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Storage allocation under processor sharing II: Further asymptotic results

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We consider a processor-sharing storage allocation model, which has m primary holding spaces and infinitely many secondary ones, and a single processor servicing the stored items. All of the spaces are numbered and ordered. An arriving customer takes the lowest available space. Dynamic storage allocation and the fragmentation of computer memory are well-known applications of this model. We define the traffic intensity ρ to be λ/μ, where λ is the customers' arrival rate and μ is the service rate of the processor. We study the joint probability distribution of the numbers of occupied primary and secondary spaces. We study the problem in two asymptotic limits: (1) $m \to \infty$ with a fixed $\rho < 1$, and (2) $\rho \uparrow 1$, $m \to \infty$ with $m(1-\rho) = O(1)$. The asymptotics yield insight into how many secondary spaces tend to be needed, and into the sample paths leading to the occupation of the two types of spaces. We show that the asymptotics lead to accurate numerical approximations.

1 Introduction

We consider the following storage allocation model. Suppose that near a restaurant there are m primary parking spaces and across the street there are infinitely many additional ones. However, the restaurant has only one server, which serves all of the customers. All of the parking spaces are numbered and ordered; the one with rank $= 1$ is closest to the restaurant and the primary spaces are numbered $\{1, 2, 3, \ldots, m\}$. We assume the following: customers arrive according to a Poisson process with rate λ, the server works at rate μ, an arriving car parks in the lowest numbered available space, and if there are N customers in the restaurant, the server serves each customer at the rate μ/N. This corresponds to a processor sharing (PS) service discipline.

We define N_1 to be the number of occupied primary spaces and N_2 to be the number of occupied secondary spaces. Then we define $\mathbf{S}$ to be the set of indices of the occupied spaces, and the 'wasted spaces' W are defined as the difference between the largest index of the occupied spaces (Max $\mathbf{S}$) and the total number of occupied spaces ($|\mathbf{S}| = N_1 + N_2$).

We next briefly survey storage allocation models, and discuss their application to computers. For this purpose we can consider a linear array of cells that are numbered

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$\{1, 2, 3, \ldots\}$. For the parking space model described above, this would correspond to treating identically all of the spaces without distinguishing between primary spaces and secondary ones. As discussed in [4], dynamic storage allocation models examine various algorithms for storing items (or list elements) in the linear array. Such models can be deterministic or probabilistic, and we shall only consider the latter. Items are added and deleted from the list, and the algorithm gives the rule where a newly arriving item is placed. The sizes of the items can vary or be constant. For example, we can assume that a new item has size ℓ with probability $f(\ell)$, then $f(\ell) = \delta_{\ell,L}$ would correspond to all items having the common size L, and each item would take L cells to store. We then need to specify how items arrive, to be added to the list and how they are ultimately deleted from it.

As items arrive and depart, at any time instant the linear array of cells will have some cells occupied, and all cells above some maximum number will be empty, because the number of cells in the linear array is taken as infinite. If there are N occupied cells (the used space) and the highest number of the occupied space is Max $\mathbf{S}$, then the wasted spaces are defined as $W = \text{Max } \mathbf{S} - N$. The wasted spaces correspond to the union of the 'holes' in the linear array, and we note that a hole may correspond to several cells.

The design and analysis of algorithms for dynamic storage allocation is a fundamental part of computer science [7], and is discussed in the classic book of Knuth [14]. The items correspond to records, files, lists *etc.* and the storage device is a set of consecutive locations or addresses. A storage device could also be a magnetic tape or disc, depending on the application. After time, items are inserted and deleted, and the linear array of cells consists of the regions of occupied cells alternating with interior holes. This is called fragmentation. Collapsing the holes and eliminating the wasted spaces correspond to, say, running a defragmentation program. Estimating the probable amount of wasted spaces can be used to decide if much is to be gained by running such a program.

Worst case studies of dynamic storage allocation go back to the 1960s (see [2, 14, 20]); average case, or probabilistic studies go back to the mid-1980s, though there are some earlier results, such as Knuth's 50% rule [14]. This states that for a certain class of models the number of holes in the linear array is approximately half the expected number of items stored.

A commonly used algorithm is called 'first-fit' (FF). Here a new item is placed in the lowest numbered set of consecutive unoccupied cells. For example, if cells 1, 2, 3, 5, 6, 10 and 13 are occupied and a new item needs to take two cells, it will be placed in cells numbered 7 and 8. Another algorithm is 'best-fit' (BF), which would place the item in the smallest hole that is large enough to hold the item. For the above example, the new item would be put in cells 11 and 12 under BF.

