Ergod. Th. & Dynam. Sys. (2002), **22**, 571–589
DOI: 10.1017/S0143385702000287
© *2002 Cambridge University Press*
Printed in the United Kingdom

Ergodic attractors as limits of hyperbolic horseshoes

FERNANDO JOSÉ SÁNCHEZ-SALAS
Departamento de Matematicas y Computacion, Facultad de Ciencias, Universidad del Zulia, Venezuela
(e-mail: fsanchez@luz.ve)

(*Received* 8 *August* 1999 *and accepted in revised form* 8 *March* 2001)

Abstract. Let $f : M \to M$ be a C^2 diffeomorphism of a compact Riemannian manifold of dimension $m \geq 2$ leaving invariant an ergodic Sinai–Ruelle–Bowen measure μ with non-zero Lyapunov exponents. We prove that μ can be approximated by ergodic measures supported on hyperbolic horseshoes with arbitrarily large unstable dimensions.

1. *Introduction*

Following [**5**] we will say that a Borel probability μ preserved by a C^2 diffeomorphism of a compact Riemannian manifold is a Sinai–Ruelle–Bowen (SRB) measure if it is absolutely continuous along unstable leaves (see [**5**] and [**6**]). Pugh and Shub [**12**] have proved that any ergodic non-atomic SRB measure with non-zero Lyapunov exponents is an *ergodic attractor* in that the set of generic points $G_\mu^+ = \{x \in M : 1/n \sum_{k=0}^{n-1} \delta_{F^k(x)} \to \mu \text{ weakly}\}$ has a positive volume.

This work is aimed at proving that SRB measures can be approximated by measures supported on hyperbolic subsets with arbitrarily large unstable dimensions, hence generalizing a similar result for surface diffeomorphisms (see [**8**]).

The work by Katok [**2**] is a landmark in this type of question. In fact we can use the methods from [**2**] and [**3**] to recover most of the dynamical structure of non-uniformly hyperbolic systems as a limiting process, generating many hyperbolic horseshoes.

So, a natural question is how large that part of the dynamics described by these invariant subsets is. In addressing this problem we will try to estimate the Hausdorff dimension of the stable (respectively unstable) Cantor sets of a horseshoe. The Hausdorff dimension is a classic and well-known fractal invariant used as a measure of the geometrical complexity of a set. We refer the reader to [**1**] for definitions and general properties.

There are several well-known formulas for the Hausdorff dimensions of conformal Cantor sets and measures in terms of such dynamical quantities as the entropy and Lyapunov exponents (see [**10**]). However, much less is known about the geometry of

https://doi.org/10.1017/S0143385702000287 Published online by Cambridge University Press

non-conformal Cantor sets, even though some progress has recently been made. Indeed, in most cases we do not know how to compute the Hausdorff dimension whether its Hausdorff measure is finite, positive or not. It is also an open question as to whether the stable and unstable dimensions of non-conformal horseshoes are preserved by the holonomy of their invariant laminations. We refer the reader to [**14**] for an update of these problems.

In this work we will use the *dynamical dimension* of an invariant set to obtain some useful bounds on the Hausdorff dimension of non-conformal horseshoes. This number is a dimensional characteristic of a Carathéodory structure defined by the volume as a set function and certain generating classes of dynamically defined coverings (see [**10**] and §4 for definitions and properties). In particular, the dynamical dimension of a Cantor set $\Lambda \subset \mathbb{R}^d$ generated by a piecewise C^2 expanding map $f : D \subset R \to R$ defined in a compact domain $R \subset \mathbb{R}^d$ is the unique solution $0 < \dim_{\mathcal{D},f}(\Lambda) < 1$ to the Bowen equation

$$P(f|\Lambda, -(\dim_{\mathcal{D},f} \Lambda) \ln Jf) = 0,$$

where $Jf(x) = |\det Df(x)|$ is the Jacobian of f and $P(f, \phi)$ is the topological pressure. If Λ is conformal, then $\dim_{\mathcal{D},f}(\Lambda) = \dim_{\mathcal{H}}(\Lambda)$. See [**4**] and [**10**].

A dynamical dimension is not a geometrical invariant of a set but a dimension-like property of the dynamics. Indeed, a hyperbolic Cantor set might have several different dynamical dimensions depending on how it is represented as an invariant set of a given transformation. This provides a complete spectra of dimensions associated with a given dynamically defined Cantor set.

Notwithstanding this lack of an intrinsical geometrical meaning, the dynamical dimension provides a useful bound for the Hausdorff dimension of non-conformal Cantor sets. In fact, suppose that the border δR of the domain R is not too fractal. We prove that there is a constant $C = C(f, R) > 0$ which only depends on f and the domain R such that

$$|d - \dim_{\mathcal{H}}(\Lambda)| \leq C|1 - \dim_{\mathcal{D},f}(\Lambda)|. \tag{1}$$

A similar estimate is valid for the unstable Cantor sets of non-conformal horseshoes. This important result is implicit in the work by Horita and Viana [**16**] and it has many interesting consequences for the geometry of non-conformal hyperbolic Cantor sets. We will use (1) to prove the following result.

MAIN THEOREM. *Let μ be an ergodic SRB measure left invariant by a C^2 diffeomorphism $f : M \to M$ on a compact Riemannian manifold of dimension $m \geq 2$. Then, there exists a sequence of ergodic measures μ_n supported on hyperbolic sets Ω_n such that $\mu_n \to \mu$ in a weak topology and*

$$\dim_{\mathcal{H}}(\Omega_n \cap W^u(x)) \to \dim W^u(x),$$

uniformly in $x \in \Omega_n$.

In fact we will find, for any given $0 < \epsilon \ll 1$, an integer $N = N(\epsilon) > 0$ such that $d^u \geq \dim_{\mathcal{H}}(\Omega_n \cap W^u(x)) \geq d^u - \epsilon$ for every $x \in \Omega_n$ and $n \geq N$. To prove this result we look for a non-invariant subset $\Omega^* \subset M$ with $\mu(\Omega^*) > 0$ with a *hyperbolic product structure*, that is, a compact subset endowed with two continuous laminations $\mathcal{F}^u$ and $\mathcal{F}^s$ of local stable and unstable manifolds of dimension d^u and d^s respectively, which

https://doi.org/10.1017/S0143385702000287 Published online by Cambridge University Press

intersects transversally at a unique point with an angle uniformly bounded away from zero and such that $\Omega^* = \bigcup \mathcal{F}^s \cap \bigcup \mathcal{F}^u$. In fact, Ω^* can be described as a type of 'horseshoe with infinitely many branches and variable return times', according to [**15**]. See §3 below.

Ω^* is the topological limit of an increasing sequence Ω_n^* of ordinary horseshoes, defined by finitely many hyperbolic branches. Moreover, we can find a $C > 0$ such that $|d^u - \dim_{\mathcal{H}}(\Omega_n^* \cap \mathcal{F}^u(x))| \leq C|1 - \dim_{\mathcal{D}}(\Omega_n^* \cap \mathcal{F}^u(x))|$. If μ is an SRB measure then $\dim_{\mathcal{D}}(\Omega_n^* \cap \mathcal{F}^u(x)) \to 1$. Let Ω_n be the f-invariant saturate of Ω_n^*. We use these remarks to prove that $\dim_{\mathcal{H}}(\Omega_n \cap W^u(x)) \to d^u$ uniformly in $x \in \Omega_n$.

2. *Pesin sets and pseudo-Markov property*

In this section we review some terminology and basic facts of Pesin theory which will be used later. Our main reference are the *Notes on Smooth Ergodic Theory* by Katok and Mendoza [**3**].

Throughout this paper $f : (M, \mu) \to (M, \mu)$ will be a C^2 diffeomorphism preserving an ergodic non-atomic hyperbolic measure μ. Using Oseledec's theorem one can define two Borel measurable f-invariant sub-bundles:

$$E_x^u = \bigoplus_{\chi_i > 0} E_i(x) \quad \text{and} \quad E_x^s = \bigoplus_{\chi_i > 0} E_i(x)$$

such that $T_x = M = E_x^s \oplus E_x^u$, where the E_i are invariant subspaces associated with the Lyapunov exponent χ_i (see [**3**, Theorem S.2.9]). Given $0 < \epsilon \ll 1$, an *ϵ-reduction* is provided by a Borel measurable family of linear coordinate changes $C_\epsilon : M \to GL_n(\mathbb{R})$ which transforms $Df(x)$ into a Lyapunov block form and such that $\|C_\epsilon\|$ is bounded by an ϵ-tempered function (see [**3**, Theorem S.2.10]).

Recall that given $\epsilon > 0$ a Borel measurable and positive function K_ϵ is called *ϵ-tempered with respect to f* or simply *ϵ-tempered* if $(1+\epsilon)^{-1} \leq K(x)/K(f(x)) \leq (1+\epsilon)$ almost surely. The tempering Kernel Lemma (see [**3**, Lemma S.2.12]) states that given a measurable function K with $\lim_{n\to\infty} 1/n \ln K(f^n(x)) = 0$ and $\epsilon > 0$ we can find an ϵ-tempered function K_ϵ which bounds K from above.

We can introduce at almost every point $p \in M$ a new metric $\langle , \rangle_p^*$ which makes E^s and E^u orthogonal and such that $Df(x) : T_xM \to T_{f(x)}M$ becomes uniformly hyperbolic with respect to $\|\cdot\|^*$. We call this a *Lyapunov metric*. We define $\langle , \rangle_p^*$ using the previously described linear Lyapunov coordinate changes. Namely, $\langle C_\epsilon(p)v, C_\epsilon(p)w\rangle_p^* \equiv \langle v, w\rangle$ for every $v, w \in \mathbb{R}^m$. The distortion of the Lyapunov metric with respect to the Riemannian metric of M is limited by an ϵ-tempered function $L_\epsilon : 1/\sqrt{2}\|v\|_p \leq \|v\|_p^* \leq L_\epsilon(p)\|v\|_p$ for every $v \in T_pM$ (see [**3**, Theorem S.2.10]).

