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An example of bifurcation and non-uniqueness in electrohydrodynamics

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The theorem of bifurcation from a simple eigenvalue is applied to prove non-uniqueness for the problem of a layer of a dielectric liquid subjected to an electric field and to injection of charges on the electrodes.

1 Introduction

Electric charge convection in a dielectric liquid subjected to unipolar injection has been investigated mainly by the British and French school of fluid mechanics (see, e.g., [1–11] and the references therein). We refer in particular to Atten *et al*. [4], where an analogy is pointed out between the problem of charge convection in a dielectric liquid and that of thermal convection in an horizontal layer heated from below, the so-called Bénard problem [12]. The experimental apparatus contains two circular plane metal electrodes covered with ion exchange membranes. With these electrodes, a controllable injection of known ions is achieved. If the dielectric liquid is initially stationary, a first bifurcation occurs when the applied voltage reaches a critical value. A structure of very stable hexagonal cells can be seen by the Schlieren method. These cells are surprisingly similar to those of the celebrated Bénard experiment. As the voltage is further increased, the regular pattern breaks down and turbulence eventually occurs. Similar results have been obtained, with different methods [13, 14]. We also mention the generalized energy method developed [7, 11, 15, 16].

The basic constitutive equations used in electrohydrodynamics are either

$$\boldsymbol{J} = -k\nabla q + q\boldsymbol{v} + MqE, \tag{1.0}$$

or

$$\boldsymbol{J} = -k\nabla q + q\boldsymbol{v} + \sigma E, \tag{1.1}$$

where $\boldsymbol{J}$ is the current density, q the charge density, $\boldsymbol{v}$ the velocity and E the electric field; k is the charge diffusion coefficient and M the ionic mobility. The diffusion term is often neglected, although it seems to play a crucial role both physically and mathematically. For a formulation of the problem based on (1.0) we refer to Deo & Richardson [7], where, however, only stability results are presented.

The goal of this paper is to prove, in a mathematically rigorous way, the occurrence of the first bifurcation from a steady, one-dimensional, hydrostatic equilibrium solution. Our formulation is based on (1.1), where $\sigma \geqslant 0$ is the constant electric conductivity. Equation (1.1) clearly gives rise to 'less nonlinear' equations, but it is physically equally acceptable, if referred to slightly conducting liquid like corn oil, which was used in the experiment of Schneider and Watson.

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We find that, from the mathematical point of view, the problem of electric charge convection in a dielectric liquid subjected to unipolar injection indeed has similarities with the Bénard problem, as stated in Atten *et al.* [4], but with a key difference: the linear operator to which the study is ultimately reduced is not always positive, as happens in the Bénard case. This is our main conclusion, and what makes the present problem considerably more difficult.

We note that our theory applies even when $\sigma = 0$. The assumptions of electrohydrodynamics are taken as valid (see: [17, Chapter 11]). In particular, all magnetic effects are neglected. This results in the existence of an electric potential u, i.e.

$$\boldsymbol{E} = \nabla u \tag{1.2}$$

which, along with the relations $\boldsymbol{D} = \epsilon \boldsymbol{E}$ and $q = \nabla \cdot \boldsymbol{D}$, gives rise to

$$\epsilon \Delta u = q \tag{1.3}$$

with ϵ the electric permittivity. We shall study a plane problem in the strip

$$S = \{(x, y); -\infty < x < \infty, 0 < y < L\}.$$

The fluid is supposed to be incompressible; therefore

$$\nabla \cdot \boldsymbol{v} = 0. \tag{1.4}$$

We introduce a stream function $\psi(x, y)$ by setting

$$v_1 = \psi_y \quad v_2 = -\psi_x, \tag{1.5}$$

where v_1 and v_2 are the components of the velocity in the x and y directions. In terms of ψ, the equation of motion

$$\nabla p - \mu \Delta \boldsymbol{v} + \rho(\boldsymbol{v}\nabla)\,\boldsymbol{v} = q\boldsymbol{E} \tag{1.6}$$

becomes

$$\mu \Delta^2 \psi + \rho(\psi_y \Delta \psi_x - \psi_x \Delta \psi_y) + (q_y u_x - q_x u_y) = 0, \tag{1.7}$$

where μ is the dynamic viscosity and ρ the mass density. Only body forces of electric origin are taken into account, and they are represented by the term $q\boldsymbol{E}$ in (1.6). The principle of conservation of charge implies

$$\nabla \cdot \boldsymbol{J} = 0. \tag{1.8}$$

Substituting (1.1), (1.2) and (1.3) into (1.8), we have, by (1.4) and (1.5),

$$-k\epsilon \Delta q + \epsilon(q_x \psi_y - q_y \psi_x) + \sigma q = 0. \tag{1.9}$$

