and} \quad \tfrac{1}{6}\pi < \mathrm{ph}\,(i\tau^{\frac{1}{2}}) < \tfrac{1}{2}\pi. \tag{6·5}$$

Then row 2 and row 4 each involve two saddle-points in the x-plane. On the determinant Δ_{00}^{**} carry out the operations

$$\text{row } 2 - \text{row } 1, \quad \text{row } 4 - \text{row } 3,$$

which leave the value of the determinant unchanged but which change the surface of integration in row 2 into $(1 \xrightarrow[x]{} 9, 5 \xrightarrow[y]{} 1)$, and the surface of integration in row 4 into $(1 \xrightarrow[x]{} 9, 5 \xrightarrow[y]{} 9)$. Each element of the modified determinant now involves one saddle-point, and each row of the modified determinant involves a different saddle-point. (Compare the discussion in I, Lemma 2.) Thus the bound (6·4) is seen to hold when (6·5) is satisfied. By an operation of this kind on the rows of the determinant Δ_{00}^{**} as σ moves along $|\sigma| = r$ and τ moves along $|\tau| = r$, each row can be made to involve a different saddle-point, and we obtain a bound like (6·4) for all σ on $|\sigma| = r$, and for all τ on $|\tau| = r$. This argument is applicable to all the determinants Δ_{00}^{**}, Δ_{10}^{**}, Δ_{01}^{**}, Δ_{11}^{**}. We have thus proved

LEMMA 3. *For all σ on $|\phi| = r$, and for all τ on $|\tau| = r$ there exists an inequality of the form*

$$|\Delta_{pq}^{**}(\sigma, \tau)| \leqslant M\left(\frac{2\pi}{N}\right)^4 |\exp\{iN\Sigma\phi^{**}(x_j^{**}, y_j^{**})\}| \tag{6·6}$$

for all the determinants $\Delta_{pq}^{**}(p = 0, 1; q = 0, 1)$.

It can now be seen that the same arguments apply to Δ_{pq}^{*} except possibly when $i\sigma^{\frac{1}{2}}$ or $i\tau^{\frac{1}{2}}$ cross a Stokes line; in fact they remain valid there because the bounds (5·23) have been shown to remain valid there.

Thus we have

LEMMA 4. (Cf. I, Lemma 3.) *For all σ on $|\sigma| = r$, and for all τ on $|\tau| = r$, and for all α, β and γ such that $|\alpha| + |\beta| + |\gamma| \leqslant \rho_3$ there exists an inequality of the form*

$$|\Delta_{pq}^{*}(N, \alpha, \beta, \gamma, \sigma, \tau)| \leqslant M\left(\frac{2\pi}{N}\right)^4 |\exp\{iN\Sigma\phi^{*}(x_j^{*}, y_j^{*})\}|, \tag{6·7}$$

where M is a constant depending on the function $G(\ )$.

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Since $\Sigma \phi^*(x_j^*, y_j^*) = \gamma^3$, (see appendix 2) this can be written in the form

$$|\Delta_{pq}^*(N, \alpha, \beta, \gamma, \sigma, \tau)| \leqslant M \left(\frac{2\pi}{N}\right)^4 |\exp(iN\gamma^3)|. \tag{6·8}$$

Now apply the maximum–modulus theorem for 2 complex variables (see Gunning and Rossi(7), p. 7) to the function $\Delta_{pq}^*(N, \alpha, \beta, \gamma, \sigma, \tau)$ which, as we know, is an analytic function of σ and τ. We thus obtain

LEMMA 5. (Cf. I, Lemma 7.)

$$\begin{aligned} |\Delta_{pq}(N, \alpha, \beta, \gamma)| &= |\Delta^*(N, \alpha, \beta, \gamma, 0, 0)| \\ &\leqslant M \left(\frac{2\pi}{N}\right)^4 |\exp(iN\gamma^3)|. \end{aligned} \tag{6·9}$$

It is also possible to find $\Delta(N, \alpha, \beta, \gamma)$ explicitly; it is shown in appendix 4 that

$$\Delta(N, \alpha, \beta, \gamma) = \left(\frac{2\pi}{N}\right)^4 \exp(iN\gamma^3). \tag{6·10}$$

These relations are sufficient for the proofs of Theorem 2 and Theorem 3.

