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A commutative Noetherian ring which possesses a dualizing complex is acceptable

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Introduction. Throughout this paper the word 'ring' will mean 'commutative Noetherian ring with non-zero identity', and A will always denote such a ring. It will only be assumed that A is local when this is explicitly stated, but when this is the case $\mathfrak{m}$ (resp. k) will always be used to denote the maximal ideal (resp. residue field) of A. It is to be understood that ring homomorphisms respect identity elements.

A new class of rings, called acceptable rings, was introduced in (**19**). The ring A is said to be *acceptable* if the following three conditions are satisfied:

(a) A is universally catenary;

(b) whenever B is a finitely generated A-algebra, the Gorenstein locus of B ((**19**), 2·1) is an open subset of Spec (B); and

(c) for every prime ideal $\mathfrak{p}$ of A, the natural (flat) ring homomorphism $A_{\mathfrak{p}} \to (A_{\mathfrak{p}})\hat{}$ is a Gorenstein ring homomorphism ((**19**), 2·4). (The completion of a local ring R is denoted by $\hat{R}$ or $(R)\hat{}$.)

These conditions are obtained from the definition of excellent ring given in ((**8**), § 34) by changing each occurrence of the word 'regular' to 'Gorenstein'. The motivation for the introduction of the concept of acceptable ring was the occurrence in the literature in recent years of several results about rings which can be expressed as homomorphic images of finite-dimensional Gorenstein rings, indicating that such rings behave well in certain respects; it therefore seemed desirable to have some sort of norm for describing good behaviour of rings with respect to Gorenstein properties. Since there exists a regular local ring which is not excellent ((**8**), (34.B)), one cannot expect every homomorphic image of a finite-dimensional Gorenstein ring to satisfy the strong regularity requirements of excellency; however, it seemed much more reasonable to hope that such rings satisfy the corresponding conditions with the word 'regular' changed to 'Gorenstein'. Indeed, the main results of (**19**) are that the class of acceptable rings contains all homomorphic images of finite-dimensional Gorenstein rings (but does not contain every (commutative Noetherian) ring), and that the class of acceptable rings is stable under the operations of formation of fractions and passage to finitely generated algebras.

The purpose of the present paper is to provide further motivation for the study of acceptable rings. The main result of the paper is that if the ring A possesses a dualizing complex, then A is acceptable. (It is well known ((**6**), chapter V, § 10) or ((**18**), (3·7) and (3·9)) that a ring which is a homomorphic image of a finite-dimensional Gorenstein ring possesses a dualizing complex.) We shall also establish the rather easier result that if

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A is a finite-dimensional ring which possesses a Gorenstein module (13) having support equal to the whole of Spec (A), then A is acceptable. Here, one cannot omit this condition on the support of the Gorenstein module, for if G is a Gorenstein module over the ring A and B is a second ring, then ((16), (3·4)), however unpleasant B may be, G can be given the structure of a Gorenstein $A \times B$-module. However, this result does show that if A is a finite-dimensional ring with connected prime spectrum (e.g. if A is local) which possesses a Gorenstein module, then A is acceptable.

This last statement is perhaps relevant to the following problem which, as far as this author is aware, is still open. Suppose that A is local. It is well known that if there exists a Gorenstein A-module, then A must be a Cohen–Macaulay ring ((13), (3·9)), and that there exists a Gorenstein A-module of rank 1 ((16), (1·8)) if and only if A is a Cohen–Macaulay homomorphic image of a Gorenstein local ring ((15), 3·1, and (11)). The outstanding question in this area is the following. Does there exist a Cohen–Macaulay local ring which possesses a Gorenstein module but which cannot be expressed as a homomorphic image of a Gorenstein local ring? The most striking result on this problem seems to be that obtained by Fossum, Foxby, Griffith and Reiten ((5), (4·9)) which shows that all Gorenstein modules over such a ring must have even rank. The results of section 1 of the present paper show that such a Cohen–Macaulay local ring must be acceptable.

However, results about dualizing complexes form the main body of the paper. In order to establish the main result, we shall need several preliminary results about dualizing complexes, including the fact that if the ring A possesses a dualizing complex and X is an indeterminate over A, then $A[X]$ possesses a dualizing complex also. Section 2 is devoted to technical results concerning dualizing complexes, and section 3 to the results which lead to the proof of the main theorem. We shall adopt the relatively elementary approach to dualizing complexes outlined in (18) (rather than the more sophisticated approach of Hartshorne ((6), chapter v)); when working with dualizing complexes, we shall use the notation and terminology of (18).

The results of this paper tend to support the author's view that acceptable rings provide a standard for the lowest level of reasonable behaviour in what might be termed the 'Gorenstein area' of commutative algebra. This view would be reinforced if we could prove that every excellent ring is acceptable. The present author certainly conjectures that this is so, but, at the time of writing, has not been able to prove this point. As is explained in ((19), 6·7), this problem reduces to showing that an excellent ring has open Gorenstein locus.

1. *On rings which possess Gorenstein modules.* The purpose of this section is to prove that if the ring A has finite dimension and there exists a Gorenstein A-module G with $\operatorname{Supp}_A(G) = \operatorname{Spec}(A)$, then A is acceptable.

(1·1) PROPOSITION. *Suppose that A is local and that there exists a Gorenstein A-module G. Then the natural homomorphism $\tau\colon A \to \hat{A}$ is a Gorenstein ring homomorphism.*

Proof. By ((17), (2·8)), $\hat{A} \otimes_A G$ is a Gorenstein $\hat{A}$-module. By ((13), (4·12)),

$$\operatorname{Supp}_A(G) = \operatorname{Spec}(A) \quad \text{and} \quad \operatorname{Supp}_{\hat{A}}(\hat{A} \otimes_A G) = \operatorname{Spec}(\hat{A}).$$

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To show that τ is a Gorenstein ring homomorphism, we argue as in the proof of ((17), (2·10)). It is enough to show that if $\mathfrak{p}$ is a prime ideal of A and $\mathfrak{q}$ is a prime ideal of $\hat{A}$ which contracts to $\mathfrak{p}$ under τ, then the fibre ring $(\hat{A})_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} k(\mathfrak{p})$ (where $k(\mathfrak{p})$ denotes the residue field of $A_{\mathfrak{p}}$) of the ring homomorphism $\tau' : A_{\mathfrak{p}} \to (\hat{A})_{\mathfrak{q}}$ (induced by τ) over the maximal ideal of $A_{\mathfrak{p}}$ is a Gorenstein local ring. However, this is immediate from ((17), (2·5) and (2·8)) and the observation that $(\hat{A} \otimes_A G)_{\mathfrak{q}}$ is a Gorenstein $(\hat{A})_{\mathfrak{q}}$-module which is isomorphic to $(\hat{A})_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} G_{\mathfrak{p}}$.

(1·2) Lemma. *Suppose there is a Gorenstein A-module G for which*

$$\operatorname{Supp}_A(G) = \operatorname{Spec}(A),$$

and that $X_1, X_2, \ldots, X_n$ are indeterminates independent over A. Then $A[X_1, \ldots, X_n] \otimes_A G$ is a Gorenstein $A[X_1, \ldots, X_n]$-module for which

$$\operatorname{Supp}_{A[X_1, \ldots, X_n]}(A[X_1, \ldots, X_n] \otimes_A G) = \operatorname{Spec}(A[X_1, \ldots, X_n]).$$

Proof. This is immediate from ((17), (2·6), (2·9) and (2·12) (i)).

(1·3) Proposition. *Suppose that A is local, that C is a second local ring, and that $\phi : A \to C$ is a surjective ring homomorphism. Assume that there exists a Gorenstein A-module G. Let n (resp. m) denote the dimension of A (resp. C), and let t denote the depth of C. Then*

(i) *$n-t$ is the greatest integer i for which $\operatorname{Ext}^i_A(C, G) \neq 0$;*

(ii) *$n-m$ is the least integer i for which $\operatorname{Ext}^i_A(C, G) \neq 0$;*

and (iii) *if C is a Cohen–Macaulay local ring, the C-module $\operatorname{Ext}^{n-m}_A(C, G)$ is a Gorenstein C-module.*

Proof. (i) As G has finite injective dimension, A is a Cohen–Macaulay local ring ((13), (3·9)), and $\operatorname{depth}_A C = t$, this part is immediate from ((7), § 2).