The system (1.3), (1.7) and (1.9) is completed with the boundary conditions

$$\psi(x, 0) = \psi(x, L) = \psi_y(x, 0) = \psi_y(x, L) = 0, \tag{1.10}$$

$$u(x, 0) = 0, \quad u(x, L) = u_2, \quad u_2 > 0, \tag{1.11}$$

$$q(x, 0) = q_1, \quad q(x, L) = q_2. \tag{1.12}$$

Instead of (1.10), others less physically obvious, but sometime analytically convenient, boundary conditions are

$$\psi(x, 0) = \psi(x, L) = \psi_{yy}(x, 0) = \psi_{yy}(x, L) = 0. \tag{1.10'}$$

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The physical situation we have in mind, i.e. an injection of charges on the electrodes, is modelled by (1.12). The problem has the 'static solution' (ψ, u, q) given by

$$\psi = 0, \quad q = \tilde{q}(y), \quad u = \tilde{u}(y),$$

where $\tilde{q}(y)$ and $\tilde{u}(y)$ are easily computed from the conditions

$$-k\epsilon\tilde{q}'' + \sigma\tilde{q} = 0, \quad \tilde{q}(0) = q_1, \quad \tilde{q}(L) = q_2,$$
$$\epsilon\tilde{u}'' = \tilde{q}, \quad \tilde{u}(0) = 0, \quad \tilde{u}(1) = u_2.$$

Remark 1.1 The governing equations of Deo & Richardson [7], based on (1.0), also possess a steady one-dimensional hydrostatic solution. However, in this case the electric potential satisfies the nonlinear equation

$$\frac{\mathrm{d}^4 u}{\mathrm{d}y^4} + \alpha \frac{\mathrm{d}^2}{\mathrm{d}y^2}\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2 = 0$$

and suitable boundary conditions. This is one of the reasons why the treatment based on (1.0) is more difficult.

In our problem, we distinguish three cases:

(I) $\quad q_2 > q_1, \quad u_2 > 0;$

(II) $\quad q_2 = q_1, \quad u_2 > 0;$

(III) $\quad q_2 < q_1, \quad u_2 > 0.$

If (ψ, q, u) is a solution to equations (1.3), (1.6) and (1.7), $(\psi, -q, -u)$ is also a solution. For this reason, the cases corresponding to $u_2 < 0$ can be reduced to one of the situations (I), (II) or (III). To rewrite the problem in non-dimensional form, we adopt the following units: L for the length, μ/ρ for the stream function, u_2 for the potential and $\bar{q}$ for the charge density, where we define $\bar{q} = q_2 - q_1$ in case (I), $\bar{q} =$ an arbitrary unit in case (II) and $\bar{q} = q_1 - q_2$ when (III) holds. If

$$\psi = \frac{\rho\psi}{\mu}, \quad \varphi = \frac{q}{\bar{q}}, \quad \omega = \frac{u}{u_2}, \quad \mathrm{x} = \frac{x}{L}, \quad \mathrm{y} = \frac{y}{L}$$

we have the non-dimensional formulation:

Problem (Pb1)

$$\left.\begin{aligned} &\Delta^2\psi + (\psi_{\mathrm{y}}\Delta\psi_{\mathrm{x}} - \psi_{\mathrm{x}}\Delta\psi_{\mathrm{y}}) = \delta(\varphi_{\mathrm{x}}\omega_{\mathrm{y}} - \varphi_{\mathrm{y}}\omega_{\mathrm{x}}), \\ &\psi(\mathrm{x}, 0) = \psi(\mathrm{x}, 1) = \psi_{\mathrm{y}}(\mathrm{x}, 0) = \psi_{\mathrm{y}}(\mathrm{x}, 1) = 0; \end{aligned}\right\} \tag{1.13}$$