Proof of Theorem 3. From (6·9) and (6·10) we have

$$|\Delta_{00}/\Delta| = |E(N, \alpha, \beta, \gamma)| \leqslant M$$

and similarly $|A(N, \alpha, \beta, \gamma)| \leqslant M$, $|B(N, \alpha, \beta, \gamma)| \leqslant M$, $|C(N, \alpha, \beta, \gamma)| \leqslant M$. This concludes the proof of Theorem 3.

Proof of Theorem 2. To prove that, for example,

$$\left| E(N, \alpha, \beta, \gamma) - \sum_0^n \left(\frac{i}{N}\right)^k E_k(\alpha, \beta, \gamma) \right| \leqslant \frac{M_{n+1}}{N^{n+1}}.$$

we note from (2·5), that

$$\begin{aligned} \Delta_{00}(N, \alpha, \beta, \gamma) - \left(E_0 + \frac{i}{N} E_1 + \ldots + \left(\frac{i}{N}\right)^n E_n\right) \Delta(N, \alpha, \beta, \gamma) \\ = \left(\frac{i}{N}\right)^{n+1} \Delta_{00}(N, n+1, \alpha, \beta, \gamma), \end{aligned} \tag{6·11}$$

where $\Delta_{00}(N, n+1, \alpha, \beta, \gamma)$ is defined like $\Delta_{00}(N, \alpha, \beta, \gamma)$, except that G_{n+1} takes the place of G. On dividing (6·11) by Δ and applying Theorem 3 to

$$\iint G_{n+1}(\,) \exp(iN\phi)\, dx\, dy$$

we find that

$$\left| E(N, \alpha, \beta, \gamma) - \sum_0^n \left(\frac{i}{N}\right)^k E_k(\alpha, \beta, \gamma) \right| \leqslant \frac{M_{n+1}}{N^{n+1}}. \tag{6·12}$$

as was to be shown.

7. *Discussion*. In the present paper the asymptotic expansion of the integral

$$\iint G(x, y, \alpha, \beta, \gamma) \exp\{iN(\tfrac{1}{3}x^3 + \tfrac{1}{3}y^3 + \alpha x + \beta y + \gamma xy)\}\, dx\, dy \tag{7·1}$$

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was found subject to two restrictions. It was assumed that the integration is taken over a two-dimensional surface which has all its boundary points at ∞, and it was assumed that α, β and γ are all sufficiently small. Comparison with the one-dimensional case suggests that the first restriction can be removed, for it was shown in I, §6 that finite limits of integration for a single variable can be treated by almost the same arguments as infinite limits. The second restriction is more fundamental; when α, β and γ are not necessarily small the problem of determining the relevant saddle-points must be solved for any prescribed surface of integration, e.g. for $S(1)$. Even for one complex variable this can be done explicitly only in simple cases (see Ursell(10)), while for two complex variables little seems to be known. Nevertheless progress could be made with (7·1) because $\phi(x, y, \sigma, \tau, 0)$ is the product of a function of x and a function of y. The function $\phi(x, y, \alpha+\sigma, \beta+\tau, \gamma)$ could then be treated by a perturbation technique, since α, β and γ were assumed to be sufficiently small. More work is needed on a method of steepest descents for two complex variables, particularly on the deformation of the two-dimensional surfaces of integration.

8. *Acknowledgments*. My thanks are due to Dr J. N. L. Connor for bringing this problem to my notice, and to Dr R. M. W. Wood for many helpful discussions.

Appendix 1. *Two-dimensional integrals*

The integrals, of the form $\iint \psi(x, y)\, dx\, dy$, considered in this paper are integrals taken over two-dimensional surfaces in the four-dimensional space of the two complex variables x and y, and must be defined more precisely. It is assumed that the function $\psi(x, y)$ is an analytic function of x and y, that the surface of integration Σ is defined parametrically by functions

$$x = \xi(\lambda, \mu), \quad y = \eta(\lambda, \mu), \tag{A 1·1}$$

which are complex-valued functions of the two real variables λ and μ, and that the points (λ, μ) cover a two-dimensional domain A with boundary curve C. Then, by definition,

$$\iint_{\Sigma} \psi(x, y)\, dx\, dy = \iint_{A} \psi[\xi(\lambda, \mu), \eta(\lambda, \mu)] \frac{\partial(\xi, \eta)}{\partial(\lambda, \mu)} d\lambda\, d\mu, \tag{A 1·2}$$

where

$$\frac{\partial(\xi, \eta)}{\partial(\lambda, \mu)} = \frac{\partial \xi}{\partial \lambda}\frac{\partial \eta}{\partial \mu} - \frac{\partial \xi}{\partial \mu}\frac{\partial \eta}{\partial \lambda}$$

is the Jacobian determinant of the mapping (A 1·1).

