THE CALCULATION OF POTENTIAL DERIVATIVES BY USING NBIE FOR ANISOTROPIC POTENTIAL PROBLEMS

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ABSTRACT

The natural boundary integral equation (NBIE) is developed to calculate potential derivatives for potential problems with anisotropic media. Firstly, the governing equation of the two-dimensional anisotropic potential problem is transformed into standard Laplace equation by a coordinate transformation method. Then a potential derivative boundary integral equation named as NBIE is extended to solve the anisotropic potential problem. The most important virtue of the NBIE is that the singularity of the integral kernel function is reduced by one order in comparison with the conventional potential derivative boundary integral equation(CDBIE). Therefore the new potential derivative boundary integral equation only contains strongly singular integrals rather than hyper-singular integrals. Thus the NBIE can calculate more accurate potential derivative results for both boundary nodes and interior points. Moreover, in combination with the analytical integral regularization algorithm of nearly singular integrals, the NBIE can obtain more accurate potential derivatives of interior points very close to the boundary than the CDBIE. Numerical examples on heat conduction in anisotropic media demonstrate the accuracy and efficiency of the NBIE.

Keywords: Anisotropic potential problems, Coordinate transformation method, Natural boundary integral equation, Nearly singular integrals.

1. INTRODUCTION

There are many natural or man-made anisotropic materials whose property coefficients vary with directions. With the rapid increase of their industrial use, the anisotropic potential problem has become an important subject in the practical engineering. Heat conduction in anisotropic media, subsurface flow in anisotropic media, and torsion in anisotropic uniform bar are all concluded as anisotropic potential problems. Therefore, the research on anisotropic potential problems is of great significance.

Many researchers have studied anisotropic potential problems systematically by the boundary element method (BEM). Earlier research on BEM in anisotropic potential problems was made by Chang *et al*. [1]. They presented a study on the fundamental Green's functions associated with heat conduction in anisotropic media, then by means of these functions and Green's formula, differential equations were transformed into integral equations. Mera *et al*. [2-4] investigated various regularization methods in order to solve the Cauchy steady-state heat conduction problem. Gao [5] proposed a new robust BEM, based on a source point isolation technique, for solving general anisotropic potential problems with varying coefficients. Zhang *et al*. [6] presented a research on the regularization of indirect boundary integral equations for anisotropic potential problems. Shiah *et al*. [7-11] proposed the direct domain mapping technique to treat 2D and 3D anisotropic problems, and they [10, 11] fully regularized the nearly singular integrals arising in the boundary integral equation for heat conduction in thin anisotropic media by the technique of integration by parts. Based on the transformation approach, Rafiezadeh [12] developed a three-dimensional boundary element solution for the seepage analysis in multi-domain general anisotropic media.

As is familiar to the BEM community, by using the conventional potential BIE, the potential and normal potential derivative of boundary node can be obtained easily. However, in some cases, the potential derivative with respect to coordinate axis direction is required. Thus many researchers sought to establish applicable derivative boundary integral equation to calculate derivative quantities. Generally, the derivative boundary integral equation contains the hyper-singular integral.

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The accuracy of the result depends on the precision of the evaluation of these integrals, and how to treat the hyper-singular integral is always an important topic in the BEM. The earlier research on the regularization of singular integrals and nearly singular integrals was reviewed in details in the References [13, 14]. Sladek *et al*. [15] brought together a comprehensive treatment of the principal strategies developed to deal with the various kinds of singular integrals. They [16] also gave some introductory notes on singular integrals in the first chapter of their book. Gray *et al*. [17] transformed the hyper-singular surface integral into the line integral along the contour of the element, and the numerical quadrature was employed to compute the line integral. The subtraction technique [18, 19] was used by subtracting and adding back a term for the hyper-singular integrand at a load point. As a result, the hyper-singular integrals were transformed into the strongly singular integrals by the Stokes' theorem. Tomioka and Nishiyama [20] used an analytical regularization of hyper-singular boundary integral for solving a two-dimensional time harmonic wave equation. The regularization was based on cancelation of the hyper-singularity by considering properties of hyper-singular elements adjacent to a singular node. Aimi *et al*. [21] introduced numerical integration schemes required for the discretization of space-time hyper-singular BIEs reformulating exterior Neumann wave propagation problems. For hyper-singular integral equations in 2D with a positive-order Sobolev space, Erath *et al*. [22] analyzed the mathematical relation between various error estimators. Chen *et al*. [23-27] proposed the dual boundary integral equation to solve crack problems etc.. They [25, 26] summarized the regularization of hyper-singular integrals occurring in dual BEM. Recently they [27] derived dual boundary integral equations for the N-dimensional Laplace problems with a smooth boundary by using the contour approach surrounding the singularity. Yu [28, 29] proposed the natural BEM which contains the hyper-singular integral operator in the Dirichlet to Neumann (DtN) map. Yu [30] discussed the natural integral operator and natural BEM for harmonic problem over exterior elliptic domain.

Although numerous schemes have been proposed to treat the hyper-singular integrals, the existence of hypersingular integral will still lead to inaccurate result in some cases. For this reason, Niu and Zhou [31] derived a kind of derivative boundary integral equation termed the natural boundary integral equation (NBIE) for two-dimensional isotropic potential problems. The major characteristic of the NBIE is that there is only strongly singular integral in the equation, thus the potential derivative calculated by the NBIE is more accurate than that by the CDBIE. Recently, Zhou *et al*. [32] extended the NBIE to solve orthotropic potential problems by using the coordinate transformation method.

