J. Fluid Mech. (2014), *vol.* 746, *pp.* 85–98. © Cambridge University Press 2014
doi:10.1017/jfm.2014.93

The temperature spectrum generated by frictional heating in isotropic turbulence

Wouter J. T. Bos†

LMFA-CNRS, Université de Lyon, Ecole Centrale de Lyon, 69134 Ecully, France

(Received 12 July 2013; revised 27 November 2013; accepted 12 February 2014;
first published online 28 March 2014)

In every turbulent flow with non-zero viscosity, heat is generated by viscous friction. This heat is then mixed by the velocity field. We consider how heat fluctuations generated in this way are injected and distributed over length scales in isotropic turbulence. A triadic closure is derived and numerically integrated. It is shown how the heat fluctuation spectrum depends on the Reynolds and Prandtl numbers.

Key words: isotropic turbulence, mixing

1. Introduction

Turbulence allows an efficient conversion of kinetic energy into heat. In general, in the presence of external heat sources, the influence of such heat production is small, and few studies have considered how exactly heat fluctuations thus generated are redistributed over length scales by the turbulent flow itself. In a recent work, De Marinis *et al.* (2013) addressed the question of how this heat generation influences the wavenumber spectrum of heat fluctuations, using the eddy-damped quasi-normal Markovian (EDQNM) model, combined with an *ad hoc* spectrally local model. Unfortunately, in the important case of a spatially uniform temperature distribution, their model predicts zero frictional heat production, which is at odds with the physics of turbulent flows, where heat fluctuations are produced as soon as the instantaneous viscous dissipation is non-uniform in space. In the present paper we derive an EDQNM model of heat-fluctuation production which is more generally applicable, since it is independent of the initial distribution of temperature fluctuations.

2. Analysis of the problem

The problem we will consider is conceptually the simplest possible setting in which a non-trivial effect of the heat generation by turbulent dissipation can be observed: statistically isotropic and stationary incompressible turbulence, governed by the Navier–Stokes equations. The viscous dissipation of kinetic energy is a heat source, and the advection of this scalar quantity, the temperature, will be considered passive, ignoring the influence of thermal expansion of the fluid elements. It will be shown later that the temperature fluctuations are small in most situations, so that this assumption seems very reasonable. The heat diffusion coefficient α, dynamic viscosity

† Email address for correspondence: wouter.bos@ec-lyon.fr

https://doi.org/10.1017/jfm.2014.93 Published online by Cambridge University Press

μ and density ρ are assumed to be uniform and constant. The temperature Θ is then governed by the equation

$$\frac{\partial \Theta}{\partial t} + u_i \frac{\partial \Theta}{\partial x_i} = \alpha \frac{\partial^2 \Theta}{\partial x_i^2} + \frac{1}{\rho c_p} \tau_{ij} D_{ij}, \tag{2.1}$$

where c_p is the specific heat, which we will assume to be constant, and the viscous stress tensor τ_{ij} is given, for a Newtonian fluid, by

$$\tau_{ij} = 2\mu D_{ij}, \tag{2.2}$$

with

$$D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \tag{2.3}$$

Combining these expressions gives (Landau & Lifschitz 1987)

$$\frac{\partial \Theta}{\partial t} + u_i \frac{\partial \Theta}{\partial x_i} = \alpha \frac{\partial^2 \Theta}{\partial x_i^2} + \frac{\nu}{c_p} \left(\frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \right), \tag{2.4}$$

where $\nu = \mu/\rho$ is the kinematic viscosity. The velocity u_i and temperature Θ are functions of space $\boldsymbol{x}$ and time t, but this dependence will be omitted from the notation unless two or more different space locations or time instants appear in the same expression. Introducing the Reynolds decomposition $\Theta = \bar{\Theta} + \theta$, with $\bar{\Theta}$ being the average temperature and θ the fluctuation, we find the following equation for the mean temperature:

$$\frac{\partial \bar{\Theta}}{\partial t} = \frac{\nu}{c_p} \overline{\frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j}} = \frac{\epsilon}{c_p}, \tag{2.5}$$

where ϵ is the mean dissipation rate and the overlined quantities denote averages. Clearly, the average temperature increases since no heat sinks exist in our system. In a statistically stationary turbulent flow, where ϵ is constant, the mean temperature will thus increase linearly in time.

The equation for the fluctuations is

$$\frac{\partial \theta}{\partial t} + u_i \frac{\partial \theta}{\partial x_i} = \alpha \frac{\partial^2 \theta}{\partial x_i^2} - \frac{\nu}{c_p} \overline{\frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j}} + \frac{\nu}{c_p} \left(\frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \right). \tag{2.6}$$

The variance of the temperature fluctuations is then given by

$$\frac{\partial \overline{\theta^2}}{\partial t} = -2\alpha \overline{\frac{\partial \theta}{\partial x_i} \frac{\partial \theta}{\partial x_i}} + 2\frac{\nu}{c_p} \left(\overline{\theta \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j}} + \overline{\theta \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i}} \right). \tag{2.7}$$

Note that the second term in expression (2.11) of De Marinis *et al.* (2013) is zero due to homogeneity, and the last term is not complete since their basic equation (2.1) is not correct. This does not change the rest of their analysis, which did not explicitly use this information, but it is important if one wants to derive a model analytically from the governing equations, as we will do in the following section.

https://doi.org/10.1017/jfm.2014.93 Published online by Cambridge University Press