Suppose now that the surface Σ is deformed while the boundary $\partial\Sigma$ is kept fixed; more precisely, suppose that the mapping (A 1·1) depends also on a parameter t,

$$x = \xi(\lambda, \mu, t), \quad y = \eta(\lambda, \mu, t), \tag{A 1·3}$$

so that A is mapped on a set $\Sigma(t)$ of two-dimensional surfaces, all with the same boundary $\partial\Sigma$. Then we have the following

THEOREM A 1·1. *If the boundary curve is kept fixed, the integral* $\iint \psi(x, y)\, dx\, dy$ *is independent of the surface spanning the boundary curve.*

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Proof. Denoting differentiation with respect to t by the suffix t we have

$$\frac{\partial}{\partial t}\iint_A \psi(\xi,\eta)\frac{\partial(\xi,\eta)}{\partial(\lambda,\mu)}\,d\lambda\,d\mu \tag{A 1·4}$$

$$= \iint\left(\xi_t\frac{\partial\psi}{\partial\xi}+\eta_t\frac{\partial\psi}{\partial\eta}\right)\frac{\partial(\xi,\eta)}{\partial(\lambda,\mu)}\,d\lambda\,d\mu+\iint\psi\left\{\frac{\partial(\xi_t,\eta)}{\partial(\lambda,\mu)}+\frac{\partial(\xi,\eta_t)}{\partial(\lambda,\mu)}\right\}d\lambda\,d\mu. \tag{A 1·5}$$

Also the first integrand in (A 1·5) is

$$\left(\xi_t\frac{\partial\psi}{\partial\xi}+\eta_t\frac{\partial\psi}{\partial\eta}\right)\frac{\partial(\xi,\eta)}{\partial(\lambda,\mu)} = \left(\xi_t\frac{\partial(\psi,\eta)}{\partial(\xi,\eta)}+\eta_t\frac{\partial(\xi,\psi)}{\partial(\xi,\eta)}\right)\frac{\partial(\xi,\eta)}{\partial(\lambda,\mu)} = \xi_t\frac{\partial(\psi,\eta)}{\partial(\lambda,\mu)}+\eta_t\frac{\partial(\xi,\psi)}{\partial(\lambda,\mu)},$$

by the multiplicative property of Jacobian determinants.
Thus (A 1·4)

$$= \iint\left\{\frac{\partial(\psi\xi_t,\eta)}{\partial(\lambda,\mu)}+\frac{\partial(\xi,\psi\eta_t)}{\partial(\lambda,\mu)}\right\}d\lambda\,d\mu$$

$$= \iint\left\{\frac{\partial}{\partial\lambda}\left(\xi_t\psi\frac{\partial\eta}{\partial\mu}-\eta_t\psi\frac{\partial\xi}{\partial\mu}\right)-\frac{\partial}{\partial\mu}\left(\xi_t\psi\frac{\partial\eta}{\partial\lambda}-\eta_t\psi\frac{\partial\xi}{\partial\lambda}\right)\right\}d\lambda\,d\mu$$

$$= -\int_{\partial\Sigma}\psi(\xi_t\,d\eta-\eta_t\,d\xi), \text{ by Stokes's theorem,}$$

$$= 0 \quad \text{since } \xi_t=\eta_t=0 \quad \text{on the fixed boundary } \partial\Sigma.$$

This concludes the proof of Theorem A 1·1.

Appendix 2. *Saddle points of* $\phi(x,y,\alpha,\beta,\gamma)$.

By definition the saddle points (x_j,y_j) satisfy the equations

$$\partial\phi/\partial x = x^2+\alpha+\gamma y = 0 \quad \text{and} \quad \partial\phi/\partial y = y^2+\beta+\gamma x = 0.$$

By elimination of y it is found that

$$(x^2+\alpha)^2 = \gamma^2y^2 = -\gamma^2(\beta+\gamma x),$$

i.e.
$$P_1(x) \equiv x^4+2\alpha x^2+\gamma^3x+\alpha^2+\beta\gamma^2 = 0; \tag{A 2·1}$$

this equation has four roots, x_1, x_2, x_3, x_4, say. Similarly it is found that

$$P_2(y) \equiv y^4+2\beta y^2+\gamma^3y+\beta^2+\alpha\gamma^2 = 0. \tag{A 2·2}$$

which has four roots, y_1, y_2, y_3, y_4, say. Since $\gamma y = -x^2-\alpha$ it follows that to each x_j there corresponds a unique y_j. Thus there are four saddle-points (x_j,y_j).