In this paper, the NBIE of isotropic potential problems is firstly developed to analyze anisotropic potential problems. The remainder of the paper is organized as follows. Section 2 presents the governing equation of potential problems with anisotropic media. Section 3 discusses a coordinate transformation method to transform the governing equation of anisotropic potential problem into standard Laplace equation. Section 4 and Section 5 introduce the conventional derivative boundary integral equation and the new natural boundary integral equation, respectively. Moreover, Section 5 gives the application flowchart of the NBIE. Section 6 investigates two numerical examples on heat conduction in anisotropic media. Section 7 summarizes the characteristics of the new method.

2. THE GOVERNING EQUATION

Assume that the medium is anisotropic. As shown in Fig. 1, the boundary conditions of the two-dimensional anisotropic potential problem can be written as follows.

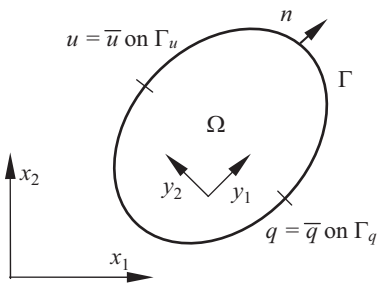


Fig. 1 Anisotropic potential problems.

$$\begin{cases} u(y) = \overline{u}(y) & \text{on } \Gamma_u \\ q(y) = \left(k_{11}\dfrac{\partial u}{\partial y_1} + k_{12}\dfrac{\partial u}{\partial y_2} \right) n_1 + \left(k_{12}\dfrac{\partial u}{\partial y_1} + k_{22}\dfrac{\partial u}{\partial y_2} \right) n_2 = \overline{q}(y) & \text{on } \Gamma_q \end{cases} \tag{1}$$

where n_1 and n_2 are the direction cosines of the outward normal n to surface Γ, and we assume that medium properties are constant. Thus the coefficients k_{ij} are independent of both the space and time variables. The domain boundary, $\Gamma = \Gamma_u \cup \Gamma_q$, $\Gamma_u, \Gamma_q \neq \varnothing$, $\Gamma_u \cap \Gamma_q = \varnothing$. The governing equation can be written as

$$k_{11}\frac{\partial^2 u}{\partial y_1^2} + 2k_{12}\frac{\partial^2 u}{\partial y_1 \partial y_2} + k_{22}\frac{\partial^2 u}{\partial y_2^2} = 0 \tag{2}$$

The fundamental solution to Eq. (2) is given as follows:

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$$u^* = \frac{1}{2\pi\sqrt{|k_{ij}|}} \ln\frac{1}{r} \tag{3}$$

where $|k_{ij}|$ is the determinant of the medium property coefficients matrix

$$|k_{ij}| = k_{11}k_{22} - k_{12}^2 = \begin{vmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{vmatrix} \tag{4}$$

Noting that

$$\begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} = \begin{pmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{pmatrix}^{-1} = \frac{1}{|k_{ij}|}\begin{pmatrix} k_{22} & -k_{12} \\ -k_{12} & k_{11} \end{pmatrix} \tag{5}$$

Then the distance function r between source point y_i^s and field point y_i can be expressed as

$$r = \sqrt{s_{11}(y_1^s - y_1)^2 + 2s_{12}(y_1^s - y_1)(y_2^s - y_2) + s_{22}(y_2^s - y_2)^2} \tag{6}$$

For the linear element, it can be obtained that

$$n_1 = (y_{F_2} - y_{I_2})/l, \quad n_2 = -(y_{F_1} - y_{I_1})/l \tag{7}$$

where y_{I_i} and y_{F_i} denote the coordinates of the initial and final nodes of a linear element, respectively; and l is the length of the element. The normal potential derivative can be given by

$$q^* = \left(k_{11}\frac{\partial u^*}{\partial y_1} + k_{12}\frac{\partial u^*}{\partial y_2}\right)n_1 + \left(k_{12}\frac{\partial u^*}{\partial y_1} + k_{22}\frac{\partial u^*}{\partial y_2}\right)n_2 \tag{8}$$

3. COORDINATE TRANSFORMATION METHOD

The concept to apply coordinate transformation to treat 2D and 3D anisotropic potential problems in the BEM was firstly proposed by Shiah *et al.* [7, 8]. In order to use the NBIE of the isotropic potential problems to solve the anisotropic potential problems, the coordinate transformation equations are introduced as follows:

$$\begin{cases} \hat{y}_1 = y_1 \\ \hat{y}_2 = -\dfrac{k_{12}}{\sqrt{|k_{ij}|}}y_1 + \dfrac{k_{11}}{\sqrt{|k_{ij}|}}y_2 \end{cases} \tag{9}$$

where the superscript "^" means that this variable belongs to the new coordinate system, similarly hereinafter.

Substituting Eq. (9) into Eq. (2) and letting $\hat{u}(\hat{y}) = u(y)$ give

$$\frac{\partial^2\hat{u}}{\partial\hat{y}_1^2} + \frac{\partial^2\hat{u}}{\partial\hat{y}_2^2} = 0 \tag{10}$$

At the same time, the boundary conditions can be rewritten as

$$\begin{cases} \hat{u}(\hat{y}) = \bar{u}(\hat{y}) & \text{on } \hat{\Gamma}_u \\ \hat{q}(\hat{y}) = \dfrac{\partial\hat{u}}{\partial\hat{y}_1}\hat{n}_1 + \dfrac{\partial\hat{u}}{\partial\hat{y}_2}\hat{n}_2 = \bar{q}(\hat{y}) & \text{on } \hat{\Gamma}_q \end{cases} \tag{11}$$

where $\hat{\Gamma} = \hat{\Gamma}_u + \hat{\Gamma}_q$.