We also introduce nonlinear local coordinates at p such that f looks like a uniform Lipschitz perturbation of a hyperbolic isomorphism. More precisely, let $\Psi_p : B(0, r(p)) \to M$ where $\Psi_p = \exp_p \circ C_\epsilon(p)$. The radius $r = r_\epsilon(p)$ of these Lyapunov charts is a Borel measurable ϵ-tempered function (see [**3**, Theorem S.3.1]). We can extend the local representative of f in these charts to a diffeomorphism $f_p : \mathbb{R}^m \to \mathbb{R}^m$ such that

$$f_p(x, y) = (A_p^s x + h_p^s(x, y), A_p^u y + h_p^u(x, y)) = A_p(x, y) + h_p(x, y)$$

https://doi.org/10.1017/S0143385702000287 Published online by Cambridge University Press

for $(x, y) \in \mathbb{R}^s \oplus \mathbb{R}^u$. $Ap = C_\epsilon(p)^{-1} \circ df(p) \circ C_\epsilon(p)$ and A^s_p (respectively A^u_p) is the stable (respectively unstable) representative of the linear part of f in p in the new coordinates. We choose the radius at p such that $\|(dh_p)_v - (dh_p)_w\| \leq \epsilon$ for every $\|v\|, \|w\| < r(p)$ and almost every $p \in M$. Furthermore, the geometrical distortion of the standard metric of $\mathbb{R}^m$ under Lyapunov charts Ψ_p is bounded by an ϵ-tempered function D_ϵ:

$$K \operatorname{dist}_M(\Psi_p(x), \Psi_p(y)) \leq \|x - y\| \leq D_\epsilon(p) \operatorname{dist}_M(\psi_p(x), \Psi_p(y)),$$

for every $x, y \in B(0, r(p))$ and a suitable $K > 0$.

Let $0 < h \leq 1$ be given. A *regular neighbourhood of size h* is defined as follows:

$$\mathsf{R}(p, h) = \Psi_p(B^s(0, hr(p)) \oplus B^u(0, hr(p))),$$

where $B^u(0, r) \subseteq \mathbb{R}^u$ and $B^s(0, r) \subseteq \mathbb{R}^s$ are the open balls of centre 0 and radius $r > 0$ with respect to the standard metric [**3**, Definition S.3.3].

Lyapunov charts can be used to introduce *unstable cones of width* $\gamma \in (0, 1)$ at every point in $p = \Psi_p(x) \in \mathsf{R}$:

$$\mathsf{K}^u_\gamma(p) \equiv D\Psi_p(x)\{(u, v) \in E^u \times E^s : \|v\| \leq \gamma \|u\|\}.$$

A family of stable cones $\mathsf{K}^s_\gamma(p)$ is similarly defined. We use these cones to define a class of *admissible unstable submanifolds* Γ^u [**3**, Definition S.3.4]. Each $\gamma^u \in \Gamma^u$ is the image under the Lyapunov chart Ψ_p of $\overline{\gamma}^u = \operatorname{Graph}(\phi)$, the graph of a C^1 map $\phi : B^u(0, r(p)) \to B^s(0, r(p))$ with tangent spaces which lie inside the unstable cones and similarly so for Γ^s. Every $\gamma^u \in \Gamma^u$ and $\gamma^s \in \Gamma^s$ intersects transversally at a single point x with an angle bounded from below.

Admissible submanifolds with dispersion γ have a bounded geometry, that is, the intrinsic Riemannian metric is comparable with the metric in M bounded by constants which only depend on γ and the radius of the Lyapunov chart. Let $\bar{g}_{ij}$ be the Riemannian metric of $\overline{\gamma}^u$, a graph with dispersion γ such that $\gamma^u = \Psi_p(\overline{\gamma}^u)$, then $\bar{g}_{ij}(x) = \delta_{ij} + o(\gamma^2)$. The property of the bounded geometry of graphs is transferred to admissible manifolds $\gamma^u = \Psi_p(\overline{\gamma}^u)$ with slowly varying corrections along the orbit. In particular, we can find $C = C(p, \gamma) > 1$ such that

$$C^{-1} \leq \inf_{\gamma^u} \operatorname{Vol}_u(\gamma^u) \leq \sup_{\gamma^u} \operatorname{Vol}_u(\gamma^u) \leq C$$

and

$$C^{-1} \leq \frac{\operatorname{dist}_{\gamma^u}(\Psi_p(x), \Psi_p(y))}{\|x - y\|} \leq C$$

for every $x, y \in \gamma^u$ and every $\gamma^u \in \Gamma^u$, where C_ϵ is a ϵ-tempered Borel measurable function.

The class of admissible submanifolds near to a hyperbolic orbit is preserved under iterations, if their dispersion γ is small enough. Indeed, we can find a universal constant $0 < \gamma_0 < 1$ such that for every $\gamma \in (0, \gamma_0]$ there exists $\epsilon = \epsilon(\gamma) \in (0, 1)$ depending on γ and an ϵ-reduction such that for every $0 < h \leq 1$ and for every $\gamma^u \in \Gamma^u(p, h/2)$ the image $f(\gamma^u) \cap \mathsf{R}(f(p), h/2)$ is an admissible unstable manifold with a dispersion strictly less than γ. This follows from the Inclination Lemma [**3**, Proposition S.3.5]. So we have a well-defined *graph transform* $f_* : \Gamma^u(p, h/2) \to \Gamma(f(p), h/2)$ which is a contraction in the

https://doi.org/10.1017/S0143385702000287 Published online by Cambridge University Press

C^1-metric: there exists $0 < \theta < 1$ such that $\mathrm{dist}_{C^1}(f_*(\gamma^u), f_*(\overline{\gamma}^u)) \leq \theta\, \mathrm{dist}_{C^1}(\gamma^u, \overline{\gamma}^u)$. The rate of contraction does not depend on p [**3**, Corollary S.3.6]. Moreover, given an admissible unstable manifold γ^u one find a disc $\mathbb{D}^u \subset \gamma^u$ such that $f : \mathbb{D}^u \to f_*\gamma^u$ expands with a nonlinear expansion coefficient $\lambda = \lambda(\epsilon, \gamma, \chi) > 1$ which depends only on the ϵ-reduction, the dispersion $\gamma \in (0, 1)$ and χ, the minimal rate of typical exponential behaviour of the system, also called the *rate of hyperbolicity of* μ [**3**, p. 695].

By Luzin–Egoroff's theorem we can find large hyperbolic non-invariant sets which are also called *Pesin sets*. A δ-Pesin set is a compact non-invariant set with $\mu(\Lambda_\delta) \geq 1 - \delta$ such that the linear Lyapunov charts $C : \Lambda_\delta \to GL(m, \mathbb{R})$, the size $r(x)$ of regular neighbourhoods, the geometrical distortion introduced by Lyapunov charts and the splittings into stable and unstable subspaces are continuous on Λ_δ [**3**, Theorem S.4.3]. The linear distortion $L_\epsilon(x)$ of the Lyapunov metric and the nonlinear distortion $D_\epsilon(x)$ of the Lyapunov charts attain maxima L_δ and D_δ over Λ_δ. Similarly, let r_δ be the minimum of the radius of the Lyapunov charts on this Pesin set. We denote $\mathsf{R}(p, h) := \Psi_p(B^s(0, hr_\delta) \oplus B^u(0, hr_\delta))$ and call it the *h-reduced regular neighbourhood* centred at p. Lyapunov charts depend continuously on points in a given hyperbolic Pesin set. Indeed, let $1 < \beta \leq 2$. We can find $\kappa = \kappa(\delta, \beta, h) > 0$ such that if $p, q \in \Omega_\delta$, $d(p, q) < \kappa$ and $\gamma^u \in \Gamma^u(\mathsf{R}(p, h))$ then $\Psi_p(\Psi_q^{-1}(\gamma^u)) \cap \tilde{\mathsf{R}}(q, h/2)$ is part of an admissible unstable submanifold near q with a dispersion $< \beta\gamma$. Thus, although changes in the centre of a Lyapunov coordinate system may increase the dispersion of the admissible manifolds, this deformation can be controlled when p runs through a hyperbolic Pesin set (see [**3**, Proposition S.4.5] and [**3**, Corollary S.4.6]).

A compact connected subset $S \subset \mathsf{R}(p)$ is an *admissible stable cylinder* if it admits a foliation by admissible stable submanifolds and if its unstable sections $S \cap \gamma^u$ are convex sets for every $\gamma^u \in \Gamma^u(\mathsf{R})$. Unstable cylinders are defined similarly. We also recall that a smooth map $C^1T : S \to U$ between stable and unstable cylinders is hyperbolic if it strictly preserves the cone families.

LEMMA 2.1. (Pseudo-Markov property) *There exists* $0 < \theta < 1$ *such that for every* $0 < \delta < 1$ *and every Pesin set* Λ_δ *and any given* $0 < h \leq 1$ *we can find* $\kappa = \kappa(\delta, h) > 0$, D_δ, C_δ *with the following property.*

Given $p, q \in \Lambda_\delta$ *and* $x \in \Lambda_\delta$ *returning to* Λ_δ *in* m *iterates and such that* $d(x, p) < \kappa$ *and* $d(f^m(x), q) < \kappa$, *then there are stable and unstable cylinders* $S \subset \mathsf{R}(p, h)$ *and* $U \subset \tilde{\mathsf{R}}(q, h)$ *containing* x *and* $f^m(x)$ *respectively such that:*

(1) $f^m : S \to U$ *is hyperbolic;*

(2) $\mathrm{diam}(f^i(S)) \leq D_\delta \max\{\mathrm{diam}(\tilde{\mathsf{R}}(p, h))\theta^i, \mathrm{diam}(\tilde{\mathsf{R}}(f^m(p), h))\theta^{m-i}\}$ *for every* $i = 0, \ldots, m$; *and*

(3) *for every* $\gamma^u \in \Gamma^u(p)$ *it holds that* $f^m : \gamma^u \cap S(x) \to f_*^n\gamma^u$ *is an expanding diffeomorphism, that is*

$$d_u(f^m(z), f^m(z')) \geq C_\delta \lambda^m d_u(z, z')$$

for every $z, z' \in \gamma^u \cap S(x)$, d_u *being the intrinsic Riemannian metric along admissible unstable manifolds.*

$S(x)$ *is the connected component of* $\mathsf{R}(p, h) \cap f^{-m}(\mathsf{R}(q, h))$ *containing* x *and* $U(x)$ *is the connected component of* $f^m(\mathsf{R}(p, h)) \cap \mathsf{R}(q, h)$ *containing* $f^m(S(x))$.

https://doi.org/10.1017/S0143385702000287 Published online by Cambridge University Press

See Definition S.4.15 and Theorem S.4.16 in [**3**]. The constant $0 < \theta < 1$ is provided by the graph transform contraction properties and $\kappa > 0$ is provided by the continuity property of the dispersion mentioned before.

