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HOW A SEMANTICS FOR TONK SHOULD BE

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Abstract. This paper explores how a semantics for Prior's infamous connective tonk should be, a connective defined by inference rules that trivialize the logic of a deductive system if that logic is supposed to be transitive. To avoid triviality, one must reject transitivity and in a relatively recent paper, Roy Cook develops a semantics for tonk with non-transitive entailment. However, I show in this paper that a cut-free sequent calculus for tonk - the arguably most natural and simplest deductive system for a non-transitive logic - can neither be complete with respect to Cook's semantics nor with respect to a semantics with non-transitive entailment based on a semantics for vagueness and transparent truth developed by Cobreros *et al*. It is argued that the failure to adequately represent tonk is connected with the fact that tonk is not uniquely defined in a cut-free sequent calculus system unless the logic is in addition non-reflexive. To remedy this, the paper develops a semantics with non-transitive and non-reflexive entailment based on the idea that complex formulae are true or false relative to them being assessed as premise or as conclusion.

§1. Introduction. This paper explores how a semantics for Prior's infamous connective tonk should be, a connective defined by inference rules that trivialize the logic of a deductive system if that logic is supposed to be transitive. To avoid triviality, one must reject transitivity and I illustrate how to accomplish this using a cut-free sequent calculus in Section 2. One might thus think that producing a semantics with non-transitive entailment is sufficient for that semantics to represent tonk as for example Roy Cook (2005) assumes. However, I show in Sections 3 and 4 that the cut-free sequent calculus for tonk can neither be complete with respect to Cook's semantics nor be complete with respect to the semantics with non-transitive entailment developed by Pablo Cobreros, Paul Egré, David Ripley and Robert van Rooij for vagueness and transparent truth adapted to tonk. In Section 5, the failure to represent tonk is connected with the fact that tonk is only uniquely defined in a cut-free sequent calculus if the logic is in addition non-reflexive. Inspired by the idea that the truth of a sentence is relative to whether it is assessed as premise or as conclusion, the paper develops in Section 6 a model-theoretic semantics for tonk with non-transitive and non-reflexive entailment. In Section 7, we prove that a sequent calculus for tonk that defines a non-transitive and non-reflexive logic is sound and complete with respect to that semantics.

§2. A cut-free sequent calculus for tonk. A.N. Prior (1960) introduced the connective tonk as part of a *reductio ad absurdum* of the thesis that the meanings of logical connectives are determined by inference rules governing their use. The idea was that if any set of inference rules are supposed to be meaning-determining, then the following pair

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of one introduction and one elimination rule for a natural deduction system would suffice to fix the meaning of his new connective tonk:

$$\frac{A}{A\text{-tonk-}B}\ \text{tonk-I} \qquad \frac{A\text{-tonk-}B}{B}\ \text{tonk-E}$$

If added to a standard natural deduction system, these rules will trivialize the logic.[1]

In response to Prior, Belnap suggested that an adequate proof-theoretic definition must be conservative: extending a deductive system with inference rules or axioms for a term t should not result in the derivation of sentences or inferences not containing t. (Belnap, 1962, p. 133) If we understand an inference as a relation between two multisets[2] of formulae, a more precise definition can be given as follows:[3] if $\mathcal{L}$ is our original language, $\mathcal{L}^t$ the language obtained by augmenting $\mathcal{L}$ with t, S the deductive system for $\mathcal{L}$ and S^t the deductive system obtained by adding inference rules or axioms for t to S, then S^t conservatively extends S only if there are no multisets of $\mathcal{L}$-formulae Γ and Δ such that $\Gamma \vdash_{S^t} \Delta$ but $\Gamma \nvdash_S \Delta$. As Belnap pointed out, the rules for tonk are clearly non-conservative if the resulting logic is non-empty and transitive.

However, if a logic can be non-transitive, we can develop deductive systems which are conservatively extended by Prior's rules. In natural deduction, one can accomplish this by requiring that every proof must be in normal form, i.e. that they contain no detours in the sense that there is some formula in the derivation which is both the conclusion of an introduction-rule and the major premise in an elimination-rule.[4] One could also adopt generalized elimination rules in which the major premise may only occur in top position in a derivation.[5]

It is more natural to obtain a non-transitive logic using sequent calculus. In a sequent calculus with two-sided sequents where the sequents represent multiple-conclusion inferences,[6] we can accomplish this by accepting only admissible instances of the structural rule CUT:[7]

$$\frac{\Gamma \vdash \Delta, A \qquad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \text{CUT}$$

By accepting only the admissible instances of CUT, we are accepting only those instances of it that hold independently of the rule in the sense that if there is a derivation of each

[1] In the following, I will use the term "logic" in the restricted sense according to which deductive systems such as natural deduction and sequent calculus, but also various semantics such as those introduced in this paper, define logics. These logics are characterized using a single turnstile $\vdash$ if defined with a deductive system and double turnstile $\models$ if defined with a notion of entailment and a semantics.

[2] Multisets are "sets with multiplicity" (Troelstra & Schwichtenberg, 2000, p. 6), that is, unordered lists or sequences in which an item can occur more than once.

[3] Cf. Gupta (2009) and Belnap (1962).

[4] For more on derivations in normal form and normalization in natural deduction, see Troelstra & Schwichtenberg (2000), in particularly Ch. 6.

[5] Schroeder-Heister suggests (but dismisses) this approach for inferential definitions of atomic formulae in Schroeder-Heister (2004).

[6] Some authors are sceptical about multiple conclusions, one recent example being Steinberger (2011). However, since nothing I say here relies on whether formal systems are supposed to capture "the way agents reason" as opposed to merely capturing what is logical, we can ignore such doubts in the following.

[7] In this and other sequent calculus rules throughout the paper, Γ and Δ are finite, possibly empty, multisets of formulae. Sequents are thus ordered pairs of finite multisets of formulae.

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of the premise-sequents, then there is also a derivation of the conclusion-sequent without using CUT.[8]

To see that this is sufficient to obtain a non-transitive logic for tonk, consider the sequent calculus rules for tonk which correspond to Prior's original natural deduction rules:[9]

$$\frac{A, \Gamma \vdash \Delta}{B\text{-tonk-}A, \Gamma \vdash \Delta}\ \text{tonkL} \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A\text{-tonk-}B}\ \text{tonkR}$$

In a sequent calculus system consisting of the rules CUT, tonkL, tonkR and every instance of ID as initial sequents where A is any formula:

$$\frac{}{\Gamma, A \vdash A, \Delta}\ \text{ID}$$

the following derivation is available which corresponds to Prior's two-step derivation in natural deduction:

$$\dfrac{\dfrac{A \vdash A}{A \vdash A\text{-tonk-}B}\ \text{tonkR} \qquad \dfrac{B \vdash B}{A\text{-tonk-}B \vdash B}\ \text{tonkL}}{A \vdash B}\ \text{CUT}$$

If we do not add CUT to the system, such a derivation is not available and the pair tonkR&tonkL are conservative over the instances of ID. This follows from the fact that the system enjoys the subformula property.[10] Notice how the rules tonkR and tonkL only introduce formulae while non-conservativeness would (in this case) require a derivation of $A \vdash B$ where A and B are two distinct atomic formulae. We cannot derive such a sequent using only tonkR and tonkL from instances of ID.

Let us thus define such a - from the perspective of conservativeness - good system for tonk:[11]

DEFINITION 2.1. *Let* $\mathrm{T}^{\mathrm{L}}_{\mathrm{R}}\mathrm{ID}$ *be the sequent calculus system consisting of* tonkR, tonkL *and* ID *for a language consisting of countably many propositional letters and the connective* tonk.