In (6·7) we need the value of

$$\sum_j \phi(x_j,y_j) = \tfrac{1}{3}\Sigma x_j^3+\tfrac{1}{3}\Sigma y_j^3+\alpha\Sigma x_j+\beta\Sigma y_j+\gamma\Sigma x_jy_j.$$

The value of each term can be found from elementary results of the theory of equations. Thus $\Sigma x_j = 0$ and $\Sigma x_j^3 = -3\gamma^3$ from (A 2·1), and $\Sigma y_j = 0$ and $\Sigma y_j^3 = -3\gamma^3$ from (A 2·2), while $\gamma\Sigma x_jy_j = -\Sigma x_j(x_j^3+\alpha\} = -\alpha\Sigma x_j-\Sigma x_j^3 = 3\gamma^3$. Thus $\sum_j \phi(x_j,y_j) = \gamma^3$.

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Appendix 3. The Bleistein sequence

THEOREM A 3·1. *Any analytic function $G(x, y, \alpha, \beta, \gamma)$ of x and y, involving parameters α, β and γ, can be written in the form*

$$G(x,y,\alpha,\beta,\gamma) = E(\alpha,\beta,\gamma) + A(\alpha,\beta,\gamma)\,x + B(\alpha,\beta,\gamma)\,y + C(\alpha,\beta,\gamma)\,xy + (x^2+\alpha+\gamma y)\,H(x,y,\alpha,\beta,\gamma) + (y^2+\beta+\gamma x)\,K(x,y,\alpha,\beta,\gamma), \quad \text{(A 3·1)}$$

where all the functions on the right-hand side of (A 3·1) are analytic functions of x and y. (For the sake of brevity write $X = x^2+\alpha+\gamma y$, $Y = y^2+\beta+\gamma x$.)

Note. It is obvious that at the saddle-points (x_j, y_j) we have

$$G(x_j, y_j, \alpha, \beta, \gamma) = E + Ax_j + By_j + Cx_j y_j \quad (j = 1, 2, 3, 4).$$

These four equations determine E, A, B, C, but the expansion (A 3·1) does not follow immediately.

Proof. First let us consider polynomials in x and y. Evidently we have

$$x^2 = -\alpha - \gamma y + X,\ y^2 = -\beta - \gamma x + Y,$$

whence

$$x^2 y = -\alpha y + \gamma(\beta+\gamma x) - \gamma Y + yX,\ xy^2 = -\beta x + \gamma(\alpha+\gamma y) - \gamma X + xY,$$

$$x^2y^2 = \alpha\beta + \alpha\gamma x + \beta\gamma y + \gamma^2 xy - (\alpha+\gamma y)\,Y - (\beta+\gamma x)\,X + XY. \quad \text{(A 3·2)}$$

Similarly any polynomial in x and y can be expressed in this form in a finite number of steps.

To represent a general analytic function we note that

$$G(x,y) = \left(\frac{1}{2\pi i}\right)^2 \iint \frac{G(\xi,\eta)}{(\xi-x)\,(\eta-y)}\,d\xi\,d\eta, \quad \text{(A 3·3)}$$

and obtain a representation of $\{(\xi-x)\,(\eta-y)\}^{-1}$ of the form

$$\frac{1}{(\xi-x)\,(\eta-y)} = \mathbf{E} + x\mathbf{A} + y\mathbf{B} + xy\mathbf{C} + X\mathbf{H} + Y\mathbf{K}, \quad \text{(A 3·4)}$$

where $\mathbf{E}$, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are functions of ξ and η, and $\mathbf{H}$ and $\mathbf{K}$ are functions of x, y, ξ and η, as well as of the parameters α, β and γ. To obtain (A 3·1), consider the product

$$\Pi = (\xi\eta - y\xi - x\eta + xy)\,(\mathbf{E} + x\mathbf{A} + y\mathbf{B} + xy\mathbf{C}), \quad \text{(A 3·5)}$$

let this be expanded and arranged in powers of x and y. The expansion contains the powers $1, x, y, xy, x^2, y^2, x^2y, xy^2, x^2y^2$, and no other powers. For the powers $x^2, y^2, x^2y, xy^2, x^2y^2$ substitute the expressions (A 3·2). Then we find that

