Note: There is an extensive collection of John Bonnycastle's textbooks in the MA Library housed in the David Wilson Library at Leicester University. In particular, there are multiple editions of his *Introduction to mensuration and practical geometry*.

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105.37 A Potter's brief

Introduction

A wealthy patron commissions a hand potter to design a cup and saucer to exacting specifications. Particularly, the cup and saucer are to be the shape of spherical caps. For the cup, the patron specifies both the distance across its aperture and the volume it is to contain. For the saucer, the distance across its aperture is also specified but since the potter will be asked to decorate and glaze the items using expensive materials, the patron insists that the surface area of the saucer should be the same as that of the cup. For the purpose of the exercise, the patron tells the potter that the cup and saucer can be treated as if they are infinitely thin so that surface areas on the inside are the same as those on the outside. In addition, the potter is told that the items will have to be put on a base at some stage to stop them rocking, but that the details concerning this aspect could be ignored at the design stage. Neither too at this stage is the potter to worry about the design of a cup handle, should the patron eventually decide to have one. Finally, the patron gives the dimensions of interest to the potter as follows: the aperture of the cup is to be 9.5 cm, its volume is to be 270 cm^3 and the diameter across the aperture of the saucer is to be 14 cm.

The potter will be throwing and moulding the clay required by hand and so weighs up these specifications, deciding that it will suffice to first construct templates for the exercise, one for the cup and one for the saucer, which can be tested against the profiles that are being shaped to see if *in situ* corrections are needed. It is anticipated that the one for the saucer will be quite shallow, but it is not yet apparent if it will be possible to meet its specification because of the area requirement. Accordingly, the potter attends the local library to garner mathematical details concerning the volume and surface area of spherical caps, in conjunction with what can be gleaned from the internet. The formulae the potter finds are discussed below.

Volume and surface area of a spherical cap

Nowadays, formulae for these parameters can be found on crib sheets and books of formulae etc. For the author, the procedure to establish them was encountered on a GCE Advanced Level Mathematics syllabus in the last century, as an exercise involving applications of the integral calculus. Thus, the formulae will be simply quoted here with reference to the schematic of

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Figure 1. In this figure, the height of the spherical cap, from base to rim, will be denoted by y and the radius of the rim will be denoted by x.

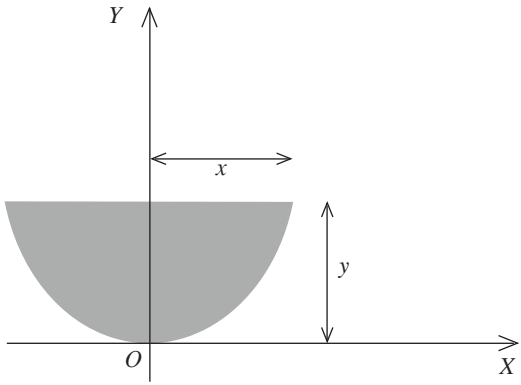


FIGURE 1: Schematic representation of the spherical cap

From this figure, it is a fairly straightforward exercise in geometry and trigonometry to establish that the radius of the cup's sphere, ρ say, is

$$\rho = \frac{x^2 + y^2}{2y}. \tag{1}$$

Moreover, for the surface area of the spherical cap, S say, it is usually established that this is the same as the intercept area on a circumscribing cylinder of the sphere, that is to say, it is simply $2\pi\rho y$. For ρ given by (1), this reduces to

$$S = \pi\left(x^2 + y^2\right). \tag{2}$$

Finally, for the volume of the spherical cap, V say, a method such as that adopted in [1] relating to solids of revolution can be employed to show that

$$V = \frac{\pi}{6}\left(3x^2y + y^3\right). \tag{3}$$

It is to these three formulae that the potter turns, as described hereunder.

The cup and the saucer

Since the design of the saucer is contingent upon the design of the cup, the potter addresses first that of the cup.

(a) *The cup*

In this instance, values for the volume and distance parameters, V and x, are given as 270 and 4.75 $\left(\frac{1}{2} \times 9.5\right)$ respectively, and so they can be substituted into (3) to determine a corresponding value for y (in cm). The ensuing equation is a cubic in y and it does not contain a term involving y^2. It is therefore in what is termed an already reduced format. The potter will

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have to obtain a solution to this cubic in order to establish the template. A solution can be obtained either in closed form or numerically.