It is important to note that accepting only admissible instances of CUT does not necessarily deprive a system of transitivity. If for example the system consisted only of standard rules for the logical vocabulary such as $\neg$ and $\rightarrow$, every instance of CUT would be admissible, and if we extend that system with the rules for tonk, CUT fails to be admissible only under certain conditions for certain formulae containing tonk.

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8 For a definition of admissibility, see Negri & von Plato (2001, p. 20).

9 These rules are for example presented in Hjortland (2009) and by Greg Restall in various presentations. See also Ripley (2014).

10 A sequent calculus system satisfies the subformula property if and only if every formula in a derivation is a subformula of a formula in the endsequent, the "conclusion", of that derivation. For an introduction to the subformula property, see Negri & von Plato (2001, p. 15). Note that it is thus not sufficient for a system to be cut-free for the subformula property to obtain. In addition, the rules must increase the complexity of a formula and not eliminate formulae from the premise-sequents.

11 There are countably many systems which are conservatively extended by tonk but which are more "admissible" than $\mathrm{T}^{\mathrm{L}}_{\mathrm{R}}\mathrm{ID}$. For example, we can envisage a natural deduction system in which we can derive any formula from any formula containing tonk but it is not the case that we can derive any tonk-free formula from any tonk-free formula. However, tonk would not be so well-behaved in such a system and it would, tonk-wise, be quite an uninteresting system. I choose thus in this paper to focus on $\mathrm{T}^{\mathrm{L}}_{\mathrm{R}}\mathrm{ID}$.

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Finally, one may also note that even if $A \vdash A\text{-tonk-}B$ and $A\text{-tonk-}B \vdash B$ holds in $\mathrm{T}^{\mathrm{L}}_{\mathrm{R}}\mathrm{ID}$, it is nevertheless the case that $A \nvdash B\text{-tonk-}A$ and $B\text{-tonk-}A \nvdash B$. This should not surprise us and is in accordance with Prior's inferential definition of tonk. Indeed, since tonk's "meaning is completely given by the rules" (Prior, 1960, p. 39) tonk-I and tonk-E, it would seem that a system which is such that $A \vdash B\text{-tonk-}A$ or $B\text{-tonk-}A \vdash B$ overgenerates since deriving those inferences would require some extra rules over and above Prior's two rules (unless the system's logic is transitive and thus trivial). Prior presented two inference *rules* and not two inferences, and it would thus be a mistake to understand them merely as necessary conditions for a connective to be tonk along the lines of Tarski's condition for the material adequacy of a formal definition of truth in Tarski (1944). It follows also that if a semantics overgenerates by validating $A \vDash B\text{-tonk-}A$ or $B\text{-tonk-}A \vDash B$, it does not represent tonk.[12]

§3. Cook's tonk-consequence. Cook (2005) develops a semantics with a notion of entailment which validates the tonk-inferences for a connective $\otimes$, i.e. that A entails $A \otimes B$ and $A \otimes B$ entails B, but without it being always the case that A entails B.

In a bivalent semantics, forward truth-preservation defined as 'if the premisses are true then the conclusion is true' is equivalent to backwards falsity-preservation defined as 'if the conclusion is false then some of the premisses are false'. This equivalence breaks down with more than two truth-values and if one, for some reason, is unwilling to give priority to either forward truth-preservation or backwards falsity-preservation, one could possibly, Cook suggests, go for a disjunctive definition of consequence.

Cook presents *Tonk Consequence* using a relational semantics where a valuation is not a function from the set of well-formed formulae of a language $\mathcal{L}$ ($\mathrm{WFF}_{\mathcal{L}}$) to a set of truth-values $\{1, 0\}$ but rather a relation $R \subseteq \mathrm{WFF}_{\mathcal{L}} \times \{1, 0\}$. Entailment is defined as follows:

DEFINITION 3.1. $\Gamma \vDash_{TC} A$ *iff either for all valuations R: if R(B,1) for all* $B \in \Gamma$*, then R(A,1) or for all valuations R: if R(A,0), then R(B,0) for some* $B \in \Gamma$.

With this notion of consequence on board, Cook augments $\mathcal{L}$ with a connective $\otimes$ and puts the following constraints on R:

DEFINITION 3.2. *R is a valuation for* $\mathcal{L}^{\otimes}$ *only if the following conditions for every sentence A, B in* $WFF_{\mathcal{L}^{\otimes}}$ *hold:*

- $R \subseteq WFF_{\mathcal{L}^{\otimes}} \times \{1, 0\}$
- $R(A \otimes B, 1)$ *if and only if* $R(A, 1)$
- $R(A \otimes B, 0)$ *if and only if* $R(B, 0)$

It follows that $A \vDash_{TC} A \otimes B$ because if A is true then $A \otimes B$ is true and $A \otimes B \vDash_{TC} B$ because if B is false then $A \otimes B$ is false. (Cook, 2005, p. 223). $\otimes$ satisfies thus the two inferences associated with tonk without it always being the case that $A \vDash_{TC} B$.

However, despite being tonk-like, $\otimes$ is clearly not tonk. In addition to $A \otimes B \vDash_{TC} B$ and $A \vDash_{TC} A \otimes B$, it is also the case that $B \vDash_{TC} A \otimes B$ because if $A \otimes B$ is false then B is false and $A \otimes B \vDash_{TC} A$ because if $A \otimes B$ is true then A is true. We would not expect this from tonk and the latter inferences do not hold for tonk in $\mathrm{T}^{\mathrm{L}}_{\mathrm{R}}\mathrm{ID}$. *Tonk Consequence* overgenerates. One may wonder whether we can find some alternative clauses for $\otimes$ which

[12] Thanks to an anonymous referee for pressing me on this issue.

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would deliver tonk. However, as we shall see in the next section, we cannot use *Tonk Consequence* to define a logic for which (a notational variant of) $\mathrm{T}^{\mathrm{L}}_{\mathrm{R}}\mathrm{ID}$ is complete.

§4. Strict-tolerant semantics. In recent work,[13] Cobreros *et al* define entailment on trivalent valuations by requiring that if the premises are true, then at least one of the conclusions is not false. On a trivalent valuation, a sentence is said to be tolerantly true if its value is in $\{1, \frac{1}{2}\}$, tolerantly false if its value is in $\{\frac{1}{2}, 0\}$, strictly true if its value is 1 and strictly false if its value is 0. An inference is thus valid if and only if, if the premises are strictly true, then at least one of the conclusions is tolerantly true. The framework was originally developed for vagueness but has subsequently been adopted for transparent truth. While the semantics has not been applied to tonk-like connectives in published work, it is nevertheless instructive for our purposes to attempt to represent tonk in it.

With I_{ST} being a function from the well-formed formulae to $\{1, \frac{1}{2}, 0\}$, entailment for *strict-tolerant semantics* is defined as follows:

DEFINITION 4.1. $\Gamma \vDash_{ST} \Delta$ *if and only if, if for every* $A \in \Gamma$, $I_{ST}(A) = 1$ *then there is some* $B \in \Delta$ *s.t.* $I_{ST}(B) \in \{1, \frac{1}{2}\}$.