A common probabilistic assumption is that new items arrive according to a Poisson process. Suppose that each item takes up exactly one cell (thus $f(\ell) = \delta_{\ell,1}$). The 'cell' could represent any unit of information, such as byte, a word, fixed-length record or a page of memory [4]. A data structure [4, 14] used in many computer algorithms deletes the items under a first-fit-first-out (FIFO) discipline. Here the stored item that arrived firstly is serviced. For this model under FF, an estimate of the wasted spaces of the

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form $E[W] = \Theta(\sqrt{N})$ is given in [4], where again N is a typical number of stored items. Analogous bounds were obtained by Coffman and Leighton [7] for models with Poisson process arrivals and deleting, who in particular showed that for any algorithm and distribution $f(\ell)$ of file sizes, the wasted spaces are $\Omega(\sqrt{N \log\log N})$. This lower bound was supplemented by an upper bound of the form $O(\sqrt{N}(\log N)^{3/4})$, for a special algorithm called BFA, which we do not define here. The lower bound is actually achieved in the case of FF with $f(\ell) = \delta_{\ell,L}$. Numerical studies in [6] suggest that the wasted spaces are $\Theta(N^{4/5})$ for the FF rule when the number of cells that an arriving item takes is uniformly distributed over some interval $1 \leqslant \ell \leqslant \ell_{\max}$. These estimates are all obtained by probabilistic upper and lower bound arguments, and suggest that the number of the wasted spaces is generally sub-linear in the number of occupied cells. If, say, $E[W] = O(\sqrt{N})$, it may not be particularly advantageous to run a defragmentation program.

Having briefly surveyed dynamic storage allocation models, we mention that it is generally very difficult to compute exactly the wasted spaces (say, its mean or probability distribution). Dynamic storage allocation is much more difficult than static scheduling problems, such as bin packing, where items are stored but not deleted [7]. However, two such models are susceptible to a more precise analysis, because they can be formulated as Markov chains of low dimension. One is the $M/M/\infty$ queue with ranked servers. This has been already studied by Kosten [13] in relation to communication systems and later by Knuth [14] in the analysis of the 50% rule. In this model the arrivals are Poisson and each stored item is held in place for an exponentially distributed holding time. Under FF and with $f(\ell) = \delta_{\ell,1}$ (each item takes one cell or server) the problem can be described as a two-dimensional Markov Chain, by breaking up the linear array into cells ranked $\leqslant m$ and cells ranked $> m$. From this two-dimensional chain one can infer the distributions of Max $\mathbf{S}$ and W. This model has been studied by many authors [1, 5, 7, 13, 15, 16]. It differs from the current model in that if there are a total of $N = N_1 + N_2$ spaces occupied, the total service rate is μN, as each customer in the restaurant is served at rate μ. For this model various asymptotic studies appear in [1, 7, 15]. In particular, Aldous [1] showed that the mean number of the wasted spaces is $E[W] \sim \sqrt{2\rho \log\log \rho}$ as $\rho = \lambda/\mu \to \infty$.

A simple derivation of the exact joint distribution of finding N_1 (resp., N_2) occupied primary (resp., secondary) spaces appears in [17] and detailed asymptotic results for this joint distribution appear in [11, 12], while the distribution of Max $\mathbf{S}$ is analysed in [10]. In [11] Knessl showed how to directly obtain asymptotic results for the infinite server model from the basic difference equation. As the present PS model does not seem amenable to exact solution, we shall employ such a direct asymptotic approach here.

The PS model we consider also corresponds to a two-dimensional Markov chain, but seems much more difficult. Here we study the joint probability distribution of the numbers of occupied spaces in the PS model, letting $\pi(k, r) = Prob[N_1 = k, N_2 = r]$ in the steady state. In part I (see [19]) we obtained exact solutions for $m = 1$ and $m = 2$, and developed a semi-analytic and semi-numerical method for general m. We also derived asymptotic results in the heavy traffic case $\rho \to 1$, but with $m = O(1)$. Here we shall obtain asymptotic results for $m \to \infty$, for the cases $\rho < 1$ and $1 - \rho = O(m^{-1})$.