Substituting Eqs. (7) and (9) into Eq. (8) and letting $\hat{u}(\hat{y}) = u(y)$ yield

$$\hat{q} = \frac{1}{\sqrt{|k_{ij}|}}\frac{ql}{\hat{l}} \tag{12}$$

4. CONVENTIONAL POTENTIAL DERIVATIVE BIE

The potential BIE has the form

$$\hat{u}(\hat{y}^s) = \int_{\hat{\Gamma}} \hat{u}^*\hat{q}\,\mathrm{d}\Gamma - \int_{\hat{\Gamma}} \hat{q}^*\hat{u}\,\mathrm{d}\Gamma, \quad \hat{y}^s \text{ in } \Omega \tag{13}$$

where $\hat{u}^*$ and $\hat{q}^*$ are the fundamental solutions.

$$\hat{u}^* = \frac{1}{2\pi}\ln\frac{1}{\hat{r}}, \quad \hat{q}^* = -\frac{1}{2\pi\hat{r}}\hat{r}_{,n} \tag{14}$$

where

$$\hat{r}_{,n} = \partial\hat{r}/\partial n = \frac{1}{\hat{r}}[(\hat{y}_1 - \hat{y}_1^s)\hat{n}_1 + (\hat{y}_2 - \hat{y}_2^s)\hat{n}_2] \tag{15}$$

When source point $\hat{y}^s$ locates at the boundary $\hat{\Gamma}$, Eq. (13) becomes

$$C(\hat{y}^s)\hat{u}(\hat{y}^s) = \int_{\hat{\Gamma}} \hat{u}^*\hat{q}\,\mathrm{d}\Gamma - \int_{\hat{\Gamma}} \hat{q}^*\hat{u}\,\mathrm{d}\Gamma, \quad \hat{y}^s \text{ on } \hat{\Gamma} \tag{16}$$

where $C(\hat{y}^s)$ is the jump coefficient, which is dependent on the boundary geometry characteristics. In the two-dimensional potential problems, $C(\hat{y}^s) = \phi/2\pi$, where ϕ is the interior angle at $\hat{y}^s$ on $\hat{\Gamma}$.

By differentiating Eq. (13), the CDBIE is obtained as follows

$$\hat{u}_{,i}(\hat{y}^s) = \int_{\hat{\Gamma}} \hat{u}_{,i}^*(\hat{y}^s)\hat{q}\,\mathrm{d}\Gamma - \int_{\hat{\Gamma}} \hat{q}_{,i}^*(\hat{y}^s)\hat{u}\,\mathrm{d}\Gamma, \quad \hat{y}^s \text{ in } \Omega \tag{17}$$

where

$$\hat{u}_{,i}^*(\hat{y}^s) = \frac{1}{2\pi\hat{r}}\hat{r}_{,i}(\hat{y}), \quad \hat{q}_{,i}^*(\hat{y}^s) = \frac{1}{2\pi\hat{r}^2}(\hat{n}_i - 2\hat{r}_{,i}\hat{r}_{,n}) \tag{18}$$

It can be obviously found that the nearly strongly singular and hyper-singular integrals occur in Eq. (17) when the source point $\hat{y}^s$ is close to the boundary $\hat{\Gamma}$. When $\hat{y}^s$ is on the boundary, Eq. (17) is written as

$$D(\hat{y}^s)\hat{u}_{,i} = \int_{\hat{\Gamma}} \hat{u}_{,i}^*\hat{q}\,\mathrm{d}\Gamma - \int_{\hat{\Gamma}} \hat{q}_{,i}^*\hat{u}\,\mathrm{d}\Gamma, \quad \hat{y}^s \text{ on } \hat{\Gamma} \tag{19}$$

where $D(\hat{y}^s)$ is the singular coefficient. It can be

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seen that kernel $1/\hat{r}$ and $1/\hat{r}^2$ can produce strongly singularity and hyper-singularity, respectively, and hence it is very difficult to deal with them directly. The method for dealing with the singular integral introduced earlier could not acquire satisfying results in some specific cases.

5. NATURAL BOUNDARY INTEGRAL EQUATION

As shown in Fig. 2, φdenotes the angle between the $\hat{y}_1$-axis and the vector r, $\overline{y}_1 // \hat{y}_1$.

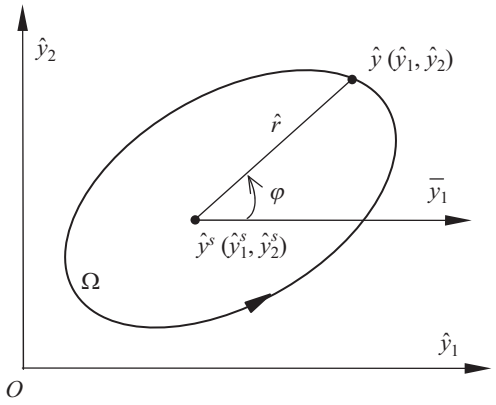


Fig. 2 Definition of angle φ.