As a suitable truth-function for $\otimes$, we cannot use the following suggestion by Cook (2005, p. 226):

	$A \otimes B$	B: 1	B: $\frac{1}{2}$	B: 0
	1	1	$\frac{1}{2}$	$\frac{1}{2}$
A	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
	0	0	0	0

While it would be the case that $A \otimes B \vDash_{ST} B$, $A \vDash_{ST} A \otimes B$ and $B \nvDash_{ST} A \otimes B$, it would still be the case that $A \otimes B \vDash_{ST} A$. The problem is that if $I_{ST}(A \otimes B) = 1$ then $I_{ST}(A) \in \{1, \frac{1}{2}\}$. The following truth-function would get all four inferences correct:

	$A \otimes B$	B: 1	B: $\frac{1}{2}$	B: 0
	1	1	$\frac{1}{2}$	$\frac{1}{2}$
A	$\frac{1}{2}$	$\frac{1}{2}$	1	0
	0	0	1	0

However, this truth-function is also problematic.

In fact, any truth-function will be problematic because it will not support the induction on the complexity of a formula required to prove completeness. Suppose that we want to prove that a notational variant of $\mathrm{T}^{\mathrm{L}}_{\mathrm{R}}\mathrm{ID}$ where "tonk" is substituted for "$\otimes$" is complete relative to $\vDash_{ST}$ where $\otimes$ is assigned a seemingly suitable truth-function. Following the recipe in Ripley (2013), one proves completeness in two steps. First, one uses reduction trees to produce an unprovable sequent $\Gamma \nvdash \Delta$ that contains for each complex formula A the subformulae of A required to derive A. For example, if $A \otimes B$ is in Γ, then B is in Γ, and if $A \otimes B$ is in Δ then A is in Δ. Then, a model is defined based on the unprovable sequent such that for each atomic formula $\alpha \in \Gamma$, $I_{ST}(\alpha) = 1$ and for every atomic formula $\alpha \in \Delta$, $I_{ST}(\alpha) = 0$. Finally, one proves by induction on the complexity of a formula that

[13] See for example Cobreros *et al.* (2012), Cobreros *et al.* (2013), Ripley (2012), and Ripley (2013).

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this is indeed a model for $\Gamma \nvdash \Delta$. In the inductive step, one shows that if it holds for formulae of complexity n, then it holds for formulae of complexity $n + 1$. Suppose now that the formula in Γ is of the form $A \otimes B$. It follows that B is also in Γ and by the induction hypothesis, $I_{ST}(B) = 1$. To secure that $I_{ST}(A \otimes B) = 1$, $I_{ST}(B) = 1$ must be sufficient for $I_{ST}(A \otimes B) = 1$. Correspondingly, if the formula in Δ is of the form $A \otimes B$, it follows that A is in Δ and by the induction hypothesis, $I_{ST}(A) = 0$. To secure that $I_{ST}(A \otimes B) = 0$, $I_{ST}(A) = 0$ must be sufficient for $I_{ST}(A \otimes B) = 0$. This combination, however, requires that the value of $A \otimes B$ is 1 whenever B is 1 and 0 whenever A is 0. This is impossible. *Strict-tolerant semantics* cannot be used to define a logic for which (a notational variant of) $\mathrm{T}^{\mathrm{L}}_{\mathrm{R}}\mathrm{ID}$ is complete.

While a completeness proof concerning *Tonk Consequence* would be much more complex than the completeness proof concerning *strict-tolerant semantics*, it is nevertheless the case that an analogous situation will arise in the case of *Tonk Consequence*: it will require for example that if $R(B, 1)$ then $R(A \otimes B, 1)$ and that if it is not the case that $R(A, 1)$ then it is not the case that $R(A \otimes B, 1)$. Under the assumption that clauses for R are supposed to be biconditionals, a completeness proof for *Tonk Consequence* is also out of the question.

§5. The problem with tonk. As it turns out, the problem with representing tonk in *Tonk Consequence* and *strict-tolerant semantics* is related to the fact that tonk is not uniquely defined in $\mathrm{T}^{\mathrm{L}}_{\mathrm{R}}\mathrm{ID}$. In the same paper that Belnap introduces conservativeness as condition for a proof-theoretic definition he also presents a second condition, uniqueness. (Belnap, 1962) A n-ary connective C is uniquely defined relative to a particular deductive system if and only if any connective C' defined with the same axioms or inference rules as C is such that C and C' are both premise-equivalent,

$$\Gamma, C(A_1, \ldots A_n) \vdash \Delta \text{ if and only if } \Gamma, C'(A_1, \ldots A_n) \vdash \Delta$$

and conclusion-equivalent,

$$\Gamma \vdash C(A_1, \ldots A_n), \Delta \text{ if and only if } \Gamma \vdash C'(A_1, \ldots A_n), \Delta$$

Uniqueness fails for tonk in $\mathrm{T}^{\mathrm{L}}_{\mathrm{R}}\mathrm{ID}$. Let $\mathrm{T}^{\mathrm{L}}_{\mathrm{R}}\mathrm{ID}$ be augmented with a connective *monk* defined with the same rules as tonk. Both tonk and monk satisfy conservativeness. However, while there is a one-line derivation of A-tonk-$B \vdash A$-tonk-B through ID, there is no corresponding derivation of A-monk-$B \vdash A$-tonk-B since the system enjoys the subformula property and a cut-free derivation of A-monk-$B \vdash A$-tonk-B for any A and B requires thus $B \vdash A$ for any A and B:

$$\dfrac{\dfrac{\Gamma, B \vdash A, \Delta}{\Gamma, A\text{-monk-}B \vdash A, \Delta}}{\Gamma, A\text{-monk-}B \vdash A\text{-tonk-}B, \Delta}$$

Tonk and monk are not premise-equivalent. Similarly, there is no derivation of A-tonk-$B \vdash A$-monk-B. Tonk and monk are also not conclusion-equivalent.

To see why uniqueness is not obtainable in $\mathrm{T}^{\mathrm{L}}_{\mathrm{R}}\mathrm{ID}$, consider the following: While the rules guarantee that if $\Gamma, B \vdash \Delta$ then Γ, A-tonk-$B \vdash \Delta$, a proof of uniqueness requires in addition that it is invertible: if Γ, A-tonk-$B \vdash \Delta$ then $\Gamma, B \vdash \Delta$. Only then can we guarantee that if Γ, A-tonk-$B \vdash \Delta$ then Γ, A-monk-$B \vdash \Delta$. For example, if we could replace A-tonk-B by B on the left-side in A-tonk-$B \vdash A$-tonk-B then we could derive A-monk-$B \vdash A$-tonk-B by one application of the left-rule for monk. However, satisfying this would trivialize $\mathrm{T}^{\mathrm{L}}_{\mathrm{R}}\mathrm{ID}$. If A-tonk-$B \vdash A$-tonk-B holds it cannot be the case

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that B and A-tonk-B are premise-equivalent and that A and A-tonk-B are conclusion-equivalent unless $B \vdash A$ because it would allow us to derive $B \vdash A$ for any A and B through substitution.

This is also how trouble arises for *strict-tolerant semantics* with regard to representing tonk: a completeness proof would require that A-tonk-B is dissected to B on the left-hand side of a sequent at the same time as $I(B) = 1$ is sufficient for $I(A\text{-tonk-}B) = 1$. This corresponds to the premise-equivalence of B and A-tonk-B. Analogously, the proof would require that A-tonk-B and A are conclusion-equivalent.

The troublemaker for uniqueness in $\mathrm{T}^{\mathrm{L}}_{\mathrm{R}}\mathrm{ID}$ is ID and we obtain uniqueness by restricting ID to the tonk-free fragment.

