ON CANONICAL THREEFOLDS

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In a paper* which is to be published shortly formulae have been obtained for the virtual characters of the system of surfaces cut out on a primal of S_4, having ordinary singularities, by adjoint primals of any order; an account of the principal results has been given elsewhere†. As a first application of these formulae we propose in the present paper to investigate the canonical systems belonging to various threefolds and thence to construct the canonical models which correspond to them.

In § 2 we consider cases in which the canonical system is compounded of a pencil. In § 5 we give examples of threefolds whose canonical surfaces have maximum geometrical genus with respect to the linear genus, and as the extension of these results, which are due to Noether and Castelnuovo, we obtain in § 5·3 the theorem:

If the canonical system of a threefold is simple, irreducible and without base-points, the geometrical and linear genera P and Ω_1 satisfy the inequality $6P \leqslant \Omega_1 + 20$; *and threefolds exist for which the upper limit for P is attained.*

The threefolds with maximum geometrical genus are constructed according to a rule which is in strict analogy with that holding for surfaces and, in fact, for varieties of any dimension; the generalization for the latter is indicated in § 5·6.

The construction of canonical threefolds in general is carried out by methods which are a natural extension of principles due to Enriques; numerous examples of these methods are given in §§ 6–7. In cases where the canonical system is compounded of an involution we obtain representations of the threefolds on multiple spaces; thus, in § 5 we obtain a threefold having genus $P = 4$, whose canonical system is compounded of an involution I_2, and in §§ 5·1, 5·4 we give examples with genera $P = 6, 7$, whose canonical system is similarly compounded. In § 8 we obtain threefolds for which $P = 4$, whose canonical systems are compounded of involutions I_4 and I_6 respectively, and we illustrate by means of a simple example the fact that the canonical system may contain fixed surfaces which are transformable into base-points.

1. We recall that a general (algebraic) threefold is specified by seven additive projective characters‡ which, if the threefold is not situated in S_4, may be con-

* Roth (4). † Roth (5). ‡ Roth (6).

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veniently defined in terms of its projection V^n on S_4 from an external point. The characters of V^n are then the following:

The order n, rank a, first class m and class w;

The order j of the cuspidal curve; its class j_1 of immersion in V^n and its class j_0 of immersion in the double surface of V^n.

The multiple loci of V^n are a double surface B, a curve C which is triple on V^n and on B, and a finite number ξ of quadruple points of C which are quadruple on V^n and sextuple on B. The characters of these loci are expressible in terms of the seven fundamental characters by means of equations given in previous work*. Thus the order μ_0, rank μ_1, class μ_2 and *ceto* ν_2 of B are respectively

$$\left.\begin{aligned} \mu_0 &= \binom{n}{2} - \tfrac{1}{2}a, \quad \mu_1 = a(n-1) - \tfrac{5}{2}j - m, \\ \mu_2 &= \tfrac{1}{2}a^2 + mn + 2j - \tfrac{1}{2}(3m + 3w + 33j_1 - 11j_0), \\ \nu_2 &= \tfrac{1}{2}(a^2 - m - w - 13j_1) + 2j_0. \end{aligned}\right\} \tag{1}$$

The order t and rank r_t of C are

$$\left.\begin{aligned} t &= \binom{n}{3} - \tfrac{1}{2}an + \tfrac{1}{2}(2a + 2j + m), \\ r_t &= a\binom{n}{2} - \tfrac{1}{2}a^2 - n(a + \tfrac{5}{2}j + m) + a + 4j + 2m + w + 14j_1 - 4j_0, \end{aligned}\right\} \tag{2}$$

and the number of quadruple points on C is

$$\xi = \binom{n}{4} - \tfrac{1}{2}a\binom{n}{2} + \tfrac{1}{8}a^2 + \tfrac{1}{3}n(2a + 2j + m) - \tfrac{1}{4}(3a + 5j + 3m + w + 13j_1) + j_0. \tag{3}$$