Assume that $\hat{n}_i$ and $\hat{\tau}_i$ are the outward normal vector and the tangential vector of the boundary, respectively. And define $\delta_{12}=\delta_{21}=0$, $\delta_{11}=\delta_{22}=1$, $\in_{12}=-\in_{21}=1$, $\in_{11}=\in_{22}=0$, there are

$$\hat{n}_i \in_{ij} = \hat{\tau}_j, \quad \hat{\tau} \in_{ij} = -\hat{n}_j, \quad i,j=1,2 \tag{20}$$

$$\hat{r}_{,1} = \frac{\partial \hat{r}}{\partial \hat{y}_1} = \cos\varphi, \quad \hat{r}_{,2} = \frac{\partial \hat{r}}{\partial \hat{y}_2} = \sin\varphi \tag{21}$$

Differentiating Eq. (21) yields

$$-\varphi_{,i}\sin\varphi = \frac{\delta_{1i} - \hat{r}_{,1}\hat{r}_{,i}}{\hat{r}}, \quad \varphi_{,i}\cos\varphi = \frac{\delta_{2i} - \hat{r}_{,2}\hat{r}_{,i}}{\hat{r}} \tag{22}$$

Then

$$\varphi_{,i} = \frac{\hat{r}_{,1}\delta_{2i} - \hat{r}_{,2}\delta_{1i}}{\hat{r}} = \frac{\in_{ji}\hat{r}_{,j}(\hat{y})}{\hat{r}} \tag{23}$$

Both sides of Eq. (23) multiplied by $\hat{\tau}_i$, it can be written

$$\frac{\partial \varphi}{\partial s} = \varphi_{,i}\hat{\tau}_i = \frac{\hat{r}_{,n}}{\hat{r}} \tag{24}$$

On the boundary $\hat{\Gamma}$, it is easy to acquire the following

$$\frac{\hat{n}_i - \hat{r}_{,i}\hat{r}_{,n}}{\hat{r}^2} = \frac{\partial}{\partial \hat{y}_i}\left(\frac{1}{\hat{r}}\hat{r}_{,n}\right) + \hat{r}_{,n}\frac{\hat{r}_{,i}}{\hat{r}^2} \tag{25}$$

Using Eqs. (24) and (25), we have

$$\frac{\hat{n}_i - 2\hat{r}_{,i}\hat{r}_{,n}}{\hat{r}^2} = \frac{\partial}{\partial \hat{y}_i}\left(\frac{1}{\hat{r}}\hat{r}_{,n}\right) = \varphi_{,is} \tag{26}$$

Substituting Eq. (26) into Eq. (18), and considering the second integral on the right side of Eq. (17) on the arbitrary segment $\hat{\Gamma}_1 = \hat{y}_1\hat{y}_2$ shown in Fig. 3, we can get the following form with the help of the integration by parts

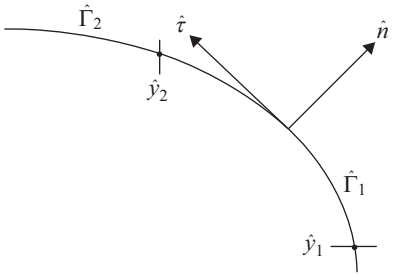


Fig. 3 The boundary $\hat{\Gamma}_1$.

$$\begin{aligned}\int_{\hat{\Gamma}_1}\frac{\hat{n}_i - 2\hat{r}_{,i}\hat{r}_{,n}}{\hat{r}^2}\hat{u}(\hat{y})\mathrm{d}\Gamma &= \int_{\hat{\Gamma}_1}\varphi_{,is}\hat{u}(\hat{y})\mathrm{d}\Gamma = \int_{\hat{\Gamma}_1}\hat{u}(\hat{y})\mathrm{d}\varphi_{,i} \\ &= \frac{1}{\hat{r}}\in_{ji}\hat{r}_{,j}\hat{u}(\hat{y})\Big|_{\hat{y}=\hat{y}_1}^{\hat{y}_2} - \int_{\hat{\Gamma}_1}\frac{1}{\hat{r}}\in_{ji}\hat{r}_{,j}\frac{\partial \hat{u}}{\partial s}\mathrm{d}\Gamma\end{aligned} \tag{27}$$

Let $\hat{\Gamma} = \hat{\Gamma}_1 + \hat{\Gamma}_2$, and substitute Eq. (27) into Eq.(17), then

$$\begin{aligned}\hat{u}_{,i}(\hat{y}^s) = &\int_{\hat{\Gamma}}\frac{\hat{r}_{,i}\hat{q}}{2\pi\hat{r}}\mathrm{d}\Gamma + \int_{\hat{\Gamma}_1}\frac{\in_{ji}\hat{r}_{,j}}{2\pi\hat{r}}\frac{\partial \hat{u}}{\partial s}\mathrm{d}\Gamma \\ &- \frac{\in_{ji}\hat{r}_{,j}}{2\pi\hat{r}}\hat{u}\Big|_{\hat{y}=\hat{y}_1}^{\hat{y}_2} - \int_{\hat{\Gamma}_2}\frac{\hat{n}_i - 2\hat{r}_{,i}\hat{r}_{,n}}{2\pi\hat{r}^2}\hat{u}\mathrm{d}\Gamma\end{aligned} \tag{28}$$