The temperature distribution over different length scales is given by the temperature spectrum, defined such that

$$\int E_\theta(k)\,\mathrm{d}k = \overline{\theta^2}, \tag{2.8}$$

which implies, due to homogeneity and isotropy,

$$E_\theta(k)\delta(\boldsymbol{k}+\boldsymbol{k}') = 4\pi k^2 \overline{\theta(\boldsymbol{k})\theta(\boldsymbol{k}')}, \tag{2.9}$$

where $\theta(\boldsymbol{k})$ is the Fourier transform of the temperature fluctuation. The equation for $E_\theta(k)$ is straightforwardly derived from (2.6) as

$$\begin{aligned}\left(\frac{\partial}{\partial t}+2\alpha k^2\right)E_\theta(k) \;=\; & -4\pi k^2 \iint \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\mathrm{i}k_i \overline{u_i(\boldsymbol{p})\theta(\boldsymbol{q})\theta(-\boldsymbol{k})}\,\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{q} \\ & -4\pi k^2 \frac{\nu}{c_p} \iint \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})p_i q_i \overline{u_j(\boldsymbol{p})u_j(\boldsymbol{q})\theta(-\boldsymbol{k})}\,\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{q} \\ & -4\pi k^2 \frac{\nu}{c_p} \iint \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})p_i q_j \overline{u_j(\boldsymbol{p})u_i(\boldsymbol{q})\theta(-\boldsymbol{k})}\,\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{q}, \end{aligned} \tag{2.10}$$

where $u_i(\boldsymbol{k})$ is the Fourier transform of the velocity fluctuation. This equation is unclosed, and three triple correlations need to be determined before we can integrate (2.10). In De Marinis *et al.* (2013) the first correlation, $\overline{u_i(\boldsymbol{p})\theta(\boldsymbol{q})\theta(-\boldsymbol{k})}$, was modelled by classic EDQNM theory (Herring *et al.* 1982; Vignon & Cambon 1980). The second correlation was modelled by the *ad hoc* model

$$-4\pi k^2 \frac{\nu}{c_p} \iint \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})p_i q_i \overline{u_j(\boldsymbol{p})u_j(\boldsymbol{q})\theta(-\boldsymbol{k})}\,\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{q} \sim \frac{\nu}{c_p} k^{5/2} E(k)\sqrt{E_\theta(k)}. \tag{2.11}$$

As stated in the introduction, model (2.11) cannot be used in the case of a uniform temperature distribution. In that case, the temperature fluctuations will stay zero forever, because the above model for their production is proportional to the spectrum $E_\theta(k)$. One could argue that the model can, perhaps, give correct results for the case in which the temperature spectrum is initially non-zero. This would, however, be fortuitous, since the temperature is supposed to be passive under our assumptions; it should therefore not have any influence on the injection rate of heat fluctuations.

We will derive in the following section an EDQNM model for the viscous production term of heat fluctuations. Subsequently we will carry out simulations to show the temperature spectrum that results from this production term. We will compare our results with those obtained by De Marinis *et al.* (2013).

3. Derivation of an EDQNM model for production of viscous heat fluctuations

We use the procedure outlined in Bos *et al.* (2012) to derive the EDQNM model for the triple correlations in (2.10): first we derive a direct interaction approximation (DIA, Kraichnan 1959) of the triple correlation; then we modify the time dependence to obtain a closure expression of EDQNM type. Note that Orszag (1970) already pointed out this relation between the Eulerian DIA equations and the EDQNM equations for isotropic turbulence. The idea behind the DIA is the following. In the triple correlations that we are interested in, the turbulent fluctuations are expanded

https://doi.org/10.1017/jfm.2014.93 Published online by Cambridge University Press

around a state $(u^{(0)}, \theta^{(0)})$ in which all but one particular triad are retained. The influence of this particular triad only is denoted by $(u^{(1)}, \theta^{(1)})$ and is treated as a perturbation around the state $(u^{(0)}, \theta^{(0)})$. The expansion is then truncated at the lowest non-vanishing order. The expressions for $u^{(1)}$ and $\theta^{(1)}$ are, in the present case,

$$u_i^{(1)}(\boldsymbol{k}, t) = -\mathrm{i} \int_0^t G(k, t, s) \left(k_a \delta_{ib} + k_b \delta_{ia} - 2 \frac{k_i k_a k_b}{k^2} \right) u_a(\boldsymbol{p}, s) u_b(\boldsymbol{q}, s) \, \mathrm{d}s \tag{3.1}$$

and

$$\begin{aligned} \theta^{(1)}(\boldsymbol{k}, t) \; = \; & -\mathrm{i} \int_0^t G_\theta(k, t, s) \left(k_a u_a(\boldsymbol{p}, s) \theta(\boldsymbol{q}, s) + k_a u_a(\boldsymbol{q}, s) \theta(\boldsymbol{p}, s) \right) \mathrm{d}s \\ & - 2 \int_0^t G_\theta(k, t, s) \frac{\nu}{c_p} p_i q_i u_j(\boldsymbol{p}, s) u_j(\boldsymbol{q}, s) \, \mathrm{d}s \\ & - 2 \int_0^t G_\theta(k, t, s) \frac{\nu}{c_p} p_i q_j u_j(\boldsymbol{p}, s) u_i(\boldsymbol{q}, s) \, \mathrm{d}s, \end{aligned} \tag{3.2}$$

where $G(k, t, s)$ and $G_\theta(k, t, s)$ are the nonlinear response functions of the velocity and temperature fluctuations, respectively. Expression (3.1) and the first line of (3.2) are the standard DIA contributions for the velocity and the scalar (Kraichnan 1959; Roberts 1961). The second and third lines of (3.2) constitute the DIA contribution to the scalar stemming from the frictional heat-source nonlinearity.