LEMMA 2.2. *Let $0 < h < 1$. Then, for every $\epsilon > 0$ there is an ϵ-tempered correction $C = C_\epsilon(x) > 1$ with the following properties.*

(1) *Let $\gamma^u \in \Gamma^u(p, h/2)$ be an admissible unstable manifold, $m > 0$ and $\mathbb{D}^u_m \subset \gamma^u$ such that $f^m : \mathbb{D}^u_m \to f^m_* \gamma^u$ is an expanding map onto an admissible unstable manifold in $\mathsf{R}(f^m(p), h/2)$. Then,*

$$\left| \frac{J(f^n|\gamma^u)(z)}{J(f^n|\gamma^u)(x')} \right| \leq \exp[C(f^n(p)) d_u(f^n(z), f^n(z'))] \quad \forall z, z' \in \mathbb{D}^u_m. \tag{2}$$

(2) *Let $\gamma^u(x)$, $\gamma^u(y) \in \Gamma^u(p, h/2)$ be two admissible unstable manifolds passing by x and y and suppose that they are in the same local stable manifold. We define*

$$h(x, y) = \prod_{i=1}^{+\infty} \frac{J(f|\gamma^u(x))(f^i(x))}{J(f|\gamma^u(y))(f^i(y))}.$$

Then, $\ln h(x, y) \leq C(p) d_s(x, y)$ and for every $x' \in \gamma^u(x)$ and every $y' \in \gamma^u(y)$ which are in the same local stable manifold it holds that

$$\ln \frac{h(x, y)}{h(x', y')} \leq C(p) \max\{d_u(x, x'), d_u(y, y')\}. \tag{3}$$

(3) *For every $z, z' \in M$ resting in the same local unstable manifold we have*

$$\Delta(z, z') = \prod_{i=1}^{+\infty} \frac{J^u f(f^{-i}(z))}{J^u f(f^{-1}(z'))} \leq \exp(K_2(p) d_u(z, z')). \tag{4}$$

Jacobians and distances d_u along unstable admissible manifolds are computed with the intrinsic Riemannian metric.

These results are well known for invariant local unstable manifolds (see, for example, [**5**]). Here we are considering *non-invariant admissible unstable manifolds*, but the principles of the proof are the same as for invariant manifolds. For this we use the expansion along admissible unstable manifolds and the graph transform properties. Compare this with [**5**, Proposition 3.1].

LEMMA 2.3. *Let $\delta \in (0, 1)$. We can find a δ-Pesin set Λ_δ and $C_\delta > 1$ such that for every $p \in \Lambda_\delta$ we have the following properties:*

(1) *for every $\gamma^u \in \Gamma^u(p, h/2)$ and $m > 0$ such that $f^m(p) \in \Lambda_\delta$*

$$\ln \left| \frac{J(f^n|\gamma^u)(z)}{J(f^n|\gamma^u)(z')} \right| \leq C_\delta d_u(f^n(z), f^n(z')),$$

for every $z, z' \in \mathbb{D}^u_m$, where $\mathbb{D}^u_m \subset \gamma^u$ is such that $f^m : \mathbb{D}^u_m \to f^m_* \gamma^u$ is an expanding map onto an admissible unstable manifold in $\mathsf{R}(f^m(p), h/2)$.

https://doi.org/10.1017/S0143385702000287 Published online by Cambridge University Press

(2) $\ln h(x, y) \leq K_\delta d_s(x, y)$, *for every* $\gamma^u(x)$, $\gamma^u(y) \in \Gamma^u(p, h)$ *such that* x *and* y *lie in the same stable manifold, i.e.* $W^s(x, \mathsf{R}(p)) = W^s(y, \mathsf{R}(p))$. *Also, for every* $x' \in \gamma^u(x)$ *and* $y' \in \gamma^u(y)$ *such that*

$$y' = W^s(x', \mathsf{R}(p, h)) \cap \gamma^u(y)$$

it holds that

$$\ln \frac{h(x, y)}{h(x', y')} \leq C_\delta \max\{d_u(x, x'), d_u(y, y')\},$$

(3) $\ln \Delta(z, z') \leq C_\delta d_u(z, z')$.

The following proposition can be proved using the pseudo-Markov property of Pesin sets and will be used in the next section to construct a fat horseshoe as claimed in the Introduction.

KATOK'S CLOSING LEMMA. *Let* Λ_δ *be a hyperbolic Pesin set and let* $\epsilon > 0$. *Then there exists* $\beta = \beta(\epsilon, \delta) < \kappa$ *with the following property: suppose that* x *and* $f^m(x)$ *belong to* Λ_δ *and that* $\operatorname{dist}(x, f^m(x)) < \beta$, *then there is a hyperbolic periodic point* $z \in M$ *such that* $\operatorname{dist}(f^j(x), f^j(z)) < \epsilon$ *for* $j = 0, \ldots, m$.

Compare [**3**, Theorem S.4.13]. This implies that hyperbolic periodic points are dense in the support of a hyperbolic non-atomic measure [**3**, Theorem S.5.1]. If these points are near enough they will be homoclinically related, so they can be embedded in a hyperbolic horseshoe [**3**, Theorem 6.5.5; **9**, Theorem 1, Chapter 2]. We will use this fact to generate large hyperbolic subsets approximating the dynamics of $f : (M, \mu) \to (M, \mu)$.

3. *Inducing fat horseshoes*

From now on we shall fix a hyperbolic Pesin set Λ_δ, $0 < h < 1$, $p \in \Lambda_\delta$ a point in the support of the measure μ and $\tilde{\mathsf{R}}(p, h)$ an h-reduced regular neighbourhood. Also let $h' = h'(\kappa, h)$ be such that $\tilde{\mathsf{R}}(p, h') \subset \tilde{\mathsf{R}}(p, h)$ and $\operatorname{diam} \tilde{\mathsf{R}}(p, h') < \kappa$. These regular neighbourhoods will be held fixed and shall be denoted by $\mathsf{R} = \tilde{\mathsf{R}}(p, h)$ and $\mathsf{Q} = \tilde{\mathsf{R}}(p, h')$, respectively. Now, let $0 < \rho < 1$ and choose $\delta > 0$ and $0 < h < 1$ such that $\mu(\Lambda_\delta \cap \mathsf{Q}) \geq (1 - \rho)\mu(\mathsf{Q})$.

COROLLARY 3.1. *Let* $E \subset \mathsf{Q}$ *be a finite subset of hyperbolic periodic points. Then there exist finite disjoint collections* $\{S(z)\}_{z \in E}$ *and* $\{U(z)\}_{z \in E}$ *and hyperbolic branches* $T_z : S(z) \to U(z)$ *generating a hyperbolic horseshoe* $\Lambda^*(E)$ *which contains* E.

Proof. By Lemma 2.1 there exists for each $z \in E$ a nested sequence of stable cylinders converging to $W^s(z, \mathsf{R})$, the local stable manifold of z. Indeed, $W^s(z, \mathsf{R}) = \bigcap_{k>0} f^{-km(z)} S_0(z)$, where $f^{m(z)} : S_0(z) \to U_0(z)$ is provided by the pseudo-Markov property in Lemma 2.1 and $m(z)$ is the prime period of z. One can choose a finite disjoint collection of stable cylinders $\{S(z)\}_{z \in E}$ mapped hyperbolically to a finite disjoint collection of unstable cylinders: we take for each $z \in E$ the least positive integer $k(z)$ such that the stable cylinders $S(z) = f^{-kn(z)} S_0(z)$ form a disjoint family. In addition, $f^{2kn(z)} : S(z) \to U(z)$ is hyperbolic. $\Lambda^*(E) = \bigcap_{n \in Z} T^{-n}\big(\bigcup_{z \in E} S(z)\big)$ is a hyperbolic horseshoe with finitely many branches as claimed. □

https://doi.org/10.1017/S0143385702000287 Published online by Cambridge University Press

PROPOSITION 3.1. *There is a maximal compact set* $\Omega^* \subset \mathsf{R}$ *with a* hyperbolic product structure *such that* $\mu(\Omega^*) \geq (1-\rho)\mu(\mathsf{Q})$.

Proof. By Katok's Closing Lemma, we can obtain a countable dense subset of hyperbolic periodic points $\{z_i\}$ such that $\mathsf{Q} \cap \Lambda_\delta \subset \overline{\{z_1, z_2, \dots\}}$. For this we let $\mathcal{P}_n$ be a sequence of decompositions of Q into compact sets with non-overlapping interiors such that $\operatorname{diam}(\mathcal{P}_n) \to 0^+$ as n goes to infinity (take, for example, images by the Lyapunov charts of decompositions into dyadical cubes in $\mathbb{R}^m$). Following the same arguments as in [**3**, Theorem S.5.1] we find, for every $P \in \mathcal{P}_n$ with $\mu(P) > 0$, a hyperbolic periodic point $z \in P$. Let $\{z_1, z_2, \dots\}$ be the set of hyperbolic closings so obtained, indexed in increasing order according to their prime period. Clearly $\{z_1, z_2, \dots\}$ is dense in $\mathsf{Q} \cap \Lambda_\delta$.