$$\left.\begin{aligned} &\beta\Delta\omega = \varphi, \\ &\omega(\mathrm{x}, 0) = 0, \quad \omega(\mathrm{x}, 1) = 1; \end{aligned}\right\} \tag{1.14}$$

$$\left.\begin{aligned} &-\Delta\varphi + \tau\varphi = \gamma(\varphi_{\mathrm{y}}\psi_{\mathrm{x}} - \varphi_{\mathrm{x}}\psi_{\mathrm{y}}), \\ &\varphi(\mathrm{x}, 0) = \frac{q_1}{\bar{q}}, \quad \varphi(\mathrm{x}, 1) = \frac{q_2}{\bar{q}}; \end{aligned}\right\} \tag{1.15}$$

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where
$$\delta = \frac{\rho \bar{q} u_2 L^2}{\mu^2}, \quad \beta = \frac{\epsilon u_2}{L^2 \bar{q}}, \quad \gamma = \frac{\mu}{k\rho}, \quad \tau = \frac{\sigma L^2}{k\epsilon}. \tag{1.16}$$

Here γ is the Prandtl number, β is an 'electrical' non-dimensional parameter and τ can be considered as a non-dimensional conductivity. The non-dimensional form of the problemn defining the static solution is given by

$$\left.\begin{aligned} -\tilde{\varphi}'' + \tau\tilde{\varphi} = 0, \quad \tilde{\varphi}(0) = \frac{q_1}{\bar{q}}, \quad \tilde{\varphi}(1) = \frac{q_2}{\bar{q}}; \\ \beta\tilde{\omega}'' = \tilde{\varphi}, \quad \tilde{\omega}(0) = 0, \quad \tilde{\omega}(1) = 1. \end{aligned}\right\} \tag{1.17}$$

As suggested by the periodic structure of the convective cells of the experiment of Atten *et al.* [4], we look for periodic solutions of problem Pb2 which are $2\pi/\alpha$ periodic in x. When $q_1 = q_2 = 0$, problem Pb1 has only the trivial solution (in any conceivable space). This is immediately seen, since from (1.15) follows $\varphi(x, y) = 0$ by the maximum principle and (1.13), (1.14) become uncoupled.

Let us define a reduced potential u(x, y) and a reduced charge density q(x, y) in the following way:

$$u(x, y) = \omega(x, y) - \tilde{\omega}(y), \quad q(x, y) = \varphi(x, y) - \tilde{\varphi}(y).$$

In terms of u and q, problem Pb1 becomes

Problem Pb2

$$\Delta^2\psi + (\psi_y \Delta\psi_x - \psi_x \Delta\psi_y) = \delta[q_x(u_y + \tilde{\omega}'(y)) - (q_y + \tilde{\varphi}'(y))\, u_x], \tag{1.18a}$$

$$\psi(x, 0) = \psi(x, 1) = \psi_y(x, 0) = \psi_y(x, 1) = 0, \tag{1.18b}$$

$$\beta\Delta u = q, \tag{1.19a}$$

$$u(x, 0) = u(x, 1) = 0, \tag{1.19b}$$

$$-\Delta q + \tau q = \gamma[(q_y + \tilde{\varphi}'(y))\,\psi_x - q_x\psi_y], \tag{1.20a}$$

$$q(x, 0) = q(x, 1) = 0. \tag{1.20b}$$

Let us consider for simplicity, the case of a perfect insulating fluid, i.e. put $\tau = 0$ in equations (1.15) and (1.17). As a consequence, we have

$$\tilde{\varphi}'(y) = \begin{cases} 1 & \text{in case (I)} \\ 0 & \text{in case (II)} \\ -1 & \text{in case (III).} \end{cases}$$

Let $R = \{(x, y); -\infty < x < \infty, 0 < y < 1\}$ and $S = \{(x, y); 0 < x < 2\pi/a, 0 < y < 1\}$.