DEFINITION 5.1. *Let* $\mathrm{T}^{\mathrm{L}}_{\mathrm{R}}\mathrm{ID}_\alpha$ *be the sequent calculus system for a language consisting of countably many atomic formulae and the connective* tonk *consisting of* tonkR, tonkL *and the following initial sequents:*

$$\frac{}{\Gamma, \alpha \vdash \alpha, \Delta}\ \mathrm{ID}_\alpha$$

where α *is an atomic formula and* Γ, Δ *are multisets of formulae.*

It should be clear that $\vdash$ is both non-reflexive and non-transitive in $\mathrm{T}^{\mathrm{L}}_{\mathrm{R}}\mathrm{ID}_\alpha$. Since $\mathrm{T}^{\mathrm{L}}_{\mathrm{R}}\mathrm{ID}_\alpha$ is cut-free and the rules for tonk are such that the subformula property is satisfied, a derivation of A-tonk-$B \vdash A$-tonk-B would be of the above form (with "monk" substituted for "tonk") but that is out of the question since it is not the case that for any A and B; $B \vdash A$. As promised, tonk is uniquely defined in $\mathrm{T}^{\mathrm{L}}_{\mathrm{R}}\mathrm{ID}_\alpha$:

THEOREM 5.2. *Let* $\mathrm{T}^{\mathrm{L}}_{\mathrm{R}}\mathrm{ID}_\alpha$ *be augmented by a connective monk characterized by the same rules as tonk. Tonk and monk are premise- and conclusion-equivalent in* $\mathrm{T}^{\mathrm{L}}_{\mathrm{R}}\mathrm{ID}_\alpha$ *augmented with monk.*

Proof. The proof is by induction on the height of a derivation where the height of a derivation is defined as usual.[14] We show here that they are premise-equivalent. The proof that they are conclusion-equivalent is analogous. The base case: Since initial sequents are restricted to atomic formula, it is clear that Γ, A-tonk-$B \vdash \Delta$ is an initial sequent if and only if Γ, A-monk-$B \vdash \Delta$ is an initial sequent. The inductive step: there are two cases to consider for each direction of the equivalence. We show here the direction from tonk to monk. The converse is analogous. Assume that there is a derivation with height $n+1$ of Γ, A-tonk-$B \vdash \Delta$ and the last applied rule is tonkL. It follows that there is a derivation with height n of Γ, $B \vdash \Delta$ and hence that there is a derivation of Γ, A-monk-$B \vdash \Delta$ with the height $n+1$. If the last applied rule is not tonkL but some other rule R, then there is a derivation with the height n of Γ', A-tonk-$B \vdash \Delta'$. By the induction hypothesis, there is a derivation with height n of Γ', A-monk-$B \vdash \Delta'$ and by one application of R, it follows that there is a derivation of height $n+1$ of Γ, A-monk-$B \vdash \Delta$. □

If we adopt a cut-free sequent calculus for tonk, we should in addition accept that $\vdash$ is non-reflexive to satisfy uniqueness.

A curious feature of $\mathrm{T}^{\mathrm{L}}_{\mathrm{R}}\mathrm{ID}_\alpha$ is that B is premise-equivalent to A-tonk-B, and A is conclusion-equivalent to A-tonk-B, but it is not the case that for any formula A and B, $A \vdash A$-tonk-B and A-tonk-$B \vdash B$. Instead, it is only the case that $A \vdash A$-tonk-B only

[14] See Negri & von Plato (2001) for a definition of the height of a derivation. Note that we are proving a stronger result than premise-equivalence and conclusion-equivalence since this shows that they are height-preserving premise- and conclusion-equivalent.

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if $A \vdash A$, and A-tonk-$B \vdash B$ only if $B \vdash B$. The subformula property guarantees this result. This means that it is not the case that each instance of the inferences we typically associate with tonk is derivable in $\mathrm{T}^{\mathrm{L}}_{\mathrm{R}}\mathrm{ID}_{\alpha}$ even if the system is defined with the sequent calculus rules that correspond to Prior's natural deduction rules. This is consistent with our commitment to the rules rather than the inferences as explained at the end of Section 2 and should thus not bother us.

§6. Premise/conclusion relativism. Since A-tonk-B is supposed to be premise-equivalent to B and conclusion-equivalent to A, it is perhaps tempting to think that tonk should be captured by two distinct truth-functions, one for premises and one for conclusions. The idea that tonk should be understood in terms of more than one truth-function was already suggested by Stevenson in a reply to Prior. In Stevenson (1961), Stevenson suggests that tonk can be safely separated into two distinct connectives governed by two truth-functions represented by the following truth-tables:

tonki	1	0
1	1	1
0	0	0

tonko	1	0
1	1	0
0	1	0

This corresponds to the following biconditionals:

(1) $V(A\text{-tonki-}B) = 1$ if and only if $V(A) = 1$
(2) $V(A\text{-tonko-}B) = 1$ if and only if $V(B) = 1$

While it is the case that tonki and tonko overgenerate with entailment as forward truth-preservation, we are not interested in two distinct connectives. The aim is rather to obtain a semantics which is such that (2) holds for tonk as premise and (1) holds for tonk as conclusion. In this section, we will accomplish this by employing Humberstone's framework for heterogeneous logics.

Humberstone (1988) suggests that entailment can be defined using pairs of valuations - one for premises and one for conclusions - in the following way: Γ entails Δ if and only if there is no pair of valuations such that all the Γ's are true on the valuation for premisses and all the Δ's are false on the valuation for conclusions. Hence, while models typically contain one interpretation or valuation of each formula of the language, a model is now an ordered pair $\langle V_p, V_c \rangle$, where V_p and V_c are valuation-functions for each formula of the language as premises and conclusions respectively.

With different valuations for premises and conclusions, we seem to be dealing with something like an alethic premise/conclusion-relativism analogous to MacFarlane's truth relativism. (MacFarlane, 2012) While MacFarlane's relativism consists roughly in the claim that the truth of sentences like "olives are tasty" are relative to the standard of the assessor, the kind of relativism involved here concerns how the truth of a sentence is relative to whether it is assessed as premise or as conclusion.

We can make sense of premise/conclusion-relativism with a broadly use-theoretic perspective on meaning. According to Dummett (1991), there are two aspects of the assertoric use of a sentence which determine the meaning of a sentence: the conditions under which asserting the sentence is licensed and the commitments of such an assertion. For Dummett, the two aspects manifest themselves in natural deduction as introduction and elimination-rules. Now, introduction- and elimination-rules correspond to left- and right introduction rules in sequent calculus, rules which arguably can be described as introducing a term as premise or as conclusion. Hence, it is not unreasonable to connect a focus on commitments

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with treating a sentence as premise, and a focus on conditions for asserting with treating the sentence as a conclusion. While Dummett required the two aspects of use to be in harmony, a condition which tonk fails to satisfy, we are here embracing the potential disharmony between them by suggesting that it give rise to a premise/conclusion relativism in model-theoretic semantics. If we look at the semantics developed in this section in this way, it is merely an attempt to make the two aspects of use explicit as two standards for evaluating the truth of a sentence in those cases where there is a disharmony between conditions and commitments.