Then, given $n > 0$ and using Lemma 3.1 one find a horseshoe with finitely many branches $\Lambda(z_1, \dots, z_n)$. The compact set $\overline{\bigcup_{n>0} \Lambda(z_1, \dots, z_n)}$ satisfies conditions (1)–(4).

By Zorn's lemma one can find a maximal element Ω^* among the family of subsets in R with a hyperbolic product structure containing $\{z_1, z_2, \dots\}$, whose invariant leaves are admissible submanifolds. Ω^* contains the closure of all the points which are transversal homoclinically related to points in this set of hyperbolic periodic closings and whose local invariant manifolds passing by R are admissible. Therefore, $\mu(\Omega^*) \geq (1-\rho)\mu(\mathsf{Q})$, since it contains $\Lambda_\delta \cap \mathsf{Q}$. □

Let T be the first return map to $\Omega^* : T(x) = f^{\tau(x)}(x)$, where $f^k(x) \notin \Omega^*$ for $0 < k < \tau(x)$. The periodic points of T are dense in Ω^*. Moreover, $T : \Omega^* \to \Omega^*$ is topologically mixing. In fact, $T : \Omega^* \to \Omega^*$ can be seen as a sort of 'horseshoe with infinitely many branches and variable return time' following the terminology introduced in [**15**]. Indeed, by the pseudo-Markov property for each $x \in \Omega^*$ there are a stable cylinder $S(x) \subset \mathsf{R}$ containing x and an unstable cylinder $U(x)$ containing $T(x)$ such that $T : S(x) \to U(x)$ is a hyperbolic diffeomorphism, where $T|S(x) = f^{\tau(x)}$ is the extension to $S(x)$ of the first return map to Ω^*. In particular, given any finite subset $E \subset \operatorname{Per}(T)$ of hyperbolic periodic points, we can form a horseshoe with finitely many hyperbolic branches $\Lambda^* = \bigcap_{n\in Z} T^{-n}\big(\bigcup_{z\in E} S(z)\big)$. Let $\wp_0 = \{S(z)\}_{z\in E}$ and $\wp_n = T^{-n}\wp_0$. We call $\wp_n$ the *stable cylinders of generation* n.

Now, let $\wp^* = \{S(x) : x \in \Omega^*\}$ denote the family of stable cylinders so obtained. Each stable leaf $\mathcal{F}^s(x)$ is a C^1-limit of a nested sequences of stable cylinders $S_n(x) \in \wp^*$, that is, $\mathcal{F}^s(x) = \bigcap_{n>0} S_n(x)$ and $\operatorname{dist}_{C^1}(\mathcal{F}^s(x), S_n(x)) \to 0^+$. This follows from the pseudo-Markov and graph transform properties. We deduce that $\wp^*$ generates the sub σ-algebra of the Borel subsets which are a reunion of stable leaves or *stable sets* of Ω^* for short.

Let $x \in \Omega^*$. We will use the following terminology: we say that $P \in \wp^*$ is a *stable cylinder of generation* n if $T^n P$ is mapped onto an unstable cylinder and $T^k P$ is a stable cylinder for every $0 \leq k < n$. We denote by $\wp_n^*$ the family of stable cylinders of generation $n > 0$ of Ω^*.

The bounded distortion properties stated in Lemma 2.3 provide the following useful volume estimates for $\wp^*$-cylinders. This will be used later.

LEMMA 3.1. *There is a constant* $C = C(\delta, \Gamma) > 1$ *depending on* Λ_δ *and the family of admissible unstable manifolds* Γ^u *such that for every* $n > 0$ *and* $x \in \Omega^*$ *the following properties hold:*

https://doi.org/10.1017/S0143385702000287 Published online by Cambridge University Press

(1) $C^{-1} \le \mathrm{Vol}_u(\wp_n^*(x) \cap \gamma^u) J(T^n|\gamma^u)(x) \le C$;
(2) $C^{-1} \le \mathrm{Vol}_u(P \cap \overline{\gamma}^u)/\mathrm{Vol}_u(P \cap \gamma^u) \le C$, *for every* $P \in \wp^*$ *and any pair of admissible unstable submanifolds* $\gamma^u, \overline{\gamma}^u \in \Gamma^u$;
(3) $C^{-1} \le \mathrm{Vol}_u(\wp_n^*(x))/\mathrm{Vol}_u(\wp_n^*(x) \cap \gamma^u) \le C$; *and*
(4) $C^{-1} \le \mathrm{Vol}_u(\wp_n^*(x)) J(T^n|\gamma^u)(x) \le C$;

where $\wp_n^*(x)$ *is a stable cylinder of generation* n *and* $\gamma^u \in \Gamma^u$ *is an admissible unstable manifold containing* x.

4. *Measure-theoretical properties of horseshoes*

Now we introduce the dynamical measure and dynamical dimension recalling some of their properties.

Given an admissible unstable manifold γ^u we define $\wp^*(\gamma^u) = \{P \cap \gamma^u : P \in \wp^*\}$. A countable family $\mathcal{U} = \{U_i\}_{i>0} \mathcal{U} \subset \wp^*$ with $\mathrm{diam}(U_i) < \delta$ is called a δ-covering by $\wp^*(\gamma^u)$-sets. Let $a > 0$; we set

$$\mathcal{D}_{a,\wp^*,\gamma^u}(X) = \lim_{\delta \to 0^+} \inf_{\mathcal{U}} \sum_{i=1}^{+\infty} \mathrm{Vol}_{\gamma^u}(U_i)^a, \tag{5}$$

where the infimum is taken over the δ-covering of $\wp^*(\gamma^u)$-sets. Hereafter Vol_{γ^u} will denote the volume on $\gamma^u \in \Gamma^u$ defined by the intrinsic Riemannian metric of an admissible unstable manifold $\gamma^u \in \Gamma^u$ and $\mathrm{Vol}_{\gamma^u}(X) = \mathrm{Vol}_{\gamma^u}(X \cap \gamma^u)$.

The outer measure $\mathcal{D}_{a,\wp^*,\gamma^u}$ so defined is a Borel regular metric measure on γ^u. Changing the Riemannian volume gives equivalent measures. Also we can enlarge the covering family $\wp^*$ using the bounded distortion property, but these changes hold inside a well-defined equivalence class of measures. We associate Carathéodory's dimensional characteristic with this fractal measure and it is called the *dynamical dimension*, namely

$$\dim_{\gamma^u,\wp^*}(X) = \inf\{a > 0 : \mathcal{D}_{\alpha,\gamma^u}(X) = 0\}. \tag{6}$$

This is a dimensional characteristic and it has the following standard properties:
(1) $\dim_{\gamma^u,\wp^*}(X) \le \dim_{\gamma^u,\wp^*}(Y)$ if $X \subset Y$;
(2) $\dim_{\gamma^u,\wp^*}(\{point\}) = 0$;
(3) $\dim_{\gamma^u,\wp^*}\left(\bigcup_{i=0}^{=\infty} X_i\right) = \sum \dim_{\gamma^u,\wp^*}(X_i)$ (see [**10**, Theorem 1.1]).
Compare [**10**, Chapters 1 and 4].

The following result collects the properties of the dynamical measure which will be used later. Some of the statements are well known (e.g. the relation with Bowen's equation) and some are new (in particular the absolute continuity property and the computation of a Jacobian for the holonomy maps).

PROPOSITION 4.1. *Let* Λ *be a* C^2 *hyperbolic horseshoe associated with a hyperbolic transformation defined by finitely many hyperbolic branches* T_i: $S_i \to U_i$ *in some domain* $\mathsf{R} \subset M$ *of a compact Riemannian manifold and* $\wp$ *the family of geometrical cylinders generating the stable sets of* Λ. *Then the following properties hold.*
(1) $\alpha = \dim_{\mathcal{D},W}(\Lambda)$ *is the unique solution to the Bowen equation*

$$P(T|\Omega, -\alpha \ln J^u T) = 0, \tag{7}$$

for every $W \in \mathcal{F}^u$.

https://doi.org/10.1017/S0143385702000287 Published online by Cambridge University Press

(2) *There is a Borel regular probability measure μ_W and a constant $C > 1$, which depend only on the bounds of the nonlinear distortion of the volume along unstable leaves and the expansion coefficient of Λ such that*

$$C^{-1} \inf_{\gamma^u \in \mathcal{F}^u} \mathrm{Vol}_u(\gamma^u) \leq \frac{\mu_W(P)}{(\mathrm{Vol}_W\ P)^{\dim_u(\Lambda)}} \leq C \sup_{\gamma^u \in \mathcal{F}^u} \mathrm{Vol}_u(\gamma^u), \tag{8}$$

for every stable cylinder $P \in \wp$, where $\mathrm{Vol}_W(P) = \mathrm{Vol}_u(P \cap W)$.

(3) *The stable lamination $\mathcal{F}^s$ is absolutely continuous with respect to the measure classes $\{\mathcal{D}_{\alpha,\gamma^u}\}_{\gamma^u \in \Gamma^u}$ and it has a continuous and bounded Jacobian.*

Before continuing, some remarks are in order. First of all let us introduce the following notation.

Notation. From now on we will use the notation $f \asymp g$ if given two positive functions f and g defined over a common domain D, there exists a constant $C > 1$ such that $f(x)/g(x) \in [C^{-1}, C]$ for every $x \in D$.

$\mathrm{Dim}_u(\Lambda)$ denotes the *unstable dimension* of Λ. This is well defined since the unstable dimension of $\Lambda \cap W^u(x, \mathsf{R})$ does not depend on the point $x \in \Lambda$.

Let $\alpha = \dim_u(\Lambda)$ and μ_α be the Gibbs measure (cf. [**4**]) associated with the potential $-\alpha \ln J^u T$. Then, μ_W is the projection of μ_α onto the local unstable manifold $W = W^u(x, \mathsf{R})$ along the stable leaves, that is

$$\mu_W(B) = \mu_a\left(\bigcup_{z \in B} \mathcal{F}^s(z)\right) \quad \text{for every Borel subset } B \subset \Lambda \cap W^u(x, \mathsf{R}).$$

In particular μ_α is equivalent to the dynamical measure $\mathcal{D}_{\alpha,W}$ along unstable leaves.