Remark 1.2 Let us multiply equation (1.18a) by ψ, equation (1.19a) by $\delta/\gamma\, u$ and equation (1.20a) by $\delta\beta/\gamma\, \Delta u$. Integrating by parts and adding, we obtain

$$\int_S |\Delta\psi|^2\, dx\, dy + \frac{\beta\delta}{\gamma}\int_S |\Delta u|^2\, dx\, dy + \frac{\beta^2\delta}{\gamma}\int_S |\nabla(\Delta u)|^2\, dx\, dy$$
$$= \beta\delta\int_S (\tilde{\varphi}'(y) - \tilde{\omega}'(y))\,\Delta u\,\psi_x\, dx\, dy. \tag{1.21}$$

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When (III) holds the right-hand side cannot be absorbed in the left-hand side to give uniqueness. On the other hand, if we are in case (I), we have, uniformly in y,

$$\lim_{\beta\to\infty}(\tilde{q}'(\mathrm{y})-\tilde{\omega}'(\mathrm{y}))=0.$$

Thus, applying the Poincaré inequality to the right-hand side of (1.21) and taking β sufficiently large (this can be done while keeping $q_1/\bar{q}$ and $q_2/\bar{q}$ fixed), we have uniqueness. Hence the solution is unique under the condition of quasi-electroneutrality. However, we believe this is true always, when case (I) holds.

Remark 1.3 When $\beta\to\infty$ and $\tau=0$, the problem studied in this paper is similar to the Rayleigh–Bénard convection [18, 19]. Case (I) corresponds to heating from above and case (III) to heating from below; however, there is no counterpart in the Bénard case to equation (1.3). If β is arbitrary and $\tau\neq 0$ the two problems are different: our equations do not have constant coefficients as in the Bénard case (they become constant when $\beta\to\infty$ and $\tau=0$). This makes the study of the linearized equations more difficult, since the Krein–Rutman [18] theorem is not directly applicable.

Remark 1.4 We note that with the boundary conditions (1.10′) and a slight change of notation, a formal truncation of the two-dimensional Fourier representation of the stream function leads to the Lorenz equations [19]. One new parameter appears over and above the usual ones r, b, but this does not appear to change the basic properties of the Lorenz system.

2 Non-uniqueness in case (III)

The result of § 1 suggests that instability and bifurcation occur when $u_2>0$ and $q_2<q_1$. The experimental results of Atten *et al*. [4] confirm this fact. We assume, in the remaining part of this paper, case (III) to hold. There exists a constant $\beta_1>0$ which can easily be computed from (1.17), such that $\tilde{\omega}'(\mathrm{y})>0$ for $\mathrm{y}\in[0,1]$ if

$$\beta>\beta_1. \tag{2.1}$$

Hereafter we assume (2.1) to hold. In this way, the electric field of the static solution, $\tilde{\mathrm{E}}=\tilde{\omega}(\mathrm{y})$, is directed toward the upper plate and an accumulation of positive charge near the lower electrode tends to be displaced upward. This promotes instability. We shall take δ as the bifurcation parameter and prove that, if β is sufficiently large, a branch of non-trivial solutions starts from the motionless solution when δ equals a critical value δ^*.

We recall the following well-known theorem of bifurcation from a simple eigenvalue.

Theorem 2.1 *Let X be a Banach space. Let $\mathbb{G}\in C^2(X,X)$, $\mathbb{B}\in C^2(X,X)$ be two operators such that $\mathbb{G}'(0)$* is compact and $\mathbb{G}(0)=\mathbb{B}(0)=\mathbb{B}'(0)=0$. *Assume $1/\delta^*\neq 0$ to be an eigenvalue of $\mathbb{G}'(0)$ such that*

$$\dim\operatorname{Ker}(\mathrm{I}-\delta^*\,\mathbb{G}'(0))=1, \tag{2.2}$$

$$\operatorname{Ker}(\mathrm{I}-\delta^*\,\mathbb{G}'(0))=\operatorname{Ker}((\mathrm{I}-\delta^*\,\mathbb{G}'(0))^2. \tag{2.3}$$

Then δ^* is a bifurcation point for the equation

$$\psi+\mathbb{B}(\psi)-\delta\,\mathbb{G}(\psi)=0.$$

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To simplify the proof of conditions (2.2), (2.3) we restrict still further the class of the admissible functions. More precisely, we require u(x, y) and q(x, y) to be even in x and $\psi(x, y)$ to be odd in x.

