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Packing Circuits into K_N

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We shall pack circuits of arbitrary lengths into the complete graph K_N. More precisely, if N is odd and $\sum_{i=1}^{t} m_i = \binom{N}{2}$, $m_i \geqslant 3$, then the edges of K_N can be written as an edge-disjoint union of circuits of lengths $m_1, \ldots, m_t$. Since the degrees of the vertices in any such packing must be even, this result cannot hold for even N. For N even, we prove that if $\sum_{i=1}^{t} m_i \leqslant \binom{N}{2} - \frac{N}{2}$ then we can write some subgraph of K_N as an edge-disjoint union of circuits of lengths $m_1, \ldots, m_t$. In particular, K_N minus a 1-factor can be written as a union of such circuits when $\sum_{i=1}^{t} m_i = \binom{N}{2} - \frac{N}{2}$. We shall also show that these results are best possible.

1. Introduction

The main result of this paper is the following.

Theorem 1.1. *Let $L = \sum_{i=1}^{t} m_i$, $m_i \geqslant 3$, with $L = \binom{N}{2}$ when N is odd and $\binom{N}{2} - \frac{N}{2} - 2 \leqslant L \leqslant \binom{N}{2} - \frac{N}{2}$ when N is even. Then we can write some subgraph of K_N as an edge-disjoint union of circuits of lengths $m_1, \ldots, m_t$.*

Note that we only guarantee the existence of circuits (even connected subgraphs) of the appropriate sizes. We have no control over their exact form; in particular, we cannot ensure that they are all cycles.

Corollary 1.2. *Let $L = \sum_{i=1}^{t} m_i$, $m_i \geqslant 3$; then we can write some subgraph of K_N as an edge-disjoint union of circuits of lengths $m_1, \ldots, m_t$ if and only if either*

(1) *N is odd, $L = \binom{N}{2}$ or $L \leqslant \binom{N}{2} - 3$, or*
(2) *N is even, $L \leqslant \binom{N}{2} - \frac{N}{2}$.*

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Proof. If $L \leqslant \binom{N}{2} - 3$ (N odd) or $L \leqslant \binom{N}{2} - \frac{N}{2} - 3$ (N even), then add an extra m_i so that $L = \binom{N}{2}$ or $L = \binom{N}{2} - \frac{N}{2}$, respectively. Using Theorem 1.1 and removing the final circuit proves the 'if'. For the 'only if', assume we have such a packing. The vertices of the packed subgraph G of K_N must be of even degree, so the vertices of the edge complement G^c of this graph must have even degree if N is odd and odd degree if N is even. In the first case G^c must have either zero or at least three edges. In the second case G^c must have at least $\frac{N}{2}$ edges, since each vertex must have degree at least one. □

A weaker result was proved in [1], which showed that for $\sum m_i \leqslant \binom{N}{2}$ the circuits can be packed into a complete graph with about $9N/2$ vertices.

If we replace 'circuits' by 'cycles' in Theorem 1.1 and add the condition that all $m_i \leqslant N$, then we obtain a conjecture of B. Alspach [4]. In general this conjecture is much stronger than Theorem 1.1; however, since all circuits of length at most five are cycles, Theorem 1.1 does imply the main result of [7], which states that this conjecture holds when all the cycle lengths m_i are three or five.

Theorem 1.1 has been proved by F. Hwang and S. Lin [9] in the special case when all the circuits are of the same length. More recently, the stronger Alspach's conjecture has also been proved when all the cycles are of equal length [5, 6].

Another application of Corollary 1.2 is in proving a conjecture of Burris and Schelp on vertex-distinguishing proper edge-colourings of graphs in the case of 2-regular graphs (see [1, 2] and especially [3] for details).

2. Notation

Write K_n for a complete graph, E_n for an empty graph and C_n for a cycle on n vertices. If we have a specific set S of vertices in mind, we shall also use notation such as K_S and E_S. Write $K'_n = K_n$ if n is odd and $K'_n = K_n \setminus I$ if n is even, where I is any 1-factor of K_n. Here $G_1 \setminus G_2$ represents the graph G_1 with the edges of the subgraph G_2 removed. The graph K'_n is the largest subgraph of K_n for which each vertex has even degree and it has $\binom{n}{2}$ or $\binom{n}{2} - \frac{n}{2}$ edges depending on whether n is odd or even. Write $P(v_1, v_2, \ldots, v_r)$ for the *trail* of length $r-1$ on the vertices v_i with edges v_iv_{i+1}. Also, write $C(v_1, \ldots, v_r)$ for the *circuit* $P(v_1, \ldots, v_r, v_1)$ of length r. Note that we do *not* require the vertices v_i to be distinct.

If G_1 and G_2 are vertex-disjoint graphs, $G_1 \cup G_2$ will denote the union of G_1 and G_2. We also write $G_1 + G_2$ for the join of G_1 and G_2, that is, the graph $G_1 \cup G_2$ with all edges connecting G_1 and G_2 included.

If G_1 and G_2 are graphs, a *packing* of G_1 into G_2 is a map $f : V(G_1) \to V(G_2)$ such that $xy \in E(G_1)$ implies $f(x)f(y) \in E(G_2)$ and the induced map on edges $xy \mapsto f(x)f(y)$ is an injection from $E(G_1)$ into $E(G_2)$. Note that f is *not* required to be injective on vertices, so if G_1 contains a cycle or path, its image in G_2 will be a circuit or trail. A packing will be called *exact* if it induces a bijection between $E(G_1)$ and $E(G_2)$. We shall write $G_1 \mapsto G_2$ to mean that an exact packing of G_1 into G_2 exists. With this notation, the problem is one of packing a disjoint union of cycles $\cup_{i=1}^{t} C_{m_i}$ into K_N.

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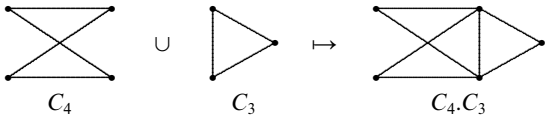


Figure 1 Example of linking graphs C_4 and C_3

We shall write $G_1.G_2$ for the union of two graphs which are *not* disjoint on vertices. In other words, it is the image of a packing of $G_1 \cup G_2$ which is injective on $V(G_1)$ and injective on $V(G_2)$, but in which some vertices of G_1 are identified with some vertices of G_2. To make this precise, we shall always make it clear which vertices of G_1 are identified with which vertices in G_2. Vertices of G_i that are identified will sometimes be called a *link* of G_i, and we shall call the identification a *linking* of G_1 and G_2.

Later on we shall define for some graphs *initial* and *final* links as (ordered) sets of vertices, possibly the same set. In these cases $G_1.G_2$ will identify the vertices of the final link of G_1 with the vertices of the initial link of G_2 (in the same order). Clearly this is only well defined if the two links are of the same size and no edge appears in both links. The initial link of the resultant graph will be that of G_1 and the final link will be that of G_2. This makes '.' an associative operation on graphs (when defined). Similarly, the initial link of $G_1 \cup G_2$ will be that of G_1 and the final link will be that of G_2. We shall write $G^{.n}$ and $G^{\cup n}$ for $G.G \cdots G$ and $G \cup \cdots \cup G$, respectively, where in each case there are n copies of G. We shall sometimes refer to $G^{.n}$ (or the image of $G^{.n}$ under some packing) as a *trail* of Gs.

Assume we have a graph of the form $C_a.C_b$ in which two cycles are linked at at least one vertex. We can pack C_{a+b} into this graph by picking such a link vertex v and going round C_a starting at v, then going round C_b. By induction, if we have a sequence of cycles $C_{a_1}.C_{a_2} \cdots C_{a_r}$ with each meeting the next in at least one vertex, we can pack a cycle of length $\sum_{i=1}^{r} a_i$ exactly into such a graph. We shall use this observation many times in what follows. We shall call a sequence of (connected) subgraphs in which each subgraph meets the next a *connected sequence* of subgraphs. A *circular connected sequence* will be a connected sequence in which the last subgraph also meets the first.

If we have an exact packing of a graph G with triangles, then each edge in G belongs to a unique triangle in the packing. We can define *trails*, *cycles*, *etc.*, of triangles as trails, cycles, *etc.*, (of edges) in which each edge belongs to a distinct triangle. The existence of a trail of triangles $T_1, \ldots, T_r$ is stronger than the existence of a connected sequence of triangles. Indeed, for such a sequence to form a trail of triangles we need $V(T_i) \cap V(T_{i+1}) = \{v_i\}$ with $v_i \neq v_{i+1}$. Note that, if we define initial and final links of a triangle as single *distinct* vertices, then this terminology fits with the previous definition of a trail of triangles as a packing of $T^{.n}$ (see Figure 4).

Define graphs $G_{n,r}$, for n and r not both odd, to be K'_{n+r} with the edges of a subgraph isomorphic to K'_r removed. More explicitly, let R be a set of r vertices and M a set of n vertices and let I_M (resp. I_R) be a 1-regular graph on M (resp. R) when n (resp. r) is even.

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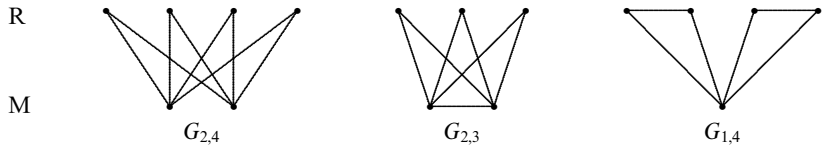


Figure 2 Examples of graphs $G_{n,r}$

Then

$$G_{n,r} = \begin{cases} K_M + E_R, & n \text{ even, } r \text{ odd,} \\ (K_M \setminus I_M) + E_R, & n \text{ even, } r \text{ even,} \\ K_M + I_R, & n \text{ odd, } r \text{ even.} \end{cases}$$

In all cases, we can write $K'_{n+r} = G_{n,r}.K'_r$ by identifying (linking) R with the vertices of K'_r, and identifying I_R with the missing 1-factor of K'_r when r is even. Whenever we have a graph of the form $G_{n,r}$, the sets of vertices M and R will be defined as above.

Given a sequence of numbers $m_1, \ldots, m_t$ with $m_i \geqslant 3$, write $L = \sum_{i=1}^{t} m_i$ for their sum. We shall define n_j to be the number of terms m_i that are equal to j, so that $L = \sum m_i = \sum j n_j$.

3. The strategy

The proof of Theorem 1.1 will be by induction on N, so for many of the lemmas we shall assume the following.

Induction Hypothesis. The conditions of Theorem 1.1 hold for N and Theorem 1.1 holds for all smaller values of N.

The idea behind the proof is to pack some cycles into $K_N \setminus K_r$ and the rest by induction into K_r to obtain a packing of all the cycles into K_N. For reasons of parity, we use the $G_{n,r}$ defined above with $n + r = N$ instead of $K_N \setminus K_r$, since every vertex of $G_{n,r}$ has even degree. We can pack $G_{n,r} \cup K_r$ (or $G_{n,r} \cup K'_r$ if n is odd) into K_N. The advantage of this approach is that we can choose the subset of cycles that we pack into $G_{n,r}$, leaving any 'awkward' cycles to be packed by induction into K_r.

One major complication is that an exact packing into $G_{n,r}$ may not exist. Indeed, there may be no subset $S \subseteq \{1, \ldots, t\}$ with $\sum_{i \in S} m_i = |E(G_{n,r})|$. To allow greater flexibility, we need to allow some 'overflow' out of $G_{n,r}$. To be precise, we shall pack our subset of cycles $\cup_{i \in S} C_{m_i}$ into graphs of the form $G_{n,r}.(\cup_{i=1}^{k} C_{l_i})$, where we have linked some extra cycles $C_{l_1}, \ldots, C_{l_k}$ to $G_{n,r}$ at some vertices of R. The extra cycles and the manner in which they are linked to $G_{n,r}$ are chosen so as to make such a packing possible.

We shall then pack $(\cup_{i=1}^{k} C_{l_i}) \cup (\cup_{i \in S^c} C_{m_i})$ – the C_{l_i} and the remaining C_{m_i} – into K_r using the Induction Hypothesis. We aim to combine these two packings into one packing into $G_{n,r}.K_r \subseteq K_N$ (or $G_{n,r}.K'_r = K_N$ when n is odd). However, when packing into K_N, we must make sure that the packed cycles C_{l_i} in K_r meet $G_{n,r}$ at the correct vertices so

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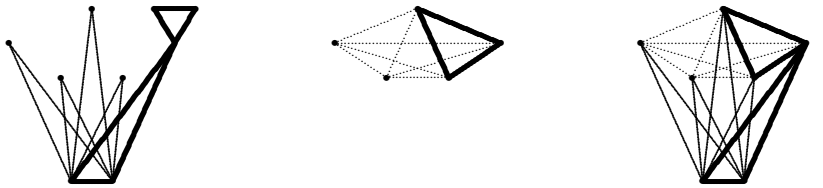


Figure 3 Example with $m_1 = 8$, $m_2 = 6$, $m_3 = 7$, $l_1 = 3$, $S = \{1, 2\}$

that we get a packing of $G_{n,r}.(\cup_{i=1}^{k} C_{l_i})$ and $\cup_{i \in S^c} C_{m_i}$ into K_N (see Figure 3). Lemma 4.1 gives some sufficient conditions that ensure that this is possible. The details are somewhat technical, but the basic idea is that the strategy works provided that there are not too many additional cycles C_{l_i} and they are not linked to $G_{n,r}$ at too many vertices.

For small values of n we can write $G_{n,r}$ as a connected sequence of squares and triangles. This and Lemma 4.1 allows us to prove some special cases of Theorem 1.1 when we either have a large number of cycles C_{m_i} of certain lengths (Lemmas 4.2 and 4.3) or if N is sufficiently small (Section 5). These results are needed to cover some special cases not handled by the general methods of later sections. However, for more general combinations of cycle lengths this approach will fail, since we would need to link too many additional cycles C_{l_i} to $G_{n,r}$. Instead, for more general combinations of cycle lengths we need to look at $G_{n,r}$ for large n. For these n, $G_{n,r}$ becomes a much more complicated graph, so we shall decompose it into simpler graphs.

Our strategy then is to decompose some $G_{n,r}$ with $n + r = N$ into connected sequences of octahedra (K_6'). We shall then devise methods of packing our cycles C_{m_i} into such connected sequences. Octahedra were chosen because they are large enough to allow a wide range of packings by cycles but sufficiently small to analyse these packings. The situation is complicated by the fact that, in general, $G_{n,r}$ cannot be packed with octahedra exactly. Indeed, we will only be able to get such packings when $N \equiv 2 \bmod 4$. For other N, we shall need to pack some other small graphs into $G_{n,r}$ to fill the 'gaps' left by the octahedra.

Sections 6 and 7 deal with the decomposition of $G_{n,r}$ into octahedra. In Section 6 we pack $G_{n,r}$ for suitable n and r with trails of triangles. In Section 7, we double up vertices to obtain a packing of some larger $G_{n,r}$ with trails of octahedra together with some other small graphs. Generating octahedra from trails of triangles ensures that the octahedra are linked together in a suitable manner.

Define O to be the graph of an octahedron, so $O = K_6' = K_{2,2,2} = E_2 + E_2 + E_2$. The first E_2 will be the initial link and the last E_2 will be the final link of O. In fact, by symmetry it does not matter which E_2s are chosen, or the order of the vertices in either link. The trail of octahedra generated in Section 7 is then a packing of $O^{\cdot r} = O.O \cdots O$ where the final link of each octahedron is linked to the initial link of the next one (recall the notation defined in Section 2 and see Figure 4).

Having packed trails of octahedra into some $G_{n,r}$, we now need to pack our cycles C_{m_i} into these trails. In general, it will not be possible to group the cycles into combinations

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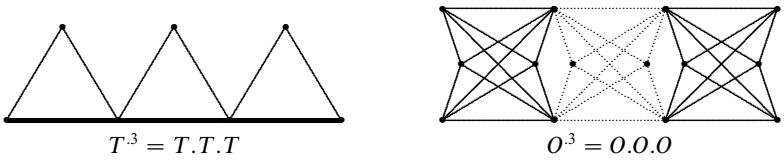


Figure 4 A trail of triangles and a trail of octahedra

with total length $|E(O)| = 12$. We therefore need to allow some 'overflow' from one octahedron to the next. This is done in Section 8, where we define the 'overflow' graphs L_n and inductively pack (almost) arbitrary combinations of cycles into trails of octahedra. The proof of Theorem 1.1 will follow, at least in the case when $N \equiv 2 \bmod 4$. For other N, $G_{n,r}$ is not packed completely with octahedra, and we need to pack cycles into some other small graphs. In the case $N \equiv 3 \bmod 4$ these extra graphs are triangles. We can pair the triangles up with octahedra into trails of the form $T.O.T.O\ldots$, and we can handle $T.O$ in very much the same way as O. The cases $N \equiv 2$ and $3 \bmod 4$ are proved in Section 9. For $N \equiv 1 \bmod 4$, the extra graphs are K_5s linked at a single vertex. These can also be packed with a wide variety of cycles. The proof of Theorem 1.1 in this case is given in Section 10. Since K_5s cannot pack every cycle length, we need to handle a number of special cases separately. These are the special cases in Section 4 described above. The most difficult case is when $N \equiv 0 \bmod 4$. For these we pack cycles into linked $G_{4,4}$s. The graph $G_{4,4}$ has 20 edges, which makes packing cycles into $G_{4,4}$ much more complicated than octahedra and K_5s. Also, $G_{4,4}$s cannot be used to pack triangles, so we run into problems when there are many triangles to be packed. In particular, we need to construct several special packings to handle the case when N is small and there are many triangles. Section 11 gives the details of the proof in this case.

4. Some packing lemmas

As described in Section 3, we shall pack some subset $\cup_{i\in S} C_{m_i}$, $S \subseteq \{1,\ldots,t\}$, of the cycles into a graph of the form $G_{n,r}.(\cup_{i=1}^{k} C_{l_i})$ and then pack the C_{l_i} and the remaining C_{m_i} into K_r using the Induction Hypothesis. We aim to combine these two packings into one packing into $G_{n,r}.K_r \subseteq K_N$ (or $G_{n,r}.K_r' = K_N$ if n is odd). However, when packing into K_N, we must make sure that the packed cycles C_{l_i} in K_r meet $G_{n,r}$ at the correct vertices so that we get a packing of $G_{n,r}.(\cup_{i=1}^{k} C_{l_i})$ and $\cup_{i\in S^c} C_{m_i}$ into K_N. The following lemma gives some sufficient conditions that ensure that this is possible.