1·1. The virtual canonical system $|K|$ of V^n is the system of surfaces, residual to B, cut out on V^n by adjoint primals of order $n-5$; such primals form a linear system whose virtual and effective dimensions are respectively P_a and P_g, the arithmetic and geometrical genera of V^n. In the usual notation, we denote by Ω_2 the arithmetic genus of a surface K, by Ω_1 the curvilinear genus and by Ω_0 the grade of the system $|K|$. Between these invariants there exist the classical relations†

$$2\Omega_1 - 2 = 3\Omega_0, \tag{4}$$

$$2P_a = \Omega_0 - \Omega_1 + \Omega_2 + 4. \tag{5}$$

Further, if ω is the arithmetic linear genus of K, then‡

$$\omega - 1 = 4\Omega_0. \tag{6}$$

In terms of the projective characters of V^n the invariants are given by†

$$24P_a = 24(1-n) + 18a - 7m + w - 4j + 5j_1 - 2j_0; \tag{7}$$

$$\Omega_0 = 48a - 12m - 64n + w - 13j + 8j_1 - 3j_0. \tag{8}$$

* Roth (6). † Roth (5).

‡ We note that the arithmetic and linear genera of a prime section of V^n are given by

$$12p_a = 12(n-1) - 8a + j + 2m, \quad p^{(1)} = j + m - 6a + 9n + 1.$$

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Unless otherwise stated, it is to be assumed in the sequel that the threefold V^n under discussion is situated in S_4 and that it is regular, with genus $P_g = P_a = P$; also that the canonical surfaces $|K|$ are regular, with genera whose actual values are represented by ω and Ω_2.

2. *Canonical systems compounded of pencils.* If $\Omega_1 = 1$, then $\Omega_0 = 0$, and conversely: in this case $|K|$ either consists of a pencil, if $P = 2$, or, if $P > 2$, is compounded of one. Consider, for example, a threefold V^6 having a double plane, so that $P = 2$. Evidently the canonical surfaces K are general quartics, for which $\omega = 1$ and $\Omega_2 = 1$, whence $\Omega_1 = 1$, $\Omega_0 = 0$. Again, consider a V^7 having a double Del Pezzo quartic surface ${}^1F^4$, so that $P = 2$. The remaining invariants may be calculated by means of the formulae given in § 1, but they may also be determined independently as follows. A surface K is cut out by an adjoint quadric V^2; and if V^2 is represented on a space S_3 by quadric surfaces passing through a fixed conic or, symbolically, by (2, 1), the double surface ${}^1F^4$ is represented by (4, 2) and K therefore corresponds to the surface $7(2, 1) - 2(4, 2) = (6, 3)$, that is, a sextic surface having a triple conic. Such a surface has invariants $\Omega_2 = 1$, $\omega = 1$, so that $\Omega_1 = 1$ and $\Omega_0 = 0$.

To obtain an example for which $P > 2$, consider a V^7 having a double cubic scroll ${}^0R^3$. This lies on a net of adjoint quadric cones V^2, so that $P = 3$. We may represent V^2 on S_3 by means of quadric surfaces passing through two intersecting lines or, symbolically, by (2; 1, 1). If in this representation ${}^0R^3$ corresponds to (3; 2, 1), a canonical surface K is given by $7(2; 1, 1) - 2(3; 2, 1) = (8; 3, 5)$. To calculate the invariants of this surface we use Noether's formulae* for surfaces having multiple curves; thus in the present case we obtain $\Omega_2 = 3$, $\omega = 1$, whence $\Omega_1 = 1$, $\Omega_0 = 0$.

A simpler example is that of a sextic threefold with a triple line. For the general V^6 is itself a canonical threefold, having $P = 5$, $\Omega_0 = 6$, and if V^6 acquires a triple line (which is simple on the adjoints) we evidently have $P = 3$, the characteristic curves of $|K|$ are plane cubics, so that $\Omega_1 = 1$, $\Omega_0 = 0$.

3. *Threefolds with genera unity.* If $P = 1$, there are two cases to distinguish, according as V^n does or does not possess a canonical surface of order greater than zero. In the second case, the unique adjoint V^{n-5} either meets V^n in the double surface only or in the double surface and one or more exceptional surfaces. V^n is characterized by the fact that all its plurigenera are unity; if $|F|$ is any linear system on V^n, free from exceptional surfaces, and $|F'|$ is its system of adjoints, then

$$|F'| = |F|.$$

Hence, if ν and π are the grade and curvilinear genus of $|F|$, we obtain

$$2\pi - 2 - \nu = \nu, \quad \text{or} \quad \pi - 1 = \nu.$$

* Noether (3).