When $\hat{y}^s$ is close to the boundary $\hat{\Gamma}$, there are the strongly singular integrals in the first and second terms on the right side of Eq. (28), and they are often computed separately, but failures always arise. Now we add them on $\hat{\Gamma}_1$ as follows

$$\begin{aligned}I_i(\hat{y}^s) &= \frac{1}{2\pi}\int_{\hat{\Gamma}_1}\frac{\hat{r}_{,i}\hat{q}}{\hat{r}}\mathrm{d}\Gamma + \frac{1}{2\pi}\int_{\hat{\Gamma}_1}\frac{\in_{ji}\hat{r}_{,j}}{\hat{r}}\frac{\partial \hat{u}}{\partial s}\mathrm{d}\Gamma \\ &= \frac{1}{2\pi}\int_{\hat{\Gamma}_1}\frac{1}{\hat{r}}[(\hat{r}_{,i}\hat{n}_1 + \in_{ji}\hat{r}_{,j}\hat{\tau}_1)\frac{\partial \hat{u}}{\partial \hat{y}_1} + (\hat{r}_{,i}\hat{n}_2 + \in_{ji}\hat{r}_{,j}\hat{\tau}_2)\frac{\partial \hat{u}}{\partial \hat{y}_2}]\mathrm{d}\Gamma \\ &= \frac{1}{2\pi}\int_{\hat{\Gamma}_1}\frac{1}{\hat{r}}(\hat{r}_{,n}\delta_{ij}\frac{\partial \hat{u}}{\partial \hat{y}_j} - \hat{r}_{,s}\in_{ij}\frac{\partial \hat{u}}{\partial \hat{y}_i})\mathrm{d}\Gamma\end{aligned} \tag{29}$$

Considering the limit analysis in Eq. (29) when $\hat{y}^s$

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on $\hat{\Gamma}_1$, then

$$\phi\frac{\partial\hat{u}}{\partial\hat{y}_i^s}=\int_{\hat{\Gamma}_1}\frac{1}{\hat{r}}(\hat{r}_{,n}\delta_{ij}\frac{\partial\hat{u}}{\partial\hat{y}_j}-\hat{r}_{,s}\in_{ij}\frac{\partial\hat{u}}{\partial\hat{y}_i})\mathrm{d}\Gamma-\frac{1}{\hat{r}}\hat{r}_{,j}\in_{ji}\hat{u}\Big|_{\hat{y}=\hat{y}_1}^{\hat{y}_2}$$
$$+\int_{\hat{\Gamma}_2}\frac{1}{\hat{r}}\hat{r}_{,i}\hat{q}\mathrm{d}\Gamma-\int_{\hat{\Gamma}_2}\frac{1}{\hat{r}^2}(\hat{n}_i-2\hat{r}_{,i}\hat{r}_{,n})\hat{u}\mathrm{d}\Gamma,\quad \hat{y}^s\ \text{on}\ \hat{\Gamma} \tag{30}$$

where ϕ is the interior angle at $\hat{y}^s$ on $\hat{\Gamma}_1$.

By using the conventional potential BIE (16), all the values of potential and normal potential derivatives on the whole boundary are known. Then substituting them into Eq. (30) obtains the potential derivatives of coordinate axis direction.

When $\hat{y}^s$ is inside the domain, Eq. (30) can be simplified as

$$\frac{\partial\hat{u}}{\partial\hat{y}_i^s}=\frac{1}{2\pi}\int\frac{1}{\hat{r}}\left(\hat{r}_{,n}\delta_{ij}\frac{\partial\hat{u}}{\partial\hat{y}_j}-\hat{r}_{,s}\in_{ij}\frac{\partial\hat{u}}{\partial\hat{y}_i}\right)\mathrm{d}\Gamma,\quad \hat{y}^s\ \text{in}\ \Omega \tag{31}$$

Eq. (30) or Eq. (31) is named as the natural boundary integral equation (NBIE). The detailed deduction of the equations can be further referred to References [31, 32]. As is well known, the CDBIE (Eq. (17)) represents the integral relationship between potential derivatives of internal points and boundary values of potentials and normal potential derivatives. Correspondingly, the NBIE (Eq. (31)) establishes the direct and concise integral relationship between potential derivatives of internal points and boundary values of potential derivatives. Furthermore, the singularity of the NBIE is reduced by one order in comparison with the CDBIE.

Figure 4 shows the detailed application flowchart of the NBIE to treat anisotropic potential problems.