Let $H_{\gamma^u,\overline{\gamma}^u}$ be the holonomy map of the local stable lamination $\mathcal{F}^s$ defined by unstable admissible unstable sections $\gamma^u, \overline{\gamma}^u \in \Gamma^u(\mathsf{R}) : H(x) = \mathcal{F}^s(x) \cap \overline{\gamma}^u$. Absolutely continuous means that the pull-back of the dynamical measure $H^*\mathcal{D}_{\alpha,\overline{\gamma}^u}$, is absolutely continuous (in fact equivalent) to $\mathcal{D}_{\alpha,\overline{\gamma}^u}$. We can prove this by using volume estimates derived from the bounded distortion property as stated in Lemma 3.1. Indeed, $\mathrm{Vol}_u(P \cap \gamma^u) \asymp \mathrm{Vol}_u(P \cap \overline{\gamma}^u)$ for every stable geometrical cylinder $P \in \wp$ and every pair of admissible unstable manifolds $\gamma^u, \overline{\gamma}^u \in \Gamma^u$. Compare this with Lemma 3.1.

Furthermore, we can explicitly compute the Jacobian of the holonomy map with respect to the dynamical measure. Indeed,

$$\frac{dH^*\mathcal{D}_{\alpha,\overline{\gamma}^u}}{d\mathcal{D}_{\alpha,\gamma^u}}(x) = \lim_{n \to +\infty} \frac{J(T^n|\gamma^u)^\alpha(x)}{J(T^n|\overline{\gamma}^u)^\alpha(H(x))}. \tag{9}$$

This was done in [**13**] using the conformality of the measure following the proof of the absolute continuity of stable foliation of a C^2 Anosov diffeomorphism in [**11**]. Compare this also with [**7**].

Now recall that there is a constant $C > 0$ such that

$$J^u(f^n)(x)/J^u(f^n)(y) \leq \exp(Cd(T^n(x), T^n(y))) \tag{10}$$

for $y \in \wp_n(x) \cap W^u(x, \mathsf{R})$ and $n \geq 0$. We use this bounded distortion property to obtain volume estimates as stated before for Ω^*. Using (10) we prove that

https://doi.org/10.1017/S0143385702000287 Published online by Cambridge University Press

$\sum_{P\in\wp_n}(\mathrm{Vol}_W\ P)^a$ is comparable to $\sum_{x\in\mathrm{Per}(n)} S_n(a\phi^u)(x)$ uniformly in $n > 0$, where $S_n\phi(x) = \sum_{k=0}^{n-1}\phi(T^k(x))$ and $\phi^u(x) = -\log J^uT^n(x)$. Hence, they should diverge to infinity or converge to zero together. This proves that $\alpha = \dim_u(\Lambda)$ is the solution to Bowen's equation.

Using (10) we find a $d > 0$ such that $J^u(f^n)(x)/J^u(f^n)(y) \le e^d$ uniformly and $b = b(\lambda_V, d, \gamma) > 0$ such that

$$e^{-d}\inf_{\gamma^u\in\Gamma^u}\mathrm{Vol}_u(\gamma^u)A^{-b} \le \sum_{P\in\wp_n}(\mathrm{Vol}_W\ P)^\alpha \le e^d\sup_{\gamma^u\in\Gamma^u}\mathrm{Vol}_u(\gamma^u)A^b \tag{11}$$

for every local unstable section $W = W^u(x,\mathsf{R})$, where $A = \sup_{e\in\Lambda} J^uT(x)$ and $\lambda_V > 1$ the minimal rate of volume expansion along invariant local unstable manifolds [**9**, Chapter IV, p. 72]. This implies $0 < \mathcal{D}_{\alpha,W}(\Omega\cap W) < +\infty$ for every $W = W^u(x,\mathsf{R})$, using the arguments in the proof of [**9**, Proposition 3, Chapter 4]. It is sufficient to find a $\kappa > 0$ such that

$$\mathrm{Vol}_W(P)^\alpha \ge \kappa^{-1}\sum_{P'\subset P:P'\in\wp(m)}\mathrm{Vol}_W(P')^\alpha$$

for every $P\in\wp_n$ and $m\ge n\ge 0$. For this we let $m\ge n\ge 0$ and $P'\in\wp_m$ contained in $P\in\wp_n$. By the bounded distortion property

$$\frac{\mathrm{Vol}_W(P')^\alpha}{\mathrm{Vol}_W\ P)^\alpha} \le e^{d\alpha}\frac{\mathrm{Vol}'_W(T^n(P'))^\alpha}{\mathrm{Vol}'_W(T^n(P))^\alpha},$$

where $W' = W^u(T^n(x),\mathsf{R})$. Thus,

$$\begin{aligned}\sum_{P'\in\wp_m:P'\subset P}\frac{\mathrm{Vol}_W(P')^\alpha}{\mathrm{Vol}_W(P)^\alpha} &\le \frac{e^{d\alpha}}{\inf_{\gamma^u\in\Gamma^u}\mathrm{Vol}_u(\gamma^u)^\alpha}\sum_{P\in\wp(m-n)}\mathrm{Vol}_{W'}(P)^\alpha\\ &\le \left(\frac{e^d}{\inf_{\gamma^u\in\Gamma^u}\mathrm{Vol}(\gamma^u)^\alpha}\right)^\alpha e^d\sup_{\gamma^u\in\Gamma^u}\mathrm{Vol}_u(\gamma^u)A^b\end{aligned}$$

and then, for every $P\in\wp_n$ and $m\ge n$,

$$\sum_{P'\in\wp(m):P'\subset P}\mathrm{Vol}_W(P')^\alpha \le \kappa\ \mathrm{Vol}_W(P)^\alpha$$

where

$$\kappa := \frac{e^{2d}A^b\sup_{\gamma^u\in\Gamma^u}\mathrm{Vol}_u(\gamma^u)}{(\inf_{\gamma^u\in\Gamma^u}\mathrm{Vol}(\gamma^u))^\alpha}.$$

Let $\Lambda\cap W^u(x,\mathsf{R})$ be an unstable Cantor set and $\tau:\Lambda\to\Lambda$ the projection of $T|W^u(x,\mathsf{R})$ along the stable lamination $\mathcal{F}^u$. It is proved that τ has a Jacobian with respect to the measure classes $\mathcal{D}_{\alpha,W}$. The map τ is a measure theoretical expanding transformation, according to [**7**, Chapter III, p. 170], when we endow the unstable Cantor set $\Lambda\cap W^u(x,\mathsf{R})$ with the non-trivial measure $\mathcal{D}_{\alpha,W}$. For this we have to prove that τ has a Jacobian with respect to the measure $\mathcal{D}_{\alpha,W}$ satisfying

$$\left|\frac{J_\alpha\tau^n(x)}{J_\alpha\tau^n(y)} - 1\right| \le Cd^*(\tau^n(x),\tau^n(y))^\theta \tag{12}$$

https://doi.org/10.1017/S0143385702000287 Published online by Cambridge University Press

whenever $\wp(n, W)(x) = \wp(n, W)(y)$, where $\wp(n, W)(x)$ denotes the connected component of the intersection $\wp_n(x) \cap W$ in $W = W^u(x, \mathsf{R})$ containing x, for a suitable constant $C = C(d, \lambda, \Gamma) > 1$ and $0 < \theta < 1$. This follows from the absolute continuity of the stable lamination $\mathcal{F}^s$ with respect to the dynamical measure and the computation of its Jacobian (9).

We get thus a unique τ-invariant ergodic measure μ_W which is absolutely continuous with respect to $\mathcal{D}_{\alpha,W}$:

$$\mu_W(B) = \lim_{n\to+\infty} \frac{\mathcal{D}_{\alpha,W}(\tau^{-n}B)}{\mathcal{D}_{\alpha,W}(\Lambda)} \tag{13}$$

for every Borel subset $B \subset \Lambda$ [7, Chapter III].

Now, let $\alpha \in (0, 1)$ be the solution to $P(T|\Lambda, -\alpha \ln J^u T) = 0$ and μ_α the Gibbs measure associated with the potential $\phi = -\alpha \ln J^u T$. By Gibbs' property $\mu_\alpha(\wp_n(x)) \asymp \mathrm{Vol}(\wp_n(x))^\alpha$ bounded by a suitable constant $C > 1$. This shows that $\mu_{\alpha,W}$, the projection of μ_α along $\mathcal{F}^u$, is equivalent to μ_W, the natural τ-equilibrium state compatible with the dynamical measure. They must be equal by ergodicity. We prove (8) using standard arguments involving the bounded distortion property and volume estimates.

Let $J_\mu\tau$ be the Jacobian of τ with respect to the asymptotic measure μ_W. Then,

$$J_\mu\tau^n = J_\alpha\tau^n \cdot \frac{h \circ \tau^n}{h},$$

where h is the density of μ_W with respect to the dynamical measure, that is, $\mu_W = h \cdot \mathcal{D}_{\alpha,W}/\mathcal{D}_{\alpha,W}(\Lambda)$. We can find a $d_0 > 0$ such that $J_\mu\tau^n(x)/J_\mu\tau^n(y) \le e^{d_0}$ and $J_\alpha\tau^n(x)/J_\alpha\tau^n(y) \le e^{d_0}$ when x and y are in the same n-cylinder.