(ii) Let $\mathfrak{c} = \ker \phi$, so that $\mathfrak{c}$ is the annihilator of C when considered as an A-module by means of ϕ. It is easy to see that the height of $\mathfrak{c}$ is $n-m$; as A is a Cohen–Macaulay ring, $\mathfrak{c}$ has grade $n-m$. However ((13), (4·13)), a sequence of elements of A is an A-sequence if and only if it is a G-sequence, and so the result follows from ((13), (1·4)).

(iii) For this, we reason as in the proof of ((15), 3·1). Let

$$0 \to G \to G^0 \to G^1 \to \ldots \to G^{n-1} \to G^n \to 0 \ldots$$

be the Cousin complex (12) for G: this provides a minimal injective resolution for G ((13), § 3). By ((12), (4·5)) and ((13), (4·12)), $\operatorname{Hom}_A(C, G^i) = 0$ for $0 \leqslant i < n-m$. Parts (i) and (ii) show that $\operatorname{Ext}^i_A(C, G) = 0$ for $i \neq n-m$. Let H denote the C-module $\operatorname{Ext}^{n-m}_A(C, G)$, so that there is an exact sequence

$$0 \to H \to \operatorname{Hom}_A(C, G^{n-m}) \to \ldots \to \operatorname{Hom}_A(C, G^n) \to 0 \qquad (*)$$

of C-modules and C-homomorphisms. By ((12), (4·4)), ((13), (2·3), (4·10), and (4·12)), and ((16), (1·8)),

$$\operatorname{Hom}_A(C, G^i) \cong \bigoplus_{\substack{\mathfrak{q} \in \operatorname{Spec}(C) \\ \operatorname{ht}\mathfrak{q} = i-(n-m)}} rE_C(C/\mathfrak{q}) \quad \text{(as } C\text{-modules)}$$

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for $n-m \leqslant i \leqslant n$, where r denotes the rank of G and 'E' is used to denote injective envelopes. It is now straightforward to see that H is a Gorenstein C-module.

(1·4) THEOREM. *Suppose that A has finite dimension and that there exists a Gorenstein A-module G with* $\operatorname{Supp}_A(G) = \operatorname{Spec}(A)$. *Let $\phi: A \to C$ be a surjective ring homomorphism from A onto a second ring C. Then* $\operatorname{Gor}(C)$ ((19), 2·1) *is an open subset of* $\operatorname{Spec}(C)$.

Proof. We use reasoning similar to that of the proof of ((19), 3·1). Let s denote the dimension of A.

Let $\mathfrak{q}$ be a prime ideal of C, and let $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$, a prime ideal of A: note that $G_{\mathfrak{p}}$ is a Gorenstein $A_{\mathfrak{p}}$-module. Now ϕ induces a surjective ring homomorphism $\phi': A_{\mathfrak{p}} \to C_{\mathfrak{q}}$ (for which $\phi'(a/s) = \phi(a)/\phi(s)$ for $a \in A$ and $s \in A - \mathfrak{p}$). For each non-negative integer i, ϕ gives rise to a natural C-module structure on $\operatorname{Ext}^i_A(C, G)$, and ϕ' gives rise to a natural $C_{\mathfrak{q}}$-module structure on $\operatorname{Ext}^i_{A_{\mathfrak{p}}}(C_{\mathfrak{q}}, G_{\mathfrak{p}})$: a straightforward technical check shows that the $C_{\mathfrak{q}}$-modules $[\operatorname{Ext}^i_A(C, G)]_{\mathfrak{q}}$ and $\operatorname{Ext}^i_{A_{\mathfrak{p}}}(C_{\mathfrak{q}}, G_{\mathfrak{p}})$ are isomorphic.

Now suppose that our prime ideal $\mathfrak{q}$ of C belongs to $\operatorname{Gor}(C)$. Let d denote

$$\dim(A_{\mathfrak{p}}) - \dim(C_{\mathfrak{q}}),$$

and let W denote the finitely generated C-module $\operatorname{Ext}^d_A(C, G)$. Let F be a finitely generated free C-module for which there is a C-epimorphism $f: F \to W$. Let K denote the kernel of f: K is a finitely generated C-module, and there is an exact sequence

$$0 \longrightarrow K \longrightarrow F \xrightarrow{f} W \longrightarrow 0.$$

Let $V = \operatorname{Spec}(C) - \operatorname{Supp}_C(\operatorname{Ext}^1_C(W, K))$, and, for each $i = 0, \ldots, s$, let

$$U_i = \operatorname{Spec}(C) - \operatorname{Supp}_C(\operatorname{Ext}^i_A(C, G)).$$

It is now straightforward to use (1·3) in conjunction with ((15), 2·1 and 4·1) to see that

$$\mathfrak{q} \in V \cap \left[\bigcap_{\substack{i=0 \\ i \neq d}}^{s} U_i\right] \subseteq \operatorname{Gor}(C).$$

It follows that $\operatorname{Gor}(C)$ is an open subset of $\operatorname{Spec}(C)$.

Remark. It would be interesting to know whether the conclusion of Theorem (1·4) remains valid if the requirement that A be finite-dimensional is deleted from the hypotheses.

We can now prove the main result of this section.

(1·5) THEOREM. *Suppose that A has finite dimension and that there exists a Gorenstein A-module G with* $\operatorname{Supp}_A(G) = \operatorname{Spec}(A)$. *Then A is acceptable.*

Proof. By ((13), (3·7) and (3·9)), A is a Cohen–Macaulay ring, and therefore universally catenary. The result is now immediate from (1·1), (1·2) and (1·4).

Two easy consequences of this Theorem are perhaps worth stating separately. The first follows from ((1·5) and) ((13), (4·15)), while the second is a special case of the first.

(1·6) COROLLARY. *Suppose that A has finite dimension and connected prime spectrum, and that there exists a Gorenstein A-module. Then A is acceptable.*

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(1·7) COROLLARY. *Suppose that A is local and that there exists a Gorenstein A-module. Then A is an acceptable (Cohen–Macaulay) ring.*

This second Corollary shows that a Cohen–Macaulay local ring which possesses a Gorenstein module is subject to certain restrictions: as was explained in the Introduction, it is still an open question whether such a Cohen–Macaulay local ring has to be a homomorphic image of a Gorenstein local ring.

2. *Technical results concerning dualizing complexes.* In this and the next section, we shall use the notation and terminology of (18) concerning dualizing complexes. Several of the preliminary results are due to Grothendieck and Hartshorne. However, as we are proposing to follow the elementary approach to dualizing complexes of (18), it seems appropriate to give some indications of proof of such results rather than a mere reference to Hartshorne's work in (6). Our first result is a fundamental fact concerning dualizing complexes for local rings.

(2·1) LEMMA. (See Hartshorne ((6), V·3·4).) *Suppose that A is local and that the injective complex $I^{\cdot}$ in $\mathscr{D}_c^b(A)$ ((18), p. 370) is a dualizing complex for A. Then there exists an integer t such that*

$$H^i(\mathrm{Hom}_A(k, I^{\cdot})) \cong \begin{cases} 0 & \text{if } i \neq t, \\ k & \text{if } i = t. \end{cases}$$

Proof. Since the field k is a Gorenstein local ring having, by ((18), (3·7)), a dualizing complex $J^{\cdot}$ with $J^0 = k$ and $J^i = 0$ for all $i \neq 0$, this is immediate from ((18), (3·9) and (4·5)).

In the course of this section, we shall establish the converse of (2·1): this will turn out to be a useful technique for recognizing dualizing complexes.

Our next result concerns an injective complex which is bounded below (but not necessarily above) and which has the property that all of its cohomology modules are finitely generated.