Upon substituting these perturbed quantities into the expression for the triple correlations, we obtain, after invoking the weak-dependence hypothesis (Kraichnan 1959),

$$\frac{\partial E_\theta(k)}{\partial t} = T_\theta(k) + P_\theta(k) - D_\theta(k), \tag{3.3}$$

with

$$\begin{aligned} T_\theta(k) \; = \; & 8\pi k^2 \iint \delta(\boldsymbol{k} - \boldsymbol{p} - \boldsymbol{q}) k_i k_j \int_0^t \Phi_{ij}(\boldsymbol{p}, t, s) \\ & \times \left[G_\theta(k, t, s) \Phi_\theta(\boldsymbol{q}, t, s) - G_\theta(q, t, s) \Phi_\theta(\boldsymbol{k}, t, s) \right] \mathrm{d}s \, \mathrm{d}\boldsymbol{p} \, \mathrm{d}\boldsymbol{q}, \end{aligned} \tag{3.4}$$

$$\begin{aligned} P_\theta(k) \; = \; & 16\pi k^2 \left(\frac{\nu}{c_p} \right)^2 \iint \delta(\boldsymbol{k} - \boldsymbol{p} - \boldsymbol{q}) \int_0^t G_\theta(k, t, s) \\ & \times \left[(p_m q_m)^2 \Phi_{ij}(\boldsymbol{p}, t, s) \Phi_{ij}(\boldsymbol{q}, t, s) + 2 p_m q_m p_i q_j \Phi_{aj}(\boldsymbol{p}, t, s) \Phi_{ia}(\boldsymbol{q}, t, s) \right. \\ & \left. + p_i p_j q_m q_n \Phi_{mn}(\boldsymbol{p}, t, s) \Phi_{ij}(\boldsymbol{q}, t, s) \right] \mathrm{d}s \, \mathrm{d}\boldsymbol{p} \, \mathrm{d}\boldsymbol{q}, \end{aligned} \tag{3.5}$$

$$D_\theta(k) \; = \; 2\alpha k^2 E_\theta(k), \tag{3.6}$$

where T, P and D denote transfer, production and diffusion, respectively. In this expression we have introduced the quantities

$$\Phi_{ij}(\boldsymbol{k}, t, s) \delta(\boldsymbol{k} + \boldsymbol{k}') = \overline{u_i(\boldsymbol{k}, t) u_j(\boldsymbol{k}', s)}, \quad \Phi_\theta(\boldsymbol{k}, t, s) \delta(\boldsymbol{k} + \boldsymbol{k}') = \overline{\theta(\boldsymbol{k}, t) \theta(\boldsymbol{k}', s)}. \tag{3.7}$$

We consider the isotropic mirror-symmetric case so that, upon introducing the time-correlation functions $R(k, t, s)$ and $R_\theta(k, t, s)$ such that

$$\Phi_{ij}(\boldsymbol{k}, t, s) = \Phi_{ij}(\boldsymbol{k}, t, t) R(k, t, s), \quad \Phi_\theta(\boldsymbol{k}, t, s) = \Phi_\theta(\boldsymbol{k}, t, t) R_\theta(k, t, s), \tag{3.8}$$

https://doi.org/10.1017/jfm.2014.93 Published online by Cambridge University Press

we can write

$$\Phi_{ij}(\boldsymbol{k}, t, s) = \frac{E(k)}{4\pi k^2}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) R(k, t, s), \quad \Phi_\theta(\boldsymbol{k}, t, s) = \frac{E_\theta(k)}{4\pi k^2} R_\theta(k, t, s). \tag{3.9}$$

The expressions for $T_\theta(k)$ and $P_\theta(k)$ can be rewritten, after some algebraic manipulations, as

$$T_\theta(k) = \iint_\Delta (1 - z^2) k^3 q^2 E(p) \left[\frac{\Theta^I(k, p, q) E_\theta(q)}{q^2} - \frac{\Theta^I(q, p, k) E_\theta(k)}{k^2}\right] \frac{\mathrm{d}p\,\mathrm{d}q}{pq}, \tag{3.10}$$

$$P_\theta(k) = 2\left(\frac{\nu}{c_p}\right)^2 \iint_\Delta \Theta^{II}(k, p, q)(4x^4 - 3x^2 + 1) k p^2 q^2 E(p) E(q) \frac{\mathrm{d}p\,\mathrm{d}q}{pq}, \tag{3.11}$$

where

$$\Theta^I(k, p, q) = \int_0^t G_\theta(k, t, s) R_\theta(q, t, s) R(p, t, s)\,\mathrm{d}s, \tag{3.12}$$

$$\Theta^{II}(k, p, q) \int_0^t G_\theta(k, t, s) R(q, t, s) R(p, t, s)\,\mathrm{d}s, \tag{3.13}$$

and Δ indicates the domain in the p–q plane in which k, p, q can form a triangle (i.e. $|p - q| \leqslant k \leqslant |p + q|$) and x, y, z are given by

$$x = -p_i q_i/(pq), \tag{3.14}$$

$$y = k_i q_i/(kq), \tag{3.15}$$

$$z = k_i p_i/(kp). \tag{3.16}$$

Assuming a fluctuation–dissipation relation for the two-time quantities and exponentially decaying time correlations

$$R(k, t, s) = G(k, t, s) = \exp(-\eta(k)(t - s)) \quad \text{for } t \geqslant s, \tag{3.17}$$

$$R_\theta(k, t, s) = G_\theta(k, t, s) = \exp(-\eta_\theta(k)(t - s)) \quad \text{for } t \geqslant s, \tag{3.18}$$

these expressions simplify, in the long-time limit, to

$$\Theta^I(k, p, q) = \frac{1}{\eta_\theta(k) + \eta_\theta(q) + \eta(p)}, \tag{3.19}$$

$$\Theta^{II}(k, p, q) = \frac{1}{\eta_\theta(k) + \eta(p) + \eta(q)}. \tag{3.20}$$

It is at this point that the choice of the form of the typical frequencies η and η_θ will determine the inertial-range scaling of the spectra. We will choose the standard forms (Herring *et al.* 1982)

$$\eta(k) = 1.3\sqrt{\int_0^k s^2 E(s)\,\mathrm{d}s} + \nu k^2 \quad \text{and} \quad \eta_\theta(k) = \alpha k^2. \tag{3.21}$$

We note here that the expression for $T_\theta(k)$ is the standard EDQNM closure for scalar fluctuations (Herring *et al.* 1982; Vignon & Cambon 1980). The expression for the

https://doi.org/10.1017/jfm.2014.93 Published online by Cambridge University Press

viscous heat-production term $P_\theta(k)$ is new. In principle, the presence of the frictional heating term could change the behaviour of the scalar frequency $\eta_\theta(k)$. This possibility is not explored in the present investigation.