Lemma 4.1. *Assume the Induction Hypothesis. Suppose that we can pack some subset of the cycles $\cup_{i\in S} C_{m_i}$ exactly into some graph of the form $G_{n,r}.(\cup_{i=1}^{k} C_{l_i})$ where $N = n + r$ and assume the links $V(C_{l_i}) \cap V(G_{n,r})$ are pairwise disjoint subsets of R (of $G_{n,r}$). Then we can pack all the cycles into some subgraph of K_N provided any one of the following conditions holds:*

(1) *n is even, $k \leqslant 9$ and $|V(C_{l_i}) \cap R| = 1$,*
(2) *n is odd, $k \leqslant 3$ and $V(C_{l_i}) \cap R = \{v_i\}$ where v_i are in distinct components of I_R,*

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(3) *n is even, $|V(C_{l_i}) \cap R| = 1$ for $i \geqslant 2$, $l_1 = 3$, 4 or 5 and $k \leqslant 5$, 2 or 1, respectively,*
(4) *n is even, $k \leqslant 6$, $|V(C_{l_1}) \cap R| = 2$ and $|V(C_{l_i}) \cap R| = 1$ for $i \geqslant 2$,*
(5) *n is even, $k = 2$, $l_1 = 3$ and $|V(C_{l_i}) \cap R| = 2$ for $i = 1, 2$.*

Proof. By the Induction Hypothesis, we can find a packing f which packs the remaining cycles $\cup_{\in S^c} C_{m_i}$ and all the C_{l_i} into K_r. The conditions on the total length of these cycles in Theorem 1.1 are satisfied since $|E(K'_N)| - |E(G_{n,r})| = |E(K'_r)|$ and N even implies r even. We shall consider the five cases separately.

(1) If we take $s \geqslant 1$ of the circuits $f(C_{l_i})$, then they contain in total at least $3s$ edges of K_r. Assume these circuits meet no more than $s-1$ vertices. By looking at the degrees of these vertices we see that $(s-1)\lfloor \frac{1}{2}(s-2) \rfloor \geqslant 3s$. This is not possible when $s \leqslant k \leqslant 9$, so the circuits meet at least s vertices. By Hall's Marriage Theorem, we can pick distinct vertices $u_1, \ldots, u_k$ with $f(C_{l_i})$ meeting u_i. By a suitable identification of the vertices of K_r with those of R, we can assume that $V(C_{l_i}) \cap R = \{u_i\}$. We now have a packing of $(G_{n,r}.(\cup_{i=1}^k C_{l_i})) \cup (\cup_{i \in S^c} C_{m_i})$ into some subgraph of $G_{n,r}.K_r \subseteq K_N$ by extending f by the identity on $G_{n,r}$. Composing with the given packing of $(\cup_{i \in S} C_{m_i})$ into $(G_{n,r}.(\cup_{i=1}^k C_{l_i}))$ gives the result.

(2) Let G be the image of f in K_r. Now r is even and N is odd, so $|E(G)| = \binom{N}{2} - |E(G_{n,r})| = \binom{r}{2} - \frac{r}{2}$. Therefore the complementary graph $I = G^c$ must have $r/2$ edges and all vertices of odd degree. Hence I is a 1-factor of K_r and $G = K_r \setminus I$. Since $(2s-2)\lfloor \frac{1}{2}(2s-3) \rfloor < 3s$ for $s \leqslant k \leqslant 3$, each set of s circuits $f(C_{l_i})$ meets at least $2s-1$ vertices and hence at least s components of I. Pick $u_1, \ldots, u_k$ in distinct components of I with $f(C_{l_i})$ meeting u_i. We can now identify G with $K_R \setminus I_R$ in such a way that I corresponds to I_R and $u_i = v_i$. The result now follows as in part (1).

(3) The image $f(C_{l_1})$ must be a cycle since $l_1 \leqslant 5$. We can therefore match the vertices of $V(C_{l_1}) \cap R$ with their images under f in K_r. To match the remaining vertices, we need to find distinct vertices $u_2, \ldots, u_k$ in K_r with $f(C_{l_i})$ meeting u_i and $u_i \notin f(C_{l_1})$. It is sufficient to ensure that $s \leqslant k-1$ of the other circuits meet at least $s + l_1$ vertices given that a l_1-cycle already uses l_1 edges. This can be checked as above by showing $(s+l_1-1)\lfloor \frac{1}{2}(s+l_1-1) \rfloor < 3s + l_1$ holds for $1 \leqslant s \leqslant k-1$ when $l_1 = 3, 4, 5$ and $k \leqslant 5, 2, 1$, respectively.

(4) We must first show that, if we pick two vertices w_i and w_j of $C_{l_1} = C(w_1, \ldots, w_{l_1})$, we can ensure their images under f are distinct. Assume this is not the case and the image under f is $u = f(w_i) = f(w_j)$. Modify f on C_{l_1} by cyclically moving its image vertices one step around the circuit, say $f_+(w_i) = f(w_{i+1})$ and $f_+ = f$ on all other cycles. If the images of w_i and w_j are still identical, say $u' = f_+(w_i) = f_+(w_j)$, then the edge uu' occurs twice in the image $f(C_{l_1})$, contradicting the definition of the packing f. The argument now proceeds as before. We check that any $s \leqslant k-1 \leqslant 5$ of the remaining circuits meet at least $s+2$ vertices (so the matching of vertices to the remaining circuits avoids the two vertices $f(w_i)$ and $f(w_j)$).

(5) It is enough to require that $|f(C_{l_2} \cap R)| = 2$ and $|f(C_{l_2} \cap R) \cap f(C_{l_1})| \leqslant 1$, since by cyclically permuting f on C_{l_1} we can then make $f(C_{l_1} \cap R)$ and $f(C_{l_2} \cap R)$ disjoint pairs of

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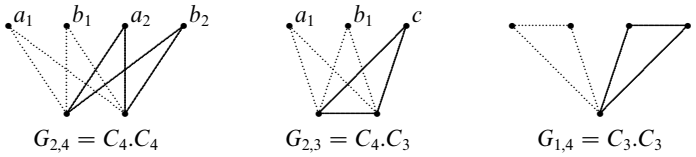


Figure 5 Packing $G_{1,r}$ and $G_{2,r}$ with C_3s and C_4s

vertices. Let $f(C_{l_1}) = \{u_1, u_2, u_3\}$ and $C_{l_2} \cap R = \{w_1, w_2\}$. Since $f(C_{l_2}) \not\subseteq \{u_1, u_2, u_3\}$, we can cyclically permute f on C_{l_2} so that $f(w_1) \notin \{u_1, u_2, u_3\}$. If $f(w_2) \neq f(w_1)$ then we are done. Otherwise let $u = f(w_1) = f(w_2)$ and cyclically permute f one step forward and backward on C_{l_2}, as in part (4), to get f_+ and f_- with $f_+(w_1) \neq f_+(w_2)$ and $f_-(w_1) \neq f_-(w_2)$. Since $f(w_1) = f(w_2)$, the points w_1 and w_2 must be distance at least three apart on C_{l_2}. Now, if $f_+(w_1) = f_-(w_2) = u'$, then we have two edges in C_{l_2} whose image under f is uu', contradicting the definition of the packing f. Similarly, we cannot have $f_+(w_2) = f_-(w_1)$. Hence, $f_\pm(w_1), f_\pm(w_2)$ are four distinct vertices, and so at least one is not in $\{u_1, u_2, u_3\}$. Replacing f by either f_+ or f_- then gives the result. □

The next two lemmas use Lemma 4.1 with small values of n to obtain packings in some special cases when we have enough cycles of lengths divisible by three or four or equal to seven. Recall that n_j is the number of cycles of length j.

Lemma 4.2. *Assume the Induction Hypothesis. If* $\sum jn_{4j} \geqslant \frac{1}{2}(N-3)$ *then the conclusion of Theorem 1.1 holds.*

Proof. The cases when $N \leqslant 3$ are trivial, so assume $N \geqslant 4$. Let $n = 2$, $r = N - 2$ and consider the graph $G_{2,r}$. Write R as the union of pairs $\{a_i, b_i\}$, $i = 1, \ldots, \lfloor r/2 \rfloor$ and an extra vertex $\{c\}$ if r is odd (see Figure 5). Then $S_i = E_M + E_{\{a_i,b_i\}} = E_2 + E_2$ are squares. Since the squares are all connected (via M), these form a connected sequence of squares $S_1.S_2 \cdots S_{\lfloor r/2 \rfloor}$. Take each cycle C_{m_i} with $m_i \equiv 0 \bmod 4$ in turn and pack them into $m_i/4$ consecutive squares in this sequence. We continue until we run out of squares. To pack the last such C_{m_i} we may need to link an extra cycle to the vertex $a_{\lfloor r/2 \rfloor}$, say, to pack all of this cycle exactly. Since the total length of all such C_{m_i} is at least $4\lfloor r/2 \rfloor$, we shall not run out of these cycles. If N is even, we have used all the edges of $G_{2,r} = E_M + E_R$, and we are done, by part (1) of Lemma 4.1 with $k \leqslant 1$.

Now assume N is odd. The remaining edges of $G_{2,r} = K_M + E_R$ form the triangle $T = K_M + E_{\{c\}} = K_2 + E_1$. If one of the remaining cycles C_{m_i}, $m_i \not\equiv 0 \bmod 4$, is of length 3 or $\geqslant 6$, we can pack this into T (linking a cycle C_{m_i-3} to vertex c if $m_i \geqslant 6$). By part (1) of Lemma 4.1 with $k \leqslant 2$, we are done in this case. This also covers the case when $\sum jn_{4j} \geqslant \frac{1}{2}(N+1)$ and not all of these cycles are squares. (Link a C_5 to c, and start by packing a cycle of length $m_i \geqslant 8$, $m_i \equiv 0 \bmod 4$, into this C_5, T and possibly some S_i first. Then pack the remaining C_{m_i} with $m_i \equiv 0 \bmod 4$ into the other S_is as above.)

Now assume there are two cycles of length 5. We can pack these into $K_{M \cup \{a_1,b_1,c\}} = K_5$ (as $C(a_1, b_1, c, u_1, u_2)$ and $C(a_1, c, u_2, b_1, u_1)$ where $M = \{u_1, u_2\}$). Now pack the the cycles

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of length $m_i \equiv 0 \bmod 4$ into $S_2, \ldots, S_{\lfloor r/2 \rfloor}$ as above. This uses all the edges of $G_{2,r}$, the triangle $C(a_1, b_1, c)$, and possibly one other cycle linked to $a_{\lfloor r/2 \rfloor}$, say. We are now done, by part (3) of Lemma 4.1 with $l_1 = 3$, $k \leqslant 2$.

The only remaining cases are when every cycle has length divisible by 4 except possibly for a single C_5. In this case $L = \binom{N}{2} \equiv 0$ or $1 \bmod 4$, so $N \geqslant 7$ and $\sum jn_{4j} \geqslant \frac{1}{4}(\binom{N}{2} - 5) \geqslant \frac{1}{2}(N+1)$. We can therefore assume we have at least $\frac{1}{2}(N+1)$ squares. Pack $\frac{1}{2}(N-5)$ squares into $S_2, \ldots, S_{\lfloor r/2 \rfloor}$, and 3 squares as $C(a_1, u_1, u_2, b_1)$, $C(a_1, u_2, c, a_2)$ and $C(b_1, u_1, c, b_2)$. This gives a packing using the edges of $G_{2,r}$ and $C(a_1, b_1, b_2, c, a_2) = C_5$. We are now done, by part (3) of Lemma 4.1 with $l_1 = 5$, $k = 1$. □

Lemma 4.3. *Assume the Induction Hypothesis with N odd. If $\sum jn_{3j} \geqslant \frac{1}{2}(N-1)$ or $n_7 \geqslant \frac{1}{2}(N-1)$, then the conclusion of Theorem 1.1 holds.*

Proof. First assume $\sum jn_{3j} \geqslant \frac{1}{2}(N-1)$ and let $n = 1$, $r = N-1$. Then $G_{1,r} = K_1 + I_R$ forms $r/2$ triangles linked by a common vertex. Pack cycles of length divisible by 3 into this graph in a manner analogous to Lemma 4.2. We may need to link a cycle to one of the vertices of R to pack the last cycle. Using part (2) of Lemma 4.1 with $k \leqslant 1$ gives the result. Now assume $n_7 \geqslant \frac{1}{2}(N-1)$. Let $n = 3$, $r = N-3$. Then $G_{3,r} = K_3 + I_R$ is an edge-disjoint union of $r/2 - 1$ graphs $E_3 + K_2$, each of which can be packed with a single C_7, and one $K_3 + K_2 = K_5$, which can be packed with a C_7 and a C_3. By permuting the vertices of this K_5, we can ensure that the C_3 meets R, so by linking a C_4 to one of the vertices of R we can pack another C_7. We now have a packing of $r/2 + 1 = \frac{1}{2}(N-1)$ C_7s into $G_{3,r}.C_4$. The result now follows, from part (2) of Lemma 4.1 with $k = 1$. □

5. Packing K_N for small N

Since the methods that we shall use in subsequent sections do not always apply for small N, it will be necessary to give alternative proofs for these N. We include these proofs first, since some of these results will be useful later. The cases treated here require long and tedious case-by-case checking. As a result, we do not include all the details. Since the methods used in these proofs are not used again (except briefly at the end of Section 11), the reader may wish to skip the proofs in this section on first reading. For $N \leqslant 10$, most of the cases are also implied by the results in [11].

Lemma 5.1. *Assume the Induction Hypothesis with $N \leqslant 11$, N odd. Then the conclusion of Theorem* 1.1 *holds.*

Proof. The cases when $N < 5$ are clear, so we shall assume $5 \leqslant N \leqslant 11$. Write each cycle length $m_i \neq 5$ as a sum of 3s and 4s. For example, $11 = 4+4+3$ and $6 = 3+3$. All cycle lengths except for 5 can be written in this way (though not necessarily uniquely). Let t_T, s_T and p_T be the total number of 3s, 4s and remaining C_5s so that $L = \binom{N}{2} = 3t_T + 4s_T + 5p_T$. The idea is to pack some of the cycles as connected sequences of triangles, squares and pentagons in certain explicit constructions.

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Assume first that $s_T \geqslant \frac{1}{2}(N-3)$. If all of the 4s occur in cycles that are written without 3s, then these cycles have lengths divisible by four and we are done, by Lemma 4.2. Hence we may assume that at least one of the cycles, C_{m_1} say, is written with both 3s and 4s. Using the construction of Lemma 4.2, we can write $G_{2,N-2}$ as a connected sequence $T.S_1 \cdots S_{\lfloor r/2 \rfloor}$, where T is the triangle $K_M + E_{\{c\}}$. Now pack the cycle $m_1 = 3 + r.4 + s.3$ into the triangle and the first r squares. Link, if necessary, a C_{3s} to the rth square at a vertex of R. Proceed in turn with each other cycle that has at least one 4 in its expansion, packing at least one square and linking a cycle to a vertex of R whenever necessary. Since $s_T \geqslant \frac{1}{2}(N-3)$, we shall pack the whole of $G_{2,r}.(\cup C_{l_i})$ where the C_{l_i} are linked to $G_{2,r}$ at single distinct vertices of R. The result now follows, from part (1) of Lemma 4.1 with $k \leqslant \frac{1}{2}(N-3) \leqslant 4$.

Now assume $t_T \geqslant \frac{1}{2}(N-1)$. Using the first construction of Lemma 4.3, we can write $G_{1,N-1}$ as a connected sequence of triangles. Pack cycles written with 3s into this connected sequence linking extra cycles C_{l_i} when necessary at vertices in R. We pack the cycles written with 3s only first. Hence, if we need to link more than one cycle to $G_{1,N-1}$, each linked cycle will arise by packing a C_{m_i} with m_i written with at least one '4'. Since we can assume $s_T \leqslant \frac{1}{2}(N-3)-1 \leqslant 3$, we shall need to link at most three cycles C_{l_i} to $G_{1,N-1}$. The cycles C_{l_i} are linked at single vertices to distinct components of I_R, so we are done, by part (2) of Lemma 4.1.

Finally, assume $p_T \geqslant \frac{1}{2}(N-1)$. Remove $\frac{1}{2}(N-1)$ C_5s, add a $C_{(N+1)/2}$ and use the Induction Hypothesis to pack these cycles into K_{N-2}. (The total number of edges is $\binom{N}{2} - \frac{5}{2}(N-1) + \frac{1}{2}(N+1) = \binom{N-2}{2}$.) By suitably labelling vertices of $K_{N-2} = K_R$ as a_i, $1 \leqslant i \leqslant \frac{1}{2}(N-1)$ and b_i, $1 \leqslant i \leqslant \frac{1}{2}(N-3)$, we can assume the cycle $C_{(N+1)/2}$ is packed as $C(x, a_1, a_2, \ldots, a_{(N-1)/2})$, where $x = b_1$ when the cycle is packed as a cycle, and $x = a_3$ in the special case when $N = 11$ and $C_{(N+1)/2} = C_6$ is packed as two connected triangles. Now pack $\frac{1}{2}(N-1)$ pentagons into $G_{2,N-2}.C_{(N+1)/2}$ as follows:

$$C(a_i, u_1, b_i, u_2, a_{i+1}), \quad 1 \leqslant i \leqslant \tfrac{1}{2}(N-3), \qquad \text{and} \qquad C(a_1, u_2, u_1, a_{(N-1)/2}, x),$$

where $M = \{u_1, u_2\}$. This uses the edges of $C_{(N+1)/2}$ and $G_{2,N-2}$ and hence gives a packing of the original cycles into $G_{2,N-2}.K_{N-2} = K_N$.

If none of these conditions hold, the total number of edges can be at most

$$3(\tfrac{1}{2}(N-1)-1) + 4(\tfrac{1}{2}(N-3)-1) + 5(\tfrac{1}{2}(N-1)-1) = 6N-22.$$

However, $L = \binom{N}{2} > 6N-22$ when $N \geqslant 5$. □

Packing K_N for even N is more difficult since we cannot use Lemma 4.3 in this case. Before we prove the result for small even N, we shall list some explicit packings that we will need. Recall that a circular connected sequence of triangles is an edge-disjoint set of triangles $T_1, \ldots, T_r$ with $V(T_i) \cap V(T_{i+1}) \neq \emptyset$ and $V(T_r) \cap V(T_1) \neq \emptyset$.

Lemma 5.2.

(1) *For $N = 8$, 10 and $4 \leqslant m \leqslant N$, or for $N = 14$ and $7 \leqslant m \leqslant N$, we can pack K_N with a circular connected sequence of triangles $T_1, \ldots, T_r$ and a cycle C_m with*

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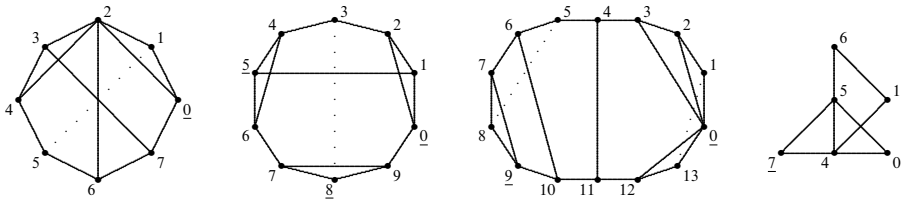


Figure 6

$3r + m \geqslant \binom{N}{2} - \frac{N}{2} - 2$, *and the cycle packed as a cycle* (*i.e., the packing is injective on the vertices of* C_m).