We use the notations $H^k(S)$ and $H^k_o(S)$ for the usual Sobolev spaces of functions which are $2\pi/\alpha$ periodic in x. Moreover, $H^k_d(S), H^k_{od}(S), (H^k_p(S), H^k_{op}(S))$ will denote the subspaces of $H^k(S)$ and $H^k_o(S)$ of functions which are even (odd) in x. Let $X = H^2_{od}(S) \cap H^3(S)$ and define the following operators:

$$\mathbb{B}_1 : X \to L^2_d(S), \quad \mathbb{B}_1(\psi) = (\psi_y \Delta\psi_x - \psi_x \Delta\psi_y);$$
$$\mathbb{B}_2 : X \to L^2_d(S), \quad \mathbb{B}_2(\psi) = q_x(u_y + \tilde{\alpha}(y)) - (q_y + \tilde{g}'(y))\, u_x,$$

where $q \in H^2_{op}(S) \cap H^3(S)$ and $u \in H^2_{op}(S) \cap H^5(S)$ are, for a given $\psi \in X$, the solutions of the following problems:

$$-\Delta q + \tau q = \gamma[(q_y + \tilde{g}'(y))\psi_x - q_x \psi_y], \quad q(x,0) = q(x,1) = 0;$$
$$\beta \Delta u = q, \quad u(x,0) = u(x,1) = 0.$$

Let $\mathbb{G}_1 : L^2_d(S) \to X$ be the inverse of the biharmonic operator with the boundary conditions $\psi(x,0) = \psi(x,1) = \psi_y(x,0) = \psi_y(x,1) = 0$. If we define

$$\mathbb{B} : X \to X, \quad \mathbb{B} = \mathbb{G}_1 \mathbb{B}_1, \quad \mathbb{G} : X \to X, \quad \mathbb{B} = \mathbb{G}_1 \mathbb{B}_2,$$

problem Pb2 can be put in the functional form

$$\psi + \mathbb{B}(\psi) - \delta\, \mathbb{G}(\psi) = 0. \tag{2.4}$$

We verify immediately that $\mathbb{G} \in C^2(X, X)$, $\mathbb{B} \in C^2(X, X)$, $\mathbb{G}(0) = \mathbb{B}(0) = \mathbb{B}'(0) = 0$, beside $\mathbb{G}'(0)$ is compact by Rellich's theorem [18], since $H^4_d(S) \supset \mathbb{G}(X)$. To prove (2.2) let us consider the linearization of (2.4), i.e.

$$(I - \delta\, \mathbb{G}'(0))\, \psi = 0, \tag{2.5}$$

and its differential counterpart:

$$\left.\begin{aligned} \Delta^2 \psi &= \delta(q_x \tilde{\alpha}'(y) - \tilde{g}'(y)\, u_x), \\ \beta \Delta u &= q, \\ -\Delta q + \tau q &= \gamma\, \tilde{g}'(y)\, \psi_x, \end{aligned}\right\} \tag{2.6}$$

$$\left.\begin{aligned} &\psi(x,0) = \psi(x,1) = \psi_y(x,0) = \psi_y(x,1) = 0, \\ &u(x,0) = u(x,1) = 0, \quad q(x,0) = q(x,1) = 0. \end{aligned}\right\} \tag{2.7}$$

All solutions of problem Pbd are regular and can be developed as Fourier series:

$$\left.\begin{aligned} \psi(x,y) &= \sum_{k=1}^{\infty} \Psi_k(y) \sin(k\alpha x), \\ u(x,y) &= U_o(y) + \sum_{k=1}^{\infty} U_k(y) \cos(k\alpha x), \\ q(x,y) &= Q_o(y) + \sum_{k=1}^{\infty} Q_k(y) \cos(k\alpha x). \end{aligned}\right\} \tag{2.8}$$

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If we insert (2.8) in (2.6) and equate similar terms, we obtain the following hierarchy of problems:

Pb_k

$$\left.\begin{aligned} \mathbb{L}_k^2 \Psi_k &= \delta(\tilde{g}'(y)\, U_k - \tilde{\alpha}'(y)\, Q_k)\, k\, a, \\ \Psi_k(0) &= \Psi_k(1) = \Psi_k'(0) = \Psi_k'(1) = 0; \end{aligned}\right\} \tag{2.9}$$

$$\left.\begin{aligned} \beta \mathbb{L}_k U_k &= -Q_k, \\ U_k(0) &= U_k(1) = 0; \end{aligned}\right\} \tag{2.10}$$

$$\left.\begin{aligned} \mathbb{M}_k Q_k &= \gamma \tilde{g}'(y)\, \Psi_k\, k\, a, \\ Q_k(0) &= Q_k(1) = 0, \end{aligned}\right\} \tag{2.11}$$

where $\mathbb{L}_k = -\mathrm{d}^2/\mathrm{d}y^2 + a^2 k^2$, $\mathbb{M}_k = \mathbb{L}_k + \tau$. As a first step in proving (2.2), we show that problem Pb_k has, for every $k \in \mathbb{N}$, a positive and simple eigenvalue (i.e. with the same algebraic and geometric multiplicity equal to one). Let $\mathscr{B}(\mathrm{y}, \xi)$, $\mathscr{G}(\mathrm{y}, \xi)$ and $\mathscr{D}(\mathrm{y}, \xi)$ be the Green's functions corresponding, respectively, to the operators $\mathbb{L}_k, \mathbb{M}_k$ with zero boundary conditions and to $\mathbb{L}_k^2$ with boundary conditions $\Psi(0) = \Psi(1) = \Psi'(0) = \Psi'(1) = 0$. We recall that the last operator is invertible with compact positive inverse (see [18, p. 116]). Define the functions

$$\mathscr{F}(\alpha, \eta) = \int_0^1 \mathscr{B}(\alpha, \zeta)\, \mathscr{G}(\zeta, \eta)\, \mathrm{d}\zeta,$$

$$\mathscr{H}(y, \eta) = -\mathrm{d}\tilde{g}'(\eta) \int_0^1 \mathscr{D}(y, \alpha)\, [\tilde{\alpha}'(\alpha)\, \beta \mathscr{G}(\alpha, \eta) + \tilde{g}'(\alpha)\, \mathscr{F}(\alpha, \eta)]\, \mathrm{d}\alpha, \tag{2.12}$$

where

$$\mathrm{d} = \frac{k^2 a^2 \gamma}{\beta}.$$

Let

$$(\mathbb{P}\Psi)(y) = \int_0^1 \mathscr{H}(y, \eta)\, \Psi(\eta)\, \mathrm{d}\eta. \tag{2.13}$$

Problem Pb_k can be written as an operator equation, in the form

$$\Psi - \delta \mathbb{P} \Psi = 0. \tag{2.14}$$

Examining (2.12) we see that, in general, $\mathbb{P}$ is not a positive operator. This is the main difference from the Bénard case. The Krein–Rutman theorem (see [18, p. 122]), is thus not directly applicable to the eigenvalue problem (2.14). This difficulty is inherent to the situation and cannot be circumvented with a different choice of signs in the applied potential and in the injected charges in (1.11) and (1.12). However, if β is sufficiently large, $\mathbb{P}$ becomes positive. To prove this result we need the following:

Lemma 2.1 *There exists a constant* $\mathrm{C} > 0$, *not depending on* a, k *or* τ, *such that*

$$\mathrm{C}\, \mathscr{G}(\alpha, \eta) - \mathscr{F}(\alpha, \eta) > 0 \quad (\alpha, \eta) \in (0, 1) \times (0, 1). \tag{2.15}$$

Proof The Green's functions $\mathscr{B}(y, \xi)$ and $\mathscr{G}(y, \xi)$ can be written down explicitly. Thus $\mathscr{F}(y, \xi)$ can also be computed. The boundedness of $\mathscr{F}(y, \xi)/\mathscr{G}(y, \xi)$, i.e. (2.15), follows from an elementary, but lengthy, analysis of the various terms of this expression. □

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From Lemma 2.1 we obtain

Lemma 2.2 *There exists a constant $\beta_2 > 0$ not depending on a, k or τ, such that the operator $\mathbb{P}$ is positive if*

$$\beta > \beta_2. \tag{2.16}$$

Proof Recalling the positivity of the inverse operator of $\mathbb{L}_k^2$ with Dirichlet boundary conditions, we need only a control on the sign of the expression in square brackets in (2.12). By (2.1) there exist two positive constants C_1 and C_2 such that

$$\tilde{u}'(\alpha) \geqslant C_1, \quad \tilde{q}'(\alpha) \geqslant -C_2.$$

Hence $$\tilde{u}'(\alpha)\beta\mathcal{G}(\alpha,\eta) + \tilde{q}'(\alpha)\mathcal{F}(\alpha,\eta) \geqslant C_2\left[\frac{\beta C_1}{C_2}\mathcal{G}(\alpha,\eta) - \mathcal{F}(\alpha,\eta)\right].$$