The equation for the energy spectrum is equivalently closed by the EDQNM model (Orszag 1970),

$$\left(\frac{\partial}{\partial t}+2\nu k^2\right)E(k)=\iint_{\Delta}\Theta(k,p,q)(xy+z^3)k^3p^2E(q)\left[\frac{E(p)}{p^2}-\frac{E(k)}{k^2}\right]\frac{\mathrm{d}p\,\mathrm{d}q}{pq}+F(k), \tag{3.22}$$

where $F(k)$ is a constant forcing term acting in the small-wavenumber range and

$$\Theta(k,p,q)=\frac{1}{\eta'(k)+\eta'(p)+\eta'(q)} \tag{3.23}$$

with

$$\eta'(k)=0.36\sqrt{\int_0^k s^2E(s)\,\mathrm{d}s}+\nu k^2. \tag{3.24}$$

It has been shown in the past that the variance of the dissipation-rate fluctuations, involving fourth-order correlations of the velocity field, was not correctly captured by DIA-related approaches (Chen *et al.* 1989). The present model involves the correlation between the dissipation-rate fluctuation and the temperature fluctuation. This is a mixed velocity–temperature third-order correlation. In general, it seems that DIA more reliably represents third-order correlations than fourth-order ones, but a validation of the model (3.11) by direct numerical simulation would be valuable, in particular since the modelled correlation involves the fluctuation of the dissipation rate.

4. Heat production at unit Prandtl number

4.1. *Preliminary analysis of the closure expression*

The goal here is to first derive expressions for the heat-fluctuation spectrum generated by frictional heating, by considering the evolution (3.3) and the EDQNM closure for the frictional heating term $P_\theta(k)$ in this equation. Then we will verify these analytical results by numerically integrating the closure equations. For our analytical considerations, we will assume the energy spectrum to be constant in time and to obey Kolmogorov scaling, $E(k)\sim\epsilon^{2/3}k^{-5/3}$, from the wavenumber corresponding to the integral scale down to the Kolmogorov wavenumber $k_\eta\sim\epsilon^{1/4}\nu^{-3/4}$.

To evaluate the convolution integral in (3.5), there are three distinct cases to consider. Firstly, the convolution integral is dominated by local interactions. Secondly, the integral is dominated by infrared non-local interactions $p\ll k\approx q$. Thirdly, the integral is dominated by ultraviolet non-local interactions $k\ll p\approx q$. We have verified the convolution integral analytically for all three types of contributions and have found that in the case of a very long inertial range, the nonlocal interactions contribute insignificantly compared to the local interactions. We will therefore consider only the approximation of (3.3) for the case of local interactions. In that case, (3.5) can be approximated by its dimensional estimate

$$P_\theta(k)\sim\left(\frac{\nu}{c_p}\right)^2k^5E(k)^2\tau(k), \tag{4.1}$$

https://doi.org/10.1017/jfm.2014.93 Published online by Cambridge University Press

where $\tau(k)$ is the local estimate of the triad interaction time $\Theta^{II}(k,p,q)$. If we suppose that

$$\tau(k) \sim \epsilon^{-1/3} k^{-2/3} \tag{4.2}$$

and assume Kolmogorov scaling, we find that

$$P_\theta(k) \sim \left(\frac{\nu}{c_p}\right)^2 \epsilon k. \tag{4.3}$$

In the case of unit and large Prandtl numbers, the diffusion will be small in the inertial-convective range, and we expect a balance to be established between the production term and the transfer term:

$$P_\theta(k) \approx T_\theta(k). \tag{4.4}$$

This balance will indeed be observed in the numerical evaluation of the closure. In this range, it was shown (Rubinstein & Clark 2013) that the scalar transfer can be qualitatively predicted by a Kovaznay-type gradient model,

$$T_\theta(k) \sim \frac{\partial}{\partial k}\left(E_\theta(k) E(k)^{1/2} k^{5/2}\right), \tag{4.5}$$

and we find from (4.3)–(4.5) that

$$E_\theta(k) \sim \left(\frac{\nu}{c_p}\right)^2 \epsilon^{2/3} k^{1/3}. \tag{4.6}$$

Integrating this spectrum up to the Kolmogorov scale, we have

$$\overline{\theta^2} \sim \frac{\epsilon \nu}{c_p^2} \sim R_\lambda^{-2}. \tag{4.7}$$

In a steady state, the total production will be balanced by the scalar dissipation, so

$$\int P_\theta(k)\,\mathrm{d}k = \epsilon_\theta \equiv \int D_\theta(k)\,\mathrm{d}k. \tag{4.8}$$

Assuming a scaling of the production spectrum of the form (4.3) up to the Kolmogorov scale k_η, we have

$$\epsilon_\theta \sim \frac{\epsilon^{3/2}\nu^{1/2}}{c_p^2}, \tag{4.9}$$

which shows that in the limit as $R_\lambda \to \infty$,

$$\epsilon_\theta \sim R_\lambda^{-1}. \tag{4.10}$$

This means that in the limit of vanishingly small viscosity, both the mean-square heat fluctuations and their dissipation tend to zero.