One might protest that sentences are not always parts of arguments in a dialogue, i.e. used as premises or as conclusion. Instead, they are sometimes simply asserted, something which would make an assessment as premise or conclusion inappropriate. For example, someone can utter "it is sunny in the summer" or perhaps more awkwardly, "it is sunny in the summer tonk Brazil won the world cup" without intending to conclude anything and without basing it on something like an observation or some premises. Nonetheless, such assertions can be considered as starting-points for dialogues or as minimal dialogues and thus as potential premises or conclusions which in turn explains why they are, despite appearance, true or false as premise and as conclusion. The speaker may after uttering "it is sunny in the summer tonk Brazil won the world cup" wonder whether or not Brazil won the world cup and validly use her first utterance to falsely conclude that they won. On the other hand, she may wonder what would justify such a claim and go on to consider whether it is the case that it is sunny in the summer.[15]

The semantics requires suitable premise- and conclusion clauses for the various connectives. As clauses for tonk, we adopt the following modifications of Stevenson's clauses:

(1) $V_c(A\text{-tonk-}B) = 1$ if and only if $V_c(A) = 1$
(2) $V_p(A\text{-tonk-}B) = 1$ if and only if $V_p(B) = 1$

While we focussed in the previous sections on a semantics for a language containing only one connective, we shall in this section focus on a language containing in addition $\neg$ and $\rightarrow$ to illustrate the interplay between the classical connectives and tonk in the semantics and hence in a non-transitive and non-reflexive logic. As it turns out, we can read off suitable clauses for $\neg$ and $\rightarrow$ from their sequent calculus rules.[16]

With the above definition of validity and since sequents are supposed to match up with validity for soundness and completeness, it is natural to associate the left-hand side of a two-sided sequent with being premise-assessed as true and its right-hand side with being conclusion-assessed as false, more or less analogous to the way Restall (2005) associates the left-hand side with assertion and the right-hand side with denial. This allows us to adopt and modify Restall's way of paraphrasing sequent calculus rules by replacing "asserting" and "an assertion" with "premise-assessing as true" and "denying" and "a denial" with "conclusion-assessing as false".

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15 Thanks to an anonymous for pressing me on this point.

16 Alternatively, we can follow a suggestion given to me by Aaron Cotnoir and introduce a procedure for translating between *strict-tolerant semantics* and clauses for the semantics developed in this section: Assuming that $V_p(A) = 1$ *if and only if* $I_{ST}(A) = 1$ and $V_c(A) = 1$ *if and only if* $I_{ST}(A) \in \{1, \frac{1}{2}\}$, one can move back and forth between clauses for my semantics and Strong Kleene valuation schemes for negation and conditional through a substitution of equivalents. Of course, this method does not give clauses for tonk.

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Consider for example the rules for ¬:

$$\frac{\Gamma \vdash \Delta, A}{\Gamma, \neg A \vdash \Delta}\ \neg\text{L} \qquad \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ \neg\text{R}$$

In Restall (2005), Restall describes them as telling us that, relative to a situation in which all the Γ's are asserted and all the Δ's are denied, "asserting $\neg A$ has the same effect as a denial of A and denying $\neg A$ has the same effect as the assertion of A" (Restall, 2005, p. 200). After substitution, the rules state that premise-assessing $\neg A$ as true is the same as conclusion-assessing A as false and conclusion-assessing $\neg A$ as false is the same as premise-assessing A as true. With the help of a classical meta-theory, we obtain the following clauses: $V_p(\neg A) = 1$ iff $V_c(A) = 0$ and $V_c(\neg A) = 1$ iff $V_p(A) = 0$.

Consider now the right introduction rule for →:

$$\rightarrow\text{R}\ \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \rightarrow B}$$

In accordance with Restall's reading of rules, this rule states that conclusion-assessing $A \rightarrow B$ as false is the same as premise-assessing A as true and conclusion-assessing B as false, i.e. $V_c(A \rightarrow B) = 0$ if and only if $V_p(A) = 1$ and $V_c(B) = 0$. This is equivalent to $V_c(A \rightarrow B) = 1$ if and only if, if $V_p(A) = 1$ then $V_c(B) = 1$.

Finally, consider the left introduction rule for →:

$$\rightarrow\text{L}\ \frac{\Gamma \vdash \Delta, A \qquad B, \Gamma \vdash \Delta}{A \rightarrow B, \Gamma \vdash \Delta}$$

This rule branches upwards and is thus presenting two alternatives: premise-assessing $A \rightarrow B$ a true is the same as either premise-assessing B as true or conclusion-assessing A a false. This amounts to that $V_p(A \rightarrow B) = 1$ iff $V_c(A) = 0$ or $V_p(B) = 1$ which is equivalent to $V_p(A \rightarrow B) = 1$ iff, if $V_c(A) = 1$ then $V_p(B) = 1$.

We can now present our semantics for a language with countably many propositional letters (the atomic formulae), tonk, negation and conditional:[17]

DEFINITION 6.1. *A PCR-model is an ordered pair $\langle V_p, V_c \rangle$ such that V_p and V_c are bivalent valuation-functions satisfying the following conditions:*

- *For each propositional letter α: $V_p(\alpha) = V_c(\alpha)$*
- $V_p(\neg A) = 1$ *iff* $V_c(A) = 0$
- $V_c(\neg A) = 1$ *iff* $V_p(A) = 0$
- $V_p(A \rightarrow B) = 1$ *iff if* $V_c(A) = 1$ *then* $V_p(B) = 1$
- $V_c(A \rightarrow B) = 1$ *iff if* $V_p(A) = 1$ *then* $V_c(B) = 1$
- $V_p(A\text{-}tonk\text{-}B) = 1$ *iff* $V_p(B) = 1$
- $V_c(A\text{-}tonk\text{-}B) = 1$ *iff* $V_c(A) = 1$

[17] For simplicity, we will not add conjunction and disjunction. For the record, they would get the following clauses:

- $V_p(A \wedge B) = 1$ iff $V_p(A) = 1$ and $V_p(B) = 1$
- $V_c(A \wedge B) = 1$ iff $V_c(A) = 1$ and $V_c(B) = 1$
- $V_p(A \vee B) = 1$ iff $V_p(A) = 1$ or $V_p(B) = 1$
- $V_c(A \vee B) = 1$ iff $V_c(A) = 1$ or $V_c(B) = 1$.

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Entailment is defined in accordance with the above remarks:

DEFINITION 6.2. *Where* Γ *and* Δ *are sets of formulae:* $\Gamma \vDash_{PC} \Delta$ *if and only if every PCR model* $\langle V_p, V_c \rangle$ *is such that if* $\forall A \in \Gamma$, $V_p(A) = 1$ *then* $\exists B \in \Delta$, $V_c(B) = 1$

As it turns out, $\vDash_{PC}$ is both non-transitive and non-reflexive:

THEOREM 6.3. *There are formulae A and B such that* $A \vDash_{PC}$ *A-tonk-B and* *A-tonk-B* $\vDash_{PC}$ *B but* $A \nvDash_{PC}$ *B*

Proof. Let α and β be two atomic formulae and define a PCR-model such that $V_p(\alpha) = V_c(\alpha) = 1$ and $V_p(\beta) = V_c(\beta) = 0$. It follows that there is a model such that $V_p(\alpha) = 1$ and $V_c(\beta) = 0$, thus $\alpha \nvDash_{PC} \beta$. It is nevertheless the case that $\alpha \vDash \alpha$-tonk-β and α-tonk-$\beta \vDash \beta$ since every PCR-model is such that if $V_p(\alpha) = 1$ then $V_c(\alpha\text{-tonk-}\beta) = 1$ and if $V_p(\alpha\text{-tonk-}\beta) = 1$ then $V_c(\beta) = 1$ because α and β are atomics and are thus assigned same value as premise and as conclusion in each model. For example, assume that $V_p(\alpha) = 1$. It follows that $V_c(\alpha) = 1$ and hence that $V_c(\alpha\text{-tonk-}\beta) = 1$ by the conclusion-clause for tonk. □

THEOREM 6.4. *There is a formula A such that* $A \nvDash_{PC} A$

Proof. Let A be the formula α-tonk-β where α and β are atomic formulae. Define a PCR-model such that $V_p(\alpha) = V_c(\alpha) = 0$ and $V_p(\beta) = V_c(\beta) = 1$. It follows that $V_p(\alpha\text{-tonk-}\beta) = 1$ since $V_p(\beta) = 1$ and $V_c(\alpha\text{-tonk-}\beta) = 0$ since $V_p(\alpha) = 0$ □

Our models are dialethic in the sense that both a sentence and its negation can be true as premise or false as conclusion. The latter is a consequence of failure of reflexivity while the former is a consequence of failure of transitivity:

THEOREM 6.5. *There are models such that for some formula A,* $V_c(A) = 0$ *and* $V_c(\neg A) = 0$.