Now, let μ_W be the transversal measure (13). Then,

$$e^{-2d_0} \le \mu_W(\wp_n(x))J_\mu\tau^n(x) \le e^{2d_0} \tag{14}$$

for every $n \ge 0$, μ-a.e. $x \in W \cap \Lambda$. Using $\mathcal{D}_{\alpha,W}(\tau^n(P)) = \int_P J_\alpha\tau^n(x)\, d\mathcal{D}_{\alpha,W}(x)$ and previous remarks we get, for every $n \ge 0$,

$$e^{-d_0}\mathcal{D}_{\alpha,W}(\Lambda) \le \mathcal{D}_{\alpha,W}(\wp_n(x))J_\alpha\tau^n(x) \le e^{d_0}\mathcal{D}_{\alpha,W}(\Lambda), \tag{15}$$

for every admissible unstable manifold $W \in \mathcal{F}^u$ and for every stable n-cylinder. Then, we obtain

$$\left(\frac{e^{3d_0}}{\mathcal{D}_{\alpha,W}(\Lambda)}\right)^{-1} \le \mu_W(\wp_n(x))/\mathcal{D}_{\alpha,W}(\wp_n(x)) \le \frac{e^{3d_0}}{\mathcal{D}_{\alpha,W}(\Lambda)}. \tag{16}$$

Then for every $n \ge 0$ and $P \in \wp_n$ we have

$$\frac{e^{-2d_0}\mathcal{D}_{\alpha,W}(\Lambda)}{\sup_{W\in\mathcal{F}^u} \mathrm{Vol}_u W} \le \frac{\mathcal{D}_{\alpha,W}(P \cap \Lambda)}{(\mathrm{Vol}_W P)^\alpha} \le \frac{e^{2d_0}\mathcal{D}_{\alpha,W}(\Lambda)}{\inf_{W\in\mathcal{F}^u} \mathrm{Vol}_u W}. \tag{17}$$

Combining (16) and (17) we obtain

$$e^{-5d_0} \inf_{W\in\mathcal{F}^u} \mathrm{Vol}_u(W) \le \frac{\mu_W(P)}{(\mathrm{Vol}_W P)^\alpha} \le e^{5d_0} \sup_{W\in\mathcal{F}^u} \mathrm{Vol}_u(W) \quad \forall P \in \wp. \tag{18}$$

The Frostman-like inequality (8) provides a universal constant $C = C(d, \lambda, \gamma) > 1$ such that $C^{-1} \le \mathcal{D}_{\alpha,W}(\Lambda) \le C$ for every unstable leaf $W \in \mathcal{F}^u$. The next result will be used later and is a straightforward consequence of this remark and Lemma 3.1.

https://doi.org/10.1017/S0143385702000287 Published online by Cambridge University Press

LEMMA 4.1. *There exists a universal constant $C = C(d, \lambda, \gamma) > 1$ which depends only on the distortion, the expansion coefficient along unstable leaves and the bounded geometry of admissible manifolds Γ^u such that, for every horseshoe $\Lambda^* \subset \Omega^*$ generated by finitely many hyperbolic branches $T_i : S_i \to U_i$ and every unstable leaf $\mathcal{F}^u(x)$ passing through $x \in \Lambda^*$, the following relation holds:*

$$C^{-1} \leq \sum_{P \in \wp_n} \mathrm{Vol}_u(P \cap \mathcal{F}^u(x))^{\dim_u(\Lambda^*)} \leq C \quad \forall n > 0, \tag{19}$$

where $\wp_n$ are the cylinders of generation n associated with Λ^* and Vol_u denotes the intrinsic Riemannian volume of $\mathcal{F}^u(x)$.

5. *Hausdorff dimension of non-conformal horseshoes*

Now we direct our attention to estimating the Hausdorff dimension of unstable Cantor sets. Our analysis will rest upon a simplified abstract model which will be described here.

Let $\mathcal{A}$ and $\mathcal{B}$ be two families of sets. We say that $\mathcal{A}$ is subordinate to $\mathcal{B}$ if every $A \in \mathcal{A}$ is contained in some $B = B(A) \in \mathcal{B}$. We denote this by $\mathcal{A} \prec \mathcal{B}$.

Let $R \subset N^d$ be the domain of a Riemannian manifold of dimension d and suppose that δR has a limit capacity less than d. The limit capacity or box dimension of a subset X of a metric space is defined by

$$c(X) = \limsup_{\epsilon \to 0+} \frac{\log n(X, \epsilon)}{|\log \epsilon|},$$

where $n(X, \epsilon) = \inf \#\{B(x_i, \epsilon) : X \subset \bigcup_i B(x_i, \epsilon)\}$ is the minimum number of ϵ balls needed to cover X.

We shall suppose that we have finitely many diffeomorphisms $g_i : R \to N, i \in S$, such that $g_i(R) \subset R$ for every $i \in S$ and that they have pairwise disjoint interiors. We define a sequence of set families $\wp_n$ as follows:

$$\wp_n = \{g_{i_1} \circ \cdots \circ g_{i_n}(R) : i_k \in S\}. \tag{20}$$

$\{\wp_n\}$ is an increasing sequence with order $\prec$. Suppose, in addition that the following hold:

(1) each $P \in \wp_n$ is a compact subset with a non-empty interior;
(2) $\mathrm{diam}(\wp_n) \to 0^+$ as $n \to +\infty$ and
(3) there are two constants $0 < \lambda_0 < \lambda_1$ such that

$$e^{-n\lambda_1} \leq \mathrm{Vol}(P) \leq e^{-n\lambda_0} \quad \forall P \in \wp_n. \tag{21}$$

Also note that $c(\partial P) < d$ for every $P \in \wp_n$ and $n > 0$. This holds since $g_{i_1} \circ \cdots \circ g_{i_n}$ is a diffeomorphism and it preserves the limit capacity of sets.

We borrow the following result from [**16**].

LEMMA 5.1. *Let R be a domain on some Riemannian manifold N of dimension d such that $c(\partial R) < d$. Let $g : N \to N$ be a diffeomorphism. Then there exists an $\epsilon > 0$ such that for every $n > 0$ there exists a covering $\mathcal{B}$ of $g^n(R)$ by open balls with $\epsilon^n \leq \mathrm{radius}(B) \leq 4\epsilon^n$ for every $B \in \mathcal{B}$ and*

$$\sum_{B \in \mathcal{B}} \mathrm{Vol}(B) \leq C_0 \,\mathrm{Vol}(g^n(R)),$$

where C_0 depends only on R and the dimension d.

https://doi.org/10.1017/S0143385702000287 Published online by Cambridge University Press

Proof. we follow the proof of [**16**, Proposition 4.1] making some minor changes to obtain the conclusion.

We start by fixing some $c(R) < d_0 < d$ and choosing $C_1 > 0$ such that ∂R is covered by, at most, $C_1\rho^{-d_0}$ balls of radius ρ. Let, in addition, $K > 1$ be an upper bound for $\|Dg\|$, $\|Dg^{-1}\|$ and the Jacobian of g and g^{-1}.

We cover ∂R with no more than $C_1(2K^{-n}\epsilon^n)^{-d_0}$ balls $B(x_i, 2K^{-n}\epsilon^n)$. Then $\partial g^n R$ will be covered by no more than $C_1(2K^{-n}\epsilon^n)^{-d_0}$ balls $B(g^n(x_i), 2\epsilon^n)$. The open set $W_n = \bigcup_i B(g^n(x_i), 3\epsilon^n)$ covers $\partial g^n R$ and

$$\operatorname{Vol}\left(\bigcup_i B(g^n(x_i), 3\epsilon^n)\right) \leq C_1(2K^{-n}\epsilon^n)^{-d_0} C_d(3\epsilon^n)^d, \tag{22}$$

where C_d depends only on the dimension d and the Riemannian metric. Indeed, W_n contains the $V_n = \{x \in N : d(x, \partial g^n R) < \epsilon_n\}$, which is easy to verify.

Now, we look for a maximal family of *closed* balls $\overline{B}(z_j, \epsilon^n)$ contained in $\partial g^n R - V_n$ with pairwise disjoint interiors. The family of open balls $\{B(z_j, 2\epsilon^n)\}$ covers $\partial g^n R - V_n$. Therefore,

$$\sum_j \operatorname{Vol}(B(z_j, 2\epsilon^n)) \leq 2^d \sum_j \operatorname{Vol}(B(z_j, \epsilon^n)) \leq 2^d \operatorname{Vol}(g^n R). \tag{23}$$

Let $\mathcal{B} = \{B(g^n(x_i), 3\epsilon^n)\} \cup \{B(z_j, 2\epsilon^n)\}$. Using (22) and (23) we obtain

$$\sum_{B\in\mathcal{B}} \operatorname{Vol}(B) \leq C_1 C_d 3^d 2^{-d_0}(K^{d_0}\epsilon^{d-d_0})^n + 2^d \operatorname{Vol}(g^n R). \tag{24}$$

If we let $0 < \epsilon = K^{-(d_0+1)/d-d_0} < 1$ then $(K^{d_0}\epsilon^{d-d_0})^n = K^{-n}$. Therefore, $K^{-n} \leq \operatorname{Vol}(g^n(r))/\operatorname{Vol}(R)$ implies that

$$\sum_{B\in\mathcal{B}} \operatorname{Vol}(B) \leq C_0 \operatorname{Vol}(g^n R), \tag{25}$$

where $C_0 = C_1 C_d 3^d 2^{-d_0}/\operatorname{Vol}(R) + 2^d$. □

PROPOSITION 5.1. *Let $\Lambda = \bigcap_{n>0} \bigcup \wp_n$ and let us suppose that there is a constant $0 < \beta_0 < 1$ such that*

$$0 < \inf_{n>0} \sum_{P\in\wp_n} \operatorname{Vol}(P)^{\beta_0} \leq \sup_{n>0} \sum_{P\in\wp_n} \operatorname{Vol}(P)^{\beta_0} < +\infty. \tag{26}$$

Let $0 < \lambda_0 < \lambda_1$ be given by (21) and $0 < \underline{\beta} < \overline{\beta} < 1$ be the unique solutions to the following equations:

$$e^{\lambda_1(1-\beta_0)}\epsilon^{d(\underline{\beta}-1)} = 1 \quad \text{and} \quad e^{\lambda_0(1-\beta_0)}\epsilon^{d(\overline{\beta}-1)} = 1. \tag{27}$$