https://doi.org/10.1017/jfm.2014.93 Published online by Cambridge University Press

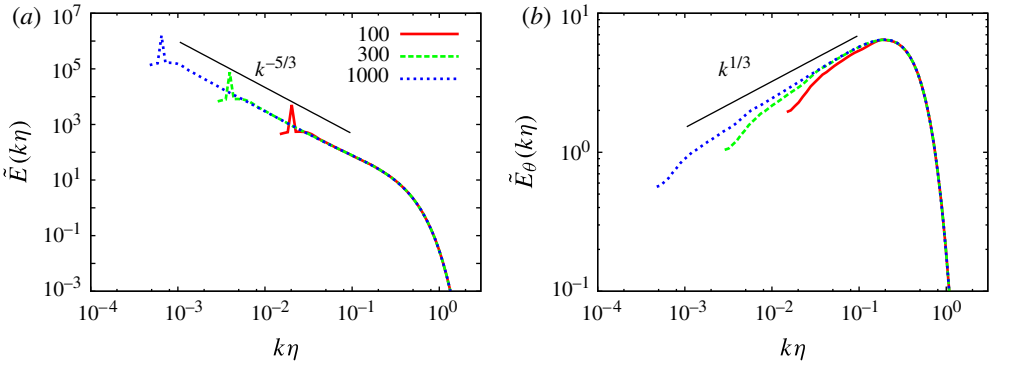


FIGURE 1. (Colour online) (*a*) Energy spectrum, normalized by Kolmogorov variables, for three different Reynolds numbers, $R_\lambda = 100$, 300 and 1000. (*b*) The corresponding temperature fluctuation spectrum generated by frictional heating (with $Pr = 1$).

4.2. *Numerical integration of the closure for unit Prandtl number*

Equations (3.3) and (3.22) are integrated numerically on a logarithmically spaced grid with 22 wavenumbers per decade for Reynolds numbers $R_\lambda = 100$, 300 and 1000. The specific heat c_p is kept constant, since its variation induces only a trivial change in the results. Indeed, since the advection equation is linear and depends linearly on c_p^{-1}, the temperature spectrum will simply be proportional to c_p^{-2}. The initial temperature spectrum is zero. All wavenumber spectra in the remainder of this section are normalized using Kolmogorov–Corrsin–Obukhov variables, i.e. using the variables ν, ϵ and ϵ_θ. The normalized spectra are indicated by a tilde. For instance, the normalized energy spectrum is given by

$$\tilde{E}(k) = \nu^{-5/4}\epsilon^{-1/4}E(k), \tag{4.11}$$

and k is normalized by the Kolmogorov length scale $\eta = \nu^{3/4}\epsilon^{-1/4}$.

Since the velocity field contains a large-scale forcing, the energy spectrum will develop towards a steady state. This long-time asymptotic energy spectrum is shown in figure 1(*a*). The energy spectra, normalized using Kolmogorov variables, collapse perfectly at high wavenumbers. The temperature spectrum which develops asymptotically due to the viscous heat production is shown in figure 1(*b*). It is an increasing function of the wavenumber, roughly proportional to $k^{1/3}$, and the spectra rapidly fall off in the far dissipation range. For different Reynolds numbers, the temperature spectra collapse upon using Kolmogorov–Corrsin–Obukhov variables,

$$\tilde{E}_\theta(k) = (\epsilon_\theta \nu^{5/4}\epsilon^{3/4})^{-1}E_\theta(k). \tag{4.12}$$

Figure 2(*a*) shows the three different contribution terms in the evolution equation of the temperature spectrum: the transfer $T_\theta(k)$, the production by frictional heating $P_\theta(k)$ and the dissipation of temperature fluctuations $D_\theta(k)$. We observe that the main part of the production is locally (in wavenumber) dissipated by the diffusion. The transfer term allows for a small transfer of temperature variance towards smaller scales. But since the production takes place mainly in the diffusive range (at unit Prandtl number), the fluctuations do not survive long enough to be strained significantly by the velocity field, and the transfer is therefore confined to a small-wavenumber

https://doi.org/10.1017/jfm.2014.93 Published online by Cambridge University Press

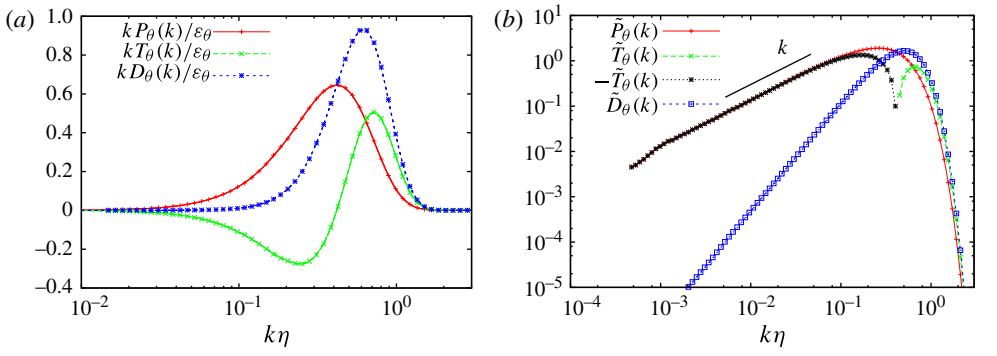


FIGURE 2. (Colour online) (*a*) Different terms in the evolution equation of the temperature spectrum (with $Pr = 1$): the transfer $T_\theta(k)$, production by frictional dissipation $P_\theta(k)$ and dissipation of temperature fluctuations $D_\theta(k)$. (*b*) Logarithmic representation of the same terms.