Proof. Let A be B-tonk-C, where B and C are distinct formulae. Let $V_p(C) = 1$ and $V_c(B) = 0$. From $V_p(C) = 1$, $V_p(B\text{-tonk-}C) = 1$ follows which in turn implies $V_c(\neg(B\text{-tonk-}C)) = 0$. From $V_c(B) = 0$, $V_c(B\text{-tonk-}C) = 0$ follows. □

THEOREM 6.6. *There are models such that for some formula A,* $V_p(A) = 1$ *and* $V_p(\neg A) = 1$.

Proof. As above. Let A be B-tonk-C, where B and C are distinct formulae. Let $V_p(C) = 1$ and $V_c(B) = 0$. It follows that $V_p(B\text{-tonk-}C) = 1$ and $V_c(B\text{-tonk-}C) = 0$, and hence that $V_p(\neg(B\text{-tonk-}C)) = 1$. □

Reflexivity for A-tonk-B is dependent on whether B entails A:

THEOREM 6.7. *A-tonk-B* $\vDash_{PC}$ *A-tonk-B only if B* $\vDash_{PC}$ *A*

Proof. Suppose that $V_p(A\text{-tonk-}B) = 0$ or $V_c(A\text{-tonk-}B) = 1$ holds in an arbitrary *PCR*-model. Since $V_p(A\text{-tonk-}B) = 0$ is equivalent to $V_p(B) = 0$ and $V_c(A\text{-tonk-}B) = 1$ is equivalent to $V_c(A) = 1$, $V_p(B) = 0$ or $V_c(A) = 1$ holds in the model. □

Unsurprisingly, the tonk-inferences do not hold unrestrictedly:

THEOREM 6.8. *A* $\vDash_{PC}$ *A-tonk-B only if A* $\vDash_{PC}$ *A*.

Proof. Assume that $V_p(A) = 0$ or $V_c(A\text{-tonk-}B) = 1$ holds in an arbitrary *PCR*-model. Since $V_c(A\text{-tonk-}B) = 1$ is equivalent to $V_c(A) = 1$, it follows that $V_p(A) = 0$ or $V_c(A) = 1$ holds in our arbitrary model, i.e. $A \vDash_{PC} A$. □

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THEOREM 6.9. *A-tonk-B* $\models_{PC}$ *B only if* $B \models_{PC} B$.

Proof. As above. □

Similarly, various classical laws do not hold unrestrictedly. For example, $\models_{PC} \neg\neg A \rightarrow A$ holds only if $A \models_{PC} A$ holds: Suppose that $V_c(\neg\neg A \rightarrow A) = 1$ holds in an arbitrary *PCR*-model. It follows that $V_p(\neg\neg A) = 0$ or $V_c(A) = 1$. With the principles for negation, $V_p(\neg\neg A) = 0$ implies $V_c(\neg A) = 1$ which in turn implies $V_p(A) = 0$. Hence, if $V_p(A) = 1$ then $V_c(A) = 1$ which means that $A \models_{PC} A$ holds.

Just as with $\mathrm{T}^{\mathrm{L}}_{\mathrm{R}}\mathrm{ID}_\alpha$, it is straight-forward to show that uniqueness holds for tonk in $\models_{PC}$. Extending the semantics with monk with the same clauses as tonk results in that *A*-tonk-*B* and *A*-monk-*B* take the same value as premise and as conclusion. In fact, we have a semantics for which $\mathrm{T}^{\mathrm{L}}_{\mathrm{R}}\mathrm{ID}_\alpha$ augmented with rules for $\rightarrow$ and $\neg$ is sound and complete.

§7. Soundness and completeness. In this section, we introduce a semantic tableaux system for the semantics, show that the semantic tableaux system is equivalent to a given sequent calculus system for tonk and finally show that the semantic tableaux system is complete with regard to the *PCR* semantics. But first, we present the sequent calculus system.

DEFINITION 7.1. *Let* $\mathrm{CNT}^{\mathrm{L}}_{\mathrm{R}}\mathrm{ID}_\alpha$ *be the system* $\mathrm{T}^{\mathrm{L}}_{\mathrm{R}}\mathrm{ID}_\alpha$ *augmented with* $\rightarrow\mathrm{L}$, $\rightarrow\mathrm{R}$, $\neg\mathrm{L}$ *and* $\neg\mathrm{R}$.

THEOREM 7.2. *The sequent calculus system* $\mathrm{CNT}^{\mathrm{L}}_{\mathrm{R}}\mathrm{ID}_\alpha$ *is sound for PCR semantics, i.e. if* $\Gamma \vdash \Delta$ *then* $\Gamma \models_{PC} \Delta$.

Proof. As usual, it suffices to show that the initial sequents and inference rules hold in the semantics. We will here consider two cases. The others are left for the reader. Let $V_p(\Gamma) = 1$ abbreviate the conjunction of $V_p(\phi) = 1$ for all $\phi \in \Gamma$ and $V_c(\Delta) = 1$ abbreviate the disjunction of $V_c(\phi) = 1$ for all $\phi \in \Delta$.

($\rightarrow$L) Let $\mathcal{M}$ be an arbitrary *PCR*-model. Suppose that *if* $V_p(\Gamma) = 1$ *then* $V_c(A) = 1$ *or* $V_c(\Delta) = 1$ and *if* $V_p(\Gamma) = 1$ *and* $V_p(B) = 1$ *then* $V_c(\Delta) = 1$. Assume $V_p(\Gamma) = 1$ and $V_p(A \rightarrow B) = 1$. It follows from the premise-principle for $\rightarrow$ that $V_c(A) = 0$ *or* $V_p(B) = 1$. Assume $V_c(A) = 0$. Since $V_c(A) = 1$ *or* $V_c(\Delta) = 1$ follows from $V_p(\Gamma) = 1$, one application of disjunctive syllogism delivers $V_c(\Delta) = 1$. Now assume $V_p(B) = 1$. Together with $V_p(\Gamma) = 1$, $V_p(B) = 1$ implies $V_c(\Delta) = 1$. Hence, by disjunction elimination, $V_c(\Delta) = 1$ holds.