$$\Pi = [\mathbf{E}\xi\eta + \mathbf{A}(\beta\gamma+\eta\alpha) + \mathbf{B}(\alpha\gamma+\xi\beta) + \mathbf{C}(\alpha\beta - \xi\alpha\gamma - \eta\beta\gamma)]_{00} \quad \text{(A 3·6)}$$

$$+ x[-\mathbf{E}\eta + \mathbf{A}(\gamma^2+\xi\eta) + \mathbf{B}(\gamma\xi-\beta) + \mathbf{C}(\alpha\beta+\xi\beta-\gamma^2\eta)]_{10} \quad \text{(A 3·7)}$$

$$+ y[-\mathbf{E}\xi + \mathbf{A}(\gamma\eta-\alpha) + \mathbf{B}(\gamma^2+\xi\eta) + \mathbf{C}(\beta\gamma+\eta\alpha-\gamma^2\xi)]_{01} \quad \text{(A 3·8)}$$

$$+ xy[\mathbf{E} - \mathbf{A}\xi - \mathbf{B}\eta + \mathbf{C}(\gamma^2+\xi\eta)]_{11} \quad \text{(A 3·9)}$$

$$+ (\text{terms containing a factor } X \text{ or a factor } Y \text{ or both factors, and involving } \mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C} \text{ linearly}). \quad \text{(A 3·10)}$$

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Now choose **E**, **A**, **B**, **C**, so that $[\ ]_{00} = 1$, and $[\ ]_{10} = [\ ]_{01} = [\ ]_{11} = 0$, where these parentheses are the parentheses appearing in (A 3·6), (A 3·7), (A 3·8) and (A 3·9). On solving these four equations it is seen that

$$\left.\begin{aligned} \mathbf{E} &= \frac{P_{00}(\xi,\eta,\alpha,\beta,\gamma)}{D(\xi,\eta,\alpha,\beta,\gamma)}, \quad \mathbf{A} = \frac{P_{10}(\xi,\eta,\alpha,\beta,\gamma)}{D(\xi,\eta,\alpha,\beta,\gamma)}, \\ \mathbf{B} &= \frac{P_{01}(\xi,\eta,\alpha,\beta,\gamma)}{D(\xi,\eta,\alpha,\beta,\gamma)}, \quad \mathbf{C} = \frac{P_{11}(\xi,\eta,\alpha,\beta,\gamma)}{D(\xi,\eta,\alpha,\beta,\gamma)}, \end{aligned}\right\} \tag{A 3·11}$$

where the expressions P_{pq} and D are polynomials; in particular, D is the determinant of the system. On substituting these values of **E**, **A**, **B**, **C** into the terms (A 3·10) we obtain

$$\begin{aligned} \Pi &= (\xi - x)(\eta - y)(\mathbf{E} + \mathbf{A}x + \mathbf{B}y + \mathbf{C}xy) \\ &= 1 - X\frac{H(x,y,\xi,\eta,\alpha,\beta,\gamma)}{D(\xi,\eta,\alpha,\beta,\gamma)} - Y\frac{K(x,y,\xi,\eta,\alpha,\beta,\gamma)}{D(\xi,\eta,\alpha,\beta,\gamma)}, \end{aligned} \tag{A 3·12}$$

where the expressions H and K (which are not written down explicitly) are also polynomials. Thus we have

$$\begin{aligned} \frac{1}{(\xi - x)(\eta - y)} &= \frac{P_{00}(\xi,\eta)}{D(\xi,\eta)} + x\frac{P_{10}(\xi,\eta)}{D(\xi,\eta)} + y\frac{P_{01}(\xi,\eta)}{D(\xi,\eta)} + xy\frac{P_{11}(\xi,\eta)}{D(\xi,\eta)} \\ &\quad + X\frac{H(x,y,\xi,\eta)}{(\xi - x)(\eta - y)D(\xi,\eta)} + Y\frac{K(x,y,\xi,\eta)}{(\xi - x)(\eta - y)D(\xi,\eta)}. \end{aligned} \tag{A 3·13}$$