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A typical threefold of this class is the general V^5 of S_4, for which $\Omega_0 = 0$, $\Omega_1 = 1$, $\Omega_2 = -1$. A second example is the V^6 with a double quadric; the adjoint prime meets V^6 in a residual quadric which is exceptional. A third example is the V^7 having for double surface a Castelnuovo ${}^2F^5$, the intersection of a quadric cone and a cubic primal residual to a plane. The adjoint quadric V^2 meets V^7 in a residual quartic scroll, as is easily verified by the representation of V^2 on S_3. In these two latter examples the invariants are $\Omega_0 = 2$, $\Omega_1 = 4$, $\Omega_2 = 0$.

In analogy with the theory* of surfaces having all their genera unity, we find that if V^n contains a system of surfaces for which $\pi = 3$, $\nu = 2$, it is representable on a double space S_3; such a threefold is the V^5 with a triple point O (which imposes no condition on the adjoints). Projecting from O we obtain the representation on a double space, the branch-surface being a non-singular octavic.

4. For the purpose of discussing later examples we note here the principal facts relating to the representation of a threefold on a double space; the results are an evident extension of those established for surfaces. If the branch-surface is of order 2ν and is without singularities, the canonical system $|K|$ is represented by the surfaces of order $\nu - 4$, so that the genus P of the double space is

$$P = (\nu - 1)(\nu - 2)(\nu - 3)/6. \tag{9}$$

The grade of $|K|$ is given by $\Omega_0 = 2(\nu - 4)^3$. It follows that

$$\Omega_1 = 1 + 3(\nu - 4)^3. \tag{10}$$

This may be obtained independently by means of the (2, 1) correspondence between the space and the threefold.

Suppose now that the branch-surface has a general curve of multiplicity $2s$; this will be of multiplicity $s - 1$ on the images of the canonical surfaces. Thus, if the order and genus of the curve are specified, the virtual reduction in P may be calculated; in fact, if the curve has order α and genus ρ, the reduction in P is found† to be

$$\frac{s(s-1)}{6}\{(3\nu - 4s - 4)\alpha - (2s - 1)(\rho - 1)\}. \tag{11}$$

5. *Maximum and minimum values of the invariants.* If the canonical system of a surface K is irreducible and without base-points, the arithmetic and linear genera of K satisfy the inequality‡

$$2\Omega_2 \leqslant \omega + 3,$$

or by § 1,

$$\Omega_2 \leqslant 2\Omega_0 + 2. \tag{12}$$

If the sign of equality holds, the canonical curves of K are hyperelliptic. That threefolds with this property exist may be seen from an example. Consider the V^8 having for double surface a Bordiga sextic ${}^3F^6$; this has $P = 4$, $\Omega_0 = 2$, $\Omega_1 = 4$,

* Enriques (2), p. 309.

† For the postulation of a multiple curve see Enriques (2), p. 206.

‡ Enriques (2), p. 390.

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$\Omega_2 = 6$. Since a single trisecant of ${}^3F^6$ passes through an arbitrary point, it follows that these trisecants define on V^8 an involution I_2 by means of which V^8 may be represented on a double space. A simpler representation is, however, obtained directly by referring the ∞^3 surfaces $|K|$ to the double planes of a space S_3; the branch-surface is evidently a general decimic. It is interesting to verify this result by means of the formula* which connects the arithmetic genera of two surfaces in (1, 2) correspondence. If the genera are p'_a, p_a and the branch-curve has genus ρ and canonical number† θ, then

$$24(p'_a+1) = 48(p_a+1) + 6(\rho-1) + 3\theta. \tag{13}$$

In the present case, we apply this result to the correspondence between a plane of S_3 and a canonical surface of V^8; thus $p'_a = \Omega_2 = 6$, $p_a = 0$, $\rho = 36$ and

$$\theta = 2 \cdot 35 - 100 = -30.$$

A threefold with the same invariants is obtained by imposing a quadruple point on a general V^6; thus P is diminished by 1 and Ω_0 by 4, whence $\Omega_1 = 4$ and $\Omega_2 = 6$.