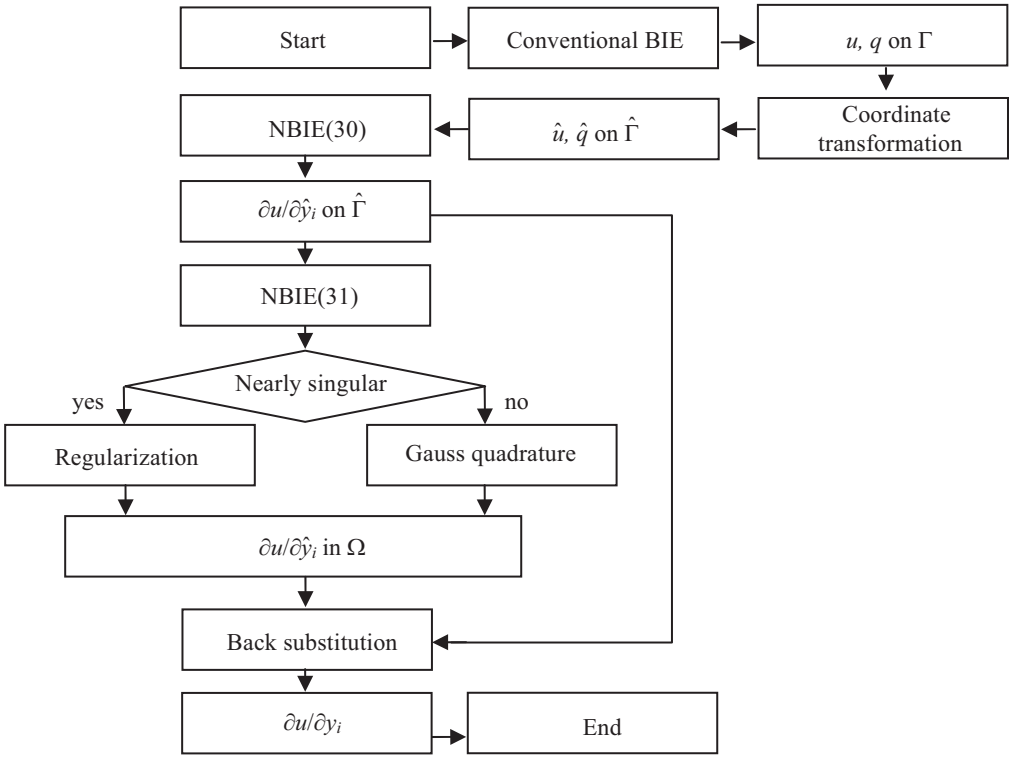


Fig. 4 Application flowchart of NBIE.

For the anisotropic potential problem, in order to use the NBIE, the conventional potential BIE is applied to obtain the potential and normal potential derivatives at first, then the coordinate transformation method in combination with the NBIE is employed to calculate the potential derivatives for boundary nodes and internal points, using Eq. (30) and Eq.(31), respectively.

When an interior point is close to the boundary, Eq. (31) only has the nearly strongly singular integral, but Eq. (17) has the nearly hypersingular integral. Obviously, Eq. (31) has weaker singular characteristics than Eq. (17). The nearly singular integrals can be evaluated by analytical integral regularization algorithm instead of conventional Gaussian quadrature [33].

6. NUMERICAL EXAMPLES

Example 1. Heat conduction in a disk with an anisotropic medium. The radius of the disk is $r = 1$. $k_{11} = 6$, $k_{12} = 1$, $k_{22} = 4$. The potential boundary conditions are only known, as shown in Fig. 5. 40 linear elements are uniformly divided on the boundary. The fluxes (potential derivatives with respect to coordinate axis direction) of boundary nodes are calculated by the NBIE and the difference method, respectively. The fluxes of 42 interior points (21 interior points close to Node A(1,0) along the axis y_1, another 21 interior points close to Node B(0,1) along the axis y_2) are calculated by the NBIE and the CDBIE.

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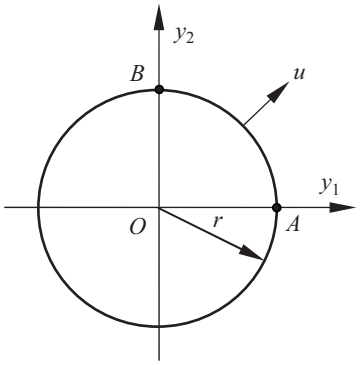


Fig. 5 Heat conduction in a disk.

The analytical solution of the temperature is $u(y_1, y_2) = \frac{1}{4}y_1^3 + \frac{3}{4}y_1^2 y_2 - \frac{3}{2}y_1 y_2^2 - \frac{1}{8}y_2^3 + \frac{3}{2}y_1 + \frac{5}{2}y_2$.

The flux results q_2 of boundary nodes are shown in Fig. 6, and the relative errors of q_2 are plotted in Fig. 7.

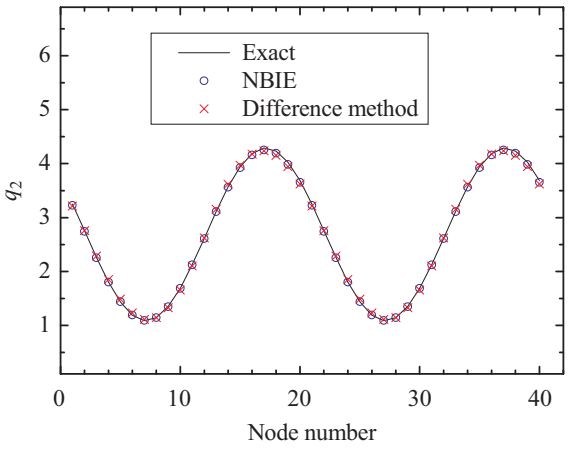


Fig. 6 q_2 of boundary nodes.

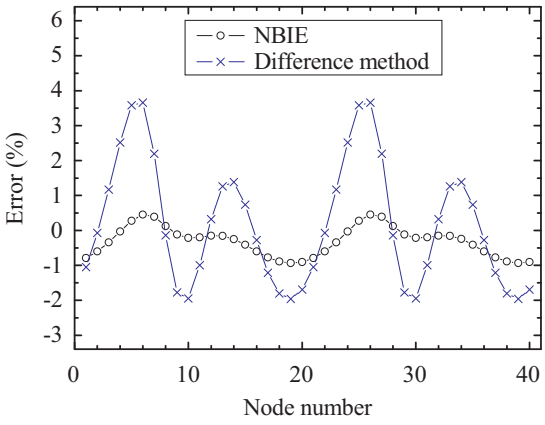


Fig. 7 Error of q_2 of boundary nodes.

In the Figs 6 and 7, "NBIE" represents that the flux results are obtained by the NBIE, "Difference method" means that the flux results are computed based on the difference method and the conventional potential BIE. After Eq. (16) is discretized and solved, all boundary potentials and normal potential derivatives become known. The tangential potential derivative q_τ can be computed by the difference method. $q_\tau = \Delta u/\Delta s$, where Δu is the potential difference between the final node and the initial node in a linear element, and Δs is the length of the element. Assume that q_n is normal potential derivative, then the potential derivatives of coordinate axis direction can be calculated by using the equations $q_1 = q_n n_1 - q_\tau n_2$, $q_2 = q_n n_2 + q_\tau n_1$. So the precision of the potential derivative by the difference method is reduced by one order in comparison with the precision of the potential.