(2·2) PROPOSITION. *Suppose that A is local and that $I^{\cdot}$ is an injective complex in $\mathscr{D}_c^+(A)$. Assume that there is an integer t with the property that $H^i(\mathrm{Hom}_A(k, I^{\cdot})) \neq 0$ if and only if $i = t$. Let M be a non-zero finitely generated A-module of dimension r. Then*

$$H^i(\mathrm{Hom}_A(M, I^{\cdot})) \neq 0 \quad \text{only when the integer } i \text{ satisfies } t - r \leqslant i \leqslant t.$$

Proof. (See Hartshorne ((6), p. 270).) Note that, whenever N is a finitely generated A-module, then ((18), (3·4) (ii)) $\mathrm{Hom}_A(N, I^{\cdot}) \in \mathscr{D}_c^+(A)$. Also, a short exact sequence of finitely generated A-modules

$$0 \to N' \to N \to N'' \to 0$$

induces an exact sequence ((9), §4·2)

$$0 \to \mathrm{Hom}_A(N'', I^{\cdot}) \to \mathrm{Hom}_A(N, I^{\cdot}) \to \mathrm{Hom}_A(N', I^{\cdot}) \to 0$$

in $\mathscr{D}(A)$; this in turn induces a long exact sequence of cohomology modules, and we shall make considerable use of such long exact sequences.

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We proceed by induction on r: the result for the case in which $r = 0$ can easily be established by induction on length and use of long exact sequences of the above-mentioned type. So assume, inductively, that j is a non-negative integer and that the desired result has been established when $0 \leqslant r \leqslant j$; consider now the case in which N is a non-zero finitely generated A-module of dimension $j+1$.

Let L be the largest submodule of N which has finite length. Then N/L has dimension $j+1$. Consideration of the exact sequence

$$0 \to L \to N \to N/L \to 0$$

shows that it is sufficient to prove the result for N/L; however, N/L has depth at least 1. We may therefore assume henceforth that $\operatorname{depth}_A N \geqslant 1$.

Let x be an element of $\mathfrak{m}$ which is a non-zerodivisor on N; then N/xN has dimension j. Consideration of the long exact sequence of cohomology modules induced from the exact sequence

$$0 \longrightarrow N \xrightarrow{x} N \longrightarrow N/xN \longrightarrow 0,$$

together with use of Nakayama's lemma and the inductive hypothesis, completes the inductive step.

The result therefore follows by induction.

Occasionally, when we are studying an injective complex $I^\cdot$ which is bounded below, it is very helpful if we can find a bounded injective complex $J^\cdot$ and a quasi-isomorphism (abbreviated *quism* ((18), p. 372)) $u^\cdot: I^\cdot \to J^\cdot$. The next proposition shows that we can do this in certain circumstances.

(2·3) PROPOSITION. *Suppose that $I^\cdot$ is an injective complex in $\mathscr{D}_c^+(A)$. Assume that there is an integer t having the property that $H^i(\operatorname{Hom}_A(M, I^\cdot)) = 0$ whenever $i > t$ and M is a finitely generated A-module. Let $J^t = \ker d_I^t$., and let $J^\cdot$ denote the complex*

$$\ldots \to I^i \to I^{i+1} \to \ldots \to I^{t-1} \to J^t \to 0 \to 0 \to \ldots$$

(in which it is to be understood that J^t is the t-th term) induced from $I^\cdot$. Then $J^\cdot$ is an injective complex in $\mathscr{D}_c^b(A)$, and there is a quism $u^\cdot: I^\cdot \to J^\cdot$.

Proof. Since $H^i(\operatorname{Hom}_A(A, I^\cdot)) = 0$ for all $i > t$, it is clear that the complex

$$0 \to J^t \to I^t \to I^{t+1} \to \ldots \to I^{t+j} \to \ldots$$

induced from $I^\cdot$ provides an injective resolution for J^t. The hypothesis ensures that $\operatorname{Ext}_A^i(M, J^t) = 0$ whenever $i > 0$ and M is a finitely generated A-module. It therefore follows from, for example ((1), (2·7)) that J^t is injective. Hence $J^\cdot$ is an injective complex in $\mathscr{D}^b(A)$.

The identity endomorphism of the injective A-module J^t may be extended to a homomorphism $u^t: I^t \to J^t$. For $i < t$, let $u^i: I^i \to I^i$ be the identity mapping, and, for $i > t$, let $u^i: I^i \to 0$ be the zero homomorphism. Then $u^\cdot: I^\cdot \to J^\cdot$ is a quism, from which it follows that $J^\cdot$ is an injective complex in $\mathscr{D}_c^b(A)$. This completes the proof.

Recall ((18), (2·4)) that an injective complex $I^\cdot \in \mathscr{D}_c^b(A)$ is a dualizing complex for A if the morphism of complexes

$$\theta(X^\cdot; I^\cdot)^\cdot: X^\cdot \to \operatorname{Hom}_A([\operatorname{Hom}_A(X^\cdot, I^\cdot)], I^\cdot)$$

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of ((18), (2·3) (ii)) is a quism whenever $X^{\cdot}$ is a complex in $\mathscr{D}_c^b(A)$. Our first theorem in this section shows that, when A is local, a sufficient (and, clearly, necessary) condition for $I^{\cdot}$ to be a dualizing complex for A is that $\theta(k; I^{\cdot})^{\cdot}$ be a quism.

(2·4) THEOREM. (See Hartshorne ((6), v·2·5).) *Suppose that A is local and that $I^{\cdot}$ is an injective complex in $\mathscr{D}_c^b(A)$. Assume that the morphism of complexes*

$$\theta(k; I^{\cdot})^{\cdot}: k \to \mathrm{Hom}_A([\mathrm{Hom}_A(k, I^{\cdot})], I^{\cdot})$$

of ((18), (2·3) (ii)) *is a quism. Then $I^{\cdot}$ is a dualizing complex for A.*

Proof. We first show that, whenever M is a non-zero finitely generated A-module, then $\theta(M; I^{\cdot})^{\cdot}$ is a quism; to achieve this we proceed by induction on $\dim_A M$. It will be convenient in this proof to denote the functor $\mathrm{Hom}_A([\mathrm{Hom}_A(-, I^{\cdot})], I^{\cdot})$ (from $\mathscr{D}(A)$ to $\mathscr{D}(A)$) by T.

Note that, whenever N is a finitely generated A-module, then ((18), (3·5) (ii)) $T(N) \in \mathscr{D}_c^b(A)$. Also ((18), (2·2) (iii)), $\mathrm{Hom}_A(I^{\cdot}, I^{\cdot})$ is a flat complex, and so it follows from ((18), (2·3) (i)) that a short exact sequence of finitely generated A-modules

$$0 \to N' \to N \to N'' \to 0$$

induces a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta(N'; I^{\cdot})^{\cdot}} & & \downarrow{\scriptstyle \theta(N; I^{\cdot})^{\cdot}} & & \downarrow{\scriptstyle \theta(N''; I^{\cdot})^{\cdot}} & & \\ 0 & \longrightarrow & T(N') & \longrightarrow & T(N) & \longrightarrow & T(N'') & \longrightarrow & 0 \end{array}$$

in $\mathscr{D}(A)$ in which both rows are exact; this diagram therefore induces a morphism of the two corresponding long exact sequences of cohomology modules. We shall make considerable use of such morphisms: for instance, one can use them to show, by induction on length, that $\theta(M; I^{\cdot})^{\cdot}$ is a quism whenever M is an A-module of finite length.

Suppose, inductively, that j is a non-negative integer and we have proved that $\theta(M; I^{\cdot})^{\cdot}$ is a quism whenever M is a non-zero finitely generated A-module with $\dim_A M \leqslant j$; consider now the case of a non-zero finitely generated A-module N having dimension $j+1$. Consideration of the largest submodule of N having finite length shows that we may assume $\mathrm{depth}_A N \geqslant 1$. So let x be an element of $\mathfrak{m}$ which is a non-zero-divisor on N. The inductive hypothesis ensures that $\theta(N/xN; I^{\cdot})^{\cdot}$ is a quism. Consideration of the exact sequence

$$0 \longrightarrow N \xrightarrow{x} N \longrightarrow N/xN \longrightarrow 0$$

therefore shows that $H^i(T(N)) = xH^i(T(N))$ for all $i \neq 0$, so that $H^i(T(N)) = 0$ for all $i \neq 0$ by Nakayama's lemma, and also that there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{x} & N & \longrightarrow & N/xN & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} & & \\ 0 & \longrightarrow & H^0(T(N)) & \xrightarrow{x} & H^0(T(N)) & \longrightarrow & H^0(T(N/xN)) & \longrightarrow & 0 \end{array}$$

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with exact rows (in which ϕ denotes $H^0(\theta(N; I^\cdot)^\cdot)$ and ψ denotes the isomorphism $H^0(\theta(N/xN; I^\cdot)^\cdot)$). A diagram-chase now shows that $\operatorname{coker}\phi = x(\operatorname{coker}\phi)$ and $\ker\phi = x(\ker\phi)$; thus ϕ is an isomorphism. Hence $\theta(N; I^\cdot)^\cdot$ is a quism, and the inductive step is complete.