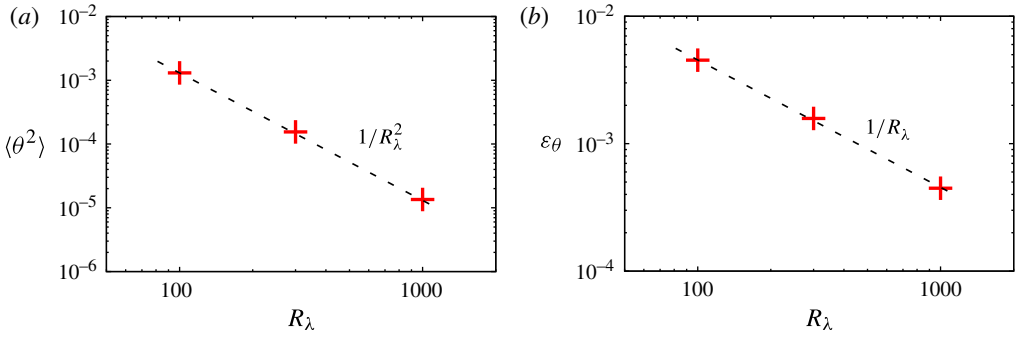


FIGURE 3. (Colour online) Reynolds number dependence at $Pr = 1$ of: (*a*) the mean-square heat fluctuations; (*b*) the dissipation of heat fluctuations.

span in the dissipation range. This picture will change at large Prandtl numbers. In figure 2(*b*) we plot the same contribution terms but in a logarithmic representation. This allows us to focus more closely on the small- and intermediate-wavenumber ranges. In the inertial range, the production term is proportional to k. The production is largely dominant over the diffusion $D_\theta(k)$ in this range, and it is therefore the transfer term which balances the production, as assumed in expression (4.4). The wavenumber dependences of $P_\theta(k)$ and $E_\theta(k)$ are as predicted in expressions (4.3) and (4.6), which were obtained by assuming that the convolution integral in (3.5) is dominated by local interactions. A further consequence of this assumption is that the mean-square temperature fluctuations should be proportional to R_λ^{-2}. The dissipation of temperature fluctuations should be inversely proportional to the Reynolds number. This is indeed observed in figure 3.

5. Temperature spectrum in the inertial-diffusive and viscous-convective range

5.1. *Phenomenology for small and large Prandtl numbers*

The energy spectrum decays rapidly for $k > k_\eta$. Since the convolution integral in (3.11) contains interactions with $E(p)$ and $E(q)$ under the constraint $|p - q| \leqslant k \leqslant |p + q|$,

https://doi.org/10.1017/jfm.2014.93 Published online by Cambridge University Press

negligible production will be observed beyond $k = 2k_\eta$. For a very high Prandtl number we therefore expect a Batchelor-type phenomenology in which the scalar fluctuations are strained by the smallest available velocity scale, around k_η (Batchelor 1959). This scaling is given by

$$E_\theta(k) \sim \epsilon_\theta \sqrt{\frac{\nu}{\epsilon}}\, k^{-1}. \tag{5.1}$$

Substituting the ϵ_θ from expression (4.9), we have

$$E_\theta(k) \sim \frac{\epsilon \nu}{c_p^2} k^{-1}. \tag{5.2}$$

For the opposite case, in which the Prandtl number is very small, the production term will be balanced by the diffusion term. The production term was approximated by

$$P_\theta(k) \sim \left(\frac{\nu}{c_p}\right)^2 k^5 E(k)^2 \tau(k), \tag{5.3}$$

assuming dominance of the local interactions in the generation of the heat fluctuations. However, in this case the triad-correlation time scale is not given by (4.2), since at these scales the dynamics is now dominated by the diffusion of heat, so that

$$\tau(k) \sim 1/(\alpha k^2), \tag{5.4}$$

and the production spectrum in the inertial-diffusive range will then be given by

$$P_\theta(k) \sim \left(\frac{\nu}{c_p}\right)^2 \frac{\epsilon^{4/3}}{\alpha} k^{-1/3}. \tag{5.5}$$

Since the diffusion spectrum is

$$D_\theta(k) = 2\alpha k^2 E_\theta(k), \tag{5.6}$$

we find from

$$P_\theta(k) \approx D_\theta(k) \tag{5.7}$$

that

$$E_\theta(k) \sim \left(\frac{\nu}{c_p}\right)^2 \frac{\epsilon^{4/3}}{\alpha^2} k^{-7/3}. \tag{5.8}$$

5.2. *Numerical integration of the closure for small and large Prandtl numbers*

The results of a large-Prandtl-number simulation ($Pr = 100$, $R_\lambda = 100$) are shown in figure 4. The k^{-1} spectrum starts to appear but is not yet fully developed at this Prandtl number. Since the peak of the production mechanism remains fixed around k_η, independently of the Prandtl number, there is no reason why it should not become more pronounced at higher Prandtl numbers. The balance of diffusion and transfer is observed in figure 4(*b*), for wavenumbers larger than k_η, where the energy spectrum drops rapidly.