Figure 6 indicates that the flux results q_2 of boundary nodes by the NBIE are more accurate than those by the difference method. Fig. 7 shows that the errors of fluxes by the NBIE are less than those by the difference method.

The flux results q_1 and q_2 of interior points corresponding to four different methods are presented in Figs. 8 and 9, compared with the exact solutions. "NBIE with regularization" and "CDBIE with regularization" mean that the nearly singular integrals(NSI) are calculated by the analytical integral algorithm [33] for the NBIE and the CDBIE, and those "without regularization" denote that the NSI are directly computed by the conventional Gauss quadrature.

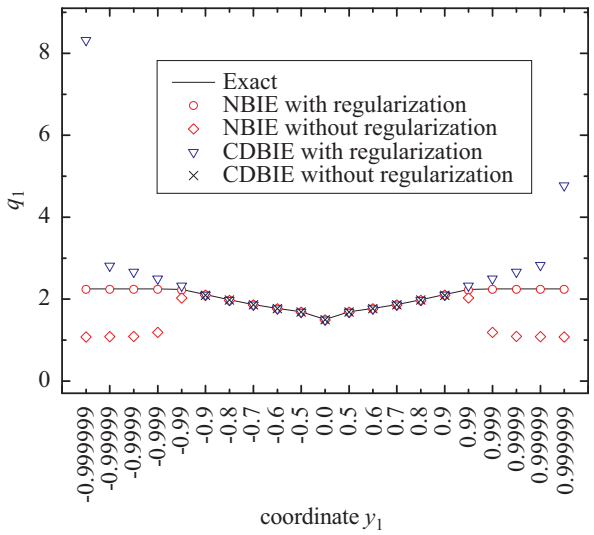


Fig. 8 q_1 of interior points.

Figures 8 and 9 compare the flux results of interior points by NBIE and CDBIE with the exact solutions. They show that the results by the NBIE with the regularization of NSI are the most accurate, and the results by the CDBIE without the regularization of NSI are the most inaccurate. Altogether, the results by the NBIE are more accurate than those by the CDBIE, and the results by the NBIE with the regularization of NSI are in better agreement with the exact solutions than those by the CDBIE with the regularization of NSI. Especially, the CDBIE without the regularization fails to obtain accurate results when the interior point is very close to the

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boundary, it is only valid for those points $-0.9 \le y_2 \le 0.9$. The results of the NBIE with regularization always coincide with the exact solutions.

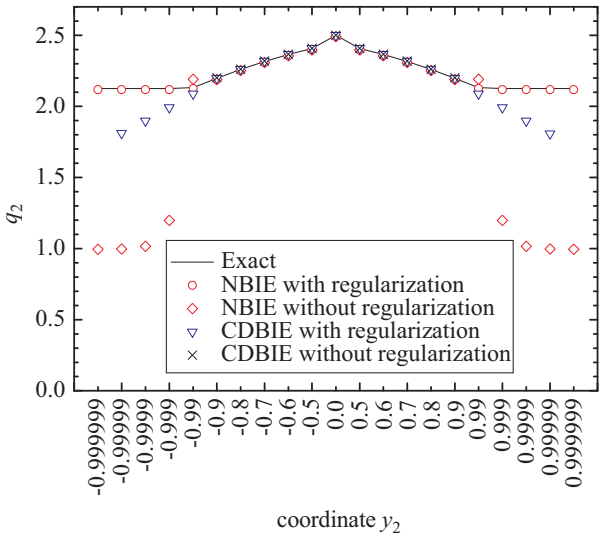


Fig. 9 q_2 of interior points.

Example 2. Heat conduction in a square with an anisotropic medium. The side length of the square is 2. $k_{11} = 4$, $k_{12} = 1$, $k_{22} = 2$.The boundary conditions are shown in Fig. 10, and 40 linear elements are uniformly generated on the boundary. The fluxes of boundary nodes are calculated by the NBIE and the difference method, respectively. The fluxes of 21 interior points close to Node C (1, 1) along the line $y_1 = y_2$ are calculated by the NBIE and the CDBIE.

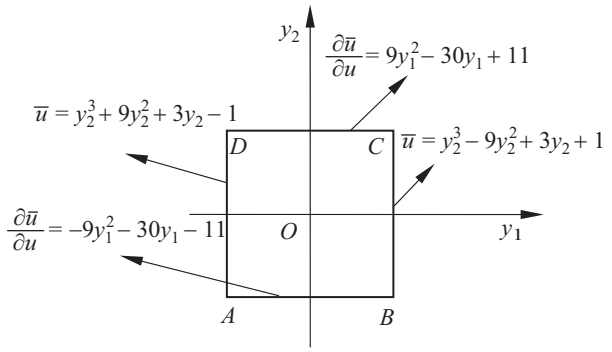


Fig. 10 Heat conduction in a square.

The analytical solution is $u(y_1, y_2) = y_1^3 + y_1^2 y_2 - 9y_1 y_2^2 + y_2^3 + 4y_1 + 5y_2$.

The flux results q_2 of boundary nodes are shown in Fig. 11, and the errors of q_2 are plotted in Fig. 12. The flux results q_1 and q_2 of interior points are presented in Figs. 13 and 14.

From Figs. 11 and 12, it can be discovered that the flux results of boundary nodes by the NBIE are more accurate than those by the difference method.