(2) *We can pack* $G_{8,4}$ *exactly with a circular connected sequence of triangles together with either one* C_5 *meeting* R *or two linked* C_4*s which meet distinct vertices of* R (*of* $G_{8,4}$).

(3) *We can pack* $C_3 \cup C_4 \cup C_5$, $C_6 \cup C_6$ *and* $C_6 \cup C_3 \cup C_3$ *into* K_6' *with the* C_6*s packed as cycles.*

(4) *We can pack* C_m *and two connected triangles into* K_6 *when* $m = 4$ *or* 5.

Proof. (We made use of a computer search to find the constructions in this proof.) Label the vertices of K_N as 0 to $N-1$ in part (1), and the vertices of $G_{8,4} = (K_M \setminus I_M) + E_R$ as $R = \{0,1,2,3\}$, $M = \{4,5,6,7,4',5',6',7'\}$ in part (2) where I_M consists of the edges nn'. Remove the following circular connected sequences of triangles from K_8, K_{10}, K_{14} and $G_{8,4}$, respectively:

from K_8	$C(\underline{0},3,5)$, $C(\underline{0},4,6)$, $C(1,3,6)$, $C(1,4,7)$, $C(2,5,7)$
from K_{10}	$C(\underline{0},3,6)$, $C(\underline{0},4,7)$, $C(0,5,\underline{8})$, $C(1,4,\underline{8})$, $C(1,3,7)$
	$C(1,6,9)$, $C(2,4,9)$, $C(2,6,8)$, $C(2,\underline{5},7)$, $C(3,\underline{5},9)$
from K_{14}	$C(\underline{0},4,6)$, $C(\underline{0},5,7)$, $C(0,8,10)$, $C(0,\underline{9},11)$, $C(1,6,\underline{9})$, $C(1,3,5)$
	$C(1,4,8)$, $C(1,7,11)$, $C(1,10,12)$, $C(2,5,10)$, $C(2,4,9)$, $C(2,6,11)$
	$C(2,7,13)$, $C(2,8,12)$, $C(3,6,12)$, $C(3,7,10)$, $C(3,8,11)$, $C(3,9,13)$
	$C(4,10,13)$, $C(4,7,12)$, $C(5,9,12)$, $C(5,11,13)$, $C(6,8,13)$
from $G_{8,4}$	$C(0,6',7')$, $C(0,4',5')$, $C(0,6,\underline{7})$, $C(1,5',\underline{7})$, $C(1,5,7')$, $C(1,4',6')$
	$C(2,4',5)$, $C(2,4',7)$, $C(2,5',7')$, $C(2,4,6)$, $C(4,5',6')$, $C(3,6',7)$
	$C(3,5',6)$, $C(3,4',5)$, $C(3,4,7')$, $C(4',6,7')$.

The resulting graphs are shown in Figure 6 (where we have removed some unimportant vertices and edges for clarity).

We shall pack H_8, H_{10} and H_{14} with C_m and some triangles. These triangles will meet the underlined vertices in Figure 6, so can be inserted in the circular connected sequences at the points where the sequences are connected via these vertices. The result follows by inspection for $N = 8$, $m = 8,7$ (with no triangles), $m = 6,5,4$ (with triangle $C(0,1,2)$), $N = 10$, $m = 10,9,8$ (with no triangles), $m = 7,6,5$ (with triangle $C(7,8,9)$), $N = 14$, $m = 14,13$ (with no triangles), $m = 12,11,10$ (with triangle $C(0,1,2)$), $m = 9$ (with triangles $C(0,1,2)$ and $C(7,8,9)$) and $m = 8,7$ (with triangles $C(0,1,2)$ and $C(0,12,13)$). For $N = 10$, $m = 4$, delete $C(2,4,9)$ from the connected sequence of triangles (leaving it

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still connected), use the three triangles in H_{10} and the square $C(2,3,4,9)$. Similarly $H_{8,4}$ can be packed with two squares or a pentagon and a triangle $C(7,4,5)$. Parts (1) and (2) now follow. Parts (3) and (4) are easy to check. □

Following the proof of Lemma 5.1, we shall write the cycle lengths m_i as a sum of 3s, 4s and 5s. However, unlike Lemma 5.1, we shall write them so as to use as many 4s as possible. For example, $9 = 4+5$, $13 = 4+4+5$, $10 = 4+3+3$. It is clear that all m_i will be written in one of the following forms:

$$3, \qquad 5, \qquad 3+3, \qquad n.4, \qquad n.4+3, \qquad n.4+5, \qquad \text{or} \qquad n.4+3+3 \qquad (n \geqslant 1).$$

As in Lemma 5.1, we write t_T, s_T and p_T for the total number of 3s, 4s and 5s so that $L = 3t_T + 4s_T + 5p_T$. As before, we shall attempt to pack some of the cycles as connected sequences of triangles, squares and pentagons.

Lemma 5.3. *Assume the Induction Hypothesis with N even, $N \leqslant 12$ or with $N = 16$ and $2s_T + p_T \leqslant 7$. Then the conclusion of Theorem 1.1 holds.*

Proof. The cases when $N < 6$ are clear, so assume $N \geqslant 6$. Let $n = 2$, $r = N-2$ and consider packings of $G_{2,r}$. As in Lemma 4.2 this can be written as a connected sequence of $r/2$ squares. If $s_T \geqslant r/2$, we are done, by a similar (but slightly simpler) argument to that in Lemma 5.1, so assume $s_T < r/2$. We shall use the following packings of t triangles, p pentagons and s squares into $G_{2,r}.C_l$ where t is even, $l = t+p \geqslant 3$ and $t+2p+2s = r$. Label the vertices of R as $a_1, \ldots, a_l, a_{l+1} = a_1$ and $b_1, \ldots, b_{r-l}$ and let $M = \{u_1, u_2\}$. Pack pairs of triangles as $C(a_i, u_1, a_{i+1})$ and $C(a_{i+1}, u_2, a_{i+2})$ with i odd, $1 \leqslant i < t$. Then pack pentagons as $C(a_i, u_1, b_j, u_2, a_{i+1})$ for $1 \leqslant j \leqslant p$, $i = j + t$. Finally, pack squares as $C(b_j, u_1, b_{j+1}, u_2)$ with $j > p$. This uses all the edges of $G_{2,r}$ and the cycle $C_l = C(a_1, \ldots, a_l)$. All the squares and pentagons are connected to every other cycle in this packing (via M) and the triangles form a connected sequence. We shall consider four cases.

(1) The cases $r/2 - 2 \leqslant s_T < r/2$, $1 \leqslant p_T \leqslant 2$, $6 \leqslant N \leqslant 12$.
Use the construction above with $s = r/2 - 2$, $p = 1$, $t = 2$, $l = 3$. Pack cycles written with 4s or 5s into the squares and pentagons of this construction. We may need to link extra cycles to the squares and pentagons in this construction to pack these cycles fully. At most $s + p \leqslant 4$ extra cycles will be needed and they can be linked to distinct vertices b_j (so avoiding C_l). If these cycles also use at least two 3s between them, we can use them to pack the remaining two triangles in the construction. Otherwise use two cycles C_{m_i} of length 3 or one cycle of length 6 (if they exist). By considering the total number of edges it can be shown that we can fill the two triangles in the construction except in three cases. The first is packing $C_7 \cup C_7 \cup C_5 \cup C_5$ into K_8. For this, split one of the C_7s into a C_3 and a C_4. Pack these and check that we can make the C_3 meet the C_4 in the final packing. The next case is $t_T = s_T = p_T = 1$ in K_6. This case follows from part (3) of Lemma 5.2 (the cycles clearly all meet in this packing since there are only six vertices). Finally there is the case $p_T = 2$, $t_T = s_T = 0$ in K_6. In this case we have a packing into K_5, which is a subgraph of K_6. In the general case we are done, by part (3) of Lemma 4.1 with $l_1 = l = 3$, $k \leqslant 5$.

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(2) The cases $r/2-2 \leqslant s_T < r/2$, $p_T = 0$, $6 \leqslant N \leqslant 12$.
Use the construction above with $s = r/2-2$, $p = 0$, $l = t = 4$. Pick a minimal subset of cycles that use at least s 4s between them and as many 3s as possible. By minimality, these cycles must use at most k 3s where $k \leqslant 2s \leqslant 6$. By counting edges, it can be shown that if $k < 4$ then there are enough cycles of lengths 3 and 6 to pack all the remaining $4-k$ triangles of the construction except in the case $s_T = 1$, $t_T = 2$ in K_6. This exceptional case can be packed into K_5, which is a subgraph of K_6. In the general case we take a minimal set of C_3s and C_6s to pack the remaining triangles. Pack these and then pack each of our minimal subset of cycles into the triangles and squares of the construction. It can be shown that we need to split at most one cycle and hence need only link an extra cycle to at most one of the squares. The result follows from part (3) of Lemma 4.1 with $l_1 = l = 4$ and $k \leqslant 2$.

(3) The cases $s_T < r/2-2$, $s_T + p_T < r/2$ or $N = 16$.
Use the construction above with $s = s_T$, $p = p_T$, $t = r-2s-2p \geqslant 0$. Pack all cycles that use 4s or 5s, then continue with the C_3s and C_6s until we have packed $G_{2,r}.C_l$. In all these packings, $t \geqslant 2(s-1)$, so at most one extra cycle needs to be linked (at most two 3s occur in each cycle written with 4s and none occur in cycles written with 5s). The linked cycle is of length 3 or 6 (if it exists) and all unpacked cycles are C_3s and C_6s. These can be packed together with C_l into K_{N-2} by part (1) of Lemma 5.2 when $N > 8$. We pack the C_3s and C_6s into the sequence of triangles. Since the sequence is circular and meets every vertex, we may start at any vertex and the cycle linked to $G_{2,r}.C_l$ can be connected up properly. For $N = 8$ use parts (3) and (4) of Lemma 5.2 instead.

(4) The cases $s_T + p_T \geqslant r/2$, $p_T \geqslant 3$, $6 \leqslant N \leqslant 12$.
Since $p_T \geqslant 3$, $N \geqslant 8$ and $r/2 \geqslant 3$. Use the construction with $t = 0$, $s = \min(s_T, r/2-3)$, $l = p = r/2-s \geqslant 3$. Pick a minimal subset of cycles that use at least s 4s and p 5s between them. Pack these into $G_{2,r}.C_l$ as before. We may have to link extra cycles to the squares and pentagons in the construction. If $l > 3$, then all the 4s are used and the only extra cycles are 3s and 5s linked to 4s. There will be at most s such cycles, but $s+l = r/2 \leqslant 5$, so if $l = 5$ then $s = 0$ and if $l = 4$, $s \leqslant 1$. Finally, if $l = 3$ we may need to link cycles to all but one of the 4s and 5s, so we may need to link $s+p-1 \leqslant 4$ cycles. (We will never need to link cycles to all the squares and pentagons, since 5s can only be linked to 4s.) These can be linked to distinct vertices b_j as in part (1). In all cases we are done, by part (3) of Lemma 4.1.

We have now covered all possible cases, so the result follows. □

6. Packing trails of triangles in $G_{2n,r}$

In this section we shall pack some graphs of the form $G_{2n,r}$ with trails of triangles. If we have an exact packing of $G_{2n,r}$ with triangles, then each edge belongs to a unique triangle. Recall that in Section 2 we defined trails and cycles of triangles as trails and cycles (of edges) in which each edge belongs to a distinct triangle. These trails will be used in two ways. One is when we have a very large number of cycles of length divisible by three. These cycles can be packed into such a trail by a method analogous to Lemma 4.3.

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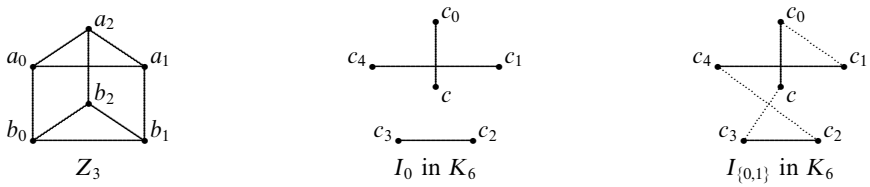


Figure 7 Examples of Z_m and I_S

(A similar method was used in [1] to prove Theorem 1.1 when all the cycle lengths are divisible by three.) However, we shall also use these trails in Section 7 to construct packings of octahedra and related graphs into larger $G_{2n,r}$. We shall then pack these trails of octahedra with cycles of arbitrary lengths.

In Section 7 we will need to be careful about where the trail ends, and we may have to pack a cycle of triangles separately. Unfortunately, this introduces many technicalities into the proofs, which could be done much more simply otherwise.

Packings of $G_{2n,r}$ with triangles exist whenever $2n > r$ and $|E(G_{2n,r})| \equiv 0 \bmod 3$. We shall not prove this here in full generality (for the cases $r \equiv 1, 3 \bmod 6$ see [8], for the general case see [10] and [7]), but restrict ourselves to some specific r and n. Let $r = r(N)$ be defined by

$$r(N) = \begin{cases} r(N+1)-1, & N \equiv 0 \bmod 2, \\ \frac{1}{2}(N-1), & N \equiv 3 \bmod 4, \\ \frac{1}{2}(N-7), & N \equiv 1 \bmod 4, N \geqslant 13, \\ 3, & N = 9, \\ \text{undefined}, & N = 0, 1, 4, 5. \end{cases}$$

For these values of r, define n by $N = 2n + r$. We shall pack triangles into $G_{2n,r}$.

Define a *prism* Z_m, $m \geqslant 3$, to be a 3-regular graph on $2m$ vertices $\{a_i, b_i : i \in \mathbb{Z}/m\mathbb{Z}\}$. The edges of Z_m are those of the form a_ib_i, a_ia_{i+1} and b_ib_{i+1}, so Z_m is two cycles of length m joined by a 1-factor. Define 1-factors of K_{2m} by labelling the vertices as $\{c\} \cup \{c_i : i \in \mathbb{Z}/(2m-1)\mathbb{Z}\}$ and setting

$$I_s = \{c_ic_j : i + j \equiv s \bmod 2m-1\} \cup \{cc_i : 2i \equiv s \bmod 2m-1\}, \qquad s \in \mathbb{Z}/(2m-1)\mathbb{Z}.$$

For $S \subseteq \mathbb{Z}/(2m-1)\mathbb{Z}$, let I_S be the edge-disjoint union of I_s for $s \in S$. In particular, $I_{\{0,\ldots,2m-2\}} = K_{2m}$.

Lemma 6.1.

(1) *For all s, $I_{\{s,s+1\}}$ is a Hamiltonian cycle of K_{2m}.*
(2) *Both $I_{\{0,1,2\}}$ and Z_m are edge-disjoint unions of three 1-factors, two of which form a Hamiltonian cycle.*
(3) *$C_{2m} + E_2$ can be packed exactly with a cycle of triangles. It can also be packed exactly with a trail of triangles ending at a vertex v of E_2 and starting with a triangle that*

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meets the other vertex of E_2. Moreover, all but one of the vertices of C_{2m} occur as interior vertices of the trail (of edges) corresponding to the trail of triangles.

(4) *Both $Z_m + E_3$ and $I_{\{0,1,2\}} + E_3$ can be packed exactly with a trail of triangles ending at a vertex v of E_3 and starting with a triangle that meets one of the other two vertices of E_3. Moreover, all but at most one of the vertices of Z_m or $I_{\{0,1,2\}}$ occur as interior vertices of the trail (of edges) corresponding to the trail of triangles.*

Proof. **(1)** By relabelling c_i as c_{i-s_0}, where $2s_0 \equiv s \bmod 2m-1$, we can assume $s = 0$. Now

$$I_{\{0,1\}} = C(c_m, c_{m-1}, c_{m+1}, c_{m-2}, c_{m+2}, \ldots, c_2, c_{2m-2}, c_1, c_0, c).$$

(2) The cycle $C(a_1, a_2, \ldots, a_{2m-1}, a_0, b_0, b_{2m-1}, \ldots, b_2, b_1)$ is a Hamiltonian cycle of Z_m, and the remaining edges $I = \{a_i b_i : 2 \leqslant i < m\} \cup \{a_0 a_1, b_0 b_1\}$ form a 1-factor. Since the cycle is of even length, it can be decomposed into two 1-factors. For $I_{\{0,1,2\}}$ the subgraph $I_{\{0,1\}}$ forms a Hamiltonian cycle and I_2 is the remaining 1-factor.

(3) Let the cycle be $C(a_1, \ldots, a_{2m})$ and let the two other vertices be u and v. The sequence of triangles $C(u, a_1, a_2), C(v, a_2, a_3), C(u, a_3, a_4), \ldots, C(v, a_{2m}, a_1)$ gives a cycle of triangles corresponding to $C(a_1, \ldots, a_{2m})$ and a trail of triangles corresponding to $P(u, a_2, \ldots, a_{2m}, v)$. All the vertices of C_{2m} except a_1 occur in the interior of the trail.

(4) Let the three vertices of E_3 be $\{u, v, w\}$. Relabel the vertices of the Hamiltonian cycle of Z_m or $I_{\{0,1,2\}}$ given in part (2) as $C(u_1, v_1, u_2, v_2, \ldots)$. We obtain triangles $C(u, u_i, v_i)$, $C(v, v_i, u_{i+1})$ and triangles formed by joining the remaining 1-factor I to w. The subgraph $G = \{uu_i\} \cup \{vv_i\} \cup I$ contains one edge from each triangle and has degree two at each vertex except possibly at u and v. We now show that it is also connected (except for the isolated vertex w). In the Z_m case, I joins $a_2 = v_1$ and $b_2 = u_m$. In the $I_{\{0,1,2\}}$ case, I_2 joins $c_0 = u_m$ and $c_2 = v_{m-2}$. In both cases, all the u_i are connected to u and all the v_i are connected to v, so G is connected. Therefore G has an Eulerian trail. If m is odd, the trail starts at u and ends at v. If m is even, we get an Eulerian circuit. For even m, remove one edge xy of I, and replace it with an edge xw. This gives a graph G' with an Eulerian trail ending at w and starting with an edge from a triangle meeting either u or v. In both cases, each edge of the trail lies in a unique triangle, so the result follows. Note that every vertex of Z_m or $I_{\{0,1,2\}}$, except possibly y, has degree two in G or G', so occurs as an interior vertex of the trail. □

Theorem 6.2. *If $N = 2n + r$ and $r = r(N)$, then for $N \geqslant 6$ there exists an exact packing of triangles into $G_{2n,r}$, which forms a trail of triangles with the last vertex $v \in R$ and the first triangle meeting $R \setminus \{v\}$. If $N \geqslant 9$, then there is also an exact packing which is an edge-disjoint union of a Hamiltonian cycle of triangles in M and such a trail.*

By a Hamiltonian cycle of triangles in M we mean that we can replace each triangle with a single edge so as to get a cycle of length $2n$ inside the subset M of vertices. We do *not* require each triangle to have all its vertices in M.