Then,

$$d\underline{\beta} \leq \dim_{\mathcal{H}}(\Lambda) \leq d\overline{\beta}.$$

Proof. We will find a sequence of coverings by open balls $\mathcal{B}_n$ with $\operatorname{diam}(\mathcal{B}_n)$ which goes to zero as $n \to +\infty$ such that $\sum_{P\in\wp_n} \operatorname{Vol}(P)^\beta \to +\infty$ as $n \to +\infty$ if $\beta < \underline{\beta}$ and such that $\sum_{P\in\wp_n} \operatorname{Vol}(P)^\beta \to 0^+$ when $\beta > \overline{\beta}$. This would be enough to prove the proposition.

https://doi.org/10.1017/S0143385702000287 Published online by Cambridge University Press

Let $\mathcal{B}_n = \bigcup_{P\in\wp_n} \mathcal{B}(P)$, where $\mathcal{B}(P)$ is the covering of $P = g_{i_1} \circ \cdots \circ g_{i_n}(R)$ given by Lemma 5.1. Here $K > 1$ will be an upper bound of $\|Dg_i\|$, $\|Dg_i^{-1}\|$ and the Jacobian of g_i and g_i^{-1} for every $i \in S$. Now, let $0 < \beta < 1$. Then

$$\sum_{B\in\mathcal{B}(P)} \mathrm{Vol}(B)^\beta \le \sum_{B\in\mathcal{B}(P)} \mathrm{Vol}(B) \sup_{B\in\mathcal{B}(P)} \mathrm{Vol}(B)^{-1}.$$

Hence,

$$\sum_{B\in\mathcal{B}(P)} \mathrm{Vol}(B)^\beta \le C_0\,\mathrm{Vol}(P)[C_d 2^d \epsilon^{nd}]^{\beta-1}. \tag{28}$$

Therefore,

$$\sum_{B\in\mathcal{B}_n} \mathrm{Vol}(B)^\beta \le [C_d 2^d]^{\beta-1} \sum_{P\in\wp_n} \mathrm{Vol}(P)[\epsilon^{nd}]^{\beta-1},$$

so that

$$\sum_{B\in\mathcal{B}_n} \mathrm{Vol}(B)^\beta \le K'_\beta [e^{\lambda_0(1-\beta_0)}\epsilon^{d(\beta-1)}]^n, \tag{29}$$

where $K'_\beta = C_0[C_d 2^d]^{\beta-1} \sup_{n>0} \sum_{P\in\wp_n} \mathrm{Vol}(P)^{\beta_0}$.

Analogously,

$$\sum_{B\in\mathcal{B}(P)} \mathrm{Vol}(B)^\beta \ge C_0\,\mathrm{Vol}(P)[C_d 8^d \epsilon^{nd}]^{\beta-1}$$

so that

$$\sum_{B\in\mathcal{B}_n} \mathrm{Vol}(B)^\beta \ge C_0[C_d 8^d]^{\beta-1} \sum_{P\in\wp_n} \mathrm{Vol}(P)^{\beta_0}[e^{\lambda_1(1-\beta_0)}\epsilon^{d(\beta-1)}]^n.$$

Hence, setting

$$K''_\beta = C_0[C_d 8^d]^{\beta-1} \inf_{n>0} \sum_{P\in\wp_n} \mathrm{Vol}(P)^{\beta_0},$$

we obtain

$$\sum_{B\in\mathcal{B}_n} \mathrm{Vol}(B)^\beta \ge K''_\beta [e^{\lambda_1(1-\beta_0)}\epsilon^{d(\beta-1)}]^n. \tag{30}$$

We reach this conclusion easily since $e^{\lambda_1(1-\beta_0)}\epsilon^{d(\beta-1)} > 1$ when $\beta < \underline{\beta}$. Similarly, if $\beta > \overline{\beta}$ then $e^{\lambda_0(1-\beta_0)}\epsilon^{d(\beta-1)} < 1$. □

LEMMA 5.2. *Let Λ be a C^2 horseshoe defined by finitely many branches $T_i\colon S_i \to U_i$ $(i \in S)$ and d^u the dimension of the unstable leaves $W^u(x)$ passing through $x \in \Lambda$. Then, there is a constant $C = C(\Lambda) > 0$ such that*

$$|d^u - \dim_{\mathcal{H}}(\Lambda \cap W^u(x))| \le C|1 - \dim_u(\Lambda)|, \tag{31}$$

for every $x \in \Lambda$.

https://doi.org/10.1017/S0143385702000287 Published online by Cambridge University Press

Proof. Let $\Lambda = \bigcap_{n\geq 0} T^{-n}\big(\bigcup_i S_i\big)$ where $T : \bigcup_i S_i \to \bigcup_i U_i$ the piecewise smooth mappings generating Λ. By definition $T_i = T|S_i$ is hyperbolic with respect to suitable cone families in R and we assume that we are given suitable families of admissible submanifolds. We in addition assume, that each $S_i \subset R$ is a stable admissible cylinder contained in a compact domain R with $c(\partial R) < d^u$ homeomorphic to $\mathbb{D}^u \times \mathbb{D}^s$, where $\dim \mathbb{D}^u = d^u$ and $\dim \mathbb{D}^s = d^s$. Also note that the local invariant manifolds $W^s(x, \mathsf{R})$ and $W^u(x, \mathsf{R})$ are admissible manifolds. Let $\wp_0 = \{S_i\}_{i\in S}$ be the standard Markov partition for Λ and $\wp_n = T^{-n}\wp_0$ the stable cylinders of generation n.

Furthermore, given $W = W^u(x, \mathsf{R})$ and $n \geq 0$ we have a family of geometrical cylinders $\mathcal{Q}_n = \{P \cap W : P \in \wp_n\}$ forming an increasing sequence which generates the unstable Cantor set $\Lambda \cap W^u(x, \mathsf{R})$.

Let $W = W^u(x, \mathsf{R})$ and $Q_i^0 \in \wp_0 \cap W$; and there is a contracting mapping $g_{i,W}^0 : W' \to W$ such that $Q_i^0 = g_{i,W}^0(W)$, where $W' = W^u(T(x), \mathsf{R})$. In this way we form a family of contracting mappings

$$\mathcal{G}_0 = \{g_{i,W}^0 : W \in \mathcal{F}^u, i \in S\}. \tag{32}$$

Furthermore, given $n > 0$ and $Q_i \in \mathcal{Q}_n$, $i \in S^n$, we can find $g_{i,W}^n : W' \to W$ such that $Q_i = g_i^n(W')$ where $W' = W^u(T^n(x), \mathsf{R})$. Clearly, for every mapping $g_{i,W}^n$ there is a sequence $g_{i_1}, \ldots, g_{i_n} \in \mathcal{G}_0$ such that $g_{i,W}^n = g_{i_n} \circ \cdots \circ g_{i_1}$. Now, let

$$K = \sup_{g\in\mathcal{G}_0, W\in\mathcal{F}^u} \sup_{x\in W} \{\|Dg(x)\|, \|Dg^{-1}(g(x))\|, Jg(x), Jg^{-1}(g(x))\} \tag{33}$$

where Jg is the Jacobian with respect to the intrinsic Riemannian volume of $W = W^u(x, \mathsf{R})$. $1 < K < +\infty$ by the bounded geometry of Γ^u the admissible unstable manifolds. Also, as $\limsup_{n\to+\infty} 1/n \ln(\mathrm{Vol}_u(\mathcal{Q}_n(x)) < 0$, we can find a $0 < \lambda_0 < \lambda_1$ such that the family $\mathcal{Q}_n$ satisfies (21) for every $W \in \mathcal{F}^u$. Using Lemma 5.1 we check that

$$C = \frac{(d - d_0)\lambda_1}{(d_0 + 1)\ln K}$$

does work for (31). □

6. *Proof of the Main Theorem: construction of the approximating measures*

LEMMA 6.1. *Let $\rho > 0$ be a small real number, $s > 0$ an integer, $\mathcal{U}$ a neighbourhood of* $\operatorname{supp}\mu$ *and $\{\phi_i\}_{i>0}$ a dense countable subset of $C^0(M)$. There is a hyperbolic f-invariant subset $\Lambda_\rho \subset \mathcal{U}$ supporting an ergodic measure μ_ρ such that*

$$\left|\int \phi_i \, d\mu_\rho - \int \phi_i \, d\mu\right| < \rho \quad i = 1, \ldots, s. \tag{34}$$

Proof. Let $\epsilon = \epsilon(\rho, \mathcal{U}, s) > 0$ such that $d(x, y) < \epsilon$ implies $|\phi_i(x) - \phi_i(y)| < \rho/4$, for $i = 1, \ldots, s$ and $B(x, \epsilon) \subset \mathcal{U}$, for every $x \in \operatorname{supp}\mu$. Now we introduce a finite decomposition ξ or R into compact domains with pairwise interiors such that $\operatorname{diam}\xi < \epsilon/4$. Let $X(n, s, \rho)$ denote the set of points $x \in \Omega^*$ which returns to $\xi(x)$ in m iterates where $n \leq m \leq (1 + \rho)m$ and such that

$$\left|\frac{1}{k}\sum_{j=0}^{k-1} \phi_i(f^j(x)) - \int \phi_i(y)\, d\mu(y)\right| < \frac{\rho}{4}, \tag{35}$$

https://doi.org/10.1017/S0143385702000287 Published online by Cambridge University Press

for every $k \geq n$ and $i = 1, \ldots, s$. Now, given $0 < \delta \ll 1$, $s > 0$ and $\rho > 0$ we find an $n > 0$ such that

$$\mu(\xi(x) \cap X(n, s, \rho)) \geq (1 - \delta)\mu(\xi(x))$$

for every $x \in \Omega^*$. This is possible since μ is ergodic. See the proof of [**3**, Theorem S.5.9] for details. So, given $x \in X(n, s, \rho)$ we find a hyperbolic periodic point $z \in \xi(x)$ such that $d(f^k(x), f^k(z)) < \epsilon$ for $k = 0, \ldots, m$, where $f^m(x) \in \xi(x)$ for some $m \in [n, (1 + \rho)n]$.