It therefore follows by induction that $\theta(M; I^\cdot)^\cdot$ is a quism whenever M is a finitely generated A-module.

In particular, $\theta(A; I^\cdot)^\cdot$ is a quism. Hence the morphism of complexes

$$\alpha^\cdot: A \to \operatorname{Hom}_A(I^\cdot, I^\cdot)$$

of ((18), (2·3)) is a quism, and so ((18), (3·6)) $I^\cdot$ is a dualizing complex for A.

The following technical lemma will be very useful.

(2·5) LEMMA. *Let $J^\cdot$ and $K^\cdot$ be complexes in $\mathscr{Y}^+(A)$ with $K^\cdot$ an injective complex. Assume that $v^\cdot: J^\cdot \to K^\cdot$ is a quism. Let T be an additive left exact covariant functor from $\mathscr{C}(A)$ ((18), p. 369) to itself, and assume that $R^iT(J^j) = 0$ for all $i > 0$ and all j. Denote by $T(J^\cdot)$ (resp. $T(K^\cdot)$) the complex obtained by applying the functor T to $J^\cdot$ (resp. $K^\cdot$). Then*

$$H^r(T(J^\cdot)) \cong H^r(T(K^\cdot)) \quad \text{for all } r.$$

Proof. Let $Mc(v^\cdot)$ denote the mapping cone ((18), p. 372) of $v^\cdot$. By ((18), p. 373), there is an exact sequence

$$0 \to K^\cdot \to Mc(v^\cdot) \to \{1\}J^\cdot \to 0$$

in $\mathscr{Y}(A)$, and ((18), (2·5)) $Mc(v^\cdot)$ is an exact complex. Note that $Mc(v^\cdot)$ is bounded below.

Since $K^\cdot$ is an injective complex, the induced sequence

$$0 \to T(K^\cdot) \to T(Mc(v^\cdot)) \to T(\{1\}J^\cdot) \to 0$$

is exact. The result will therefore follow from the induced long exact sequence of cohomology modules if we can show that the complex $T(Mc(v^\cdot))$ is exact. However, the hypotheses ensure that $R^iT(Mc(v^\cdot)^j) = 0$ for all $i > 0$ and all j; also, $Mc(v^\cdot)$ is an exact complex bounded below; consequently, consideration of the short exact sequences obtainable from $Mc(v^\cdot)$ shows that $T(Mc(v^\cdot))$ is exact. The result follows.

One argument utilizing (2·5) will occur sufficiently frequently for it to be worth our while to record the argument as a separate corollary.

(2·6) COROLLARY. *Suppose that $I^\cdot$ is an injective complex in $\mathscr{Y}^+(A)$. Let B be a second ring, and $f: A \to B$ a flat ring homomorphism. Suppose that $K^\cdot$ is an injective complex in $\mathscr{Y}^+(B)$ for which there is a quism*

$$v^\cdot: B \otimes_A I^\cdot \to K^\cdot$$

in $\mathscr{Y}(B)$. Then, whenever M is a finitely generated A-module,

$$H^r(\operatorname{Hom}_B(B \otimes_A M, K^\cdot)) \cong B \otimes_A H^r(\operatorname{Hom}_A(M, I^\cdot)) \quad (\text{as } B\text{-modules}) \quad \text{for all } r.$$

Proof. Let M be a finitely generated A-module. Since B is A-flat,

$$\operatorname{Ext}^i_B(B \otimes_A M, B \otimes_A I^j) \cong B \otimes_A \operatorname{Ext}^i_A(M, I^j) = 0$$

for all $i > 0$ and all j. We may therefore apply (2·5) to the functor $\operatorname{Hom}_B(B \otimes_A M, —)$ (from $\mathscr{C}(B)$ to itself) to see that

$$H^r(\operatorname{Hom}_B(B \otimes_A M, K^\cdot)) \cong H^r(\operatorname{Hom}_B(B \otimes_A M, B \otimes_A I^\cdot)) \quad \text{for all } r.$$

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However, as B is A-flat, there is an isomorphism of complexes

$$\mathrm{Hom}_B(B \otimes_A M, B \otimes_A I^{\cdot}) \cong B \otimes_A \mathrm{Hom}_A(M, I^{\cdot})$$

in $\mathscr{Y}(B)$. The result follows.

In the following we shall use the very satisfactory theory of direct decompositions of injective modules which is outlined briefly in ((12), (5·1)). We shall say that *a prime ideal* $\mathfrak{p}$ *of* A *occurs in an injective* A*-module* I if I has a submodule isomorphic to $E_A(A/\mathfrak{p})$.

Suppose now that A is local and that I is an injective A-module. It will be convenient to use the notation $L_{\mathfrak{m}}$ of ((14), 2·1) for the local cohomology functor (w.r.t. the maximal ideal $\mathfrak{m}$ of A) from $\mathscr{C}(A)$ to itself. It is easy to see that $L_{\mathfrak{m}}(I)$ is an injective submodule of I, so that there exists an injective submodule J of I for which $I = L_{\mathfrak{m}}(I) \oplus J$. Note that $\mathfrak{m}$ does not occur in J, while $\mathfrak{m}$ is the only prime ideal of A which can occur in $L_{\mathfrak{m}}(I)$.

(2·7) Proposition. *Suppose that* A *is local, and that* $I^{\cdot}$ *is an injective complex in* $\mathscr{Y}^b_c(A)$. *Assume that there is an integer* t *such that*

$$H^i(\mathrm{Hom}_A(k, I^{\cdot})) \cong \begin{cases} 0 & \text{if } i \neq t, \\ k & \text{if } i = t. \end{cases}$$

Then there exists an injective complex $K^{\cdot}$ *in* $\mathscr{Y}^b_c(A)$ *such that*

$$\mathrm{Hom}_A(k, K^i) \cong \begin{cases} 0 & \text{if } i \neq t, \\ k & \text{if } i = t, \end{cases}$$

together with a quism $v^{\cdot}: I^{\cdot} \to K^{\cdot}$.

Proof. It follows from (2·2), (2·3), (2·5), and the fact that the composition of quisms is again a quism that we may assume $I^i = 0$ for all $i > t$.

Note that, by ((1), (2·7)), $\mathfrak{m}$ occurs in an injective A-module I if and only if

$$\mathrm{Hom}_A(k, I) \neq 0.$$

It is clear that $\mathfrak{m}$ occurs in I^t; since $I^{\cdot}$ is bounded, there must exist a least integer s such that $\mathfrak{m}$ occurs in I^s; let us suppose that $s < t$.

Let $L^s = L_{\mathfrak{m}}(I^s)$, and select a (necessarily injective) submodule Q^s of I^s such that $I^s = L^s \oplus Q^s$. Let $p^s_1: I^s \to L^s$ and $p^s_2: I^s \to Q^s$ be the canonical projections, and let $u^s_1: L^s \to I^s$ and $u^s_2: Q^s \to I^s$ be the canonical injections. Since $\mathrm{Hom}_A(k, I^{s-1}) = 0$ and $H^s(\mathrm{Hom}_A(k, I^{\cdot})) = 0$, the sequence

$$0 \to \mathrm{Hom}_A(k, I^s) \to \mathrm{Hom}_A(k, I^{s+1}) \qquad (*)$$

is exact. It follows that $L^s \cap \ker d^s_{I^{\cdot}} = 0$, for if not there would be, by ((20), 2·16), a non-zero element of $\ker d^s_{I^{\cdot}}$ which is annihilated by $\mathfrak{m}$, and this would contradict the exactness of (*).