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Proof. **(1)** The cases $N \equiv 2, 3 \bmod 4$, $N \geqslant 6$.
In these cases $r = 2n-1$ if N is odd and $r = 2n-2$ if N is even. Label the vertices of R as $v_0, \dots, v_{r-1}$ and write $G_{2n,r}$ as an edge-disjoint union of $I_s + E_{\{v_s\}}$, where the I_s are the 1-factors of K_M constructed above (with $m = n$). If N is even, I_{2n-2} will not be used, so this will be the missing 1-factor I_M of K'_M in $G_{2n,r} = K'_M + E_R$. For even N, use part (3) of Lemma 6.1 to pack $I_{\{0,1\}} + E_{\{v_0,v_1\}}$ with a trail of triangles. For odd N, use part (4) of Lemma 6.1 to pack $I_{\{0,1,2\}} + E_{\{v_0,v_1,v_2\}}$ with a trail of triangles. In both cases, pair up the remaining subgraphs as $I_{\{s,s+1\}} + E_{\{v_s,v_{s+1}\}}$ and use part (3) of Lemma 6.1 to pack them with Hamiltonian cycles of triangles in M. By inserting cycles before the last triangle of the trail, we can always combine any surplus Hamiltonian cycles of triangles with the trail to get a longer trail starting and ending with the same triangles. For $N \geqslant 9$ (and $N \neq 13$), $r > 2$ and we have at least one such Hamiltonian cycle.

(2) The cases $N \equiv 0, 1 \bmod 4$, $N \geqslant 13$.
Split $M = M_a \cup M_b$ into two subsets of n vertices and label the vertices as $M_a = \{a_i : i \in \mathbb{Z}/n\mathbb{Z}\}$ and $M_b = \{b_i : i \in \mathbb{Z}/n\mathbb{Z}\}$. For $s \in \mathbb{Z}/n\mathbb{Z}$, let I'_s be the 1-factor $\{a_i b_{i+s} : i \in \mathbb{Z}/n\mathbb{Z}\}$. If n is odd, define 2-factors $S_s = \{a_i a_{i+s}, b_i b_{i+s} : i \in \mathbb{Z}/n\mathbb{Z}\}$ for $1 \leqslant s < n/2$. If n is even, we can decompose $K_n = K_{2m}$ into a 1-factor and $\frac{1}{2}(n-2)$ Hamiltonian cycles as above. Combining 1-factors for M_a and M_b gives a 1-factor I_* of M. Combining corresponding Hamiltonian cycles gives 2-factors S_s, $1 \leqslant s < n/2$ of M (each a union of two n-cycles). In both n even and n odd cases we may assume that S_1 consists of cycles $C(a_0, a_1, \dots, a_{n-1})$ and $C(b_0, b_1, \dots, b_{n-1})$. The edge-disjoint union of S_1, I'_1, I'_{n-1}, I'_2 and I'_{n-2} can be packed with the triangles

$$C(a_i, a_{i+1}, b_{i+2}), \quad C(b_i, b_{i+1}, a_{i+2}), \qquad i \in \mathbb{Z}/n\mathbb{Z}.$$

These triangles form a Hamiltonian cycle of triangles in M corresponding to the cycle $C(a_0, b_2, a_1, b_3, \dots)$.

If n is odd, then the edge union of any S_s with any I'_t is a vertex-disjoint union of n/m prisms Z_m, where $n/m = \gcd(s, n)$. It is therefore a union of three 1-factors. There are $\frac{1}{2}(n-3)$ remaining S_ss and $n-4$ remaining I'_ts. Since $n \geqslant 5$ when $N \geqslant 13$, $n-4 \geqslant \frac{1}{2}(n-3)$ and so all remaining edges can be decomposed into 1-factors. These then form $2n-7$ 1-factors, and three of these 1-factors can be chosen to form a prism Z_n made up of S_s and I_0 for some s with $\gcd(s, n) = 1$ (e.g., $s = 2$).

If n is even, then S_s is a union of two 1-factors. These and the remaining I'_t and I_* give $2n-7$ 1-factors. Three of these 1-factors can be chosen to form a prism Z_n made up from S_s and I_0 for some s, as before.

If N is odd, part (4) of Lemma 6.1 gives a trail of triangles in $Z_n + E_3$ where E_3 is any three vertices in R. If N is even, write Z_n as a union of a Hamiltonian cycle and a 1-factor. Remove this 1-factor: it will be the missing 1-factor I_M of K'_M. The remaining Hamiltonian cycle when joined to two vertices in R gives a trail of triangles, by part (3) of Lemma 6.1.

In both cases, the remaining edges of $G_{2n,r}$ can be decomposed into cycles of triangles by pairing the remaining 1-factors of M, joining each pair to a pair of points in R and using part (3) of Lemma 6.1. Then the cycles (of edges) corresponding to these cycles of

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triangles each meet an interior vertex of the trail corresponding to the trail of triangles (since all but at most one of the vertices of M are such a vertex). Hence we can insert the cycles into the trail, giving a longer trail with the same initial and final triangles. Doing this with each excess cycle in turn gives the result.

(3) The cases $N = 8$ and $N = 9$.
Label the vertices $M = \{a_0, a_1, a_2, b_0, b_1, b_2\}$ and $R = \{v_0, v_1\}$ ($N = 8$) or $R = \{v_0, v_1, v_2\}$ ($N = 9$). We can pack $G_{6,3}$ with triangles $T_a = C(a_0, a_1, a_2)$, $T_b = C(b_0, b_1, b_2)$ and $T_{ijk} = C(a_i, b_j, v_k)$ for all $j \equiv i + k \bmod 3$. If we remove all T_{ij2}s, we get a packing of $G_{6,2}$. For $N = 8$, we have the trail of triangles $T_{000}T_{110}T_{011}T_{121}T_aT_{220}T_bT_{201}$, corresponding to $P(a_0, v_0, b_1, v_1, a_1, a_2, b_2, b_0, v_1)$. For $N = 9$, we have a cycle $T_aT_{000}T_{102}T_{121}T_bT_{212}$ and a trail $T_{110}T_{011}T_{201}T_{220}T_{022}$, corresponding to the cycle $C(a_2, a_0, b_0, a_1, b_2, b_1)$ and the trail $P(v_0, b_1, v_1, a_2, b_2, v_2)$, respectively. □

Corollary 6.3. *If $N \geqslant 6$, $r = r(N)$, $N = 2n + r$ and $T_S = \frac{1}{3}|E(G_{2n,r})|$, then, for any s with $0 < s \leqslant T_S$, it is possible to pack $G_{2n,r}$ exactly with triangles so that s of them form a trail of triangles $C(v_i, u_i, v_{i+1})$, $1 \leqslant i \leqslant s$ corresponding to a trail $P(v_1, \ldots, v_{s+1})$ with $v_{s+1} \in R$. Also:*

(a) *if $N = 7$ or $N \geqslant 9$ and $s \geqslant 2n$, then we can make $v_1, \ldots v_{2n}$ a permutation of vertices of M,*
(b) *if $s < T_S$, then one of the remaining triangles can be made to meet $R \setminus \{v_{s+1}\}$,*
(c) *if $N \geqslant 9$ and $2n \leqslant s < T_S$, then both* (a) *and* (b) *can be made to hold simultaneously.*

Proof. Use the first part of Theorem 6.2 to get a trail of triangles and take the last s triangles in this trail. This gives the first part and (b) since, if $s < T_S$, then the first triangle of the original trail is not used, and meets $R \setminus \{v_{s+1}\}$. For $N \geqslant 9$, use Theorem 6.2 to obtain a Hamilton cycle and a trail. Use the last $s - 2n$ triangles of the trail (or just the last one if $s = 2n$). The first of these triangles must meet M in at least two vertices, so we can attach the Hamiltonian cycle (or a Hamiltonian path if $s = 2n$) onto the front of this trail. Parts (a) and (c) now follow provided $N \neq 7$. Finally, for $N = 7$ we have the following packing of $G_{4,3}$ with triangles where $M = \{a_0, a_1, a_2, a_3\}$ and $R = \{v_0, v_1, v_2\}$:

$$C(a_0, a_1, v_0),\ C(a_2, a_3, v_0),\ C(a_0, a_2, v_1),\ C(a_1, a_3, v_1),\ C(a_0, a_3, v_2),\ C(a_1, a_2, v_2).$$

The following trails give suitable trails of triangles in (a) for $N = 7$:

$$P(a_0, a_1, a_2, a_3, v_1, a_0, v_2) \quad \text{for } s = T_S = 6,$$
$$P(a_0, a_1, a_2, a_3, a_0, v_1) \quad \text{for } s = 5, \quad \text{and}$$
$$P(a_0, a_1, a_2, a_3, v_1) \quad \text{for } s = 4.$$ □

7. Packing trails of octahedra into $G_{2n,r}$

We shall now pack some graphs of the form $G_{2n,r}$ with trails of octahedra and related graphs. First, however, we need to define how these graphs are linked up.

Recall that for some graphs we can define initial and final links as (ordered) sets of

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vertices, and $G_1.G_2$ will identify the final link of G_1 with the initial link of G_2. The initial link of the resultant graph is that of G_1 and the final link is that of G_2. Similarly, the initial link of $G_1 \cup G_2$ is that of G_1 and the final link is that of G_2. We also write $G^{.n}$ for $G.G\cdots G$ and $G^{\cup n}$ for $G \cup \cdots \cup G$ where there are n copies of G.

We have also defined O to be the graph of an octahedron, so $O = K_6' = E_2 + E_2 + E_2$. The first E_2 will be the initial link and the last E_2 will be the final link of O. By symmetry, it does not matter which E_2s are chosen, or the order of the vertices in either link. Define $K_5 = K_4 + E_1$ to have initial and final link both equal to the same single vertex E_1. Define $W = G_{4,4} = C_4 + E_4$. The E_4 will be both the initial and final link (with the vertices in the same order). Define the triangle $T = K_2 + E_1$ with K_2 both the initial and final link (with vertices in the same order). Note that this differs from the links of a triangle as described in Section 3.

Theorem 7.1.

(1) *If $N \equiv 2 \bmod 4$ and $N \geqslant 14$, then there are exact packings of $O^{\cup a} \cup O^{.b}$ into $G_{4n,2r}$ with $4n + 2r = N$, $r = r(\frac{1}{2}N) \geqslant 3$ and the final link mapped into R.*

(2) *If $N \equiv 3 \bmod 4$ and $N \geqslant 15$, then there are exact packings of $O^{\cup a} \cup (T.O)^{.2n}.O^{.b}$ into $G_{4n,2r+1}$ with $4n + 2r + 1 = N$, $r = r(\frac{1}{2}(N-1)) \geqslant 3$ and the final link mapped into R. In addition, the packing maps the non-link vertices of each T to a single vertex in R.*

(3) *If $N \equiv 1 \bmod 4$ and $N \geqslant 13$, then there are exact packings of $K_5^{.n} \cup O^{\cup a} \cup O^{.b}$ into $G_{4n,2r+1}$ with $4n + 2r + 1 = N$, $r = r(\frac{1}{2}(N-1)) \geqslant 2$ and the final link mapped into R. Also, the link vertex of $K_5^{.n}$ is packed as a vertex of R distinct from the image of the vertices of any O.*

(4) *If $N \equiv 0 \bmod 4$ and $N \geqslant 16$, then there are exact packings of $W^{.n} \cup O^{\cup a} \cup O^{.b}$ into $G_{4n,2r+4}$ with $4n + 2r + 4 = N$, $r = r(\frac{1}{2}(N-4)) \geqslant 2$ and the final link mapped into R. Also, the link vertices of $W^{.n}$ are packed as vertices of R disjoint from the image of the vertices of any O.*

In each case such packings exist for all a, b with $a + b = \frac{1}{3}|E(G_{2n,r})|$, $a, b \geqslant 0$. Also, if $a > 0$, $N \neq 15$, then at least one O in the $O^{\cup a}$ can be required to have its final link in R and disjoint from the final link of $O^{.b}$ (or of $(T.O)^{.2n}$ in part (2) if $b = 0$).

Proof. **(1)** Set $r = r(\frac{1}{2}N)$, $2n + r = \frac{1}{2}N$. By Corollary 6.3, we can pack $G_{2n,r}$ with a trail of triangles with the final vertex in R. Replace each vertex v of $G_{2n,r}$ by a pair of vertices v_0, v_1, and each edge uv by four edges u_iv_j. The resulting graph is just $G_{4n,2r}$. The triangles become octahedra and a trail of triangles becomes a packing of linked octahedra $O^{.m}$. The result now follows from Corollary 6.3.

(2) As in (1), construct $G_{4n,2r}$ with $2n + r = \frac{1}{2}(N-1)$. Add one vertex v to R and join it to the missing 1-factor I_M of M in $G_{4n,2r}$. This gives $G_{4n,2r+1}$. The extra triangles can be linked to the first $2n$ links in the O-trail by part (a) of Corollary 6.3 since $\frac{1}{2}(N-1) = 7$ or $\frac{1}{2}(N-1) \geqslant 9$. Provided $N \neq 15$ (so $\frac{1}{2}(N-1) \geqslant 9$) we can also ensure that one of the isolated Os has a link that maps into R as a set disjoint from the final link of the trail.

(3) As in (1), construct $G_{2n,r}$ and double up vertices. This time r is even, so to obtain $G_{4n,2r+1}$ we must join the extra vertex v to n vertex-disjoint copies of K_4 in $K_M = K_{4n}$.

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(Each of these K_4s comes from doubling vertices in a component of I_M.) This gives K_5^n. The rest of the proof is similar to part (1).

(4) As in (3), except that we add the four extra vertices to R and join them to n vertex-disjoint copies of $K_4' = C_4$ in $K_M = K_{4n}$. This gives n copies of $C_4 + E_4$ with the E_4s linked. In other words $W^{\cdot n}$. □

We now have the decompositions of $G_{n,r}$ into octahedra and other small graphs as described in Section 3.

8. Packing cycles into O and $T.O$

Having packed trails of octahedra into suitable $G_{n,r}$, we now need to pack cycles into these octahedra. In general it will not be possible to group the cycles into combinations of length $|E(O)| = 12$. Therefore we shall attach cycles to the link vertices of O to allow overlaps from one octahedron to the next. These overlaps are the L_n defined below.

For a path P_n of length n with endpoints u and v, we shall make (u,v) both the initial and final link of P_n. Write $C_n' = C_n \cup E_1$ to denote a cycle of length n together with an extra independent vertex. The pair (u,v) will be both the initial and final link of C_n' where u is the independent vertex in E_1 and v is any vertex of the cycle C_n. The graph $P_{a_1,\dots,a_r} = P_{a_1}.P_{a_2}\cdots P_{a_r}$ will be a graph with specified link vertices (u,v) consisting of r internally vertex-disjoint paths of lengths $a_1,\dots,a_r$ from u to v. In the special case when $r = 0$ we write $P_\emptyset$ for the empty graph E_2 on $\{u,v\}$. We write $S_{a,b,c,d}$ for a cycle with initial link (u,v), final link (u',v') and four internally vertex-disjoint paths connecting these four vertices as follows. A path of length a connects u and v, a path of length b connects u and u', a path of length c connects v and v' and a path of length d connects u' and v' (see Figure 8)

Definition. The graphs L_n are defined as

$$L_0 = P_\emptyset, \quad L_3 = C_3', \quad L_4 = P_{2,2}, \quad L_5 = P_{2,3}, \quad \text{and} \quad L_n = P_{4,n-4} \text{ for } n \geqslant 7.$$

The graph L_6 will be defined as *either* $P_{3,3}$ *or* $P_{4,2}$. By this we mean that, whenever we pack a graph involving L_6 into another graph, we require that packings exist for both choices of L_6. On the other hand, if we pack a graph into L_6, then we only require that a packing exists for some choice of L_6. We define the triangle $T = P_{1,2}$ as before. Note that L_n is a cycle for $n \geqslant 3$, and we can always pack $C_n' \mapsto L_n$ with initial and final links matching.

Lemma 8.1. *The following can be packed into O with initial and final links matching:*

$$P_{2,2,2,2} \cup P_{2,2}, \quad P_\emptyset \cup P_{3,3,3,3}, \quad S_{4,1,1,3}.C_3', \quad S_{4,1,2,2}.C_3',$$
$$L_n.C_3' \cup L_{9-n}, \ (4 \leqslant n \leqslant 6) \quad \textit{and} \quad L_n \cup L_{12-n}, \ (3 \leqslant n \leqslant 9).$$

Proof. These packings are trivial to find, but their descriptions are somewhat tedious.