If $C > 0$ is the constant given by Lemma 2.1 and $\beta \geqslant C_2 C/C_1 = \beta_2$, the operator $\mathbb{P}$ is positive. □

When $\beta > \beta_2$ it is possible to apply the Krein–Rutman theorem to problem Pb_k which, therefore, has a positive, smallest and simple eigenvalue $\bar{\delta}_{k\,a}$. Let β_1 and β_2 be given by (2.1) and (2.16). Then we have

Theorem 2.2 *If $\beta > \max\{\beta_1, \beta_2, 1/2\pi^2\}$, there exists a bifurcation point $\bar{\delta}_a$ for problem* Pb1.

The proof is largely a repetition of the arguments used in the Bénard problem. To make the paper self-contained we collect them in the appendix.

In Theorem 2.2 the condition on β can be realized, e.g. taking the thickness of the liquid layer sufficiently small. Moreover, the critical value $\bar{\delta}_a$, whose existence is guaranteed by Theorem 2.2, can be reached varying the injected charge density q_1, q_2 or the applied potential u_2.

3 Conclusion

The problem of the stability of a horizontal layer of dielectric, slightly conducting liquid, with the simultaneous action of an electric field and injection of charges is proved to be similar to that of Rayleigh–Bénard convection. This, however, is only true when the electric field corresponding to the static solution is nearly constant. The main difference compared to the mathematical treatment of the Rayleigh–Bénard problem lies in the non-applicability of the Krein–Rutman theorem in proving the existence of a simple eigenvalue of the linearized problem. It would be interesting to find similar result taking as constitutive relation (1.0).

Appendix

Let $\bar{\Psi}_1(y)$ be the eigenfunction with unit norm, corresponding to $\bar{\delta}_a$ and $\bar{Q}_1(y), \bar{U}_1(y)$ the related functions given by (2.10) and (2.11). The two-dimensional problem (2.6)–(2.7) has $\bar{\delta}_a$ as eigenvalue with eigenfunction

$$\bar{\psi}_1(x,y) = \bar{\Psi}_1(y)\sin(ax), \quad \bar{u}_1(x,y) = \bar{U}_1(y)\cos(ax), \quad \bar{q}_1(x,y) = \bar{Q}_1(y)\cos(ax).$$

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We claim that no other eigenfunction of (2.6) and (2.7) corresponds to $\bar{\delta}_a$. Assume first

$$\bar{\delta}_a \neq \bar{\delta}_{ka} \quad \text{for all} \quad k \geqslant 2. \tag{A.1}$$

Let $(\bar{\psi}, \bar{u}, \bar{q})$ be any set of eigenfunctions corresponding to $\bar{\delta}_a$. Consider the Fourier development of $\bar{\psi}$ and substitute in Pbd with $\delta = \bar{\delta}_a$. Equating similar terms we have

$$\left.\begin{aligned} &\mathbb{L}_k^2 \Psi_k = \bar{\delta}_a(\tilde{q}' U_k - \tilde{u}' Q_k)\, k a, \\ &\mathbb{M}_k Q_k = \gamma \tilde{q}' \Psi_k k a, \\ &\beta \mathbb{L}_k U_k = -Q_k. \end{aligned}\right\} \tag{A.2}$$

If (A.1) holds, $\bar{\delta}_a$ is not an eigenvalue of (A.2) when $k \geqslant 2$. Therefore, $\Psi_k(y) = U_k(y) = Q_k(y) = 0$ for $k \geqslant 2$. Hence $\bar{\psi} = \bar{\psi}_1$, $\bar{q} = \bar{q}_1$, $\bar{u} = \bar{u}_1$. To prove that there exist values of a for which (A.1) holds, we need