For a closed form solution, a method due to Tartaglia and Cardano [2] can be employed. To this end the cubic is written first in the form

$$y^3 + 3Hy + G = 0, \text{ where } H = x^2 \text{ and } G = -\frac{6V}{\pi}. \tag{4}$$

The cubic root required using this method will be presented in recipe format drawn directly from the reference. A quantity of special interest is a parameter, ε say, where $\varepsilon = \sqrt{G^2 + 4H^3}$. The term under the square root is necessarily positive and the root of interest is $y = u + v$, with $u = \sqrt[3]{\xi}$ and $v = -\frac{H}{\sqrt[3]{\xi}}$, where the parameter $\xi = \frac{1}{2}(-G + \varepsilon)$. There are of course two other roots in the procedure, and the interested reader can delve into the text of the above reference to appreciate and determine them, but they are complex quantities that are not required for the purpose of this exercise. For the present brief, involving the above input values for the parameters V and x, the solution thus obtained can be shown to be $y = 5.352627$ cm and the surface area S of the inside of the cup is thus determined from (2) using this value and the above value for x. It can be shown to be 160.890752 cm^2. The radius of the spherical cap for the cup is, from (1), 4.783923 cm.

For those who would prefer to see a purely numerical approach, rather than an analytic one that involves the solution to a cubic, it is possible to plot the cubic using spreadsheet facilities. This is shown in the figure below. Thereafter, it can be appreciated from the figure that the real root lies between the values of $y = 5$ and $y = 6$, and a numerical technique such as the Newton-Raphson method can be employed to determine the root to greater accuracy and confirm thereby the earlier result. As a matter of interest regarding this cubic, there is a point of inflexion at the point $(0, -6V/\pi)$. It is now a simple matter to exploit the above information and use spreadsheet facilities to plot the profile of the cup. This is shown in Figure 3.

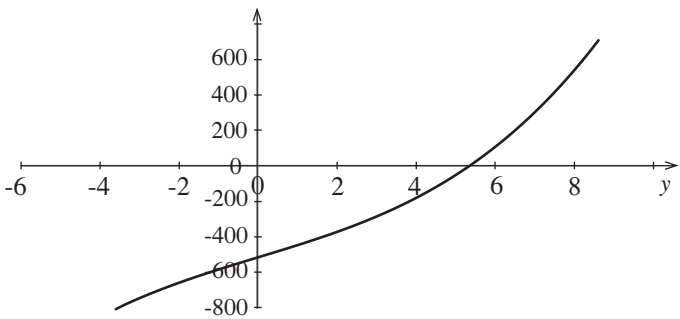


FIGURE 2: A plot of the cubic for the cup

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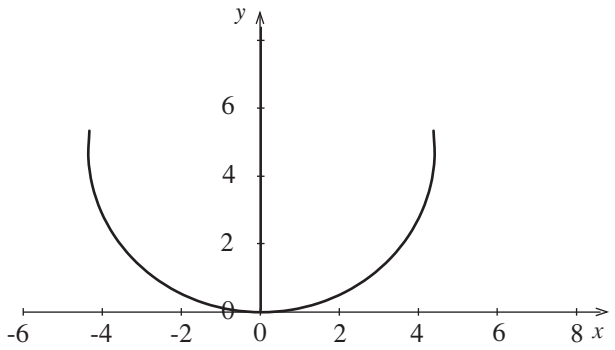


FIGURE 3: The shape of the cup (dimensions in cm)

Having determined the surface area and the shape of the cup (from which a template can be constructed), attention is turned to the requirement for the saucer. To this end, various approaches can be adopted.