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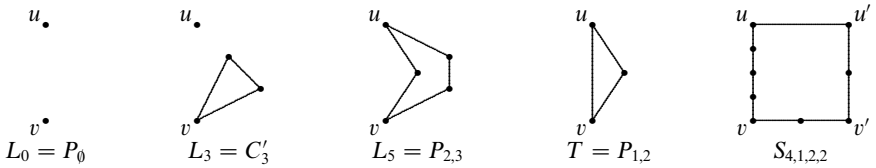


Figure 8 Graphs L_n, T and $S_{a,b,c,d}$

Number the vertices of O from 0 to 5 so that $O = E_{\{0,1\}} + E_{\{2,3\}} + E_{\{4,5\}}$ with $(0,1)$ the initial link and $(4,5)$ the final link. We pack the paths and cycles as follows:

$$\begin{aligned}
P_{2,2,2,2} \cup P_{2,2} &\mapsto \{P(0,2,1),\ P(0,3,1),\ P(0,4,1),\ P(0,5,1);\ P(4,2,5),\ P(4,3,5)\}\\
P_\emptyset \cup P_{3,3,3,3} &\mapsto \{;\ P(4,0,2,5),\ P(4,2,1,5),\ P(4,1,3,5),\ P(4,3,0,5)\}\\
S_{4,1,1,3}.C_3' &\mapsto \{P(0,2,4,3,1),\ P(0,4),\ P(1,5),\ P(4,1,2,5);\ P(5,0,3,5)\}\\
S_{4,1,2,2}.C_3' &\mapsto \{P(0,2,4,3,1),\ P(0,4),\ P(1,2,5),\ P(4,1,5);\ P(5,0,3,5)\}\\
P_{4,2}.C_3' \cup C_3' &\mapsto \{P(0,2,4,3,1),\ P(0,4,1);\ P(1,2,5,1);\ P(5,0,3,5)\}\\
P_{3,3}.C_3' \cup C_3' &\mapsto \{P(0,4,3,1),\ P(0,2,4,1);\ P(1,2,5,1);\ P(5,0,3,5)\}\\
P_{2,3}.C_3' \cup P_{2,2} &\mapsto \{P(0,3,1),\ P(0,2,4,1);\ P(1,2,5,1);\ P(4,3,5),\ P(4,0,5)\}\\
P_{2,2}.C_3' \cup P_{2,3} &\mapsto \{P(0,3,1),\ P(0,4,1);\ P(1,2,5,1);\ P(4,3,5),\ P(4,2,0,5)\}\\
P_{4,5} \cup C_3' &\mapsto \{P(0,2,4,3,1),\ P(0,4,1,2,5,1);\ P(5,0,3,5)\}\\
P_{4,4} \cup P_{2,2} &\mapsto \{P(0,2,4,3,1),\ P(0,3,5,2,1);\ P(4,1,5),\ P(4,0,5)\}\\
P_{4,3} \cup P_{2,3} &\mapsto \{P(0,2,4,3,1),\ P(0,5,2,1);\ P(4,1,5),\ P(4,0,3,5)\}\\
P_{4,2} \cup P_{4,2} &\mapsto \{P(0,4,2,5,1),\ P(0,2,1);\ P(4,1,3,0,5),\ P(4,3,5)\}\\
P_{3,3} \cup P_{4,2} &\mapsto \{P(0,2,4,1),\ P(0,3,5,1);\ P(4,3,1,2,5),\ P(4,0,5)\}\\
P_{3,3} \cup P_{3,3} &\mapsto \{P(0,4,3,1),\ P(0,3,5,1);\ P(4,1,2,5),\ P(4,2,0,5)\}.
\end{aligned}$$

In each case the edge union of the trails on the right is O. In most cases, the decomposition is a minor variant of a preceding one, so can be checked easily. Note that by symmetry, a packing of $L_3 \cup L_9$, for example, follows from a packing of $L_9 \cup L_3$. Also note that, whenever L_6 is used, both versions have been checked. □

Theorem 8.2. *Suppose that either* $m + \sum m_i \geqslant 15$ *or* $m + \sum m_i = 12$ *with* $m \geqslant 0$, $m \neq 1, 2$, $m_i \geqslant 5$, $m_i \neq 6$. *For some subset* S *and some* m' *we can pack* $L_m \cup (\cup_{i \in S} C_{m_i})$ *into* $O.L_{m'}$ *exactly with initial link matching, except in the cases when* $m \in \{0, 4, 5, 9\}$ *and all the* $m_i = 5$. *If* $m = 8$ *we can also pack* $P_{2,2,2,2} \cup (\cup_{i \in S} C_{m_i})$ *in the same way.*

Proof. See Table 1 for the packings used. In each case we can pack graph (A) into (B) by linking up suitable paths, with the initial link of L_m in (A) mapped to the initial link in (B). (For ease of checking, the underlined cycles in (A) are packed into the underlined paths and cycles in (B).) We then pack (B) into (C) by Lemma 8.1. In the case †, the P_3 part of $L_7 = P_{4,3}$ meets vertex 5 of O (see proof of Lemma 8.1) so can be linked with

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Table 1 Packings into $O.L_n$ used in Theorem 8.2

(A)	(B)	(C)	Conditions
L_m	$S_{4,1,2,2}.C'_3.C'_{m-12}$	$O.L_{m-12}$	$m \geqslant 15$
$L_{14} \cup C_{\underline{n}}$	$S_{4,1,2,\underline{2}}.C'_3.P_{4,\underline{n-2}}$	$O.L_{n+2}$	$n \geqslant 5$
$L_{13} \cup C_{\underline{n}}$	$S_{4,1,1,\underline{3}}.C'_3.P_{4,\underline{n-3}}$	$O.L_{n+1}$	$n \geqslant 5$
L_{12}	$S_{4,1,2,2}.C'_3$	$O.L_0$	
$L_{11} \cup C_{\underline{n}}$	$S_{4,1,2,\underline{2}}.C'_{\underline{3}}.P_{4,\underline{n-5}}$	$O.L_{n-1}$	$n \geqslant 7$
$L_{11} \cup C_{\underline{5}}$	$S_{4,1,1,\underline{3}}.C'_{\underline{3}}.P_{2,\underline{2}}$	$O.L_4$	
$L_{10} \cup C_{\underline{n}}$	$S_{4,1,1,\underline{3}}.C'_{\underline{3}}.P_{4,\underline{n-6}}$	$O.L_{n-2}$	$n \geqslant 8$
$L_{10} \cup C_{\underline{7}}$	$S_{4,1,2,\underline{2}}.C'_{\underline{3}}.P_{\underline{2},3}$	$O.L_5$	
$L_{10} \cup C_{\underline{5}}$	$L_7 \cup L_{\underline{5}}.C'_3$	$O.L_3$	see text †
$L_m \cup C_{\underline{n}}$	$L_m \cup L_{\underline{12-m}}.C'_{\underline{n+m-12}}$	$O.L_{n+m-12}$	$3 \leqslant m \leqslant 9, n+m \geqslant 15$
$L_m \cup C_{\underline{n}}$	$L_m \cup L_{\underline{n}}$	$O.L_0$	$3 \leqslant m \leqslant 9, n+m = 12$
$L_8 \cup C_{\underline{5}} \cup C_5$	$L_8 \cup P_{\underline{2},2}.P_{\underline{3},3}$	$O.L_6$	
$P_{2,2,2,2} \cup C_{\underline{n}}$	$P_{2,2,2,2} \cup P_{\underline{2},\underline{2}}.C'_{\underline{n-4}}$	$O.L_{n-4}$	$n \geqslant 7$
$P_{2,2,2,2} \cup C_{\underline{5}} \cup C_5$	$P_{2,2,2,2} \cup P_{\underline{2},2}.P_{\underline{3},3}$	$O.L_6$	
$L_7 \cup C_{\underline{7}} \cup C_n$	$L_7 \cup P_{\underline{2},\underline{3}}.P_{\underline{4},n-2}$	$O.L_{n+2}$	$n \geqslant 5$
$L_6 \cup C_{\underline{8}} \cup C_n$	$L_6 \cup P_{\underline{4},2}.P_{\underline{4},n-2}$	$O.L_{n+2}$	$n \geqslant 5$
$L_6 \cup C_{\underline{7}} \cup C_n$	$L_6 \cup P_{\underline{3},3}.P_{\underline{4},n-3}$	$O.L_{n+1}$	$n \geqslant 5$
$L_6 \cup C_{\underline{5}} \cup C_5$	$L_6 \cup P_{\underline{3},3}.P_{\underline{2},2}$	$O.L_4$	
$L_5 \cup C_{\underline{9}} \cup C_n$	$L_5 \cup C'_{\underline{3}}.P_{\underline{2},2}.P_{\underline{4},n-2}$	$O.L_{n+2}$	$n \geqslant 5$
$L_5 \cup C_{\underline{8}} \cup C_n$	$L_5 \cup P_{\underline{4},3}.P_{\underline{4},n-3}$	$O.L_{n+1}$	$n \geqslant 5$
$L_4 \cup C_{\underline{10}} \cup C_n$	$L_4 \cup P_{\underline{2},\underline{2},\underline{2},2}.P_{\underline{4},n-2}$	$O.L_{n+2}$	$n \geqslant 5$
$L_4 \cup C_{\underline{9}} \cup C_n$	$L_4 \cup C'_{\underline{3}}.P_{\underline{2},3}.P_{\underline{4},n-3}$	$O.L_{n+1}$	$n \geqslant 5$
$L_4 \cup C_{\underline{7}} \cup C_7$	$L_4 \cup P_{\underline{4},4}.P_{\underline{3},3}$	$O.L_6$	
$L_4 \cup C_{\underline{5}} \cup C_n$	$L_4 \cup C'_{\underline{3}}.L_{\underline{5}}.C'_{n-3}$	$O.L_{n-3}$	$n \geqslant 7$
$L_3 \cup C_{\underline{11}} \cup C_n$	$L_3 \cup C'_{\underline{3}}.P_{\underline{4},2}.P_{\underline{4},n-2}$	$O.L_{n+2}$	$n \geqslant 5$
$L_3 \cup C_{\underline{10}} \cup C_n$	$L_3 \cup C'_{\underline{3}}.P_{\underline{3},3}.P_{\underline{4},n-3}$	$O.L_{n+1}$	$n \geqslant 5$
$L_3 \cup C_{\underline{8}} \cup C_n$	$L_3 \cup C'_{\underline{3}}.P_{\underline{4},2}.P_{\underline{4},n-5}$	$O.L_{n-1}$	$n \geqslant 7$
$L_3 \cup C_{\underline{7}} \cup C_n$	$L_3 \cup C'_{\underline{3}}.P_{\underline{3},3}.P_{\underline{4},n-6}$	$O.L_{n-2}$	$n \geqslant 8$
$L_3 \cup C_{\underline{7}} \cup C_7$	$L_3 \cup C'_{\underline{3}}.P_{\underline{4},2}.P_{\underline{2},\underline{3}}$	$O.L_5$	
$L_3 \cup C_{\underline{5}} \cup C_n$	$L_{\underline{5}}.C'_3 \cup L_4.C'_{n-4}$	$O.L_{n-4}$	$n \geqslant 7$
$L_3 \cup C_{\underline{5}} \cup C_5 \cup C_{\underline{5}}$	$L_{\underline{5}}.C'_3 \cup P_{\underline{2},2}.P_{\underline{3},3}$	$O.L_6$	

the C'_3 to pack $L_{10} = P_{4,6}$. It is easy to check that, if $m > 0$ and we are not in one of the exceptional cases, then we must have a subset of one of the forms in Table 1. If $m = 0$, pack some $C_{m_{i_0}}$ into $L_{m_{i_0}}$ first and then use the result with $m > 0$. If there are C_5s, make sure to use one as our $C_{m_{i_0}}$. The only additional case that cannot be packed is when all the $m_i = 5$, which is included in the list of exceptional cases. □

We now repeat the previous two results with O replaced by $T.O$.

Lemma 8.3. *The following can be packed into $T.O$ with initial and final links matching:*

$$L_8 \cup P_{5,2}, \quad L_8 \cup P_{4,3}, \quad L_7 \cup P_{6,2}, \quad L_7 \cup P_{5,3}, \quad P_{3,3} \cup P_{7,2}, \quad P_{3,3} \cup P_{6,3},$$
$$P_{3,3} \cup P_{5,4}, \quad L_5 \cup P_{4,3}.C'_3, \quad L_5 \cup P_{5,2}.C'_3, \quad L_4 \cup P_{4,3,2,2}, \quad L_4 \cup P_{5,2,2,2}.$$

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Proof. Number the vertices of O 0 to 5 as before and let $T = C(0,1,v)$ with initial and final link equal to $(0,1)$. We pack the paths as follows:

$$P_{4,4} \cup P_{5,2} \mapsto \{P(0,2,4,3,1),\ P(0,3,5,2,1);\ P(4,0,1,v,0,5),\ P(4,1,5)\}$$
$$P_{4,4} \cup P_{4,3} \mapsto \{P(0,2,4,3,1),\ P(0,3,5,2,1);\ P(4,1,v,0,5),\ P(4,0,1,5)\}$$
$$P_{4,3} \cup P_{6,2} \mapsto \{P(0,2,4,3,1),\ P(0,5,2,1);\ P(4,0,1,v,0,3,5),\ P(4,1,5)\}$$
$$P_{4,3} \cup P_{5,3} \mapsto \{P(0,2,4,3,1),\ P(0,5,2,1);\ P(4,1,v,0,3,5),\ P(4,0,1,5)\}$$
$$P_{3,3} \cup P_{7,2} \mapsto \{P(0,4,3,1),\ P(0,5,2,1);\ P(4,2,0,1,v,0,3,5),\ P(4,1,5)\}$$
$$P_{3,3} \cup P_{6,3} \mapsto \{P(0,2,4,1),\ P(0,5,2,1);\ P(4,0,1,v,0,3,5),\ P(4,3,1,5)\}$$
$$P_{3,3} \cup P_{5,4} \mapsto \{P(0,4,3,1),\ P(0,5,2,1);\ P(4,1,v,0,3,5),\ P(4,2,0,1,5)\}$$
$$P_{2,3} \cup P_{4,3}.C_3' \mapsto \{P(0,v,1),\ P(0,4,2,1);\ P(4,1,0,2,5),\ P(4,3,1,5);\ P(5,0,3,5)\}$$
$$P_{2,3} \cup P_{5,2}.C_3' \mapsto \{P(0,v,1),\ P(0,4,2,1);\ P(4,3,1,0,2,5),\ P(4,1,5);\ P(5,0,3,5)\}$$
$$P_{2,2} \cup P_{4,3,2,2} \mapsto \{P(0,v,1),\ P(0,3,1);\ P(4,1,2,0,5),\ P(4,0,1,5),\ P(4,2,5),\ P(4,3,5)\}$$
$$P_{2,2} \cup P_{5,2,2,2} \mapsto \{P(0,v,1),\ P(0,3,1);\ P(4,0,1,2,0,5),\ P(4,1,5),\ P(4,2,5),\ P(4,3,5)\}.$$

□

Theorem 8.4. *Suppose that either* $m + \sum m_i \geqslant 18$ *or* $m + \sum m_i = 15$, *with* $m \geqslant 0$, $m \neq 1, 2$ *and* $m_i \geqslant 7$. *For some subset* S *and some* m', *we can pack* $L_m \cup (\cup_{i \in S} C_{m_i})$ *into* $T.O.L_{m'}$ *exactly with initial link matching.*

Proof. We can assume $m > 0$ by first packing some $C_{m_{i_0}}$ into $L_{m_{i_0}}$. If $m = 3$ we pack L_m into T and use Theorem 8.2 with $m = 0$. If $m \geqslant 9$ we pack $L_m = P_{4,m-4}$ into $T.L_{m-3} = P_{1,2,4,m-7}$ and use Theorem 8.2. If $m = 6$ and $L_6 = P_{4,2}$ we pack it into $P_{1,2}.C_3' = T.L_3$ and use Theorem 8.2. Since there are no C_5s the result follows in each of these cases. Otherwise $4 \leqslant m \leqslant 8$ and if $m = 6$ we can assume $L_m = P_{3,3}$. In each of these remaining cases we can use one of the packings listed in Table 2. As before we can pack graph (A) into (B) by linking up suitable paths, with the initial link of L_m in (A) mapped to the initial link in (B). We can then pack (B) into (C) by Lemma 8.3. □

Corollary 8.5.

(1) *If* $\sum m_i = 12$, $m_i \geqslant 3$, *then we can pack* $\cup C_{m_i}$ *into* O.
(2) *If* $\sum m_i = 15$, $m_i \geqslant 3$, *then we can pack* $\cup C_{m_i}$ *into* $T.O$.

Proof. **(1)** This follows from Lemma 5.3 since we can pack a subgraph of K_6 of size 12 and this subgraph must be isomorphic to $K_6' = O$.

(2) If $m_j = 3$ we can pack C_{m_j} into T and then use part (1) to pack the remaining cycles. If $m_j \geqslant 6$ then we can pack $C_{m_j-3} \cup (\cup_{i \neq j} C_{m_i})$ into O. Since the vertices of O are equivalent, we may assume C_{m_j-3} meets T and hence forms a circuit of length m_j. The only remaining case is $C_5 \cup C_5 \cup C_5 \mapsto P_{1,2}.(P_{4,3} \cup L_5) \mapsto T.O$ (using Lemma 8.1). □

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Table 2 Packings into $T.O.L_n$ used in Theorem 8.4

(A)	(B)	(C)	Conditions
$L_8 \cup C_n$	$L_8 \cup P_{5,2}.C'_{n-7}$	$T.O.L_{n-7}$	$n \geqslant 10$
$L_8 \cup C_9 \cup C_n$	$L_8 \cup P_{5,2}.P_{4,n-2}$	$T.O.L_{n+2}$	$n \geqslant 7$
$L_8 \cup C_8 \cup C_n$	$L_8 \cup P_{4,3}.P_{4,n-3}$	$T.O.L_{n+1}$	$n \geqslant 7$
$L_8 \cup C_7$	$L_8 \cup P_{4,3}$	$T.O.L_0$	
$L_7 \cup C_n$	$L_7 \cup P_{6,2}.C'_{n-8}$	$T.O.L_{n-8}$	$n \geqslant 11$
$L_7 \cup C_{10} \cup C_n$	$L_7 \cup P_{6,2}.P_{4,n-2}$	$T.O.L_{n+2}$	$n \geqslant 7$
$L_7 \cup C_9 \cup C_n$	$L_7 \cup P_{5,3}.P_{4,n-3}$	$T.O.L_{n+1}$	$n \geqslant 7$
$L_7 \cup C_8$	$L_7 \cup P_{5,3}$	$T.O.L_0$	
$L_7 \cup C_7 \cup C_7$	$L_7 \cup P_{5,3}.P_{4,2}$	$T.O.L_6$	
$L_6 \cup C_n$	$P_{3,3} \cup P_{7,2}.C'_{n-9}$	$T.O.L_{n-9}$	$n \geqslant 12$
$L_6 \cup C_{11} \cup C_n$	$P_{3,3} \cup P_{7,2}.P_{4,n-2}$	$T.O.L_{n+2}$	$n \geqslant 7$
$L_6 \cup C_{10} \cup C_n$	$P_{3,3} \cup P_{6,3}.P_{4,n-3}$	$T.O.L_{n+1}$	$n \geqslant 7$
$L_6 \cup C_9$	$P_{3,3} \cup P_{6,3}$	$T.O.L_0$	
$L_6 \cup C_8 \cup C_n$	$P_{3,3} \cup P_{5,4}.P_{4,n-5}$	$T.O.L_{n-1}$	$n \geqslant 7$
$L_6 \cup C_7 \cup C_7$	$P_{3,3} \cup P_{5,4}.P_{2,3}$	$T.O.L_5$	
$L_5 \cup C_n$	$L_5 \cup P_{5,2}.C'_3.C'_{n-10}$	$T.O.L_{n-10}$	$n \geqslant 13$
$L_5 \cup C_{12} \cup C_n$	$L_5 \cup P_{5,2}.C'_3.P_{4,n-2}$	$T.O.L_{n+2}$	$n \geqslant 7$
$L_5 \cup C_{11} \cup C_n$	$L_5 \cup P_{4,3}.C'_3.P_{4,n-3}$	$T.O.L_{n+1}$	$n \geqslant 7$
$L_5 \cup C_{10}$	$L_5 \cup P_{4,3}.C'_3$	$T.O.L_0$	
$L_5 \cup C_9 \cup C_n$	$L_5 \cup P_{5,2}.C'_3.P_{4,n-5}$	$T.O.L_{n-1}$	$n \geqslant 7$
$L_5 \cup C_8 \cup C_n$	$L_5 \cup P_{5,2}.C'_3.P_{3,n-5}$	$T.O.L_{n-2}$	$n = 7, 8$
$L_5 \cup C_7 \cup C_7$	$L_5 \cup P_{5,2}.C'_3.C'_4$	$T.O.L_4$	
$L_4 \cup C_n$	$L_4 \cup P_{4,3,2,2}.C'_{n-11}$	$T.O.L_{n-11}$	$n \geqslant 14$
$L_4 \cup C_{13} \cup C_n$	$L_4 \cup P_{4,3,2,2}.P_{4,n-2}$	$T.O.L_{n+2}$	$n \geqslant 7$
$L_4 \cup C_{12} \cup C_n$	$L_4 \cup P_{4,3,2,2}.P_{4,n-3}$	$T.O.L_{n+1}$	$n \geqslant 7$
$L_4 \cup C_{11}$	$L_4 \cup P_{4,3,2,2}$	$T.O.L_0$	
$L_4 \cup C_{10} \cup C_n$	$L_4 \cup P_{5,2,2,2}.P_{4,n-5}$	$T.O.L_{n-1}$	$n \geqslant 7$
$L_4 \cup C_9 \cup C_n$	$L_4 \cup P_{5,2,2,2}.P_{4,n-6}$	$T.O.L_{n-2}$	$n \geqslant 8$
$L_4 \cup C_8 \cup C_8$	$L_4 \cup P_{5,2,2,2}.P_{3,2}$	$T.O.L_5$	
$L_4 \cup C_7 \cup C_n$	$L_4 \cup P_{5,2,2,2}.C'_{n-4}$	$T.O.L_{n-4}$	$n \geqslant 7$

9. Packing cycles into trails of *O*s and *T.O*s

Now we put the results of the previous sections together to get a proof of Theorem 1.1 in the cases when $N \equiv 2$ or 3 mod 4. We use the results of the last section to pack (almost) arbitrary cycles into trails of Os and $T.O$s; then we use Theorem 7.1 to pack these trails into some $G_{n,r}$ and then Lemma 4.1 to pack everything into K_N. The details, as usual, are somewhat more complicated.