Hence $d^s_{I^{\cdot}} . u^s_1: L^s \to I^{s+1}$ is monomorphic: let M^{s+1} be the image of this monomorphism. Since L^s is injective, we see that M^{s+1} is an injective submodule of I^{s+1}; therefore there exists a (necessarily injective) submodule N^{s+1} of I^{s+1} such that $I^{s+1} = M^{s+1} \oplus N^{s+1}$. Let $p^{s+1}_1: I^{s+1} \to M^{s+1}$ and $p^{s+1}_2: I^{s+1} \to N^{s+1}$ be the canonical

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projections and let $u_1^{s+1}\colon M^{s+1} \to I^{s+1}$ and $u_2^{s+1}\colon N^{s+1} \to I^{s+1}$ be the canonical injections. Observe that

$$p_1^{s+1}.\, d_I^s.\, u_1^s\colon L^s \to M^{s+1} \quad \text{is an isomorphism,}$$
$$p_2^{s+1}.\, d_I^s.\, u_1^s\colon L^s \to N^{s+1} \quad \text{is zero,}$$
and
$$d_I^{s+1}.\, u_1^{s+1}\colon M^{s+1} \to I^{s+2} \quad \text{is zero.}$$

(The last fact follows from the inclusion $M^{s+1} \subseteq \operatorname{im} d_I^s.$.)

The aim of the next stage of the proof is to show that one can omit the isomorphic modules L^s and M^{s+1} from $I^\cdot$ to obtain an injective complex $J^\cdot$ in $\mathscr{D}_c^b(A)$ which has the properties that the only terms J^i in which $\mathfrak{m}$ could conceivably occur are those for which $s+1 \leqslant i \leqslant t$, and that there exists a quism $w^\cdot\colon I^\cdot \to J^\cdot$. More precisely, let $J^i = I^i$ for all integers i different from s and $s+1$, let $J^s = Q^s$, and let $J^{s+1} = N^{s+1}$; define $d_J^i.$ to be $d_I^i.$ for all integers i different from $s-1$, s and $s+1$, define d_J^{s-1} to be $p_2^s.\, d_I^{s-1}$, define $d_J^s.$ to be $p_2^{s+1}.\, d_I^s.\, u_2^s$, and define d_J^{s+1} to be $d_I^{s+1}.\, u_2^{s+1}$; and, lastly, let w^i be the identity endomorphism of I^i for all integers i different from s and $s+1$, let w^s be p_2^s, and let w^{s+1} be p_2^{s+1}. A rather lengthy series of straightforward diagram-chases will now show that $J^\cdot$ is a complex and that $w^\cdot\colon I^\cdot \to J^\cdot$ is a morphism of complexes which is even a quism. It is clear from the construction of $J^\cdot$ that it is an injective bounded complex (so that it must belong to $\mathscr{D}_c^b(A)$) and that the only terms J^i in which $\mathfrak{m}$ could conceivably occur are those for which $s+1 \leqslant i \leqslant t$. It follows from (2·5) that

$$H^i(\operatorname{Hom}_A(k, J^\cdot)) \cong \begin{cases} 0 & \text{if} \quad i \neq t, \\ k & \text{if} \quad i = t. \end{cases}$$

Since the composition of quisms is again a quism, we see that, after repeating the argument of the preceding three paragraphs a finite number of times, we shall arrive at an injective complex $K^\cdot$ in $\mathscr{D}_c^b(A)$ with the properties that

$$H^i(\operatorname{Hom}_A(k, K^\cdot)) \cong \begin{cases} 0 & \text{if} \quad i \neq t, \\ k & \text{if} \quad i = t, \end{cases}$$

that K^t is the only term of $K^\cdot$ in which $\mathfrak{m}$ occurs, and that there is a quism $v^\cdot\colon I^\cdot \to K^\cdot$. This $K^\cdot$ has all the desired properties, and so the proof is complete.

We are now in a position to prove the converse of (2·1) which was promised earlier.

(2·8) THEOREM. (See Hartshorne ((6), v·3·4).) *Suppose that A is local and that $I^\cdot$ is an injective complex in $\mathscr{D}_c^b(A)$. Assume that there exists an integer t such that*

$$H^i(\operatorname{Hom}_A(k, I^\cdot)) \cong \begin{cases} 0 & \text{if} \quad i \neq t, \\ k & \text{if} \quad i = t. \end{cases}$$

Then $I^\cdot$ is a dualizing complex for A.

Proof. By (2·7), we can find an injective complex $K^\cdot$ in $\mathscr{D}_c^b(A)$ such that t is the unique integer i for which $\mathfrak{m}$ occurs in K^i, that $\operatorname{Hom}_A(k, K^t) \cong k$, and that there exists a quism $v^\cdot\colon I^\cdot \to K^\cdot$. By ((18), (4·3) (iii)), it is sufficient to show that $K^\cdot$ is a dualizing complex for A. By (2·4), it is sufficient to show that the morphism of complexes

$$\theta(k; K^\cdot)^\cdot\colon k \to \operatorname{Hom}_A([\operatorname{Hom}_A(k, K^\cdot)], K^\cdot)$$

of ((18), (2·3) (ii)) is a quism.

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It is clear that there is an isomorphism of complexes

$$\mathrm{Hom}_A(k, K^\cdot) \to \{-t\}k;$$

this induces a further isomorphism of complexes

$$\mathrm{Hom}_A(\{-t\}k, K^\cdot) \to \mathrm{Hom}_A([\mathrm{Hom}_A(k, K^\cdot)], K^\cdot).$$

These facts show that the complex $\mathrm{Hom}_A([\mathrm{Hom}_A(k, K^\cdot)], K^\cdot)$ is actually isomorphic to k, and has 0-th term equal to $\mathrm{Hom}_A(\mathrm{Hom}_A(k, K^t), K^t)$. It is therefore sufficient to show that the A-homomorphism

$$\theta(k; K^\cdot)^0 : k \to \mathrm{Hom}_A(\mathrm{Hom}_A(k, K^t), K^t)$$

is non-zero. However, an examination of the definition of this homomorphism in ((18), (2·3)) shows that, for $x \in k$ and $g \in \mathrm{Hom}_A(k, K^t)$, we have

$$[\theta(k; K^\cdot)^0(x)](g) = (-1)^{\tau(t)} g(x),$$

where τ is the mapping defined in ((18), (2·3) (ii)). Since $\mathrm{Hom}_A(k, K^t) \neq 0$, it follows that $\theta(k; K^\cdot)^0$ cannot be zero. This completes the proof.

Our last lemma in this section effectively shows that if a dualizing complex $I^\cdot$ for A is exact at I^i for all i less than some integer s, then a new dualizing complex $J^\cdot$ may be constructed for A with the property that $J^i = 0$ for all $i < s$.

(2·9) Lemma. *Let $I^\cdot$ be an injective complex in $\mathscr{D}^+(A)$. Let s be an integer such that $H^i(I^\cdot) = 0$ for all $i < s$. Let $B^s = \mathrm{im}\, d_{I^\cdot}^{s-1}$, and let $J^s = I^s/B^s$. Let $J^\cdot$ denote the complex*

$$\ldots \to 0 \to 0 \to J^s \to I^{s+1} \to I^{s+2} \to \ldots \to I^i \to \ldots$$

(in which it is to be understood that J^s is the s-th term) induced from $I^\cdot$. (Thus, in the notation of the proof of ((18), (3·1)), $J^\cdot$ is the truncation $\sigma'_{\geqslant s}(I^\cdot)$.) Then $J^\cdot$ is an injective complex in $\mathscr{D}^+(A)$, and there is a quism $u^\cdot : I^\cdot \to J^\cdot$.

Proof. Since $I^\cdot$ is bounded below, there exists an integer m ($< s$) such that $I^i = 0$ for all $i < m$. The exact sequence

$$0 \to I^m \to I^{m+1} \to \ldots \to I^{s-1} \to B^s \to 0$$

shows that B^s is injective; B^s is therefore a direct summand of I^s, and so J^s is also injective. For $i > s$, let $u^i : I^i \to I^i$ be the identity mapping; let $u^s : I^s \to J^s = I^s/B^s$ be the natural epimorphism; and for $i < s$, let $u^i : I^i \to 0$ be the zero homomorphism. Then $u^\cdot$ is a quism, and the lemma is proved.

3. *Properties of rings which possess dualizing complexes.* The purpose of this section is to prove that if A possesses a dualizing complex, then A is acceptable. We begin with the easy results that a ring which possesses a dualizing complex has open Gorenstein locus and is catenary.