(tonkL) Let $\mathcal{M}$ be an arbitrary *PCR*-model. Suppose that *if* $V_p(\Gamma) = 1$ *and* $V_p(A) = 1$ *then* $V_c(\Delta) = 1$. Assume $V_p(\Gamma) = 1$ *and* $V_p(B\text{-tonk-}A) = 1$. From the premise-principle for tonk, $V_p(A) = 1$ follows. $V_c(\Delta) = 1$ immediately follows. □

The semantic tableaux system for the *PCR* semantics will have certain similarities with the semantic tableaux for FDE in Priest (2007). Following the terminology used there, a tableaux will consist of nodes, branches and an initial list. Each node contains one or more pairs of the form $A, p+$ or $A, c-$, where intuitively, $A, p+$ is understood as "A holds as premise" and corresponds to $V_p(A) = 1$ while $A, c-$ is understood as "A fails as conclusion" and corresponds to $V_c(A) = 0$. We shall not introduce rules involving $A, p-$ and $A, c+$ because they are not necessary to determine the set of valid inferences for the vocabulary we are interested in. The initial list represents the claim we would like

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to test. If we are interested in whether $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$, the initial list is of the following form:

$$\begin{array}{c} A_1, p+ \\ \vdots \\ A_n, p+ \\ B_1, c- \\ \vdots \\ B_m, c- \end{array}$$

DEFINITION 7.3. *A PCR-tableaux is constructed from an initial list using the following tableaux rules:*

$$\begin{array}{c} A \to B, p+ \\ \hline B, p+ \quad A, c- \end{array} \qquad \begin{array}{c} A \to B, c- \\ | \\ A, p+ \\ B, c- \end{array} \qquad \begin{array}{c} \neg A, p+ \\ | \\ A, c- \end{array} \qquad \begin{array}{c} \neg A, c- \\ | \\ A, p+ \end{array}$$

$$\begin{array}{c} A\text{-}tonk\text{-}B, p+ \\ | \\ B, p+ \end{array} \qquad \begin{array}{c} A\text{-}tonk\text{-}B, c- \\ | \\ A, c- \end{array}$$

DEFINITION 7.4. *A branch in a PCR-tableaux is closed if and only if $\alpha, p+$ and $\alpha, c-$ occurs on a branch, where α is atomic. A branch in a PCR-tableaux is open if and only if it is not closed.*

DEFINITION 7.5. *An inference from Γ to Δ holds in the semantic tableaux, written as $\Gamma \vdash \Delta$ where Γ and Δ are finite multisets, if and only if all branches of a tableaux commencing with $A, p+$ for all $A \in \Gamma$ and $B, c-$ for all $B \in \Delta$ are closed.*

If we were to add the following rule corresponding to CUT in our tableaux system, our system would be trivial:

$$\begin{array}{c} \vdots \\ \hline A, p+ \quad A, c- \end{array}$$

To demonstrate, here is a proof of $\alpha \vdash \beta$ where α and β are atomic formulae:

$$\begin{array}{c} \alpha, p+ \\ \beta, c- \\ \hline \alpha\text{-tonk-}\beta, p+ \quad \alpha\text{-tonk-}\beta, c- \\ \beta, p+ \qquad\qquad \alpha, c- \end{array}$$

Both branches in this tableaux are closed.

The occurrence of $\alpha, p+$ and $\alpha, c-$ on a branch is used as condition for a closed branch as a result of the Theorems 6.5 and 6.6. Together, they imply that we cannot use the occurrence of $\neg A, c-$ and $A, c-$ or the occurrence of $\neg A, p+$ and $A, p+$ on a branch

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as conditions for it being closed. However, because atomic formulae have the same value as premise and as conclusion, $V_p(\alpha) = 1$ and $V_c(\alpha) = 0$ is an inconsistent valuation.

THEOREM 7.6. *$\Gamma \vdash \Delta$ is derivable in $\text{CNT}^{\text{L}}_{\text{R}}\text{ID}_\alpha$ if and only if $\Gamma \vdash \Delta$ is derivable with a PCR-tableaux.*

Proof. We show this by presenting a procedure to convert a tableaux with every branch closed into a proof in sequent calculus and conversely.

To this purpose, some terminology is useful. Every $\text{CNT}^{\text{L}}_{\text{R}}\text{ID}_\alpha$-proof consists of one or more topsequents and one endsequent. The endsequent is derived from the topsequent(s) by one or more applications of the rules. The formula in a sequent containing the connective introduced by the rule is the principal formula of that sequent.[18]

The following is an informal description of the method to convert a *PCR*-tableaux with every branch closed to a $\text{CNT}^{\text{L}}_{\text{R}}\text{ID}_\alpha$-proof.

Let the initial list $A_1, p+; \ldots; A_n, p+; B_1, c-; \ldots; B_m, c-$ be the endsequent $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$. Every new sequent or pair of sequents is added above the previous sequent with the endsequent at the bottom according to the following instructions:

- If a branch with $C \to D, p+; A_1, p+; \ldots; A_n, p+; B_1, c-; \ldots; B_m, c-$ splits into two nodes, $D, p+$ and $C, c-$, add the sequents $A_1, \ldots, A_n, \vdash B_1, \ldots, B_m, C$ and $D, A_1, \ldots, A_n \vdash B_1, \ldots, B_m$ above the previous sequent.
- If a branch with $A_1, p+; \ldots; A_n, p+; B_1, c-; \ldots; B_m, c-; C \to D, c-$ is extended with one new node consisting of $C, p+$ and $D, c-$, add the sequent $A_1, \ldots, A_n, C \vdash B_1, \ldots, B_m, D$.
- If a branch with C-tonk-$D, p+; A_1, p+; \ldots; A_n, p+; B_1, c-; \ldots; B_m, c-$ is extended with one new node consisting of $D, p+$, add the sequent $A_1, \ldots, A_n, D \vdash B_1, \ldots, B_m,$.
- If a branch with $A_1, p+; \ldots; A_n, p+; B_1, c-; \ldots, B_m; c-; C$-tonk-$D, c-$ is extended with one new node consisting of $C, c-$, add the sequent $A_1, \ldots, A_n \vdash B_1, \ldots, B_m, D$.
- If a branch with $\neg C, p+; A_1, p+; \ldots; A_n, p+; B_1, c-; \ldots, B_m, c-$ is extended with one new node consisting of $C, p+$, add the sequent $A_1, \ldots, A_n, C \vdash B_1, \ldots, B_m,$.
- If a branch with $A_1, p+; \ldots; A_n, p+; B_1 c-; \ldots; B_m, c-; \neg C, c-$ is extended with one new node consisting of $C, p+$, add the sequent $A_1, \ldots, A_n, C \vdash B_1, \ldots, B_m$.

A quick inspection shows that each clause converts an application of a tableaux rule into an application of a rule in $\text{CNT}^{\text{L}}_{\text{R}}\text{ID}_\alpha$. Since each branch is closed, the last sequent added will be an instance of ID_α.[19]

To convert a $\text{CNT}^{\text{L}}_{\text{R}}\text{ID}_\alpha$-proof into a *PCR*-tableaux with every branch closed, one simply mirrors the above instructions.

Let the endsequent $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$ be the initial list $A_1, p+; \ldots; A_n, p+; B_1, c-; \ldots; B_m, c-$. Build the *PCR*-tableaux downwards from the initial list by going upwards in the $\text{CNT}^{\text{L}}_{\text{R}}\text{ID}_\alpha$-proof according to the following instructions, starting with the endsequent:

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[18] Cf. Negri & von Plato (2001).