Lemma A.1 *Let δ be any eigenvalue of problem* Pb_k. *If $\beta \geqslant 1/2\pi^2$, then*

$$\delta \geqslant \mathrm{C}_3\, a^2 k^2. \tag{A.3}$$

Proof Let $\lambda = a k \sqrt{\delta}$ and $\sqrt{\delta}\,\xi = \Psi$. Problem Pb_k can be written as follows:

$$\left.\begin{aligned} &\mathbb{L}_k^2 \xi = \lambda(\tilde{q}' U - \tilde{u}' Q), \\ &\xi(0) = \xi(1) = \xi'(0) = \xi'(1) = 0; \end{aligned}\right\} \tag{A.4}$$

$$\left.\begin{aligned} &\beta \mathbb{L}_k U = -Q, \\ &U(0) = U(1) = 0; \end{aligned}\right\} \tag{A.5}$$

$$\left.\begin{aligned} &\mathbb{L}_k Q + \tau Q = \gamma \lambda \tilde{q}' \xi, \\ &Q(0) = Q(1) = 0. \end{aligned}\right\} \tag{A.6}$$

Let is multiply (A.4) by ξ, (A.5) by U and (A.6) by Q. By addition and integration by parts we have, applying the one-dimensional Poincaré inequality,

$$a^2 k^2 \int_0^1 (\xi^2 + U^2 + Q^2)\,\mathrm{d}y + \pi^2 \int_0^1 (U^2 + Q^2)\,\mathrm{d}y \leqslant \mathrm{C}_4 \lambda \int_0^1 (\xi^2 + U^2 + Q^2)\,\mathrm{d}y + \frac{1}{2\beta}\int_0^1 (U^2 + Q^2)\,\mathrm{d}y,$$

with C_4 a suitable constant. If $\beta \geqslant 1/2\pi^2$, we obtain (A.3), choosing $\mathrm{C}_3 = \mathrm{C}_4^{-2}$. □

Lemma A.2 *There exists a period $2\pi/a$ such that* (A.1) *holds.*

Proof Let ℓ be any positive number and $\bar{\delta}_{k\ell}$ be the corresponding minimal eigenvalue of problem Pb_k. From (A.3) it follows, in particular, that

$$\bar{\delta}_{k\ell} \geqslant \mathrm{C}_3 \ell^2 k^2.$$

Hence, $\bar{\delta}_\ell$ can be an eigenvalue for Pb_k only for those integers k which satisfy

$$\bar{\delta}_\ell \geqslant \mathrm{C}_3 \ell^2 k^2. \tag{A.7}$$

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Let m be the greatest of the integer k which satisfies (A.7). If we choose $\alpha = m\ell$, condition (A.1) is satisfied. □

The only condition which remains to be proved is (2.3). Let $\mathbb{D} = I - \bar{\delta}_\alpha \mathbb{G}'(0)$ and assume, by contradiction, that there exists $\psi^o \in X$ such that

$$\mathbb{D}^2 \psi^o = 0, \quad \mathbb{D}\psi^o = \psi^1 \neq 0. \tag{A.8}$$

Since ψ^1 is an eigenfunction corresponding to $\bar{\delta}_\alpha$ and $\dim \mathrm{Ker}(\mathbb{D}) = 1$, we have $\psi^1(x, y) = \Psi_1(y)\sin(\alpha x)$. On the other hand, by substituting the Fourier development of $\psi^o(x, y)$, $u^o(x, y)$ and $q^o(x, y)$ into the differential counterpart of (A.8), we have:

$$\Psi_1^o = \bar{\delta}_\alpha \mathbb{P}\Psi_1^o + \Psi_1, \tag{A.9}$$

with $\mathbb{P}$ defined in (2.13). The adjoint equation of (A.9) is

$$\chi_1^o(y) = \bar{\delta}_\alpha \int_0^1 \mathscr{H}(\eta, y)\, \chi_1^o(\eta)\, \mathrm{d}\eta. \tag{A.10}$$

Now the necessary condition for the solvability of (A.10), i.e.

$$\int_0^1 \Psi_1 \chi_1^o \, \mathrm{d}y = 0$$

is violated, since $\chi_1^o(y) > 0$ and $\Psi_1(y) > 0$ by the Krein–Rutman theorem. Thus (2.3) holds. □

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