(3·1) Lemma. *Suppose that A is local and the injective complex $I^\cdot$ in $\mathscr{D}^b_c(A)$ is a dualizing complex for A. Then A is a Gorenstein ring if and only if there exists an integer s such that*

$$H^i(I^\cdot) = 0 \quad \textit{whenever } i \neq s,$$

and

$$H^s(I^\cdot) \cong A \quad (\textit{resp. is free}).$$

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Proof. ($\Rightarrow$) If A is a Gorenstein ring, then it is immediate from ((18), (3·7) and (4·5)) that there is an integer s such that $H^i(I^\cdot) = 0$ whenever $i \neq s$ and $H^s(I^\cdot) \cong A$.

($\Leftarrow$) Assume that there is an integer s such that $H^i(I^\cdot) = 0$ whenever $i \neq s$ and $H^s(I^\cdot)$ is free. It is clear from (2·9) that $H^s(I^\cdot)$ cannot be zero, since the zero complex cannot be a dualizing complex for A. Thus $H^s(I^\cdot)$ is a non-zero finitely generated free A-module.

By (2·9) and ((18), (4·3) (iii)), we may assume that $I^i = 0$ for all $i < s$. In this situation, there is an exact sequence

$$0 \to H^s(I^\cdot) \to I^s \to I^{s+1} \to \ldots \to I^i \to \ldots;$$

since $I^\cdot$ is bounded, it follows that $H^s(I^\cdot)$ has finite injective dimension. Therefore A has finite injective dimension, and so must be a Gorenstein ring.

(3·2) PROPOSITION. *If A has a dualizing complex, then* $\mathrm{Gor}(A)$ *is an open subset of* $\mathrm{Spec}(A)$.

Proof. Suppose the injective complex $I^\cdot$ in $\mathscr{D}_c^b(A)$ is a dualizing complex for A. There exists a positive integer u such that $I^i = 0$ whenever $i < -u$ or $i > u$. For each i, let $U_i = \mathrm{Spec}(A) - \mathrm{Supp}_A(H^i(I^\cdot))$: since $H^i(I^\cdot)$ is finitely generated, each U_i is an open subset of $\mathrm{Spec}(A)$.

Let $\mathfrak{p}$ be a prime ideal in $\mathrm{Gor}(A)$. By ((18), (4·2)), $(I^\cdot)_\mathfrak{p}$ is a dualizing complex for $A_\mathfrak{p}$. Therefore, by (3·1), there is an integer s such that $[H^i(I^\cdot)]_\mathfrak{p} = 0$ whenever $i \neq s$ and $[H^s(I^\cdot)]_\mathfrak{p} \cong A_\mathfrak{p}$. Let F be a finitely generated free A-module for which there is an epimorphism $f\colon F \to H^s(I^\cdot)$. Let K denote the kernel of f. It is easy to see that, if $\mathfrak{q}$ is a prime ideal of A, then $[H^s(I^\cdot)]_\mathfrak{q}$ is a free $A_\mathfrak{q}$-module if and only if

$$[\mathrm{Ext}^1_A(H^s(I^\cdot), K)]_\mathfrak{q} = 0.$$

So, if V denotes the open subset

$$\mathrm{Spec}(A) - \mathrm{Supp}_A(\mathrm{Ext}^1_A(H^s(I^\cdot), K))$$

of $\mathrm{Spec}(A)$, it follows easily from (3·1) and ((18), (4·2)) that

$$\mathfrak{p} \in V \cap \left[\bigcap_{\substack{i=-u \\ i \neq s}}^{u} U_i\right] \subseteq \mathrm{Gor}(A).$$

It follows that $\mathrm{Gor}(A)$ is an open subset of $\mathrm{Spec}(A)$.

If $\mathfrak{p}$ is a prime ideal of A, then $k(\mathfrak{p})$ (or $k_A(\mathfrak{p})$) will denote the residue field of the local ring $A_\mathfrak{p}$. If the injective complex $I^\cdot$ in $\mathscr{D}_c^b(A)$ is a dualizing complex for A, then ((18), (4·2)) $(I^\cdot)_\mathfrak{p}$ is a dualizing complex for $A_\mathfrak{p}$. Hence, by (2·1), there exists exactly one integer i for which $H^i(\mathrm{Hom}_{A_\mathfrak{p}}(k(\mathfrak{p}), (I^\cdot)_\mathfrak{p})) \neq 0$: for the time being we shall denote this integer by $t(\mathfrak{p}; I^\cdot)$. From (2·1), we see that

$$H^i(\mathrm{Hom}_{A_\mathfrak{p}}(k(\mathfrak{p}), (I^\cdot)_\mathfrak{p})) \cong \begin{cases} 0 & \text{if } i \neq t(\mathfrak{p}; I^\cdot), \\ k(\mathfrak{p}) & \text{if } i = t(\mathfrak{p}; I^\cdot). \end{cases}$$

It is not difficult to see from ((10), §3·8, Propositions 19 and 20) that if S is a multiplicatively closed subset of A which does not meet $\mathfrak{p}$ (so that, by ((18), (4·2)), $S^{-1}(I^\cdot)$ is a dualizing complex for $S^{-1}A$), then $t(S^{-1}\mathfrak{p}; S^{-1}(I^\cdot)) = t(\mathfrak{p}; I^\cdot)$.

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(3·3) LEMMA. (See Hartshorne ((6), v·7·1).) *Suppose that $I^\cdot$ is a dualizing complex for A and that $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals of A such that $\mathfrak{p} \subset \mathfrak{q}$ and there are no prime ideals strictly between $\mathfrak{p}$ and $\mathfrak{q}$. (The symbol '$\subset$' is reserved to denote strict inclusion.) Then*

$$t(\mathfrak{q}; I^\cdot) = t(\mathfrak{p}; I^\cdot) + 1.$$

Proof. By virtue of the sentence immediately preceding the statement of the lemma, we may assume that A is local and that $\mathfrak{q} = \mathfrak{m}$, the maximal ideal of A.

Let $x \in \mathfrak{m} - \mathfrak{p}$. Let $\mathfrak{b}$ be the ideal $\mathfrak{p} + Ax$ of A. Then there is an exact sequence

$$0 \longrightarrow A/\mathfrak{p} \xrightarrow{x} A/\mathfrak{p} \longrightarrow A/\mathfrak{b} \longrightarrow 0,$$

and $A/\mathfrak{b}$ is an A-module of finite length.

Let i be an integer different from $t(\mathfrak{q}; I^\cdot)$. Thus $H^i(\mathrm{Hom}_A(k, I^\cdot)) = 0$. Induction on length shows that $H^i(\mathrm{Hom}_A(M, I^\cdot)) = 0$ whenever M is an A-module of finite length; in particular $H^i(\mathrm{Hom}_A(A/\mathfrak{b}, I^\cdot)) = 0$.

Now the above exact sequence induces an exact sequence

$$0 \to \mathrm{Hom}_A(A/\mathfrak{b}, I^\cdot) \to \mathrm{Hom}_A(A/\mathfrak{p}, I^\cdot) \to \mathrm{Hom}_A(A/\mathfrak{p}, I^\cdot) \to 0$$

in $\mathscr{D}(A)$; this in turn induces a long exact sequence of cohomology modules which shows that

$$H^{i-1}(\mathrm{Hom}_A(A/\mathfrak{p}, I^\cdot)) = xH^{i-1}(\mathrm{Hom}_A(A/\mathfrak{p}, I^\cdot)).$$

However ((18), (3·5) (ii)), $H^{i-1}(\mathrm{Hom}_A(A/\mathfrak{p}, I^\cdot))$ is finitely generated, and so this module is zero by Nakayama's Lemma. Hence $i - 1 \neq t(\mathfrak{p}; I^\cdot)$. The result follows.

(3·4) COROLLARY. (See Hartshorne ((6), v·7·2).) *If A has a dualizing complex, then A is finite-dimensional and catenary.*

Proof. This is immediate from (3·3) and the fact that a dualizing complex is bounded.

Thus, if A has a dualizing complex, it follows from (3·2) and (3·4) that A has open Gorenstein locus and is catenary. Of course, we have to prove that whenever B is a finitely generated A-algebra, then B has open Gorenstein locus and is catenary: we shall achieve this by showing that such a B has a dualizing complex. To prove this latter point it is sufficient, since ((18), (3·9)) a homomorphic image of a ring which possesses a dualizing complex also has a dualizing complex, to show that the assumption that A possesses a dualizing complex implies that, if X is an indeterminate over A, then $A[X]$ has a dualizing complex.