Theorem 9.1. *Suppose $\sum m_i \geqslant 12a+9$ with $m_i \geqslant 3$. We can pack some subset of the cycles exactly into some graph of the form*

$$O^{\cdot a} \qquad or \qquad O^{\cdot a}.P_{c,d} \qquad or \qquad O^{\cdot a-1}.P_{2,2} \cup O.T.$$

The last form is only needed if $n_5 \geqslant 7$ and $a \geqslant 3$.

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Proof. By Corollary 8.5, we can pack O with any combination of cycles of total length exactly 12. If we can pack the first O in the trail in this way, then we are done, by induction on a. Hence we may assume no combination of cycles has total length 12. In particular, $n_4 \leqslant 2$ and $n_3 + 2n_6 \leqslant 3$. We shall now try to pack any remaining C_3s, C_4s and C_6s. In general, not all of these can be packed, so we may need to discard some. Provided we do not discard cycles of total length more than 6, we shall still have $\sum m_i \geqslant 12a + 3$. We can pack $C_4 \cup C_4 \mapsto P_{2,2,2,2}$, $C_3 \mapsto L_3$ and $C_4 \mapsto L_4$. Packing the maximum length like this will leave the remaining C_3s, C_4s and C_6s with total length at most 6 except in the cases when $n_3 + 2n_6 = 3$ and $n_4 = 1$ or 2. In these cases we can assume $n_5 = n_8 = n_9 = 0$ (since $5 + 3 + 4 = 8 + 4 = 9 + 3 = 12$). Since the total length of all cycles is at least $12 + 9 > 17 \geqslant 3n_3 + 4n_4 + 6n_6$, we must have at least one cycle C_n with $n = 7$ or $n \geqslant 10$. Pack C_n into $C_3.C_{n-3}$ and $C_3 \cup C_3 \cup C_6$ or $C_3 \cup C_3 \cup C_3 \cup C_3$ into O. Since at least one of the C_3s meets vertex number 5 of O, we can pack the C_3s, C_6s and C_n into $O.C'_{n-3}$. If $n_4 = 1$ or $a = 1$, discard the remaining C_4s and pack $O.C'_{n-3} \mapsto O.L_{n-3}$. Otherwise pack the C'_{n-3} into $C'_4.C'_{n-7}$ (if $n \geqslant 10$) and $C_4 \cup C_4 \cup C_4 \mapsto O$. We can assume one of these C_4s meets both vertex 1 and 5 of O, so we have a packing $C'_{n-3} \cup C_4 \cup C_4 \mapsto O.C'_{n-7}$ with initial links matching. We therefore obtain a packing into $O.O.L_{n-7}$ with no cycles discarded.

Now we pack the other cycles. We can assume that we have already packed an $O^{\cdot b}.L_n$ or $O^{\cdot b}.P_{2,2,2,2}$ with $0 \leqslant b < a$. If there are some C_5s remaining, we can also assume $n \in \{0, 3, 4, 6\}$. We pack the remaining cycles inductively into graphs of the same form with larger values of b. If we have enough remaining C_5s use the packings

$$L_3 \cup C_5 \cup C_5 \cup C_5 \mapsto O.L_6,$$
$$L_6 \cup C_5 \cup C_5 \mapsto O.L_4,$$
$$P_{2,2,2,2} \cup C_5 \cup C_5 \mapsto O.L_6,$$
$$C_5 \cup C_5 \cup C_5 \cup C_5 \mapsto P_\emptyset \cup P_{3,3,3,3}.P_{2,2,2,2} \mapsto O.P_{2,2,2,2},$$
$$L_4 \cup C_5 \cup C_5 \cup C_5 \cup C_5 \mapsto P_{2,2} \cup P_{2,2,2,2}.P_{3,3,3,3} \mapsto O.O.$$

The first three of these packings come from Theorem 8.2; the last two follow from Lemma 8.1. In each case the initial links match, so we can pack $O^{\cdot b}.L_n$ or $O^{\cdot b}.P_{2,2,2,2}$ into $O^{\cdot b+1}.L_{n'}$ or $O^{\cdot b+1}.P_{2,2,2,2}$ or $O^{\cdot b+2} = O^{\cdot b+2}.L_0$. When $b + 1 = a$, we must avoid the last two forms if we are to pack graphs of the type listed in the statement of the theorem.

Assume we have enough C_5s to reach a total length of at least $12a + 3$. We shall use up all the Os except in the cases when we have packed $O^{\cdot a-1}.L_n$ with $n = 0$ or 4. If $n = 0$ we can use

$$C_5 \cup C_5 \cup C_5 \mapsto L_5 \cup P_{4,3}.P_{1,2} \mapsto O.P_{1,2}.$$

However, for $n = 4$ we cannot pack $L_4 \cup C_5 \cup C_5 \cup C_5$ into the final O. If the L_4 came from an original C_4, we can discard the C_4 and pack $C_5 \cup C_5 \cup C_5 \mapsto O.P_{1,2}$ for the last O. Otherwise we must have had at least seven C_5s and three Os (the minimum number occurs if we started with $P_{2,2,2,2}$ in the first O). For this case, pack $C_5 \cup C_5 \cup C_5 \mapsto O.T$. We then get a graph of the form $O^{\cdot a-1}.L_4 \cup O.T$.

Now assume we do not have enough C_5s to pack all the octahedra. After packing as many C_5s as possible, we shall have at most two C_5s left, or three C_5s if we have

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packed $O^{\cdot b}.L_0$ or $O^{\cdot b}.L_4$. Use the $L_0 \cup C_5 \mapsto L_5$ and $L_4 \cup C_5 \cup C_n \mapsto O.L_{n-3}$ packings from Theorem 8.2 to ensure we have at most two C_5s left. Now continue packing the remaining cycles using Theorem 8.2. Whenever we have packed $O^{\cdot b}.L_n$ or $O^{\cdot b}.P_{2,2,2,2}$ with $b < a$, we have enough extra cycles to pack L_n or $P_{2,2,2,2}$ and some cycles into $O.L_{n'}$ with initial link matching by Theorem 8.2, and hence we can pack $O^{\cdot b+1}.L_{n'}$. The only exception is when we try to pack $L_9 \cup C_5 \cup C_5$ into the last O (since there are at most two C_5s and the other combinations not allowed by Theorem 8.2 have too few edges). Since $9 + 5 + 5 < 12 + 9$, even this case does not occur unless there are some discarded cycles. Since $n_5 \neq 0$, we have $n_3 n_4 = 0$ and the only possible discarded cycles are C_3s and C_6s. In this case use the discarded cycles with the packings

$$L_9 \cup C_3 \mapsto L_9 \cup L_3 \mapsto O.L_0 \quad \text{and} \quad L_9 \cup C_6 \mapsto L_9 \cup L_3.C_3' \mapsto O.P_{1,2}.$$

In the general case we finish by packing the last L_n as $P_{1,2}$ if $n = 3$. □

Theorem 9.2. *Suppose $L = \sum m_i \geqslant 12a + 15m + 11$, with $m_i \geqslant 3$, m even and $L - 4n_4 \geqslant 15m + 3$. Then we can pack some subset of the cycles exactly into some graph of the form*

$$O^{\cup a-b} \cup (T.O)^{\cdot m}.O^{\cdot b}, \qquad O^{\cup a-b} \cup (T.O)^{\cdot m}.O^{\cdot b}.P_{c,d},$$
$$\text{or} \qquad O^{\cup a-b} \cup (T.O)^{\cdot m}.O^{\cdot b-1}.P_{2,2} \cup O.T,$$

where the non-link vertices of the Ts in $(T.O)^{\cdot m}$ are identified. The last form only occurs with $a \geqslant b \geqslant 3$.

Proof. By discarding C_4s, we may assume either $n_4 = 0$ or $L \leqslant 12a + 15m + 14$. In either case $n_4 < 3a + 3$. We set $b = a - \lfloor n_4/3 \rfloor \geqslant 0$ and pack C_4s into the isolated Os of $O^{\cup a-b}$ using $C_4 \cup C_4 \cup C_4 \mapsto O$. Therefore, without loss of generality, we may assume $n_4 \leqslant 2$ and set $b = a$.

By Corollary 8.5 we can pack $C_5 \cup C_5 \cup C_5 \mapsto T.O$. Doing this twice and using induction on m we can assume either $m = 0$ or $n_5 \leqslant 5$. The case $m = 0$ follows immediately from Theorem 9.1. Since the Ts in the $(T.O)^{\cdot m}$ are all linked via a common vertex, we can pack these triangles with C_6s, and, if necessary, with C_3s. If we run out of Ts, then we must have exactly used them all (m is even). We are then done, by Theorem 9.1, noting that $n_5 \leqslant 5$, so the last form in Theorem 9.1 will not occur.

If we run out of C_3s and C_6s, drop the assumption that m is even and assume $n_3 = n_6 = 0$. By using $C_5 \cup C_5 \cup C_5 \mapsto T.O$ again, we may also assume either $m = 0$ or $n_5 \leqslant 2$. If $m = 0$ we are done, by Theorem 9.1 again. Discarding any C_4s (of total length at most 8) we may now assume $\sum m_i \geqslant 12a + 15m + 3$ with $n_3 = n_4 = n_6 = 0$, $n_5 \leqslant 2$ and without the condition that m is even.

We can pack $C_5 \cup C_n \mapsto P_{1,2}.P_{4,n-2} = T.L_{n+2}$ for $n \geqslant 5$, so that we now have no C_5s left. (Choose $n = 5$ if $n_5 = 2$ and $n \geqslant 7$ if $n_5 = 1$. There must be at least one cycle of length at least 7 since $\sum_{j<7} jn_j \leqslant 5 + 5 < 15 + 3$.) Now inductively pack the remaining cycles C_{m_i}, $m_i \geqslant 7$ according to Theorems 8.4 and 8.2 in a manner analogous to the proof of Theorem 9.1. □

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Theorem 9.3. *Assume the Induction Hypothesis with $N > 12$, $N \equiv 2$ or $3 \bmod 4$. Then Theorem 1.1 holds for N.*

Proof. **(1)** The case $N \equiv 2 \bmod 4$, $N \geqslant 14$.
By part (1) of Theorem 7.1, there are exact packings of $O^{\cdot a}$, and $O \cup O^{\cdot a-1}$ into some $G_{4n,2r}$, $4n + 2r = N$, $2r \geqslant 6$, with the final link in R. The isolated O in $O \cup O^{\cdot a-1}$ can be required to have its final link in R and disjoint from the final link of $O^{\cdot a-1}$. Now $L \geqslant |E(K'_N)| - 2 = |E(G_{4n,2r})| + |E(K'_{2r})| - 2 \geqslant 12a + 10$. Using Theorem 9.1 we can pack some cycles into $O^{\cdot a}$, $O^{\cdot a}.P_{c,d}$ or $O.T \cup O^{\cdot a-1}.P_{2,2}$. These can then be packed into $G_{4n,2r}.(\cup C_{l_i})$ with at most two attached cycles C_{l_i} meeting R in two vertices each. The cycles C_{l_i} are just the $P_{c,d}$ or the T and $P_{2,2}$ linked to the ends of the trails of octahedra. If there are two such cycles then one is a triangle (T) and their intersections with R are disjoint. Therefore we are done, by either part (4) or (5) of Lemma 4.1.

(2) The case $N \equiv 3 \bmod 4$, $N \geqslant 15$.
The proof is similar, using part (2) of Theorem 7.1 and Theorem 9.2. We have $4n+2r+1 = N$, $m = 2n$, $n \geqslant 2$, $r \geqslant 3$ and $L = |E(G_{4n,2r+1})| + |E(K_{2r+1})| \geqslant 12a + 15m + 21$. In the case $N = 15$ we also use the fact that $a = 2$ in Theorem 9.2 to avoid getting a packing of the last type. The result follows whenever $L - 4n_4 \geqslant 15m + 3 = 30n + 3$. By Lemma 4.2 we may assume $n_4 < \frac{1}{2}(N-3)$, so $4n_4 \leqslant 2N - 10$. It can be checked that $\binom{N}{2} - (2N - 10) \geqslant 30n + 3$ for $N \geqslant 15$, so the result follows. □

10. Packing cycles into linked K_5s

In this section we prove Theorem 1.1 for $N \equiv 1 \bmod 4$. The idea is the same as in the previous section, but we also need to be able to pack (almost) arbitrary cycles into linked K_5s.

Lemma 10.1. *Assume $a \geqslant 2$, $m_i \geqslant 3$ and $\sum m_i - \max(3n_3 + 6n_6 + 9n_9, 7n_7) - 4n_4 - 8n_8 \geqslant 10a + 3$. Then we can pack some subset of the cycles exactly into a graph of the form*

$$K_5^{\cdot a} \qquad \text{or} \qquad K_5^{\cdot a}.C_m \qquad \text{or} \qquad C_5.K_5^{\cdot a}.C_m,$$

where the K_5s are all linked at a common vertex v and the extra cycles are linked to distinct K_5s at single vertices which may be chosen arbitrarily (either or both possibly equal to v).

Proof. We first prove the result with the extra cycles linked at v. Since Theorem 1.1 holds for $N = 5$, we can pack any collection of cycles of total length 10 into a K_5. If we run out of K_5s we are done. We can pack $C_3 \cup C_7$ into K_5 so that the C_3 meets v. Since all the K_5s are linked at v, if we remove C_7s from the K_5s we get triangles linked at a common vertex v. Therefore we can pack C_6s and C_9s with C_7s into K_5s rather than just C_3s. Since we use C_6s and C_9s to pack the triangles, we may need to link an extra C_3 or C_6 to v to pack the last C_6 or C_9 if we run out of C_7s or K_5s. We may also link C_5 to v to pack any remaining C_5 (we can assume there is no more than one C_5 since two C_5s will pack into K_5). Assume we have not used all the K_5s but we have packed a graph of

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one of the forms $K_5^{.b}$, $K_5^{.b}.C_m$ or $C_5.K_5^{.b}.C_m$ with $0 \leqslant b < a$ and the cycles linked at v. Pack cycles C_n with $n \geqslant 11$ inductively as follows:

$$\begin{aligned}
&\text{(A)} && K_5^{.b} \cup C_n \mapsto K_5^{.b}.C_n, && \\
&\text{(B)} && K_5^{.b}.C_m \mapsto C_5.K_5^{.b}.C_{m-5}, && \text{if } m \geqslant 8,\\
&\text{(C)} && K_5^{.b}.C_m \cup C_n \mapsto K_5^{.b+1}.C_{n+m-10}, && \text{if } m \leqslant 7,\\
&\text{(D)} && C_5.K_5^{.b}.C_m \mapsto K_5^{.b+1}.C_{m-5}, && \text{if } m \geqslant 8,\\
&\text{(E)} && C_5.K_5^{.b}.C_m \cup C_n \mapsto C_5.K_5^{.b+1}.C_{n+m-10}, && \text{if } m \leqslant 7.
\end{aligned}$$

In each case, we link the cycles at the vertex v. In (B) and (D) we have split C_m as $C_5.C_{m-5}$ and packed the two C_5s in (D) into a K_5. In (C) and (E) we have split C_n as $C_{10-m}.C_{n+m-10}$ and packed the C_m and C_{10-m} into K_5. In each of these cases, we can ensure both packed cycles meet v. Note that (A) is used at most once at the beginning of the process and we can assume that we finish with one of (C), (D) or (E). The operations above can pack all the K_5s provided the sum of the lengths of the cycles that can be packed is at least $10a + 3$. We now prove that we shall not run out of such cycles. If $n_3 + 2n_6 + 3n_9 \geqslant n_7$, then we can assume that we have run out of C_7s in the first part of this proof. The only remaining cycles that we cannot pack into K_5s are C_ns with $n = 3, 6, 9, 4, 8$. Similarly, if $n_3 + 2n_6 + 3n_9 \leqslant n_7$, then we can assume we have run out of C_3s, C_6s and C_9s. The only cycles that we cannot pack into K_5s are C_ns with $n = 7, 4, 8$. In both cases, the total length of the others is at least $10a + 3$, and we are done.

We now deal with the requirement that we may have to link the cycles at different vertices. If we have just one cycle C_m linked to $K_5^{.a}$, then it will be either one of the original cycles (in which case we can link it anywhere) or part of one of the original cycles which is split as $C_n.C_m$ with the C_n packed inside $K_5^{.a}$. In this case, we can link C_m to any of the vertices of the image of C_n in $K_5^{.a}$. In particular, C_m can be linked to some vertex other than v. By permuting the K_5s and the vertices of $K_5 \setminus \{v\}$, we can therefore link C_m to any desired vertex. The same arguments apply when we have two cycles C_5 and C_m, except that in addition we have to ensure that, if both are part of larger cycles, then the other parts of these two larger cycles are packed into $K_5^{.a}$ in such a way that they meet at least two distinct K_5s at vertices other than v. For this to fail, we must have finished our packing above with operation (E) in which the C_5 on the left-hand side is not part of a cycle partially packed into the K_5s. (Recall that we never finish with operation (B).) If the C_5 was an original cycle C_{m_i}, then we can link C_5 anywhere and we are done. We may therefore assume that the C_5 and C_m of (E) came from splitting a C_{5+m} which was one of the original cycles C_{m_i}. This means that we used operation (B) with the C_m on the left-hand side equal to an original C_{m_i}. This only happens if we previously used (A), and so all but one of the K_5s were packed exactly at the beginning of this proof. We may assume all the remaining cycles are of lengths 11 or 12 (any cycle of length at least 13 could be packed using (A), (B) and (D)) and there are at least two of them (to get a total length of at least $10a + 3$). Hence we may assume that $K_5^{.a-1}$ has been packed exactly and there are two remaining cycles C_{m_1} and C_{m_2} with $11 \leqslant m_1, m_2 \leqslant 12$. Since $a \geqslant 2$, there is at least one K_5 that has been packed already.