5·1. Suppose, in the second place, that the canonical system of K is simple, irreducible and without base-points; then it has been shown by Castelnuovo‡ that

$$3\Omega_2 \leqslant \omega + 6,$$

or

$$3\Omega_2 \leqslant 4\Omega_0 + 7. \tag{14}$$

If the sign of equality is to hold, the minimum possible solution is

$$\Omega_0 = 8, \quad \Omega_1 = \Omega_2 = 13, \quad P = 6.$$

It is clear that the corresponding canonical threefold must be multiple; in fact it can only be a double Del Pezzo threefold V_3^4, the intersection of two quadrics in S_5. Since the characteristic curves of $|K|$ are represented by elliptic curves, the branch-surface must be of order 24, and is the intersection of V_3^4 with a sextic primal. We may verify by (13) that the surfaces corresponding to the prime sections of V_3^4 have arithmetic genus 13. For in the present case $p_a = 0$, $\rho - 1 = 60$ and $\theta = 2 \cdot 60 - 4 \cdot 36 = -24$, so that $p'_a = 13$.

The representation of V^8 on a double space S_3 follows from the fact that prime sections of V_3^4 correspond to cubic surfaces of S_3 passing through a quintic of genus 2, or symbolically, (3, 1). These surfaces are the images of $|K|$; the branch-surface is (18, 6), so that, according to § 4, the canonical surfaces should be represented by (5, 2). Such surfaces break up into (3, 1) together with the unique quadric (2, 1) which is exceptional for the representation.

We may obtain a simple model with these invariants by imposing an *oscnode* on the V^8 considered in § 5·5, which has $P = 7$, $\Omega_0 = 10$; as remarked in § 8, such a singularity reduces P by unity and Ω_0 by 2.

* Severi (9). † If ν is the virtual grade of the curve then $\theta = 2\rho - 2 - \nu$.

‡ Castelnuovo (1).

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5·2. Again, suppose that the characteristic curves of $|K|$ are irreducible and without base-points. Since the characteristic series of the system, of order Ω_0 and dimension $P-3$, is special, it follows, by Clifford's theorem, that

$$2P \leqslant \Omega_0 + 6, \tag{15}$$

whence

$$\Omega_2 \leqslant \Omega_1 + 2.$$

The sign of equality holds only when the characteristic curves are hyperelliptic. A threefold having this property is the V^8 considered above, in §5; another example is given later in §6·1.

5·3. In general a lower limit for P can, however, be found, analogous to that obtained for Ω_2 in (14).

Suppose that the characteristic system of $|K|$ is simple, irreducible and without base-points. Take three sets G_1, G_2, G_3 of the characteristic series $g_{\Omega_0}^{P-3}$ having $P-4$ points in common. By the Riemann-Roch theorem, the set G_1 imposes $\Omega_0-(P-3)$ conditions on the canonical sets which contain it; hence G_2 imposes at most $\Omega_0-P+3-(P-4)$, i.e. Ω_0-2P+7, conditions on the canonical sets which already contain G_1; and, similarly, G_3 presents at most

$$\Omega_0-2P+7-(P-4),$$

i.e. $\Omega_0-3P+11$, conditions on the canonical sets which contain G_1 and G_2. But since, by (4), $G_1+G_2+G_3$ form a canonical set, it follows that

$$\Omega_0-P+3+\{\Omega_0-2P+7\}+\{\Omega_0-3P+11\} \geqslant \Omega_1-1,$$

or

$$3\Omega_0-6P+21 \geqslant \Omega_1-1,$$

whence, by (4),

$$6P \leqslant \Omega_1+20. \tag{16}$$

5·4. That the limit for P can be attained is seen from familiar examples: thus, the general V^6 has $P=5$, $\Omega_1=10$, and the intersection of a quadric and quintic primal of S_5 has $\Omega_0=10$, $\Omega_1=16$, $P=6$ (§6·4). For $P>6$ there exists a general rule for the construction of canonical threefolds having $6P=\Omega_1+20$. In the first place we observe that the order of such a threefold, situated in S_{P-1}, is

$$\Omega_0=4P-14,$$

and that such a threefold has curve sections of maximum sectional genus* lying on rational scrolls whose generators meet the corresponding curves in 5 points. We now prove that

If $P>6$, a canonical threefold for which $6P=\Omega_1+20$ is obtained as the inter-

* It is at once verified that a curve of S_r, of order n and genus π such that $6r=\pi+2$ and $3n=2\pi-2$, has genus π given by the formula

$$\pi=\chi\{n-r-\tfrac{1}{2}(\chi-1)(r-1)\}, \quad \text{where} \quad \chi=\begin{bmatrix} n-1 \\ r-1 \end{bmatrix}=4,$$

and is therefore of maximum genus.

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section of a rational [3]*-form* V_4^{P-4} *of* S_{P-1} *and a quintic primal* V_{P-2}^5, *residual to* $P-6$ *spaces* [3]. *Such a threefold has curve sections of maximum genus in* S_{P-4}.