(3·5) THEOREM. (See Hartshorne ((6), VI·3·5).) *Suppose A has a dualizing complex. If X is an indeterminate over A, then $A[X]$ has a dualizing complex.*

Proof. Let the injective complex $I^\cdot$ in $\mathscr{D}^b_c(A)$ be a dualizing complex for A. By ((18), (3·5) (iii)), the complex $A[X] \otimes_A I^\cdot \in \mathscr{D}^b_c(A[X])$. Let u be a positive integer such that $A[X] \otimes_A I^i = 0$ whenever $i < -u$ or $i > u$. By ((3), chapter XVII, § 1) (and with the terminology thereof), we may find, over the ring $A[X]$, an injective resolution $Y^{\cdot\cdot}$ of the complex $A[X] \otimes_A I^\cdot$ which is such that, for each integer q, $Y^{i,q} = 0$ whenever $i < -u$ or $i > u$. However, ((2), 4·5), the injective dimension over $A[X]$ of each

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$A[X] \otimes_A I^i$ is at most 1. We may therefore modify the double complex $Y^{\cdot\cdot}$ to obtain, again over $A[X]$, a right double complex $K^{\cdot\cdot}$ over $A[X] \otimes_A I^{\cdot}$ which is such that

(*a*) $K^{i,q} = 0$ whenever $|i| > u$ or $q > 1$ (or $q < 0$, of course);

(*b*) $K^{i,q}$ is an injective $A[X]$-module for all i and q; and

(*c*) for each integer i, the induced complex

$$0 \to A[X] \otimes_A I^i \to K^{i,0} \to K^{i,1} \to 0$$

in $Y(A[X])$ is exact.

Let $K^{\cdot}$ denote the single complex associated with $K^{\cdot\cdot}$; thus $K^{\cdot}$ is a bounded injective complex in $\mathscr{Y}(A[X])$. Next, we note that the ideas of Douglas ((4), pp. 117–119) show that the augmentation morphism induces a quism

$$v^{\cdot}: A[X] \otimes_A I^{\cdot} \to K^{\cdot}.$$

It follows from (2·6) that $K^{\cdot} \in \mathscr{Y}_c^b(A[X])$.

We aim to show that $K^{\cdot}$ is a dualizing complex for $A[X]$. To do this it is, by ((18), (4·2)), sufficient to show that if $\mathfrak{n}$ is an arbitrary maximal ideal of $A[X]$, then $(K^{\cdot})_{\mathfrak{n}}$ is a dualizing complex for $A[X]_{\mathfrak{n}}$. Let $\mathfrak{p} = A \cap \mathfrak{n}$, a prime ideal of A, and let $S = A - \mathfrak{p}$, a multiplicatively closed subset of both A and $A[X]$. We may use ((10), § 3·8, propositions 19 and 20) in conjunction with (2·1) and (2·8) to see that it is enough to show that $(S^{-1}(K^{\cdot}))_{S^{-1}\mathfrak{n}}$ is a dualizing complex for $(S^{-1}(A[X]))_{S^{-1}\mathfrak{n}}$. We therefore concentrate on the ring $S^{-1}(A[X])$.

Note that the induced injective ring homomorphism $h: S^{-1}A \to S^{-1}(A[X])$ makes $S^{-1}(A[X])$ free over $S^{-1}A$ $(= A_{\mathfrak{p}})$, and that under h the maximal ideal $S^{-1}\mathfrak{n}$ contracts to $\mathfrak{p}A_{\mathfrak{p}}$. Also, $S^{-1}(A[X])/\mathfrak{p}S^{-1}(A[X]) \cong k(\mathfrak{p})[X]$, a principal ideal domain. (Here, $\mathfrak{p}S^{-1}(A[X])$ denotes the extension of $\mathfrak{p}$ to $S^{-1}(A[X])$.) Furthermore, by ((18), (2·8) (ii) and (2·10) (iii)), $v^{\cdot}$ induces a quism $S^{-1}(A[X] \otimes_A I^{\cdot}) \to S^{-1}K^{\cdot}$ in $\mathscr{Y}(S^{-1}(A[X]))$. However, it is not difficult to see that $S^{-1}(A[X] \otimes_A I^{\cdot})$ and $S^{-1}(A[X]) \otimes_{A_{\mathfrak{p}}} (I^{\cdot})_{\mathfrak{p}}$ are isomorphic complexes in $\mathscr{Y}(S^{-1}(A[X]))$; thus there exists a quism

$$S^{-1}(A[X]) \otimes_{A_{\mathfrak{p}}} (I^{\cdot})_{\mathfrak{p}} \to S^{-1}(K^{\cdot})$$

in $\mathscr{Y}(S^{-1}(A[X]))$; moreover, $(I^{\cdot})_{\mathfrak{p}}$ is a dualizing complex for $A_{\mathfrak{p}}$, and $S^{-1}(K^{\cdot})$ is an injective complex in $\mathscr{Y}_c^b(S^{-1}(A[X]))$.

It is therefore enough for us to solve the following problem. A is a local ring (with maximal ideal $\mathfrak{m}$), B is a second ring, $h: A \to B$ is an injective ring homomorphism which makes B free over A, $\mathfrak{n}$ is a maximal ideal of B which contracts to $\mathfrak{m}$, and $B/\mathfrak{m}B$ is a principal ideal domain (but not a field); $I^{\cdot}$ is a dualizing complex for A, and $J^{\cdot}$ is an injective complex in $\mathscr{Y}_c^b(B)$ for which there is a quism

$$u^{\cdot}: B \otimes_A I^{\cdot} \to J^{\cdot}$$

in $\mathscr{Y}(B)$; and we must show that $(J^{\cdot})_{\mathfrak{n}}$ is a dualizing complex for $B_{\mathfrak{n}}$.

Since B is A-flat, it follows from (2·6) and (2·1) that there is an integer t such that

$$H^i(\mathrm{Hom}_B(B/\mathfrak{m}B, J^{\cdot})) \cong \begin{cases} 0 & \text{if } i \neq t, \\ B/\mathfrak{m}B & \text{if } i = t. \end{cases}$$

Now there is an element y of $\mathfrak{n}$ such that $\mathfrak{m}B + yB = \mathfrak{n}$. The exact sequence

$$0 \longrightarrow B/\mathfrak{m}B \xrightarrow{y} B/\mathfrak{m}B \longrightarrow B/\mathfrak{n} \longrightarrow 0$$

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induces an exact sequence

$$0 \to \mathrm{Hom}_B(B/\mathfrak{n}, J^\cdot) \to \mathrm{Hom}_B(B/\mathfrak{m}B, J^\cdot) \to \mathrm{Hom}_B(B/\mathfrak{m}B, J^\cdot) \to 0$$

in $\mathscr{Y}(B)$, and this induces a long exact sequence of cohomology modules; this in turn shows that $H^i(\mathrm{Hom}_B(B/\mathfrak{n}, J^\cdot)) = 0$ whenever i is different from t and $t+1$, and that there is an exact sequence

$$0 \to H^t(\mathrm{Hom}_B(B/\mathfrak{n}, J^\cdot)) \to H^t(\mathrm{Hom}_B(B/\mathfrak{m}B, J^\cdot)) \xrightarrow{y} H^t(\mathrm{Hom}_B(B/\mathfrak{m}B, J^\cdot))$$
$$\to H^{t+1}(\mathrm{Hom}_B(B/\mathfrak{n}, J^\cdot)) \to 0.$$

Since $H^t(\mathrm{Hom}_B(B/\mathfrak{m}B, J^\cdot)) \cong B/\mathfrak{m}B$, and y is a non-zerodivisor on $B/\mathfrak{m}B$, we see that $H^t(\mathrm{Hom}_B(B/\mathfrak{n}, J^\cdot)) = 0$ and

$$H^{t+1}(\mathrm{Hom}_B(B/\mathfrak{n}, J^\cdot)) \cong [B/\mathfrak{m}B]/y[B/\mathfrak{m}B] \cong B/\mathfrak{n}.$$

Consequently,

$$H^i(\mathrm{Hom}_B(B/\mathfrak{n}, J^\cdot)) \cong \begin{cases} 0 & \text{if } i \neq t+1, \\ B/\mathfrak{n} & \text{if } i = t+1. \end{cases}$$

Therefore,

$$H^i(\mathrm{Hom}_{B_\mathfrak{n}}(k_B(\mathfrak{n}), (J^\cdot)_\mathfrak{n}) \cong \begin{cases} 0 & \text{if } i \neq t+1, \\ k_B(\mathfrak{n}) & \text{if } i = t+1. \end{cases}$$

It therefore follows from (2·8) that $(J^\cdot)_\mathfrak{n}$ is a dualizing complex for $B_\mathfrak{n}$.