If we have packed some K_5 with a C_{10}, then we can unpack this K_5 and pack the C_{10}

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and C_{m_1} into $C_5.K_5.C_{m_1-5}$ as $(C_5.C_5).(C_5.C_{m_1-5}) \mapsto C_5.K_5.C_{m_1-5}$. Now proceed as above and use C_{m_2} and operation (E) to pack the remaining K_5. This time the two linked cycles are part of larger cycles meeting two K_5s between them, and we are done.

If no K_5 was packed with a C_{10}, pick one packed K_5 and remove all except one of the original cycles, C_{m_3} say, that was packed entirely within it. If we choose m_3 to be minimal then we can assume $m_3 \leqslant 5$. Pack C_{m_3} and a length $10 - m_3$ of C_{m_1} into this K_5. We have now packed $K_5^{a-1}.C_{m_1+m_3-10}$ with $4 \leqslant m_1 + m_3 - 10 \leqslant 7$, and have C_{m_2} left to pack. We proceed by packing C_{m_2} using operation (C). This time we have only one cycle linked to K_5^a, and we are done. □

The reason we need to link the cycles in Lemma 10.1 to arbitrary vertices is because we shall pack the linked cycles C_5 and C_m into the trail of octahedra and hence may wish to link them to K_5^a at vertices in M. We need to do this for two reasons. One is that we cannot have two cycles linked at the same vertex v of R in Lemma 4.1. The other is that we have no control over the size of C_m. Indeed, C_m may be so large that there are not enough remaining cycles to pack the trail of octahedra completely.

Theorem 10.2. *Assume the Induction Hypothesis with $N > 12$, $N \equiv 1 \bmod 4$. Then Theorem 1.1 holds for N.*

Proof. We use the construction in Theorem 7.1 to pack $K_5^n \cup O^{\cdot a}$ or $K_5^n \cup O \cup O^{\cdot a-1}$ into $G_{4n,2r+1}$ where $4n + 2r + 1 = N$, $n \geqslant 2$ and $2r + 1 \geqslant 5$. By Lemma 4.2 we can assume $4n_4 + 8n_8 \leqslant 2N - 10$, and by Lemma 4.3 we can assume $\max(3n_3 + 6n_6 + 9n_9, 7n_7) \leqslant \frac{7}{2}(N-3)$. It can be checked that $\binom{N}{2} - \frac{7}{2}(N-3) - (2N-10) \geqslant 10n + 3$, so we can use Lemma 10.1 to pack some cycles into K_5^n with at most two extra cycles C_{l_i} attached. Since $\binom{N}{2} - 10n = |E(G_{4n,2r+1})| - 10n + |E(K_{2r+1})| \geqslant 12a + 10$, we can use Theorem 9.1 to pack some subset of the C_{l_i} and the remaining C_{m_i} into $O^{\cdot a}$, $O^{\cdot a}.P_{c,d}$ or $O^{\cdot a-1}.P_{2,2} \cup O.T$. This last form occurs only when there are at least seven C_5s, so by the algorithm of Lemma 10.1 we can assume there are no C_{l_i} attached to K_5^n in this case (we use up all except at most one of the C_5s before linking any cycles to K_5^n). If there are no C_{l_i}, we are done, by the same proof as in Theorem 9.3. Assume now that there is precisely one cycle $C_m = C_{l_1}$ linked to K_5^n. If this cycle was packed in the $O^{\cdot a}$, it must have at least one vertex in M when packed into $G_{4n,2r+1}$. It therefore meets some K_5, and so we can assume that this cycle is linked to the correct vertex by Lemma 10.1. (Recall from the proof of Theorem 7.1 that every vertex of M meets precisely one K_5.) If the cycle was not packed, link it at v. We now have at most one cycle linked at v and possibly one cycle $P_{c,d}$ linked at two other vertices of R. The result now follows using part (1) or (4) of Lemma 4.1 with $k \leqslant 2$.

Now assume there are two cycles C_5 and C_m which must be linked to K_5^n. If precisely one of C_5 and C_m is not packed into the trail of octahedra, we link it to v and link the other to a vertex in M as above. If both are packed and the C_5 has an edge in M, then this C_5 must meet two K_5s, and so we can link the C_m and C_5 to distinct K_5s via vertices in M. If C_5 does not have an edge in M, then it must have an even number of edges in $G_{4n,2r+1}$ since this graph has no edges in R. Therefore the C_5 must be partially packed with a path of odd length leaving the O-trail. We can assume this path is of length 3, since the only

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case where we have a pentagon with one edge in R is when we are packing three of them as $O.T$. In this case, permute the C_5s and assume our C_5 is entirely within the O. In the case when the C_5 has a path of length 3 in R, we can permute the vertices of R in any final packing so that this path meets v, and we are done. (We can permute all the vertices of R except the two link vertices of the cycle $P_{c,d}$ that may be linked to the trail of octahedra.)

Finally, we need to deal with the case when neither C_5 nor C_m is packed at all in O^a. If $r = 2$, then these cycles can be packed together with any remaining cycles exactly into $K_R = K_5$. Hence $m = 5$ and both C_5 and C_m meet v. In all other cases $r \geqslant 4$ and $|E(K_R)| \geqslant |E(K_9)| = 36$. Therefore, when using Theorem 9.1, we have $\sum m_i \geqslant 12a + 36$. Looking at the proof of Theorem 9.1, we see that, in general, we can discard C_3s, C_4s and C_6s so as to make the algorithm pack a C_5 as L_5 before packing the longer cycles. This is because the maximum length of discarded cycles will be at most 17 and $12a + 36 - 17 \geqslant 12a + 9$. So, if our C_5 could not be packed, we must have already packed all the Os exactly with combinations of cycles with lengths totalling 12 at the beginning of the proof of Theorem 9.1. Pick the last O packed. We can assume it is not packed with any cycle of length 5, m, 7, $12 - m$ or $7 - m$, since in these cases we can either swap C_5 or C_m with this packed cycle or swap the other cycles packed into this O with C_5 and/or C_m. We can also assume it is not packed with any cycle C_r with $r \geqslant 8$, since we could replace the C_r with C_{r-5} and our C_5 and get a packing of $O.C_5' \mapsto O.L_5$ with our C_5 inside the O. Hence, without loss of generality, this O is packed as $C_3 \cup C_3 \cup C_3 \cup C_3$, $C_3 \cup C_3 \cup C_6$, $C_6 \cup C_6$ or $C_4 \cup C_4 \cup C_4$, and C_m is of length $m \geqslant 5$ in all except the $C_6 \cup C_6$ case. We now use the following packings in which the C_5 is used to pack this last O:

$$C_3 \cup C_3 \cup C_5 \cup C_m \mapsto C_3' \cup C_3'.P_{3,3}.P_{2,m-3} \mapsto O.P_{2,m-3},$$
$$C_6 \cup C_5 \cup C_6 \mapsto L_6 \cup P_{3,3}.P_{2,3} \mapsto O.P_{2,3},$$
$$C_4 \cup C_4 \cup C_5 \cup C_m \mapsto P_{2,2,2,2} \cup P_{2,2}.P_{3,m-2} \mapsto O.P_{3,m-2}.$$

We can now assume the C_5 is packed within the octahedra, and so we are done. □

11. Packing cycles into linked Ws

In this section we complete the proof of Theorem 1.1 by handling the cases $N \equiv 0 \bmod 4$. In this case we shall pack cycles into linked Ws where $W = G_{4,4} = C_4 + E_4$ and the vertices of the E_4 are both the initial and final links of W (in the same order). Let these link vertices of W be (a, b, c, d). We shall use notation such as $[l_1^a l_2^b l_3^{bc} \cdots]$ to denote the vertices $\{a, b, c, d\}$ and a collection of cycles C_{l_i} with with C_{l_1} meeting a, C_{l_2} meeting b, C_{l_3} meeting both b and c, *etc*. The initial and final links of this graph will be (a, b, c, d). If a cycle meets more than one vertex, this notation can be ambiguous so, to be precise, say that the vertices occur at intervals two apart around the cycle. Define $\mathscr{D}$ to be the family of graphs

$$[\,], \quad [l_1^u], \quad [l_1^u l_2^v] \text{ with } l_2 \leqslant 5, \quad [3^u 3^v 3^w], \quad [4^u 4^v 4^w] \quad \text{or} \quad [4^a 4^b 4^c 4^d],$$

where u, v and w are distinct elements of $\{a, b, c, d\}$.

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Lemma 11.1. *We can pack any of the following exactly into W with link matching:*

$$[l_1^a l_2^b l_3^c l_4^d],\quad [l_1^a l_2^b l_3^c],\quad [l_1^a l_2^b],\quad [l_1^a] \qquad \text{with } \sum l_i = 20,$$
$$[3^a3^b3^c8^d3^a],\quad [3^a3^b3^c6^d5^a],\quad [4^a4^b4^c4^d4^a] \quad \text{and} \quad [4^a4^b4^c5^d3^a].$$

Proof. Write the non-link vertices of $W = C_4 + E_4$ as $C_4 = C(1,2,3,4)$. We have the following packings into W:

(A)	$[3^a3^a\underline{3}^b\underline{3}^c4^{bd}4^{cd}]$	$C(a,1,2)$, $C(a,3,4)$, $\underline{C(b,2,3)}$, $\underline{C(c,1,4)}$, $C(b,1,d,4)$, $C(c,2,d,3)$
(B)	$[\underline{3}^a3^a\underline{4}^{bd}5^{bc}5^{cd}]$	$\underline{C(a,1,2)}$, $C(a,3,4)$, $\underline{C(b,3,d,4)}$, $C(b,1,c,3,2)$, $C(c,2,d,1,4)$
(C)	$[5^{ab}5^{bd}5^{ac}5^{cd}]$	$C(a,3,b,2,1)$, $C(b,1,d,3,4)$, $C(a,4,c,3,2)$, $C(c,2,d,4,1)$
(D)	$[\underline{3}^a3^b\underline{3}^c5^{ad}6^{bcd}]$	$\underline{C(a,1,2)}$, $C(b,4,1)$, $\underline{C(c,3,4)}$, $C(a,4,d,2,3)$, $C(b,2,c,1,d,3)$
(E)	$[4^{ad}4^{bc}4^{bc}5^d3^a]$	$C(a,2,d,3)$, $C(b,1,c,3)$, $C(b,2,c,4)$, $C(d,1,2,3,4)$, $C(a,1,4)$
(F)	$[4^a4^b4^{cd}4^{cd}4^{ab}]$	$C(a,1,2,3)$, $C(b,3,4,1)$, $C(c,1,d,2)$, $C(c,3,d,4)$, $C(a,2,b,4)$.

In each of these packings, every cycle meets every other, except in (A), (B) and (D), where the two underlined cycles do not meet. We now combine cycles to give some of the packings in the statement of the lemma. In each case we indicate one of the above packings that can be used.

$[3^a\underline{3}^b\underline{3}^c11^d]$	(A)	$[3^a3^b7^c7^d]$	(A)	$[3^a5^b5^c7^d]$	(B)	$[4^c4^d6^a6^b]$	(A)
$[3^a3^b4^c10^d]$	(A)	$[3^b4^c4^d9^a]$	(A)	$[3^b5^a6^c6^d]$	(D)	$[4^d5^b5^c6^a]$	(B)
$[3^b3^c5^d9^a]$	(D)	$[3^a4^b5^d8^c]$	(E)	$[4^a4^b4^c8^d]$	(E)	$[5^a5^b5^c5^d]$	(C)
$[3^a3^b6^c8^d]$	(A)	$[3^a4^b6^c7^d]$	(A)	$[4^a4^b5^d7^c]$	(E)	$[3^a\underline{3}^b\underline{3}^c8^d3^a]$	(A)

In all the cases in the table above, every cycle meets every other one in the packing except for the underlined ones in the first and last cases. Together with (D), (E) and (F), these give all the cases in the statement of the lemma with at least four cycles. (We can clearly permute the link vertices by symmetry.) If we have fewer than four cycles, one of these cycles must have length at least 6, and we can split it into two smaller cycles. These cases therefore follow by combining cycles in the above table that are connected. We can always avoid combining underlined cycles by using different triangles. □

Theorem 11.2. *If $\sum m_i \geqslant 20n + 29$ and $n_3 = n_6 = 0$, then there is an exact packing of some subset of the cycles into a graph $W^{\cdot n}.D$ for some $D \in \mathscr{D}$.*

Proof. By packing four C_5s into a W (using the $[5^a5^b5^c5^d]$ packing of Lemma 11.1), we can assume by induction on n that $n_5 \leqslant 3$. If $n_5 \geqslant 2$ and $n_7 \geqslant 2$, we can pack two C_7s and two C_5s as $[7^a5^b5^c7^d] \mapsto [3^a5^b5^c7^d].[4^a] \mapsto W.[4^a]$. Pack C_4s by adding them to each link vertex in turn to get $[4^a]$, $[4^a4^b]$, $[4^a4^b4^c]$ and $[4^a4^b4^c4^d]$. If we have an extra C_4 we can pack this into a W using $[4^a4^b4^c4^d4^a] \mapsto W$. Continue until we run out of C_4s. We can now assume we have packed $W^{\cdot r}.D$ with $r < n$ and $D = [], [4^a], [4^a4^b], [4^a4^b4^c]$ or

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$[4^a4^b4^c4^d]$. Now pack C_7s inductively using the following packings:

$$\begin{array}{lcll}
[\,].[7^a7^b7^c7^d] & \mapsto & [3^a3^b7^c7^d].[4^a4^b] & \mapsto & W.[4^a4^b] \\
[4^a].[7^a7^b7^c7^d] & \mapsto & [3^b3^c7^a7^d].[4^a4^b4^c] & \mapsto & W.[4^a4^b4^c] \\
[4^a4^b].[7^a7^c7^d] & \mapsto & [4^a4^b4^c4^d4^a].[3^a3^c3^d] & \mapsto & W.[3^a3^c3^d] \\
[4^a4^b4^c].[7^a7^d] & \mapsto & [4^a4^b4^c4^d4^a].[3^a3^d] & \mapsto & W.[3^a3^d] \\
[4^a4^b4^c4^d].[7^a] & \mapsto & [4^a4^b4^c4^d4^a].[3^a] & \mapsto & W.[3^a] \\
[3^a].[7^b7^c7^d] & \mapsto & [3^a3^b7^c7^d].[4^b] & \mapsto & W.[4^b] \\
[3^a3^d].[7^b7^c] & \mapsto & [3^a3^d7^b7^c] & \mapsto & W \\
[3^a3^c3^d].[7^a7^b] & \mapsto & [3^c3^d7^a7^b].[3^a] & \mapsto & W.[3^a].
\end{array}$$

(We may need to permute the link vertices at some points.) Now discard any remaining C_5s and C_7s. The maximum length of discarded cycles is $\max(5+3\times 5, 3\times 5+7) = 26$. Hence, we can assume that the total length of cycles is at least $20n+3$, all the remaining cycles are of length at least eight and we have already packed some $W^{.r}.D$ with $r < n$ and $D \in \mathscr{D}$. We now proceed inductively in a manner similar to before, using Lemma 11.1. At each stage add cycles to any unused link vertices until the total length to be packed is at least 23, or all four link vertices have been used.

Suppose first that we have cycles of total length $s \geqslant 23$ linked to the link vertices. If the longest cycle is l_1 and $s - l_1 \leqslant 17$, then we can split C_{l_1} as $C_{l_1+20-s}.C_{s-20}$ (both linked to the same link vertex v as C_{l_1}). Now pack the C_{l_1+20-s} and the remaining cycles into W using Lemma 11.1. We now have a packing of $W^{.r+1}.[(s-20)^v] = W^{.r+1}.D'$, where $D' \in \mathscr{D}$. If $s - l_1 \geqslant 18$, then there must be another linked cycle, C_{l_2} say, with $l_2 \geqslant 6$ (since there are at most three remaining linked cycles). Also $l_1 \geqslant 8$ (since it must be one of the original C_{m_i}). If $s - l_1 \leqslant 20$, split C_{l_2} as $C_{l_2-3}.C_3$ and split C_{l_1} as $C_{l_1+23-s}.C_{s-23}$. This time use Lemma 11.1 to obtain a packing into $W^{.r+1}.[(s-23)^v3^u] = W^{.r+1}.D'$. If $s - l_1 = 21$ then we may assume $l_2 \geqslant 7$, and if $s - l_2 = 22$ we may assume $l_2 \geqslant 8$. We can treat these similarly and get packings into $W^{.r+1}.[(s-24)^v4^u]$ and $W^{.r+1}.[(s-25)^v5^u]$, respectively. Hence, in all cases we get a packing into $W^{.r+1}.D'$ for some $D' \in \mathscr{D}$.

The remaining cases are when we run out of link vertices. For this to happen, we must have cycles attached as $[3^a3^b8^c8^d]$, $[3^a3^b3^cl^d]$ with $8 \leqslant l \leqslant 13$ or $[4^a4^b4^cl^d]$ with $l = 4$, 9 or 10. We must also have another cycle C_{m_1}, say, with $m_1 \geqslant 8$. Now use the following packings:

$$\begin{array}{lcll}
[3^a3^b8^c8^d].[m_1^a] & \mapsto & [3^a3^b3^c8^d3^a].[(m_1-3)^a5^c] & \\
[3^a3^b3^cl^d].[m_1^a] & \mapsto & [3^a3^b3^c8^d3^a].[(m_1-3)^a(l-8)^d], & l = 8, 12, 13 \\
[3^a3^b3^cl^d].[m_1^a] & \mapsto & [3^a3^b3^c6^d5^a].[(m_1-5)^a(l-6)^d], & l = 9, 10, 11 \\
[4^a4^b4^c4^d].[m_1^a] & \mapsto & [4^a4^b4^c4^d4^a].[(m_1-4)^a] & \\
[4^a4^b4^cl^d].[m_1^a] & \mapsto & [4^a4^b4^c5^d3^a].[(m_1-3)^a(l-5)^d], & l = 9, 10.
\end{array}$$

In each case we obtain a packing into some graph of the form $W^{.r+1}.D'$, $D' \in \mathscr{D}$, using one of the packings (A)–(F) from the proof of Lemma 11.1.