Figure 5(*a*), shows that at low Prandtl number, the temperature spectrum is indeed approximately proportional to $k^{-7/3}$, in agreement with expression (5.8). The balance between the diffusion term and the heat-fluctuation source term holds, as can be seen in figure 5(*b*). The production spectrum is not exactly given by a power law proportional to $k^{-1/3}$, which is consistent with the fact that the temperature spectrum is not described by a pure power law either.

https://doi.org/10.1017/jfm.2014.93 Published online by Cambridge University Press

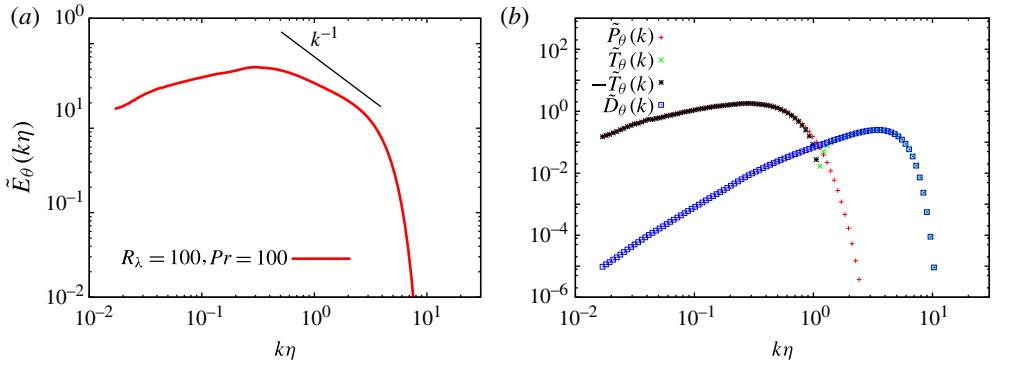


FIGURE 4. (Colour online) (*a*) $E_\theta(k)$ for $R_\lambda = 100$ and $Pr = 100$. (*b*) The transfer $T_\theta(k)$, production $P_\theta(k)$ and dissipation spectrum $D_\theta(k)$.

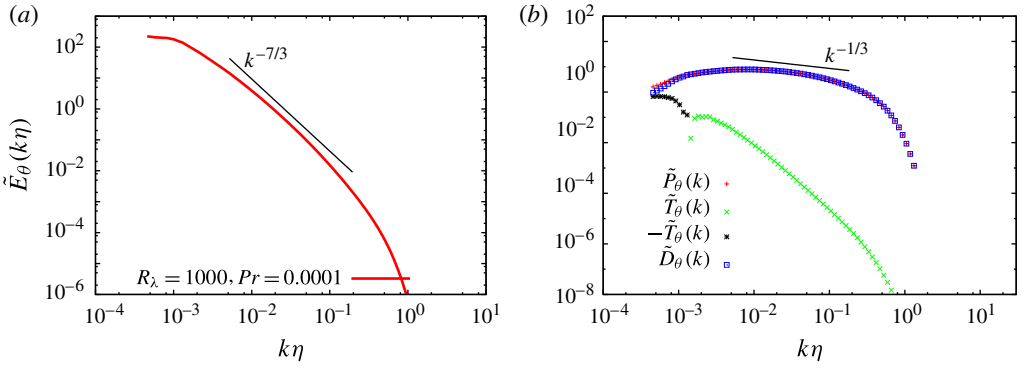


FIGURE 5. (Colour online) (*a*) $E_\theta(k)$ for $R_\lambda = 1000$ and $Pr = 10^{-4}$. (*b*) The transfer $T_\theta(k)$, production $P_\theta(k)$ and dissipation spectrum $D_\theta(k)$.

6. Comparison with previous model predictions

It was shown in the previous sections that according to the closure we derived, the generation of heat fluctuations is dominated by local interactions in wavenumber space. This allowed us to show that a simple phenomenological model of the production is

$$P_\theta(k) \sim \left(\frac{\nu}{c_p}\right)^2 k^5 E(k)^2 \tau(k), \tag{6.1}$$

with

$$\tau(k) \sim \begin{cases} \epsilon^{-1/3} k^{-2/3} & \text{for } Pr \geqslant 1,\ k\eta \ll 1, \\ 1/(\alpha k^2) & \text{for } Pr \ll 1,\ k\eta \ll 1. \end{cases} \tag{6.2}$$

Comparing the form of the above model to the model of De Marinis *et al.* (2013), the main difference is the absence of the heat-fluctuation spectrum in (6.1). In the case of the model of De Marinis *et al.* (2013), the production term cannot be determined independently of the temperature-fluctuation distribution. All cases considered in the present work would thus yield zero temperature fluctuations using

https://doi.org/10.1017/jfm.2014.93 Published online by Cambridge University Press

that model. However, in the particular case of an initial heat-fluctuation spectrum, in the absence of further heat sources and for very long times, it is possible to compare the predictions of the model of De Marinis *et al.* (2013) with the present case. For instance, in the inertial-convective range we expect a balance between nonlinear transfer of scalar fluctuations and production. Combining (4.4), (4.5) and (2.11), we find that in this particular case the temperature-fluctuation spectrum in the inertial-convective range (assuming Kolmogorov scaling) satisfies

$$E_\theta(k) \sim \left(\frac{\nu}{c_p}\right)^2 \epsilon^{2/3} k^{1/3}. \tag{6.3}$$

If we consider the low-Prandtl-number case, the balance (5.7) should hold in the inertial range and the temperature spectrum will be proportional to

$$E_\theta(k) \sim \left(\frac{\nu}{c_p}\right)^2 \alpha^{-2} \epsilon^{4/3} k^{-7/3}, \tag{6.4}$$

as was also proposed by De Marinis *et al.* (2013). The predictions of their model for these particular cases are therefore in agreement with the results of our EDQNM model. This might seem surprising, given that the model does not give physical results for the basic case in which no initial fluctuations are present. We note, however, that all models of the form

$$P_\theta(k) \sim E_\theta(k)^\gamma \left(\frac{\nu}{c_p}\right)^{2(1-\gamma)} E(k)^{3/2-\gamma} k^{7/2-2\gamma} \tag{6.5}$$

will give the same steady-state inertial-convective-range behaviour, independent of the value of γ. The only choice predicting a non-zero production in the absence of initial fluctuations is $\gamma = 0$. The value that will give a correct scaling in the inertial-diffusive range at low Prandtl number is $\gamma = 1/2$, corresponding to the model of De Marinis *et al.* (2013). It is therefore not possible to find a model of the form (6.5) which produces behaviour in agreement with both (4.6) and (5.8) and which predicts non-zero production in the absence of initial fluctuations. The simplest phenomenological model that satisfies these criteria is (6.1), involving a time scale which can mimic both diffusive and inertial behaviour.