[19] For illustration, here is a tableaux proof of $\alpha \to (\alpha \to \beta) \vdash \alpha \to \beta$, where α and β are atomic formulae:

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- If the sequent is $C \to D, A_1, \ldots, A_n \vdash B_1, \ldots, B_m$ where $C \to D$ is the principal formula and the sequent is derived using $\to$ L, split the branch downwards into two nodes, $D, p+$ and $C, c-$.
- If the sequent is $A_1, \ldots, A_n \vdash B_1, \ldots, B_m, C \to D$ with $C \to D$ as principal formula and it is derived using $\to$ R, add one node consisting of $C, p+$ and $D, c-$.
- If the sequent is $\neg C, A_1, \ldots, A_n \vdash B_1, \ldots, B_m$ with $\neg C$ as principal formula and it is derived using $\neg$L, add the node $C, c-$.
- If the sequent is $A_1, \ldots, A_n \vdash B_1, \ldots, B_m, \neg C$ with $\neg C$ as principal formula and it is derived using $\neg$R, add the node $C, p+$
- If the sequent is C-tonk-$D, A_1, \ldots, A_n \vdash B_1, \ldots, B_m$ with C-tonk-D as principal formula and it is derived using tonkL, add the node $B, p+$
- If the sequent is $A_1, \ldots, A_n \vdash B_1, \ldots, B_m, C$-tonk-$D$ with C-tonk-D as principal formula and it is derived using tonkL, add the node $D, c-$

Again, each clause converts an application of a rule in a $\mathrm{CNT}^{\mathrm{L}}_{\mathrm{R}}\mathrm{ID}_\alpha$-proof to an application of a rule in a PCR-tableaux. Since every $\mathrm{CNT}^{\mathrm{L}}_{\mathrm{R}}\mathrm{ID}_\alpha$-proof starts with one or more instances of ID_α, every branch will end up containing an atomic formula α such that $\alpha, p+$ and $\alpha, c-$. This establishes the equivalence. □

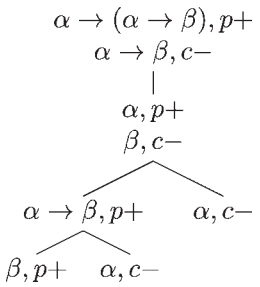
Finally, we prove completeness.

THEOREM 7.7. *The PCR-tableaux is complete for the PCR-semantics: if* $\Gamma \models_{PC} \Delta$ *then there is a PCR-tableaux for* $\Gamma \vdash \Delta$.

Proof. The proof proceeds as usual by showing the contrapositive, namely if $\Gamma \nvdash \Delta$ then $\Gamma \not\models_{PC} \Delta$. This amount to the claim that if a tableaux commencing with $\phi, p+$ for all $\phi \in \Gamma$ and $\psi, c-$ for all $\psi \in \Delta$ has an open branch $\mathcal{B}$, then there is a model $\langle V_p^{\mathcal{B}}, V_c^{\mathcal{B}} \rangle$ such that $V_p^{\mathcal{B}}(\phi) = 1$ for all $\phi \in \Gamma$ and $V_c^{\mathcal{B}}(\psi) = 0$ for all $\psi \in \Delta$.

Let $\mathcal{B}$ be an open branch and let $\langle V_p^{\mathcal{B}}, V_c^{\mathcal{B}} \rangle$ be a PCR-model defined such that for any atomic formula α occurring on $\mathcal{B}$, the following conditions hold:

- if $\alpha, p+$ is on $\mathcal{B}$ then $V_p^{\mathcal{B}}(\alpha) = V_c^{\mathcal{B}}(\alpha) = 1$
- if $\alpha, c-$ is on $\mathcal{B}$ then $V_p^{\mathcal{B}}(\alpha) = V_c^{\mathcal{B}}(\alpha) = 0$
- For every atomic formula α *not* occurring on $\mathcal{B}$, let $V_p^{\mathcal{B}}(\alpha) = 0$ and $V_c^{\mathcal{B}}(\alpha) = 0$.



The corresponding proof in sequent calculus is now constructed as follows:

- Add the sequent $\alpha \to (\alpha \to \beta) \vdash \alpha \to \beta$ as endsequent in the proof.
- Add the sequent $\alpha \to (\alpha \to \beta), \alpha \vdash \beta$
- Branch upwards with two sequents, $\alpha \vdash \beta, \alpha$ and $\alpha \to \beta, \alpha \vdash \beta$
- Branch upwards with $\alpha \vdash \beta, \alpha$ and $\alpha, \beta \vdash \beta$.

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Since $\mathcal{B}$ is an open branch, we know that there is no atomic formula α such that $\alpha, p+$ and $\alpha, c-$ holds.

To prove that $\langle V_p^{\mathcal{B}}, V_c^{\mathcal{B}} \rangle$ is a model for $\mathcal{B}$, we must show that for every formula A,

(a) if $A, p+$ is on $\mathcal{B}$ then $V_p^{\mathcal{B}}(A) = 1$

(b) if $A, c-$ is on $\mathcal{B}$ then $V_c^{\mathcal{B}}(A) = 0$

We prove this by induction on the complexity of a formula, where the complexity of a formula is defined in the usual way. Since the base case follows from the definition of $\langle V_p^{\mathcal{B}}, V_c^{\mathcal{B}} \rangle$, it is left to show that if (a) and (b) hold for any formula A with the complexity $\leq n$, then they hold for every formula of complexity $n + 1$. For each of our three connectives, we have two cases. I will go through one case for each connective:

- Suppose $A \rightarrow B, p+$ is on $\mathcal{B}$. With the $p+$-rule for $\rightarrow$, it follows that $A, c-$ or $B, p+$ is on $\mathcal{B}$. By the induction hypothesis, it follows that $V_c^{\mathcal{B}}(A) = 0$ or $V_p^{\mathcal{B}}(B) = 1$ and it follows from the premise-clause for $\rightarrow$ that $V_p^{\mathcal{B}}(A \rightarrow B) = 1$.
- Suppose $\neg A, c-$ is on $\mathcal{B}$. With the $c-$-rule for $\neg$, it follows that $A, p+$ is on $\mathcal{B}$. By our inductive hypothesis, it follows that $V_p^{\mathcal{B}}(A) = 1$. By the conclusion-clause for $\neg$, $V_c^{\mathcal{B}}(\neg A) = 0$ follows.
- Suppose A-tonk-$B, c-$ is on $\mathcal{B}$. It follows by the $c-$-rule for tonk that $A, c-$ is on $\mathcal{B}$. By the inductive hypothesis, $V_c^{\mathcal{B}}(A) = 0$ holds and it follows by the conclusion-clause for tonk that $V_c^{\mathcal{B}}(A\text{-tonk-}B) = 0$.

The desired result follows. □

§8. Concluding remarks. In this paper, we have seen that neither Cook's semantics nor Cobreros *et al*'s semantics for non-transitive entailment can adequately represent tonk as it appears in a cut-free sequent calculus, that the failure to do so is connected with the fact that tonk is not uniquely defined in a cut-free sequent calculus unless that logic is also non-reflexive, and that we can develop a suitable semantics for tonk with both non-transitive and non-reflexive entailment. The semantics was based on premise/conclusion relativism, the idea that the truth of a sentence is relative to whether it is assessed as premise or assessed as conclusion. Needless to say, this is a very radical idea and adopting it merely to accommodate tonk is probably crazy. I am not suggesting that.

Nevertheless, the results in this paper are relevant for the ongoing debate on transparent truth and semantic paradoxes in two ways. First of all, both Cook's and Cobreros *et al*'s framework can be used to give a semantics for transparent truth which delivers non-transitive entailment, but as we have seen, they cannot adequately accommodate tonk. Even if both tonk and truth can conservatively extend a system if transitivity is rejected, we have seen that tonk is still misbehaving. Adopting non-transitive entailment for truth is not tantamount to anything goes. Secondly, we can construe the liar paradox as a case in which we reason from something being true as premise to it being false as conclusion: if we start off by assuming that the liar sentence is true, we seem to derive classically that the liar sentence is false, so if the liar sentence is true (as premise), it must be false (as conclusion). If one extends the PCR semantics to a first-order language with transparent truth, reflexivity (but not transitivity) fails for that notion of entailment: paradoxical sentences do not entail themselves. While elaborating on this would of course be outside the scope of this paper and is left for future work, it nevertheless suggests that developing a semantics for tonk is perhaps not as philosophically useless as one might first think.

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