Hence, by induction we can pack $W^{.n}.D$ for some $D \in \mathscr{D}$. □

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Theorem 11.3. *Assume the Induction Hypothesis with $N > 12$, $N \equiv 0 \bmod 4$. Then Theorem 1.1 holds whenever $n_3 + 2n_6 \leqslant T_W$. Here $T_W = \lfloor (\binom{N}{2} - \frac{N}{2} - 20n - 31)/3 \rfloor$, and n is given in part* (4) *of Theorem 7.1.*

Proof. We use the construction in Theorem 7.1 to pack $W^{\cdot n} \cup O^{\cup a-b} \cup O^{\cdot b}$ into $G_{4n,2r+4}$ where $4n + 2r + 4 = N$, $2r + 4 \geqslant 8$ and $r = r(\frac{1}{2}(N-4))$. Since $L \geqslant \binom{N}{2} - \frac{N}{2} - 2$ and $n_3 + 2n_6 \leqslant T_W$, we have $L - 3n_3 - 6n_6 \geqslant 20n + 29$. Hence we can use Theorem 11.2 to pack some cycles into $W^{\cdot n}$ with at most four extra cycles C_{l_i} attached (in the form of some $D \in \mathscr{D}$). Let the largest such cycle be C_{l_1} (if such cycles exist). From the definition of $\mathscr{D}$ we see that the remaining extra cycles have total length at most 12. Since $L - 20n - 12 \geqslant 12a + |E(K'_{2r+4})| - 14 \geqslant 12a + 10$, we can use Theorem 9.1 to pack some subset of C_{l_1} and the remaining C_{m_i} into $O^{\cdot a}$, $O^{\cdot a}.P_{c,d}$ or $O^{\cdot a-1}.P_{2,2} \cup O.T$. From the algorithm of Theorem 11.2 we can assume this last form occurs only when there are no C_{l_i} (since all the Ws would be packed with C_5s).

If there are no extra cycles C_{l_i} then we are done, by the proof of Theorem 9.3. If C_{l_1} is not packed into the trail of octahedra, then we are done, by part (1) or (4) of Lemma 4.1 (depending on whether we have packed $O^{\cdot a}$ or $O^{\cdot a}.P_{c,d}$). Now assume that C_{l_1} has been packed into the octahedra. When this trail is packed into $G_{4n,2r+4}$ the cycle C_{l_1} will meet M (since there are no edges in R). If the C_{l_1} was part of the larger original cycle C_{m_1}, say, then the other part, C_k say, meets some non-link vertex of some W when C_{m_1} is packed in $W^{\cdot n}.(\cup C_{l_i})$. Every vertex of M meets some W when $W^{\cdot n}$ is packed into $G_{4n,2r+4}$. Hence, by permuting the Ws and cyclically permuting the non-link vertices of some W, we can assume that the image of C_{l_1} in the trail of octahedra meets the image of C_k in $W^{\cdot n}$ when everything is packed into $G_{4n,2r+4}$. Hence we have a packing of some subset of cycles into $G_{4n,2r+4}.(\cup C_{l'_i})$ satisfying one of the conditions of Lemma 4.1. The result now follows. □

We have now proved Theorem 1.1 for $N \equiv 0 \bmod 4$ when there are not too many cycles of lengths 3 or 6. However, if we have many cycles of lengths 3 or 6 we will need to use other packings. For large N, the packings of Section 6 are sufficient. However, there are several smaller N for which this is not sufficient. For these cases we need to give special packings. These special packings are given by the next two lemmas.

Lemma 11.4. *For all $n \geqslant 1$, there exist exact packings*

$$\begin{aligned} K'_{4n} \cup K'_{4n} \cup O^{\cdot 4n^2} &\mapsto G_{8n,4n}, \\ G_{4n,4}.G_{4n,4} \cup O^{\cdot 4n^2} &\mapsto G_{8n,4n+4}, \\ K'_{4n+4} \cup K'_{4n+4} \cup O^{\cup 2n+1} \cup O^{\cdot 2n(2n+1)} &\mapsto G_{8n+4,4n+4}. \end{aligned}$$

In each case, the initial and final links of the trail of Os are disjoint pairs of vertices of R (of the image). Also, the Rs of the $G_{4n,4}$s of the second packing are linked together, and their image is a subset of the R of $G_{8n,4n+4}$ which is disjoint from the image of any O.

Proof. We shall first prove that triangles can be packed into $G = K_{n,n,n}$ with a suitably long trail. Write $G = E_A + E_B + E_C$, where $A = \{a_i : i \in \mathbb{Z}/n\mathbb{Z}\}$, $B = \{b_i : i \in \mathbb{Z}/n\mathbb{Z}\}$

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and $C = \{c_i : i \in \mathbb{Z}/n\mathbb{Z}\}$. The existence of a packing of G with triangles follows from the existence of $n \times n$ latin squares. If the cells (i, j) contain entries k_{ij}, then the corresponding triangles are given by $(a_i, b_j, c_{k_{ij}})$. Alternatively, a packing can be constructed from first principles as follows.

If n is odd and for $i, j \in \mathbb{Z}/n\mathbb{Z}$, let T_{ij} be the triangle $C(a_i, b_{i+j}, c_{i+2j})$. It is an easy exercise to show that the T_{ij} pack G exactly. If n is even, use induction to pack $K_{n/2,n/2,n/2}$ with triangles and double up the vertices. The result is a packing of $K_{n,n,n}$ with octahedra. Since each octahedron can be packed with four triangles, we have exact packings of $K_{n,n,n}$ with triangles for all n.

We now find a long trail of triangles. Let $G[A, B] = K_{n,n}$ be the subgraph induced by the vertices of $A \cup B$. Each edge in this subgraph is contained in a unique triangle in the packing of G. There exists an Eulerian circuit of $K_{n,n}$ when n is even, and one of $K_{n,n}$ with a 1-factor removed when n is odd, $n \geqslant 1$. Hence we have a circuit of length n^2 (n even) or $n(n-1)$ (n odd, $n > 1$) in this subgraph. By replacing the edges with triangles we get a circuit of triangles in G. By breaking this circuit at some vertex $v \in A$, say, we get a trail from some vertex of C to another vertex of C. (Each triangle contains one vertex from each of A, B and C, and if two triangles both meet v then they must meet distinct vertices in C.)

Doubling up vertices, we get a packing of $O^{.n^2}$ or $O^{\cup n} \cup O^{.n(n-1)}$ into $K_{2n,2n,2n}$ with the trail of linked Os starting and ending at disjoint pairs of vertices in C.

Now let A, B and C have $2n$ vertices, and let S have 0, 2 or 4 vertices. We can decompose $G_{A\cup B,S\cup C} = K'_{A\cup B} + E_{S\cup C}$ as an edge-disjoint union of $K_{A,B,C}$, $G_{A,S} = K'_A + E_S$ and $G_{B,S} = K'_B + E_S$. The result follows in the first two cases by replacing n by $2n$ and setting $|S| = 0$ or $|S| = 4$. For $|S| = 0$ note that $G_{4n,0} = K'_{4n}$. The last case follows by replacing n with $2n + 1$ and setting $|S| = 2$, noting that $G_{4n+2,2} = K'_{4n+4}$. □

Lemma 11.5. *We can pack $G_{24,20}$ exactly with* 124 *pairs of linked triangles.*

Proof. Write $M = A \cup B \cup C$, where $A = \{a_i : 0 \leqslant i \leqslant 7\}$, $B = \{b_i : 0 \leqslant i \leqslant 7\}$ and $C = \{c_i : 0 \leqslant i \leqslant 7\}$ are disjoint sets of vertices of size 8. Decompose K_A into seven 1-factors and combine each 1-factor with a set of edges $I_j = \{b_i c_{i+j} : 0 \leqslant i \leqslant 7\}$ with $1 \leqslant j \leqslant 7$ to get seven 1-factors of $K_{A\cup B\cup C} = K_M$. Cyclically permuting A, B and C now gives a total of 21 such 1-factors. The remaining edges of K_M form 8 triangles $T_i = C(a_i, b_i, c_i)$. Discard one of the 1-factors and join the others, one to each of the 20 vertices of R. This gives a decomposition of $G_{24,20} = K'_M + E_R$ into 20 sets of 12 triangles linked at a vertex of R and 8 disjoint triangles T_i. Form 8 pairs of linked triangles by taking T_i and pairing it with the triangle containing the edge $b_i c_{i+1}$. All these triangles meet the same vertex $v \in R$. There are four remaining triangles meeting v which we can pair up to form two pairs of linked triangles. For each of the other vertices in R, we have 12 linked triangles which can be paired up to form six pairs of linked triangles. We have now paired up all the triangles into 124 pairs which pack the whole of $G_{24,20}$ exactly. □

Finally, we give the proof of Theorem 1.1 for $N \equiv 0 \bmod 4$.

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Table 3 Packings of K'_N used in Theorem 11.6

N	T_W	T_S	T_X	Isolated components	O-trail
16	13	36	–	$G_{4,4}.G_{4,4} = W^{\cdot 2}$	$O^{\cdot 4}$
20	29	56	28	$K'_8 \cup K'_8 \cup O^{\cup 3}$	$O^{\cdot 6}$
24	57	80	16	$K'_8 \cup K'_8$	$O^{\cdot 16}$
28	77	108	–	$G_{8,4}.G_{8,4}$	$O^{\cdot 16}$
32	123	140	60	$K'_{12} \cup K'_{12} \cup O^{\cup 5}$	$O^{\cdot 20}$
36	153	176	40	$K'_{12} \cup K'_{12}$	$O^{\cdot 36}$
40	209	216	72	$G_{12,4}.G_{12,4}$	$O^{\cdot 36}$
44	251	260	248	$G_{24,20}$	–

Theorem 11.6. *Assume the Induction Hypothesis with $N > 12$, $N \equiv 0 \bmod 4$. Then Theorem 1.1 holds for N.*

Proof. If $n_3 + 2n_6 \geqslant T_S$ where $T_S = \frac{1}{3}|E(G_{2n,r})|$, $r = r(N) = \frac{1}{2}(N-8)$ and $2n+r = N$, then we can pack the C_3s and C_6s into the trail of T_S triangles in $G_{2n,r}$ given by Theorem 6.2. This trail ends at a vertex of R, so by linking a C_3 if necessary at this vertex we can pack a set of C_3s and C_6s exactly. The result will then follow from part (1) of Lemma 4.1 with $k \leqslant 1$.

Also, if $n_3 + 2n_6 \leqslant T_W$ where T_W is given by Theorem 11.3, then we are done, by Theorem 11.3. Since $T_W \geqslant T_S$ for all $N \geqslant 48$, the proof of the theorem is practically complete; all we have to do is to check the cases $N < 48$.

In most of these remaining cases we use the packings of Lemma 11.4. Pack C_3s and C_6s into the 'isolated components' in Lemma 11.4. These isolated components are listed in Table 3. Isolated components $O = G_{4,2}$, $K'_8 = G_{6,2}$, $K'_{12} = G_{10,2}$ and $G_{12,4}$ can all be packed with trails of triangles by Theorem 6.2 with $N = 6$, 8, 12 and 14, respectively. In all these cases the number of triangles is even, so we can pack any C_6s exactly (pack all the C_6s before packing the C_3s). For $N = 44$, pack $G_{24,20}$ with pairs of linked triangles using Lemma 11.5. These can also pack both C_3s and C_6s, and if we have enough we are done, by induction from $N = 20$ ($G_{24,20}.K_{20} \subseteq K_{44}$). The total number of triangles T_X needed to pack the isolated components is listed in Table 3. By Theorem 11.3 we can assume that $n_3 + 2n_6 > T_W$. In all cases where we can pack the isolated components exactly with triangles, $n_3 + 2n_6 > T_W \geqslant T_X$, so we have enough C_3s and C_6s to pack these exactly. Now pack the trails of octahedra with the remaining cycles and use Lemma 4.1 as in part (1) of the proof of Theorem 9.3. Note that the O-trails in Lemma 11.4 have initial and final links packed into *disjoint* pairs of vertices in R, so packing $O.T \cup O^{\cdot a-1}.P_{2,2}$ does not cause problems if we use the first O from the trail $O^{\cdot a}$ for the $O.T$.

The only remaining cases now are $N = 16$ and $N = 28$.

(1) The case $N = 16$.

Writing the cycle lengths as sums of 3s, 4s and 5s, as described before Lemma 5.3, we can assume $2s_T + p_T \geqslant 8$ by Lemma 5.3. Pick a minimal set of cycles which involve s 4s and

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p 5s with $2s + p \geqslant 8$. From the above table we can assume $n_3 + 2n_6 > T_W = 13$. Consider the following cases separately.

(a) The case $0 \leqslant s \leqslant 2$.
We use the $W^{\cdot 2} \cup O^{\cdot 4} \mapsto G_{8,8}$ packing of Lemma 11.4. Pack s of the Ws with the $[\underline{3}^a 3^a \underline{4}^{bd} 5^{bc} 5^{cd}]$ packing (B) from the proof of Lemma 11.1. Pack the other $2 - s$ Ws with the $[5^a 5^b 5^c 5^d]$ packing of Lemma 11.1. All the squares in this packing are connected via b and all the triangles are connected via a. Also, each square is connected to a pentagon and a connected sequence of two triangles within its own W. Since each cycle is written with at most one 5 or two 3s, we can pack every cycle of our subset that is written with at least one 4 entirely into the triangles, squares and pentagons of this packing. Since $2s + p \geqslant 8$, there are enough remaining cycles written with 5s to cover the remaining pentagons in the packing. However, each of these cycles is precisely a pentagon (since a 5 cannot be written together with a 3 or another 5), so they pack entirely too. Since $n_3 + 2n_6 > 13$, there are enough C_3s and C_6s to pack any remaining triangles of the packing. At worst we may need to link a single C_3 to vertex a to pack some C_6. We therefore have an exact packing of some of our original cycles into $W^{\cdot 2}$ or $W^{\cdot 2}.C_3$.

(b) The case $s \geqslant 3$ and $p \geqslant 2$.
First assume $s = 3$ and $p = 2$. Pack one W using the $[\underline{3}^a 3^a \underline{4}^{bd} 5^{bc} 5^{cd}]$ packing (B) and the other W using the $[3^a 3^a \underline{3}^b \underline{3}^c 4^{bd} 4^{cd}]$ packing (A) from the proof of Lemma 11.1. Every square and pentagon is connected to every other square and pentagon in this packing. The triangles can be paired up into connected sequences of length 2 with one end meeting a and the other meeting a square. Hence we can pack our subset of cycles entirely into this packing with any remaining triangles all linked to a in disjoint connected sequences, each of length at most two. Packing the remaining triangles with C_3s and C_6s as in (a) allows us to exactly pack the whole of $W^{\cdot 2}$ or $W^{\cdot 2}.C_3$ with some of our original cycles.

Now assume that either $s > 3$ or $p > 2$. No cycle in the minimal subset can be a pentagon since we could remove it and still have $2s + p \geqslant 8$. Hence each cycle written with a 5 contributes at least three to the sum $2s + p$. By minimality, there cannot be more than three such cycles, so $p \leqslant 3$. If we remove one of the cycles written with a 5, then $s \leqslant 3$ and $p \leqslant 2$ for the remaining cycles. Hence we can split a cycle written with a 5 into two parts so that one part and the remaining cycles of the subset have $s = 3$ and $p = 2$. We can pack these as above and attach the other part to a suitable vertex in $\{b, c, d\}$ so as to connect it with the part that was packed. In this case we have packed $W^{\cdot 2}.(C_r)$ or $W^{\cdot 2}.(C_r \cup C_3)$ with the C_r linked at one of $\{b, c, d\}$ and the C_3 linked at a.

(c) The case $p \leqslant 1$.
Hence $s \geqslant 4$. First assume $s = 4$ and $p = 0$. Pack both Ws using $[3^a 3^a \underline{3}^b \underline{3}^c 4^{bd} 4^{cd}]$. All the squares are connected via d, so we can pack the cycles of our subset into this, and pack C_3s and C_6s into any remaining triangles as before. If $s > 4$ or $p > 0$, then we can split one of the cycles as in (b). At worst we will need to attach one cycle to one of $\{b, c, d\}$, and possibly a C_3 to a to pack the original cycles fully.

In all cases we have an exact packing of some of the original cycles into $W^{\cdot 2}.(\cup C_{l_i})$, where there are at most two cycles linked to distinct link vertices of $W^{\cdot 2}$, and if there

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are two such cycles then one of them is a C_3. Pack the longer one of these and the remaining cycles into the O-trail using Theorem 9.1. The total length of these cycles is at least $|E(K'_{16})|-2-40-3 = 12a+17$ with $a = 4$, so this succeeds. Now continue with the argument used in the proof of Theorem 11.3 to get the desired packing.

(2) The case $N = 28$.
From Table 3, we may assume $77 = T_W < n_3 + 2n_6 < T_S = 108$. First assume $n_5 \geqslant 2$. We can pack $G_{8,4}$ with a C_5 and a circular connected sequence of triangles by Lemma 5.2. Hence we can pack $G_{8,4}.G_{8,4}$ with two C_5s and a connected sequence of 34 triangles, which in turn can be packed exactly with C_3s and C_6s. We are now done, as before by packing the O-trail and using Lemma 4.1 as in Theorem 9.3. Assume now that $n_5 \leqslant 1$. From Lemma 5.2, we can pack $G_{8,4}.G_{8,4}$ with four squares and a trail of 32 triangles. We can make the four squares form a connected sequence meeting three link vertices, or two connected pairs meeting all four link vertices (by suitable choice of link vertices met by the pair of squares in each $G_{8,4}$). Pack the trail of triangles with C_3s and C_6s as above. Write the cycles of length 4 or $\geqslant 7$ as sums of 3s, 4s and 5s, as described before Lemma 5.3. Since each 4 can be written together with at most one 5 or two 3s, the total length of such cycles is at most $10s_T$. However, since $n_3 + 2n_6 < T_S$, the total length of cycles of lengths four or at least seven is at least $|E(K'_{28})|-2-5n_5-3(T_S-1) \geqslant 36$. Hence $s_T \geqslant 4$. Pack a minimal set of cycles written with at least four 4s into the packing $G_{8,4}.G_{8,4}$ described above. To do this we may need to attach up to four additional cycles C_{l_i} to the link R of $G_{8,4}.G_{8,4}$. By considering separately the cases when each of the packed cycles is written with one 4, or one is written with more than one 4, we can ensure that the linked cycles C_{l_i} meet *distinct* link vertices. At most one of these cycles C_{l_i} needs to contain a 4, so removing the longest of these gives a total length of additional cycles of at most $3 \times 6 = 18$. Pack the longest C_{l_i} and all remaining cycles into the O-trail using Theorem 9.1. This succeeds since these cycles have total length at least $12a + |E(K'_{12})| - 2 - 18 = 12a + 30$. Since there are at most two C_5s, the last form in Theorem 9.1 does not occur and we will only need to attach at most one cycle $P_{c,d}$ to the final link of this trail. We are now done, by the same argument as in Theorem 11.3. □

Theorem 1.1 now follows by induction on N using Lemmas 5.1 and 5.3 for the cases $N \leqslant 12$ and Theorems 9.3, 10.2 and 11.6 for the cases $N > 12$.

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