Substituting in (A 3·2) we find the representation (A 3·1) provided that $D(\xi,\eta)$ does not vanish on the surface of integration. We have

$$E(\alpha,\beta,\gamma) = \left(\frac{1}{2\pi i}\right)^2 \iint \frac{G(\xi,\eta,\alpha,\beta,\gamma)\,P_{00}(\xi,\eta,\alpha,\beta,\gamma)}{D(\xi,\eta,\alpha,\beta,\gamma)}\,d\xi\,d\eta, \tag{A 3·14}$$

with similar expressions for A, B and C, and we also have

$$H(x,y,\alpha,\beta,\gamma) = \left(\frac{1}{2\pi i}\right)^2 \iint \frac{G(\xi,\eta,\alpha,\beta,\gamma)\,H(x,y,\xi,\eta,\alpha,\beta,\gamma)}{(\xi - x)(\eta - y)D(\xi,\eta,\alpha,\beta,\gamma)}\,d\xi\,d\eta, \tag{A 3·15}$$

with a similar expression for $K(x,y,\alpha,\beta,\gamma)$. It remains to be shown that $D(\xi,\eta,\alpha,\beta,\gamma)$ does not vanish when ξ and η are bounded away from 0, and $|\alpha| + |\beta| + |\gamma| \leqslant \rho$. This follows since $D(\xi,\eta,0,0,0) = \xi^4\eta^4$, as is easily seen. In fact $D(\xi,\eta,\alpha,\beta,\gamma)$ can be found explicitly, for by applying the factors $\xi - x$ and $\eta - y$ successively it can be seen that the polynomials $P_1(\xi)$ and $P_2(\eta)$ (see A 2·1, A 2·2) must be factors of $D(\xi,\eta,\alpha,\beta,\gamma)$. It follows, on comparing the coefficients of $\xi^4\eta^4$ on both sides, that

$$D(\xi,\eta,\alpha,\beta,\gamma) = P_1(\xi)\,P_2(\eta). \tag{A 3·16}$$

Appendix 4. *The determinant* $\Delta(N,\alpha,\beta,\gamma)$.

This can be found explicitly, cf. I, Lemma 6. It will be shown that

$$\Delta(N,\alpha,\beta,\gamma) = (2\pi/N)^4 \exp(iN\gamma^3). \tag{A 4·1}$$

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By definition, see (3·11) above,

$$\Delta(N,\alpha,\beta,\gamma) = \begin{vmatrix} J_{00}(1), & J_{10}(1), & J_{01}(1), & J_{11}(1) \\ J_{00}(2), & \dots & \dots & \dots \\ J_{00}(3), & \dots & \dots & \dots \\ J_{00}(4), & \dots & \dots & J_{11}(4) \end{vmatrix}$$

$$= [J_{00}(1), J_{10}(2), J_{01}(3), J_{11}(4)], \quad \text{cf. (3·10)},$$

$$= [J_{00}, J_{10}, J_{01}, J_{11}], \quad \text{say}, \tag{A. 4·2}$$

where $J_{pq}(j)$ stands for

$$J_{pq}(j,N,\alpha,\beta,\gamma) = \iint_{S(j)} x^p y^q \exp\{iN(\tfrac{1}{3}x^3+\tfrac{1}{3}y^3+\alpha x+\beta y+\gamma xy)\}\,dx\,dy.$$

It is shown in appendix 3, that $x^p y^q$ can be expressed in the form (A 3·1). By repeated integration by parts, (see (2·4)) it follows that every J_{pq} is a linear combination of J_{00}, J_{10}, J_{01} and J_{11}. In particular, it can be shown that

$$J_{20} = -\alpha J_{00} - \gamma J_{01}, \quad J_{02} = -\beta J_{00} - \gamma J_{10}, \tag{A 4·3}$$

$$J_{21} = \beta\gamma J_{00} + \gamma^2 J_{10} - \alpha J_{01}, \quad J_{12} = \alpha\gamma J_{00} + \gamma^2 J_{01} - \beta J_{10}, \tag{A 4·4}$$

$$J_{22} = \alpha\beta J_{00} + \alpha\gamma J_{10} + \beta\gamma J_{01} + \gamma^2 J_{11} + \frac{i\gamma}{N} J_{00}. \tag{A 4·5}$$