The rational V_4^{P-4} may be represented on S_4 by projecting it from $P-5$ points of itself, so that its generating spaces correspond to spaces with a common plane ϖ, and its prime sections to rational plane-forms of order $P-4$ having a multiple plane ϖ of order $P-5$, and $P-5$ simple planes ϖ_i each of which is cospatial with ϖ. Hence the intersection of V_4^{P-4} and V_{P-2}^5 residual to $P-6$ spaces [3] corresponds to a V_3^{4P-14} of S_4, for which the plane ϖ is multiple of order $4P-19$ and the planes ϖ_i are quintuple. If, however, the projection is made from $P-5$ points of the threefold itself, its image in S_4 is instead a V_3^{3P-9} for which ϖ is $(3P-14)$-ple and ϖ_i are quadruple. The canonical system of V_3^{3P-9} is cut out by threefolds of order $3P-14$ for which ϖ is $(3P-15)$-ple and ϖ_i are triple. To calculate the geometrical genus we note that a single threefold of the system passes through $P-1$ given points of S_4; thus, if one of these points is taken to lie on each of the $P-5$ planes ϖ_i, the required threefold breaks up into the $P-5$ spaces (ϖ, ϖ_i) counted three times, and the space passing through the remaining four points; so that the primes of S_{P-1} cut out canonical surfaces on V_3^{4P-14}. Thus, in conclusion, V_3^{4P-14} has order $4P-14$, sectional genus $6P-20$, and geometrical genus P, and is obviously regular; it is therefore a canonical threefold of the desired type.

5·5. It may be remarked that there exist threefolds for which $|K|$ is compounded of an involution and for which P is intermediate between the values given by (15) and (16). Consider the V^8 with a double quartic ${}^0F^4$, the projection of a Veronese surface. From the formulae given in § 1 we find that

$$P=7, \quad \Omega_0=10, \quad \Omega_1=\Omega_2=16.$$

From these values of the invariants it is clear that the canonical threefold must be multiple. As in § 5, the trisecants of ${}^0F^4$ set up an involution I_2 on V^8, by means of which V^8 may be represented on a double S_3; the branch-surface is the section by S_3 of the primal generated by the trisecants of ${}^0F^4$ which touch V^8 elsewhere. To find its order consider the trisecants of ${}^0F^4$ which meet a given line; they generate a cubic scroll, meeting ${}^0F^4$ in a curve of order 6, and V^8 in a residual curve of order $3 \cdot 8 - 2 \cdot 6 = 12$, on which the trisecants define an involution. The number of coincidences of the correspondence is $2 \cdot 12 - 2 \cdot 3 = 18$, the order of the branch-surface. The section of ${}^0F^4$ by S_3 is sextuple on this surface.

Alternatively, we may represent the system $|K|$ by the prime sections of a Del Pezzo V_3^5 of S_6; evidently the branch-surface is the intersection of V^5 with a sextic primal. (We verify, as in § 5, that the correspondence gives $\Omega_2=16$.) Now V_3^5 may be represented on S_3 by cubic surfaces (3, 1) passing through a rational quartic, so that the branch-surface on V_3^5 corresponds to a surface (18, 6); we thus return to the representation found above. The system $|K|$ corresponds to the

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double surfaces (5, 2), which break up into surfaces (3, 1) together with the exceptional quadric (2, 1).

5·6. The theorem given in § 5·3, which is the analogue of a classical result for surfaces, may be generalized so as to apply to a variety V_r^n of dimension r. If the canonical system of V_r^n has grade Ω_0 and curvilinear genus Ω_1, then*

$$2\Omega_1 - 2 = r\Omega_0. \tag{17}$$

The r-dimensional analogue of § 5·3 is given by the following theorem:

If the characteristic curves of the canonical system of V_r^n are simple, irreducible and without base-points, then the genus P of V_r^n satisfies the inequality

$$r(r+1)P \leqslant 2\Omega_1 + (r-1)(r+1)(r+2), \tag{18}$$

where the sign of equality holds for the following canonical varieties:

(i) *the general V_r^{r+3} of S_{r+1} $(P = r+2)$;*

(ii) *the intersection of a V_{r+1}^2 and a V_{r+1}^{r+2} of S_{r+2} $(P = r+3)$;*

(iii) *the intersection of a rational $[r]$-form V_{r+1}^{P-r-1} of S_{P-1} and a V_{P-2}^{r+2}, residual to $P-r-3$ spaces $[r]$ $(P > r+3)$.*

All such varieties have maximum sectional genus Ω_1.