We note that as soon as initial heat fluctuations are present in the considered flow, which is a prerequisite for (2.11) to generate heat, the distribution of these fluctuations will determine the instantaneous heat generation and all transients will be affected by this. The influence of these unphysical transients can be very long if the initial heat fluctuations are present in the largest flow scales, which decay slowly.

7. Conclusion

An EDQNM model for the heat-production term due to viscous friction is derived. Temperature wavenumber spectra are deduced from this closure. In particular, it is shown that the heat production is mainly generated through local triad interactions. The heat-production spectrum is then proportional to k. Balancing this production spectrum against nonlinear transfer, we find the heat spectrum for unit and large Prandtl numbers in the inertial-convective subrange. In the very diffusive case

https://doi.org/10.1017/jfm.2014.93 Published online by Cambridge University Press

(i.e. very small Prandtl number), the production is in equilibrium with the diffusivity spectrum, leading to a different wavenumber dependence. The predictions are

$$E_\theta(k) \sim \begin{cases} \left(\dfrac{\nu}{c_p}\right)^2 \epsilon^{2/3} k^{1/3} & \text{for } Pr \geqslant 1,\ k\eta \ll 1; \\ \left(\dfrac{\nu}{c_p}\right)^2 \dfrac{\epsilon^{4/3}}{\alpha^2} k^{-7/3} & \text{for } Pr \ll 1,\ k\eta \ll 1; \\ \dfrac{\epsilon\nu}{c_p^2} k^{-1} & \text{for } Pr \gg 1,\ k\eta \gg 1. \end{cases} \tag{7.1}$$

Now that we have obtained these expressions for the heat-fluctuation spectra, it is possible to estimate how large the fluctuations are in reality. Expression (4.7) shows that, assuming Kolmogorov scaling, we have

$$\epsilon \sim \frac{\overline{\theta^2}\, c_p^2}{\nu}. \tag{7.2}$$

Let us consider typical values of c_p and ν. For a gas such as air, $c_p \approx 10^3$ J kg^{-1} K^{-1} and $\nu \approx 10^{-5}$ m^2 s^{-1}. In order to observe heat fluctuations of the order of 10^{-3} K, a dissipation rate of the order of 10^5 W kg^{-1} is needed. Together with (2.5), this implies that the mean temperature of the gas increases at a rate of 10^2 K s^{-1}. For a typical liquid such as water, this increase would be even larger. This example illustrates the smallness of the temperature variance in practice. It does not seem to us that in a practical situation where the temperature of a fluid increases by 100 K every second, fluctuations of the order of a millikelvin would be of major importance.

Note that the scalings (7.1) were observed approximately from the numerically integrated closure expressions. We stress that these are results which are obtained within the framework of the EDQNM assumptions, and the model remedies a defect in a previous model proposition which excluded heat-fluctuation generation in the absence of initial fluctuations (De Marinis *et al.* 2013). Validation of the EDQNM model for heat production and verification of the proposed scalings (7.1) using high-resolution direct numerical simulations of isotropic turbulence would certainly be valuable, particularly since DIA and EDQNM may have difficulties in representing correlations that involve fluctuations of the dissipation rate.

Acknowledgements

The author thanks Robert Rubinstein for his invaluable comments and acknowledges discussions with Marcello Meldi, Sergio Chibbaro and Pierre Sagaut.

REFERENCES

BATCHELOR, G. K. 1959 Small-scale variation of convected quantities like temperature in turbulent fluid. Part 1. General discussion and the case of small conductivity. *J. Fluid Mech.* **5**, 113–133.

BOS, W. J. T., RUBINSTEIN, R. & FANG, L. 2012 Reduction of mean-square advection in turbulent passive scalar mixing. *Phys. Fluids* **24**, 075104.

CHEN, H., HERRING, J. R., KERR, R. M. & KRAICHNAN, R. H. 1989 Non-Gaussian statistics in isotropic turbulence. *Phys. Fluids* A **1**, 1844–1854.

DE MARINIS, D., CHIBBARO, S., MELDI, M. & SAGAUT, P. 2013 Temperature dynamics in decaying isotropic turbulence with Joule heat production. *J. Fluid Mech.* **724**, 425–449.

https://doi.org/10.1017/jfm.2014.93 Published online by Cambridge University Press

HERRING, J. R., SCHERTZER, D., LESIEUR, M., NEWMAN, G. R., CHOLLET, J. P. & LARCHEVEQUE, M. 1982 A comparative assessment of spectral closures as applied to passive scalar diffusion. *J. Fluid Mech.* **124**, 411–437.

KRAICHNAN, R. H. 1959 The structure of isotropic turbulence at very high Reynolds numbers. *J. Fluid Mech.* **5**, 497–543.

LANDAU, L. D. & LIFSCHITZ, E. M. 1987 *Fluid Mechanics*. 2nd *(Course of Theoretical Physics, vol. 6)*,. Pergamon Press.

ORSZAG, S. A. 1970 Analytical theories of turbulence. *J. Fluid Mech.* **41**, 363–386.

ROBERTS, P. H. 1961 Analytical theory of turbulent diffusion. *J. Fluid Mech.* **11**, 257–283.

RUBINSTEIN, R. & CLARK, T. T. 2013 Reassessment of the classical closures for scalar turbulence. *J. Turbul.* **14**, 71–98.

VIGNON, J.-M. & CAMBON, C. 1980 Thermal spectral calculation using eddy-damped quasi-normal Markovian theory. *Phys. Fluids* **23**, 1935–1937.

https://doi.org/10.1017/jfm.2014.93 Published online by Cambridge University Press