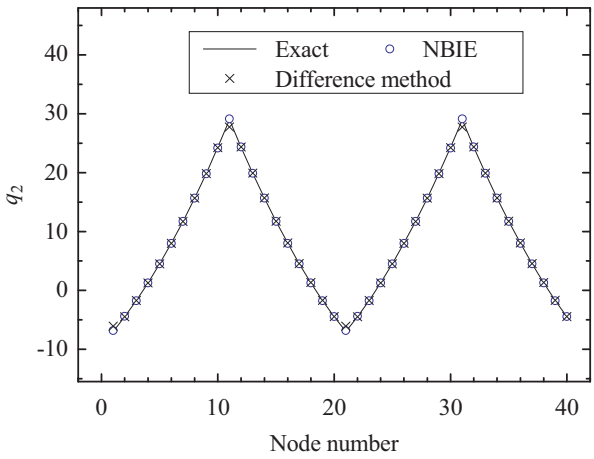


Fig. 11 q_2 of boundary nodes.

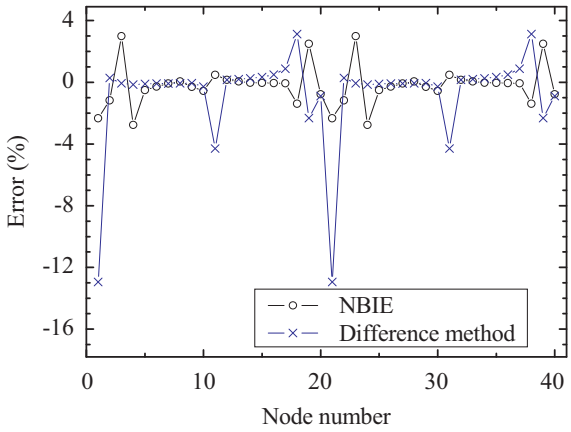


Fig. 12 Error of q_2 of boundary nodes.

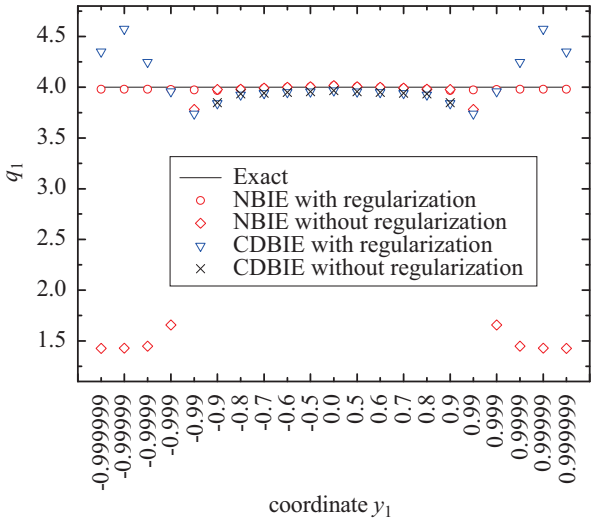


Fig. 13 q_1 of interior points.

From Figs. 13 and 14, it can be seen that the NBIE with regularization makes the best performance for solving the fluxes of interior points very close to the boundary, and the results by the NBIE with regularization are obviously more accurate than the results by the other three methods. It can be also found that the NBIE with

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regularization is always effective to compute the flux results of the interior points very close to the boundary, but the CDBIE without regularization is invalid for those points $|y_2| > 0.9$.

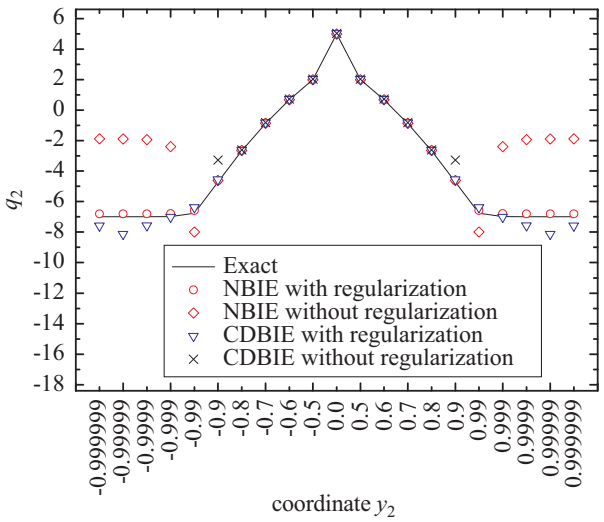


Fig. 14 q_2 of interior points.

7. CONCLUSIONS

In this paper, the NBIE of potential problems is firstly extended to solve anisotropic potential problems based on a coordinate transformation method. The NBIE is a novel potential derivative boundary integral equation. The most important virtue of the NBIE is that the singularity of the integral kernel function is reduced by one order in comparison with the CDBIE. Therefore the new potential derivative boundary integral equation only contains strongly singular integrals rather than hyper-singular integrals. The paper presents a new approach to calculate potential derivatives in the field of the boundary element method. The numerical examples on steady heat conduction with anisotropic media show that the NBIE can calculate more accurate flux results for both boundary nodes and interior points. Moreover, by combining the NBIE with the regularization algorithm of nearly singular integrals, the potential derivative of the interior points which are very close to the boundary can be computed more precisely, but the CDBIE without the regularization fails to obtain accurate results when an interior point just begins to be close to the boundary. The results by the NBIE with the regularization algorithm are also more accurate than those by the CDBIE with the regularization algorithm. The NBIE can be further extended to treat time-dependent potential problems such as transient heat conduction problem.

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