6. *The construction of canonical threefolds.* Having dealt with various special types of canonical threefolds we now exemplify some general methods of construction. If the canonical system $|K|$ is simple, irreducible and of dimension greater than three, it may be represented on the prime sections of a threefold V, normal in S_{P-1}, whose order, sectional genus and sectional arithmetic genus are respectively Ω_0, Ω_1 and Ω_2; the arithmetic linear genus of a prime section is ω, where, by (6), $\omega = 4\Omega_0 + 1$. The determination of the projective characters of V from those of the original threefold will be illustrated by particular cases (§§ 6·4, 6·6).

6·1. Suppose in the first place that $P = 5$; then the minimum possible value of Ω_0 is 4, and in this case the canonical threefold is a double quadric V^2 $(\Omega_1 = 7, \Omega_2 = 9)$, the branch-surface being the intersection of V^2 with a primal of order 8, as may be verified by (13). If V^2 is represented on S_3 by quadric surfaces (2, 1) passing through a fixed conic, the branch-surface corresponds to a surface (16, 8); the plane (1, 1) is, of course, exceptional for the representation.

6·2. The next case, $\Omega_0 = 6$, gives $\Omega_1 = 10$, $\Omega_2 = 10$; the canonical threefold is the general V^6 of S_4, as we have already remarked. Consider now the case $\Omega_0 = 8$, for which $\Omega_1 = 13$, $\Omega_2 = 11$. We may enquire whether it is possible to construct a canonical model having no triple curve on the double surface: with this assumption the formulae of § 1 give $\mu_0 = 8$, $\mu_1 = 32$, so that the double surface is an octavic with maximum section genus, and is therefore the complete intersection of a

* B. Segre (7).

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quadric and a quartic primal. It is easy to see that such a model exists; for if the equations of the quadric and quartic are $A=0$ and $B=0$ respectively, then the equation of the required threefold is

$$LA^2+MAB+B^2=0,$$

where L and M are arbitrary quartic and quadratic polynomials in the coordinates.

In order to construct the canonical threefold for $P=5$, $\Omega_0=10$, $\Omega_1=16$, we must determine a V^{10} with a double surface F^{20} along which V^{10} is touched by a V^4; F^{20} cannot lie on a V^5, else we should have $P>5$. Since a general V^4 contains only complete intersections, F^{20} will have a triple curve, double on V^4; and even if this can be discovered, the question of existence for V^{10} has still to be settled.

The V^{10} with a double surface F^{20} which is the complete intersection of a V^4 and a V^5 is, however, the projection of a canonical threefold in higher space; among the adjoint quintics there is the system consisting of V^4 and the prime sections of V^{10}. We have, from § 1, $P=6$, $\Omega_0=10$, so that V^{10} is the projection of the type described in § 6·4.

6·3. Canonical threefolds in higher space may be constructed by means of a general principle* which applies to manifolds of any dimension. If in S_{P-1} we take $P-4$ general primals of orders n_i $(i=1, 2, \ldots, P-4)$, their complete intersection is a threefold whose canonical system is cut out by primals of order $\Sigma n_i - P$. Hence, in order to obtain canonical threefolds of a given genus P we have only to choose n_i so that

$$\Sigma n_i - P = 1. \tag{19}$$

6·4. Consider now the case $P=6$. The minimum value of Ω_0 for which the canonical threefold is simple is evidently 10 (the case $\Omega_0=8$ has been discussed in § 5·1). The required model is the intersection V^{10} of a quadric and quintic primal of S_5 $(n_1=2, n_2=5)$, for which we have $\Omega_1=16$, $\Omega_2=14$. This example has already been mentioned in § 5·4. A non-canonical model with the same invariants is the V^7 of S_4 having a double quadric. This may be established from first principles; for if, as in § 2 we represent an adjoint quadric by surfaces (2, 1) of S_3, we find that a canonical surface corresponds to a surface (10, 5), which has arithmetic genus $\Omega_2=14$ and linear genus $\omega=41$; and it is evident that $P=6$.