This completes the proof.

The last major preliminary result about dualizing complexes which we need is that if A is local and possesses a dualizing complex, then the natural ring homomorphism $A \to \hat{A}$ is a Gorenstein ring homomorphism. This result is due to Grothendieck and Hartshorne ((6), p. 300), but we give a proof which is consistent with our approach.

(3·6) Proposition. (See Hartshorne ((6), v·3·5).) *Suppose A is local and that the injective complex $I^\cdot$ in $\mathscr{Y}^b_c(A)$ is a dualizing complex for A. Then there is a dualizing complex $J^\cdot$ for $\hat{A}$ for which there exists a quism $u^\cdot$: $\hat{A} \otimes_A I^\cdot \to J^\cdot$ (in $\mathscr{Y}(\hat{A})$).*

Proof. We begin as we began the proof of (3·5). By ((18), (3·5)(iii)), the complex $\hat{A} \otimes_A I^\cdot \in \mathscr{Y}^b_c(\hat{A})$. Let u be a positive integer such that $\hat{A} \otimes_A I^i = 0$ whenever $|i| > u$. By ((3), chapter XVII, §1) (and with the terminology thereof), we may find, over the ring $\hat{A}$, an injective resolution $K^{\cdot\cdot}$ of the complex $\hat{A} \otimes_A I^\cdot$ which is such that $K^{i,q} = 0$ whenever $|i| > u$ (or $q < 0$, of course).

Let $K^\cdot$ be the single complex associated with $K^{\cdot\cdot}$, so that $K^\cdot$ is an injective complex in $\mathscr{Y}^+(\hat{A})$. Note that we do not know whether $K^\cdot$ is bounded, and we shall, in fact, modify $K^\cdot$ in order to construct a dualizing complex for $\hat{A}$. The ideas of Douglas ((4), pp. 117–119) show that the augmentation morphism induces a quism

$$v^\cdot: \hat{A} \otimes_A I^\cdot \to K^\cdot$$

in $\mathscr{Y}(\hat{A})$. Next, (2·6) shows firstly that $K^\cdot \in \mathscr{Y}^+_c(\hat{A})$, and secondly, when used in conjunction with (2·1), that there exists an integer t such that

$$H^i(\mathrm{Hom}_{\hat{A}}(\hat{A}/\hat{\mathfrak{m}}, K^\cdot)) \cong \begin{cases} 0 & \text{if } i \neq t, \\ \hat{A}/\hat{\mathfrak{m}} & \text{if } i = t, \end{cases}$$

where $\hat{\mathfrak{m}}$ denotes the maximal ideal of $\hat{A}$.

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We may now use (2·2) and (2·3) to see that there is a bounded injective complex $J^\cdot$ in $\mathscr{D}_c^b(\hat{A})$ for which there is a quism $w^\cdot: K^\cdot \to J^\cdot$. Therefore there is a quism

$$u^\cdot: \hat{A} \otimes_A I^\cdot \to J^\cdot;$$

use of (2·6) and (2·1) now shows that

$$H^i(\mathrm{Hom}_{\hat{A}}(\hat{A}/\hat{\mathfrak{m}}, J^\cdot)) \cong \begin{cases} 0 & \text{if } i \neq t, \\ \hat{A}/\hat{\mathfrak{m}} & \text{if } i = t. \end{cases}$$

It follows from (2·8) that $J^\cdot$ is a dualizing complex for $\hat{A}$.

(3·7) THEOREM. (See Hartshorne ((6), p. 300).) *Suppose that A is local and possesses a dualizing complex. Then the natural ring homomorphism $A \to \hat{A}$ is a Gorenstein ring homomorphism.*

Proof. Let $\mathfrak{p}$ be a prime ideal of A, and let $T = A - \mathfrak{p}$, a multiplicatively closed subset of both A and $\hat{A}$. We must show that $\hat{A} \otimes_A k(\mathfrak{p})$ is a Gorenstein ring. By ((17), (2·2) (i)), $\hat{A} \otimes_A k(\mathfrak{p})$ is isomorphic to $T^{-1}\hat{A}/T^{-1}(\mathfrak{p}\hat{A})$.

Suppose that $I^\cdot$ is a dualizing complex for A. By (3·6), there exists a dualizing complex $J^\cdot$ for $\hat{A}$ for which there exists a quism $u^\cdot: \hat{A} \otimes_A I^\cdot \to J^\cdot$ in $\mathscr{D}(\hat{A})$. By ((18), (2·8) (ii) and (2·10) (iii)), $u^\cdot$ induces a quism

$$T^{-1}(\hat{A} \otimes_A I^\cdot) \to T^{-1}(J^\cdot)$$

in $\mathscr{D}(T^{-1}\hat{A})$. However, it is easy to see that there is an isomorphism of complexes

$$T^{-1}\hat{A} \otimes_{A_\mathfrak{p}} (I^\cdot)_\mathfrak{p} \to T^{-1}(\hat{A} \otimes_A I^\cdot)$$

in $\mathscr{D}(T^{-1}\hat{A})$. Note that, by ((18), (4·2)), $(I^\cdot)_\mathfrak{p}$ is a dualizing complex for $A_\mathfrak{p}$ and $T^{-1}(J^\cdot)$ is a dualizing complex for $T^{-1}\hat{A}$. Since $T^{-1}\hat{A}$ is flat over $A_\mathfrak{p}$, it follows from (2·6) and (2·1) that there is an integer t such that

$$H^i(\mathrm{Hom}_{T^{-1}\hat{A}}(T^{-1}\hat{A} \otimes_{A_\mathfrak{p}} k(\mathfrak{p}), T^{-1}(J^\cdot)) \cong \begin{cases} 0 & \text{if } i \neq t, \\ T^{-1}\hat{A} \otimes_{A_\mathfrak{p}} k(\mathfrak{p}) & \text{if } i = t. \end{cases}$$

However, $T^{-1}\hat{A} \otimes_{A_\mathfrak{p}} k(\mathfrak{p})$ and $T^{-1}\hat{A}/T^{-1}(\mathfrak{p}\hat{A})$ are isomorphic $T^{-1}\hat{A}$-modules. Hence

$$H^i(\mathrm{Hom}_{T^{-1}\hat{A}}(T^{-1}\hat{A}/T^{-1}(\mathfrak{p}\hat{A}), T^{-1}(J^\cdot)) \cong \begin{cases} 0 & \text{if } i \neq t, \\ T^{-1}\hat{A}/T^{-1}(\mathfrak{p}\hat{A}) & \text{if } i = t. \end{cases}$$

But ((18), (3·9)) $\mathrm{Hom}_{T^{-1}\hat{A}}(T^{-1}\hat{A}/T^{-1}(\mathfrak{p}\hat{A}), T^{-1}(J^\cdot))$ is a dualizing complex for

$$T^{-1}\hat{A}/T^{-1}(\mathfrak{p}\hat{A}).$$

It therefore follows from (3·1) that $T^{-1}\hat{A}/T^{-1}(\mathfrak{p}\hat{A})$ is a Gorenstein ring.

We are now in a position to prove the theorem of the title of the paper.

(3·8) THEOREM. *Suppose the ring A possesses a dualizing complex. Then A is an acceptable ring.*

Proof. Let B be a finitely generated A-algebra. By (3·5) and ((18), (3·9)), B possesses a dualizing complex. By (3·2), $\mathrm{Gor}(B)$ is an open subset of $\mathrm{Spec}(B)$. Also, by (3·4),

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B is catenary; it follows that A is universally catenary. Also, if $\mathfrak{p}$ is a prime ideal of A, then ((18), (4·2)) $A_{\mathfrak{p}}$ possesses a dualizing complex, and so (3·7) shows that the natural ring homomorphism $A_{\mathfrak{p}} \to (A_{\mathfrak{p}})^{\wedge}$ is a Gorenstein ring homomorphism.

Therefore A is an acceptable ring.

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