(b) *The saucer*

In the first instance, one can return to solve the cubic of (4) in an interactive manner, but now with reference to inputs for the saucer. Specifically, the input value for x is now $7 (= 14/2)$ and a value for the volume can be input and varied until the specified surface area of 160.890752 cm^2 is met. This is an interesting exercise for the reader to entertain. However, it is not strictly necessary because one can return to (2) and (3) to establish yet another cubic in the reduced form,

$$y^3 + 3Hy + G = 0, \text{ where } H = -\frac{S}{2\pi} \text{ and } G = \frac{3V}{\pi}. \tag{5}$$

Then, an elimination of the cubed term in (4) and (5) produces the result

$$\frac{V}{y} = \frac{\pi}{3}\left(x^2 + \frac{S}{2\pi}\right). \tag{6}$$

The terms on the right-hand side of (6) are known inputs for the saucer and hence a substitution for V from this equation into, say, (5) produces after cancellations a y–value for the saucer's edge. Denoting y/V by γ, this can be shown to be

$$y = \sqrt{\frac{3S}{2\pi} - \frac{3}{\gamma\pi}}. \tag{7}$$

Provided $\gamma S > 2$, this gives real values for the edge parameter y. With the specified inputs, it can be shown that $y = 1.487655$ cm and the corresponding volume of the saucer is $V = 116.227207$ cm^3. The radius of the spherical cap for the saucer can be obtained then from (1) with the relevant values for x and y, and it is 17.212701 cm.

For completeness, it should be appreciated that solutions to the cubic of (5) can also be written down using the nomenclature of [2, p. 182], which is concerned with the "Trigonometrical Solution of a Cubic with Real Roots".

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Again, in recipe format a quantity $k = \sqrt{-4H}$ is introduced along with a trigonometric angle $\varphi = \frac{1}{3}\cos^{-1}\frac{-4G}{k^3}$. The real roots for y are given by the formulae $k\cos\varphi$, $k\cos\left(\varphi + \frac{2}{3}\pi\right)$ and $k\cos\left(\varphi + \frac{4}{3}\pi\right)$. The last of these turns out to be the root y of interest here, that is the one given in the paragraph above. As before with the cubic for the cup it is possible to display, for a value of the input parameter given in the preceding paragraph, a plot of this one for the saucer. This is shown in Figure 4.

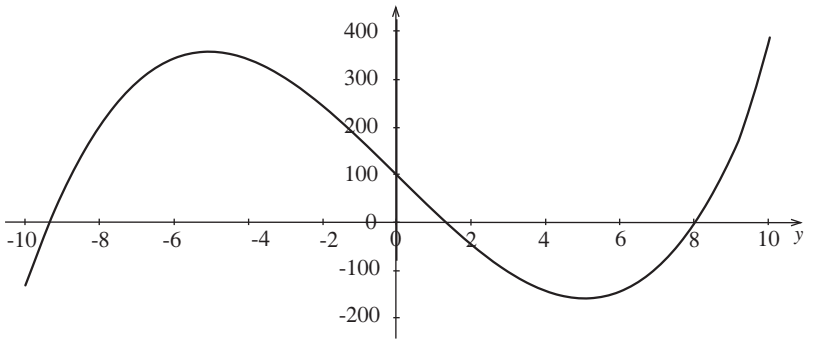


FIGURE 4: A plot of the cubic for the saucer

It can be appreciated from the figure that there are indeed three real roots. Other values for the input volume could be considered vis-à-vis this cubic and it would look marginally different to that in the figure, but an interactive response could be adopted to home in on the value required to meet the surface area requirement. This is left as an exercise for the reader.

In the manner described earlier, it is possible to plot the shape of the saucer on a spreadsheet and combine it with that of the cup. This is shown in Figure 5.

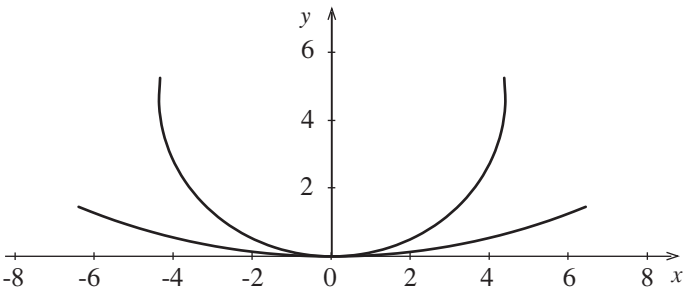


FIGURE 5: The combined shapes of the cup and the saucer (dimensions in cm)

Conclusion

It would appear from what has gone before that the potter has managed to meet the specifications laid down by the patron who commissioned the work. In the process, the mathematics has led to the notion of having to solve two different

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types of reduced cubic, either numerically or in other ways. This material could be of interest to those looking in the field of mathematics for an application involving spherical surfaces and solutions to higher order polynomials.

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Reminiscent of the quotation on page 340.

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