The characters of V^{10} which are obtained from the formulae for the intersection of two primals are

$$n=10, \quad a=50, \quad j=40, \quad m=210, \quad w=826.$$

To find the remaining characters j_1 and j_0, we substitute in the formulae (7) and (8), obtaining two equations, the solutions of which are

$$j_1=136, \quad j_0=208.$$

If we represent the quadric containing V^{10} by quadrics V_3^2 of S_4 passing through

* Severi (10).

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a fixed quadric surface, evidently V^{10} will correspond to a V_3^{10} having a quintuple quadric; we thus obtain a second non-canonical model of V^{10}. If V_3^{10} breaks up into the S_3 containing the quadric, counted three times, and a residual V^7, we re-obtain the model previously considered.

The type $\Omega_0 = 12$ ($\Omega_1 = 19$, $\Omega_2 = 15$) is obtained as the intersection of a cubic and a quartic primal of S_5 ($n_1 = 3$, $n_2 = 4$).

6·5. Similarly, if $P = 7$, we obtain the canonical threefold V^{16} for which $\Omega_0 = 16$ ($\Omega_1 = 25, \Omega_2 = 19$) by taking the intersection of two quadrics and a quadric primal of S_6 ($n_1 = n_2 = 2$, $n_3 = 4$). If the intersection of the quadrics is represented on S_4 by cubic primals V_3^3 passing through a surface ${}^2F^5$, then V^{16} corresponds to a V_3^{12} on which ${}^2F^5$ is quadruple, or symbolically, (12, 4). This threefold may break up into the quadric (2, 1) counted twice, and a residual (8, 2); we thus obtain for a non-canonical model having ordinary singularities a V^8 with a double surface ${}^2F^5$.

The type $P = 7$, $\Omega_0 = 18$ ($\Omega_1 = 28$, $\Omega_2 = 20$) is given by the intersection of a quadric and two cubic primals of S_6 ($n_1 = 2$, $n_2 = n_3 = 3$).

For $P = 8, \Omega_0 = 24$ ($\Omega_1 = 37, \Omega_2 = 25$) we have for the canonical model the intersection of three quadrics with a cubic primal of S_7.

6·6. Finally, we discuss the case in which $P = 9$, $\Omega_0 = 22$ ($\Omega_1 = 34$, $\Omega_2 = 26$); this is an example in which P attains the maximum value with respect to Ω_1 (§ 5·3), and the canonical model V^{22} is the intersection of the rational [3]-form V_4^5 of S_8 and a quintic primal V_7^5, residual to three spaces [3]. To calculate the characters of V^{22} we use the fact that a non-canonical model in S_4 is the V^7 having a double plane, as may be verified from first principles, by the method used in § 2. From § 1 the characters of V^7 are

$$n = 7, \quad a = 40, \quad j = 10, \quad m = 215, \quad w = 1060,$$

and the corresponding characters of V^{22} are

$$n' = 22, \quad a' = 110, \quad j' = 160, \quad m' = 390,$$

since the prime section of V^{22} has invariants $\omega = 89$, $\Omega_2 = 26$. To find the class w' of V^{22} we first calculate for V^7 the Zeuthen-Segre invariant

$$I = w - 2i - 2\pi, \tag{20}$$

in which π is the sectional genus and i is the corresponding invariant for a prime section, given by

$$i = m - 2a + 3n - 4. \tag{21}$$

Since in the present case $\pi = 14$ and $i = 152$, we have $I = 728$. For V^{22} we have $\pi' = 34$, $i' = 232$, so that from (20) we obtain $w' = 1260$.

The remaining characters j_1' and j_0' are determined from equations (7) and (8) which give

$$j_1' = 750, \quad j_0' = 1450$$

Thus from the known characters of a non-canonical model may be deduced the projective characters of its canonical threefold.

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7. *Other methods of construction.* Suppose that W is a fourfold and that $|V|$ is its system of prime sections, supposed non-singular. Let us determine the nature of W if it is to possess the property that its section by a quadric should be a canonical threefold; in this case the system adjoint to $|2V|$ must coincide with $|V|$, i.e.

$$|(2V)'| = |V|.$$

Hence the system $|V'|$ must be of order zero, and therefore V is characterized by the fact that all its genera are unity (§ 3).

Canonical threefolds obtained in this manner are the intersection of a quadric and a quintic primal of S_5 (§ 6·4) and the intersection of two quadrics and a quartic primal of S_6 (§ 6·5).