Obviously

$$\frac{\partial}{\partial\alpha} J_{pq} = iNJ_{p+1,q}, \quad \frac{\partial}{\partial\beta} J_{pq} = iNJ_{p,q+1},$$

and

$$\frac{\partial}{\partial\gamma} J_{pq} = iNJ_{p+1,q+1}. \tag{A 4·6}$$

Differentiating Δ column by column we find that

$$\frac{\partial\Delta}{\partial\gamma} = iN[J_{11}, J_{10}, J_{01}, J_{11}] \tag{A 4·7}$$

$$+ iN[J_{00}, J_{21}, J_{01}, J_{11}] \tag{A 4·8}$$

$$+ iN[J_{00}, J_{10}, J_{12}, J_{11}] \tag{A 4·9}$$

$$+ iN[J_{00}, J_{10}, J_{01}, J_{22}] \tag{A 4·10}$$

Clearly (A 4·7) vanishes since two columns are identical. In (A 4·8), substitute for J_{21} from (A 4·4); thus (A 4·8) $= iN\gamma^2\Delta$, and similarly (A 4·9) $= iN\gamma^2\Delta$. In (A 4·10), substitute for J_{22} from (A 4·5); thus (A 4·10) $= iN\gamma^2\Delta$. It follows that

$$\frac{\partial\Delta}{\partial\gamma} = 3iN\gamma^2\Delta. \tag{A 4·11}$$

Similar but simpler calculations show that

$$\frac{\partial\Delta}{\partial\alpha} = \frac{\partial\Delta}{\partial\beta} = 0. \tag{A 4·12}$$

It follows that

$$\Delta(N,\alpha,\beta,\gamma) = \Delta(N,0,0,0)\exp(iN\gamma^3), \tag{A 4·13}$$

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and it only remains to find $\Delta(N, 0, 0, 0)$. (It would actually be sufficient to show that $\Delta(N, 0, 0, 0) \neq 0$.)

When $\gamma = 0$, each element of Δ is a product of Airy integrals, and we have seen in (4·4) above that

$$J^{**}_{pq}(5 \to 1,\ 5 \to 9) = J_p(5 \to 1)\, J_q(5 \to 9);$$

thus we need only the values of $J_0(5 \to 1), J_1(5 \to 9), J_0(5 \to 1), J_1(5 \to 9)$ at $\sigma = 0$ in order to evaluate $\Delta(N, 0, 0, 0)$. These are

$$J_0(5 \to 1) = 2\pi\, N^{-\frac{1}{3}}\mathrm{Ai}(0) = \lambda, \quad \text{say,} \tag{A 4·14}$$

$$J_1(5 \to 1) = -2\pi\, N^{-\frac{2}{3}}\mathrm{Ai}'(0) = \mu, \quad \text{say} \tag{A 4·15}$$

$$J_0(5 \to 9) = a\lambda,\ J_1(5 \to 9) = b\mu, \quad \text{say} \tag{A 4·16}$$

where $a = -e^{2\pi i/3}$ and $b = -e^{4\pi i/3}$. It follows that

$$\Delta(N, 0, 0, 0) = (\lambda\mu)^4 \begin{vmatrix} \mathfrak{M} & \mathfrak{M} \\ a\mathfrak{M} & b\mathfrak{M} \end{vmatrix} = (\lambda\mu)^4 \delta, \quad \text{say,} \tag{A 4·17}$$

where $\mathfrak{M} = \begin{pmatrix} 1 & 1 \\ a & b \end{pmatrix}$ is a 2×2 matrix. (A 4·18)

It is known that

$$\lambda\mu = -\frac{4\alpha^2 i}{N}\,\mathrm{Ai}\,(0)\,\mathrm{Ai}'(0)$$

$$= \frac{2\pi i}{3^{\frac{1}{3}}N}, \text{ cf. Jeffreys, ((8) p. 30);} \tag{A 4·19}$$

also the determinant δ is readily evaluated by the operation column 3-column 1, column 4-column 2,

whence $$\delta = \begin{vmatrix} \mathfrak{M} & 0 \\ a\mathfrak{M} & (b-a)\mathfrak{M} \end{vmatrix} = (b-a)^4, \tag{A 4·20}$$

and $(b-a)^4 = (e^{4\pi i/3} = e^{2\pi i/3})^4 = (2i \sin \frac{1}{3}\pi)^4 = 9$, whence $\Delta(N, 0, 0, 0) = 9(\lambda\mu)^4 = (2\pi/N)^4$. The result (A 4·1) is thus established.