Let $E \subset \Omega^*$ be the set of hyperbolic periodic points so obtained and $\Lambda_\rho^* \subset \Omega^*$, the horseshoe associated with E. Let Λ_ρ be the f-invariant saturate of Λ_ρ^* and μ_ρ the Gibbs measure for the potential $\phi = -\dim_u \Lambda \log J^u f$. We use (35) and the fact that the orbits in Λ_ρ are ϵ-shadowed by points in $X(n, s, \rho)$. Therefore, $\Lambda_\rho \subset \mathcal{U}$. Likewise, we prove (34) using (35) and the arguments in the proofs of Theorems S.5.5 and S.5.9 in [**3**]. □

6.1. *Proof of the Main Theorem: the Hausdorff dimension of* Ω_n. Let $\rho(n)$ be a sequence decreasing to zero and let $\Omega_n^* = \bigcup_{0 \leq k \leq n} \Omega_{\rho(k)}^*$ be an increasing sequence of ordinary horseshoes contained in Ω^*. Clearly, $\Omega^* = \overline{\bigcup_{n>0} \Omega_n^*}$. Each Ω_n^* supports a unique T-invariant Gibbs measure μ_n^* such that $\mu_n^*(P) \asymp \mathrm{Vol}(P)^{\dim_u(\Omega_n^*)}$ for every stable cylinder of the family associated with Ω_n^*, bounded by a constant $C = C(d, \lambda, \gamma) > 1$ which only depends on the distortion, the expansion coefficient and the bounded geometry property of the admissible unstable manifolds of Ω^*. We can assume that $\mu_n^* \to \mu^*$. So, passing to the limit we get an ergodic measure μ^* supported on Ω^* such that $\mu^*(P) \asymp \mathrm{Vol}(P)^\alpha$ uniformly for $P \in \wp^*$, where $\alpha = \sup_{n>0} \dim_u(\Omega_n^*)$. This proves that $\alpha = \dim_u(\Omega^*)$ since in this case $0 < \mathcal{D}_{\alpha, W}(\Omega^*) < +\infty$ for every local unstable manifold $W \in \mathcal{F}^u$, using the Frostman Lemma argument and the absolute continuity of the stable lamination. Hence, $\dim_u(\Omega_n^*) \to \dim_u(\Omega^*)$.

Let Ω_n be the f-invariant saturate of Ω_n^* and μ_n its natural equilibrium state. By Lemma 6.1 $\mu_n \to \mu$.

Now, let us suppose that μ is absolutely continuous along unstable leaves. Then $\mathrm{Vol}_u(\Omega^* \cap \mathcal{F}^u(x)) > 0$ uniformly. In particular, $\dim_u(\Omega^*) = 1$. Thus, $\dim_u(\Omega_n^*) \to 1$. By Lemma 5.2,

$$|d^u - \dim_{\mathcal{H}}(\Omega_n^* \cap \mathcal{F}^u(x))| \leq C_n |1 - \dim_u(\Omega_n^*)|, \tag{36}$$

where $C_n = (d - d_0)\lambda_1/(d_0 + 1) \ln K_n$ and $K_n > 1$ is the least upper bound for $\|Dg\|$, $\|D(g^{-1})\|$ and the Jacobian of g and g^{-1}, where $g \in \mathcal{G}_{\mathcal{F}_\epsilon^u(x)}$, the family of contractions generating the geometrical cylinders of the unstable Cantor set $\Omega_n^* \cap \mathcal{F}^u(x)$.

Moreover, we can extend this estimate to

$$\left| d^u - \dim_{\mathcal{H}} \left(\bigcup_{x \in \Omega_n^*} \mathcal{F}^s(x) \cap \gamma^u \right) \right| \leq C_n |1 - \dim_u(\Omega_n^*)|, \tag{37}$$

for every admissible unstable manifold $\gamma^u \in \Gamma^u$. This is a consequence of the absolute continuity of the stable lamination. To prove this we use the arguments in the proof of Lemma 5.2 and verify that $\sum_{P \in \wp_n} \mathrm{Vol}_u(P \cap \gamma^u)^{\alpha(n)} \asymp 1$ uniformly, as follows from Lemmas 4.1 and 3.1, where $\alpha(n) = \dim_u(\Omega_n^*)$.

https://doi.org/10.1017/S0143385702000287 Published online by Cambridge University Press

Let $\mathcal{G}_{0,n}$ be the family of contracting mappings which generates the unstable Cantor sets of Ω_n^* as defined in (32) (Lemma 5.2). Using equation (33) we see that

$$K_n \geq \sup_{g\in\mathcal{G}_{0,n}, W\in\mathcal{F}^u} \sup_{x\in W} \{\|Dg(x)\|, \|Dg^{-1}(g(x))\|, Jg(x), Jg^{-1}(g(x))\}$$

does work as a bound to define C_n satisfying (37).

Now, for every $g \in \mathcal{G}_{0,n}$ there is an integer $b = b(g)$ such that $g = f^{-b}|W'$ and $g^{-1} = f^b|g(W')$. Moreover, $b(g) \leq N_n$ for every $g \in \mathcal{G}_{0,n}$ and a suitable $N_n > 0$. Therefore, we look for

$$K_n \geq \sup_{k\in[-N_n,N_n]} \sup_{x\in W, W\in\mathcal{F}^u} \{\|Df^k|E^u(x)\|, J(f^k|W)(x)\} > 1.$$

$\|Df^k|E^u(x)\| \geq C_1\lambda^k$ uniformly in x independently of $n > 0$, because Ω^* is hyperbolic. On the other hand, $\liminf_{n\to+\infty} 1/n \ln J(f^n|W^u(x)) \geq a > 0$ for every $x \in \Omega^*$, so there is a $C_2 > 0$ such that $J(f^k|\mathcal{F}^u(x)) \geq C_2 e^{ka}$ uniformly in $x \in \Omega^*$. Hence, there exists a $\kappa > 1$ and $C_0 > 0$ such that $K_n \geq C_0\kappa^n$ and we can find a $C > 0$ such that $C_n \leq C$, since C_n is of the order of $1/n$ for $n > 0$ large enough.

Now, let $x \in \Omega_n$. Clearly, $\dim_{\mathcal{H}}(W^u(x) \cap \Omega_n) \geq \dim_{\mathcal{H}}(W^u(x) \cap \Omega_n^*)$. $W^u(x) \cap \mathsf{R}$ decomposes into at most countably many connected components $W_i \subset W^u(x) \cap \mathsf{R}$ such that each W_i is an admissible unstable manifold and $\dim_{\mathcal{H}}(W^u(x) \cap \Omega_n^*) = \sup_i \dim_{\mathcal{H}}(W_i \cap \Omega_n^*)$.

By (37) $\dim \mathcal{H}(W_i \cap \Omega_n^*) \geq d^u - C|1 - \dim_u(\Omega_n^*)|$ for every $i > 0$. Hence,

$$d^u \geq \inf_{x\in\Omega_n} \dim_{\mathcal{H}}(W^u(x) \cap \Omega_n) \geq d^u - C|1 - \dim_u(\Omega_n^*)|.$$

Therefore, $\dim_{\mathcal{H}}(\Omega_n \cap W^u(x)) \to d^u$ uniformly in $x \in \Omega_n$. □

Acknowledgements. This paper is an extension of my doctoral thesis. I would like to thank my adviser Jacob Palis Jr and IMPA for their support and kind hospitality during that time.

References

[1] J. Falconer. *The Geometry of Fractal Sets (Cambridge Tracts in Mathematics, 85)*. Cambridge University Press, 1985.

[2] A. Katok. Lyapunov exponents, entropy and periodic orbits of diffeomorphisms. *Publ. Math. I.H.E.S.* **51** (1984).

[3] A. Katok and L. Mendoza. Dynamical systems with non-uniformly hyperbolic behavior. Supplement to: A. Katok and B. Hasselblat, *Introduction to the Modern Theory of Dynamical Systems (Encyclopedia of Mathematics and its Applications, 54)*. Cambridge University Press, 1995.

[4] G. Keller. *Equilibrium States in Ergodic Theory (London Mathematical Society Student Texts, 42)*. Cambridge University Press, 1998.

[5] F. Ledrappier. Propietés ergódiques des mesures de Sinai. *Publ. Math. I.H.E.S.* **59** (1984).

[6] F. Ledrappier and L.-S. Young. Entropy of diffeomorphisms I and II. *Ann. Math.* **122** (1985), 509–574.

[7] R. Mañé. *Ergodic Theory and Differentiable Dynamics*. Springer, New York, 1987.

https://doi.org/10.1017/S0143385702000287 Published online by Cambridge University Press

[8] L. Mendoza. Ergodic attractors for diffeomorphisms of surfaces. *J. London Math. Soc.* **37**(2) (1988), 362–374.
[9] J. Palis and F. Takens. *Hyperbolic Dynamics and Homoclinic Bifurcations*. Cambridge University Press, 1993.
[10] Ya. Pesin. *Dimension Theory and Dynamical Systems (Chicago Lectures in Mathematics*). The University of Chicago Press, Chicago, 1997.
[11] C. Pugh and M. Shub. Ergodicity of Anosov actions. *Inventiones Math.* **15** (1972), 1–23.
[12] C. Pugh and M. Shub. Ergodic attractors. *Trans. Amer. Math. Soc.* **312**(1) (1989), 1–54.
[13] F. J. Sánchez-Salas. Some geometric properties of ergodic attractors. *Informes de Matemática*, Série F-112-Janeiro/99, I.M.P.A.
[14] Y. Peres and B. Solomyak. Problems on self-similar sets and self-affine sets: an update. *Preprint*, University of Washington, 1999.
[15] L. S. Young. Statistical properties of dynamical systems with some hyperbolicity. *Ann. Math.* **147** (1998), 585–650.
[16] V. Horita and M. Viana. The Hausdorff dimension of non-hyperbolic repellers, I: Maps with holes. *Preprint*, I.M.P.A.

https://doi.org/10.1017/S0143385702000287 Published online by Cambridge University Press