7·1. Next, we determine W so that its section by a cubic primal should be canonical; this requires that

$$|(3V)'| = |V|,$$

whence the system $|(2V)'|$ is of order zero, and the threefolds $|2V|$ have unit genera. If W is of order n and sectional genus π, then $n = 2\pi - 2$.

Examples of this method of construction are the intersection of a cubic and quartic primal of S_5 (§ 6·4), the intersection of a quadric and two cubic primals of S_6, and the intersection of three quadrics and a cubic primal of S_7 (§ 6·5).

7·2. Similarly, if the section of W by a quartic primal is canonical, then the system $|(3V)'|$ is of order zero, and therefore W has elliptic curve sections. Some of the examples given in § 6 may be obtained in this way; the method also gives a number of canonical threefolds not otherwise obtainable, the intersections of Del Pezzo fourfolds (which may be cones) with quartic primals; thus we have the type $P = 8, \Omega_0 = 20$ represented by the intersection of the W^5 (section of the locus representing the lines of S_4) with a quartic primal, and the type $P = 9, \Omega_0 = 24$ as the intersection of the Segre variety W^6 (representing the pairs of points of two planes) with a quartic primal. Since W^6 may be represented on S_4 by means of quadrics passing through two skew lines, it follows that the canonical threefold corresponds in S_4 to a primal V^8 with two skew quadruple lines.

7·3. For $P = 11, \Omega_0 = 32$, the threefold is given by the intersection of a quartic primal of S_{10} with the cone W^8 which projects a Del Pezzo threefold V_3^8 from a point O. Now W^8 may be represented on S_4 by the quadrics which touch a fixed space α at a fixed point A; thus the canonical threefold corresponds in S_4 to an octavic primal V^8 having a quadruple point A at which the four sheets touch the space α; the adjoint primals cutting out the canonical system on V^8 consist of the space α and the quadrics touching α at A. The primal V^8 may break up into α and a residual V^7 having a point of contact of three sheets at A, the canonical system of V^7 being cut out by the quadrics touching α at A. Since the representation of W^8 on S_4 is obtained by projecting it from one of its Veronese surfaces, it follows

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that V^7 is the projection of the intersection of W^8 with a quartic primal containing the Veronese surface in question.

8. *The case* $P=4$. If $P=4$, the canonical threefold, if it exists, must be a multiple space of order Ω_0. The type $\Omega_0=2$ has been considered in § 5; since Ω_0 is necessarily even, the next possibility is $\Omega_0=4$. A model with these invariants is obtained from the V^8 discussed in § 6·2, which has for double surface the intersection of a quadric and quartic primal ($P=5$, $\Omega_0=8$), by imposing a general quadruple point; the canonical surfaces are the prime sections passing through this point.

A second example is the V^6 with an oscnode ($P=4$, $\Omega_0=4$, $\Omega_1=7$, $\Omega_2=7$). The canonical system is cut out by primes passing through the oscnode; in verification we note* that an oscnode on the prime section reduces the arithmetic genus Ω_2 by 3 and the linear genus ω by 7, and that these reductions are precisely those which occur in the corresponding invariants of the general V^6.

If V^6 acquires a second oscnode ($P=3$, $\Omega_0=2$) the canonical surfaces form a net passing through the two points where the line joining the oscnodes meets V^6 again; we have thus a simple example of a canonical system with base-points, which may be transformed into (exceptional) surfaces contained in the canonical system of the transformed threefold.

The model for which $P=4$, $\Omega_0=6$ is a sextuple space: a non-canonical model is found by imposing an oscnode on the V^8 considered above (§ 6·2). To show that such a threefold exists we note that, if the oscnode is taken as origin of coordinates (x, y, z, t) and $x=0$ is the tangent space there, then* the equation of the primal is

$$(x+\phi_2)(x+\phi_2+\phi_3)+x\phi_4+\phi_6=0, \tag{22}$$

where ϕ_i is homogeneous of degree i in the coordinates.

The equation of V^8 is, by § 6·2,

$$LA^2+MAB+B^2=0, \tag{23}$$

where $A=0$, $M=0$ represent quadric, and $B=0$, $L=0$ quartic primals. In order that the equation (23) should be of the form (22), it is sufficient to put

$$A\equiv\psi_0+\psi_2, \quad B\equiv\psi_3+\psi_4, \quad L\equiv(x+\phi_2)^2, \quad M\equiv x\psi_1,$$

where ψ_i is homogeneous of degree i in the coordinates.

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