Appendix 5. *Curves and surfaces of steepest descent*

It will be shown that curves and surfaces of steepest descent do not in general remain curves and surfaces of steepest descent under an analytic transformation of complex variables. In the integral (1·1) let us write $x = x_1 + ix_2$, $y = x_3 + ix_4$, and

$$f(x, y) = U(x_1, x_2, x_3, x_4) + iV(x_1, x_2, x_3, x_4), \tag{A 5·1}$$

where U and V are the real and imaginary parts of $f(x, y)$, and let us consider the three-dimensional surfaces $U(x_1, x_2, x_3, x_4) = \text{const}$. Then the direction of steepest descent at any point is the direction in which U decreases most rapidly, and is thus evidently the normal direction given by

$$\frac{dx_1}{\partial U/\partial x_1} = \frac{dx_2}{\partial U/\partial x_2} = \frac{dx_3}{\partial U/\partial x_3} = \frac{dx_4}{\partial U/\partial x_4}. \tag{A 5·2}$$

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By integration we obtain the curve of steepest descent through an arbitrary initial point. (This construction is the analogue of the construction of the curve of steepest descent in the plane of a single complex variable.) Curves of steepest descent through an arbitrary initial curve define a surface of steepest descent.

Let us now consider an analytic transformation from the complex variables x and y to new complex variables z and w, where $z = y_1 + iy_2$ and $w = y_3 + iy_4$, say. Evidently the real part of f transforms into the real part of f, i.e. surfaces $U(\ ) = \text{const.}$ transform into surfaces $U(\ ) = \text{const.}$ In (z, w) space the curves of steepest descent are given by

$$\frac{dy_1}{\partial U/\partial y_1} = \frac{dy_2}{\partial U/\partial y_2} = \frac{dy_3}{\partial U/\partial y_3} = \frac{dy_4}{\partial U/\partial y_4}, \tag{A 5·3}$$

and surfaces of steepest descent can be constructed from these. In general the curves and surfaces of steepest descent in (z, w) space differ from the curves and surfaces of steepest descent in (x, y) space. To show this it is sufficient to consider an example. Let us take

$$\begin{aligned} f(x,y) &= -x^2 - y^2 = -(x_1 + ix_2)^2 - (x_3 + ix_4)^2 \\ &= -(x_1^2 - x_2^2 + x_3^2 - x_4^2) - 2i(x_1 x_2 + x_3 x_4). \end{aligned}$$

Thus $$U(x_1, x_2, x_3, x_4) = -(x_1^2 - x_2^2 + x_3^2 - x_4^2),$$

and the curves of steepest descent are given by

$$\frac{dx_1}{x_1} = -\frac{dx_2}{x_2} = \frac{dx_3}{x_3} = -\frac{dx_4}{x_4},$$

whence $$x_1 x_2 = A, \quad x_2 x_3 = B, \quad x_3 x_4 = C,$$

on a curve of steepest descent. In particular, on curves of steepest descent through the saddle-point $(0, 0)$ we have $x_1 x_2 = x_2 x_3 = x_3 x_4 = 0$, and since U is to decrease away from $(0, 0)$ it is easy to see that $x_2 = x_4 = 0$. Thus on the curves of steepest descent through $(0, 0)$ we have

$$\frac{dx_1}{x_1} = \frac{dx_3}{x_3},$$

i.e. $x_1/x_3 = \text{const.}$, and the surface of steepest descent is the real plane $x_2 = x_4 = 0$.

Now consider the linear transformation

$$z = x\cosh\alpha + iy\sinh\alpha, \quad w = -ix\sinh\alpha + y\cosh\alpha,$$

where α is a real constant. Then evidently

$$x^2 + y^2 = z^2 + w^2,$$

and the surface of steepest descent in (z, w) space is the plane $y_2 = y_4 = 0$ which is not the same as the plane $x_2 = x_4 = 0$, as is easily verified. It follows that curves and surfaces of steepest descent do not in general remain invariant. (Note that on the contrary a curve of steepest descent in the plane of a single complex variable remains a curve of steepest descent under an analytic transformation.)

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It is well known that for a single complex variable the ideas of the method of steepest descents remain applicable when the contour of integration is composed of curves of descent which are not necessarily curves of steepest descent. However, when curves of steepest descent are chosen the contour is in general unique. Similarly for two complex variables it is sufficient that the surface of integration should be composed of surfaces of descent, while surfaces of steepest descent are not uniquely defined. It is interesting to note that on all the curves and surfaces of steepest descent through a given saddle-point (x_0, y_0) we have

$$\begin{aligned} V &\equiv \operatorname{Im} f(x, y) \\ &= \operatorname{Im} f(x_0, y_0), \quad \text{a constant.} \end{aligned}$$

This can be shown by applying the Cauchy–Riemann relations to (A 5·2).

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