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The only known previous results on the PS model consist of a rough estimate of the wasted spaces, as Coffman *et al.* [3] have shown that

$$E[W] = \Theta\left(\sqrt{\frac{1}{1-\rho}\log\left(\frac{1}{1-\rho}\right)}\right), \quad \rho \uparrow 1, \tag{1.1}$$

where $E[W]$ is the expected value of the wasted spaces. Here $E[W] = \Theta(f(\rho))$ means that there exists positive constants c, c' such that $c'f(\rho) \leqslant E[W] \leqslant cf(\rho)$. Also when $\rho \to 1$ (the heavy traffic case) Coffman & Mitrani [8] obtained upper and lower bounds on $E[W]$ in the form

$$\frac{1}{2}\sqrt{\frac{\pi}{1-\rho}} \leqslant E[W] \leqslant \frac{1}{1-\rho}\left(\frac{\pi^2}{6} - 1\right). \tag{1.2}$$

Here we consider only the joint distribution $\pi(k,r)$, but we believe this can ultimately be used to obtain asymptotic results also for the wasted spaces W.

The paper is organised as follows. In Section 2 we state the problem and obtain the basic difference equations for $\pi(k,r)$. In Section 3 we summarise our main results. In Section 4 we consider $m \to \infty$ with a fixed ρ $(0 < \rho < 1)$ and study $\pi(k,r)$ for various ranges of (k,r). In Section 5 we consider the double limit $m \to \infty$ and $\rho \uparrow 1$, with $m(1-\rho) = O(1)$. Some numerical studies and comparisons appear in Section 6, and we conclude with a discussion of the results in Section 7.

2 Statement of the problem

We let $N_1(t)$ (resp., $N_2(t)$) denote the number of primary (resp., secondary) spaces occupied at time t. The joint steady state distribution function is

$$\pi(k,r) = \pi(k,r;m) = \lim_{t\to\infty} Prob[N_1(t) = k, N_2(t) = r], \ 0 \leqslant k \leqslant m, \ r \geqslant 0.$$

Let $\rho = \lambda/\mu$ be the traffic intensity and we assume the stability condition $\rho < 1$. The pair (N_1, N_2) forms a Markov chain whose transition rates are sketched in Figure 1. The state space is the lattice strip $\{(k,r) : 0 \leqslant k \leqslant m, \ r \geqslant 0\}$ and the balance equations are

$$\begin{aligned}(I_{[k+r>0]} + \rho)\ \pi(k,r) = \rho\ \pi(k-1,r)I_{[k\geqslant 1]} + \frac{k+1}{k+r+1}\ \pi(k+1,r)I_{[k<m]} \\ + \frac{r+1}{k+r+1}\ \pi(k,r+1) + \rho\ \pi(m,r-1)I_{[k=m,\, r\geqslant 1]}.\end{aligned} \tag{2.1}$$

Here I is an indicator function. The normalisation condition is

$$\sum_{r=0}^{\infty}\sum_{k=0}^{m}\pi(k,r) = 1. \tag{2.2}$$

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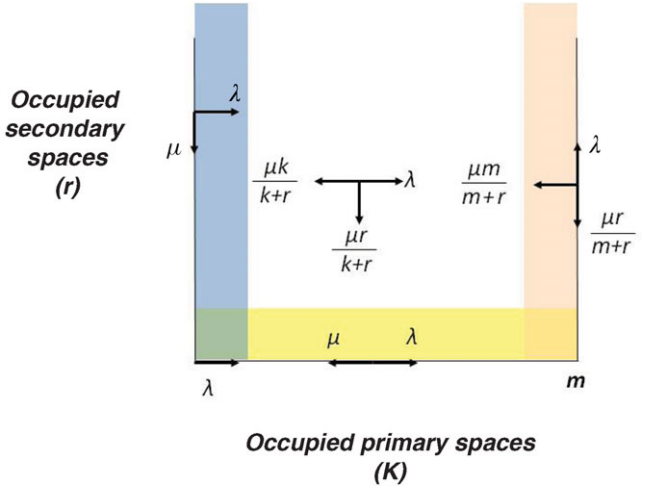


FIGURE 1. (Colour online) A sketch of the transition rates.

From our viewpoint we need to consider explicitly the boundary conditions inherent in (2.1), so we rewrite the main equation as

$$(1+\rho)\ \pi(k,r) = \rho\ \pi(k-1,r) + \frac{k+1}{k+r+1}\ \pi(k+1,r) + \frac{r+1}{k+r+1}\ \pi(k,r+1),$$
$$0 \leqslant k < m,\ r \geqslant 0,\ k+r > 0, \tag{2.3}$$

and the boundary condition at $k = m$ is

$$(1+\rho)\ \pi(m,r) = \rho\ \pi(m-1,r) + \frac{r+1}{m+r+1}\ \pi(m,r+1) + \rho\ \pi(m,r-1),\ r \geqslant 1. \tag{2.4}$$

There are also the two corner conditions

$$\rho\pi(0,0) = \pi(1,0) + \pi(0,1) \tag{2.5}$$

and

$$(1+\rho)\pi(m,0) = \rho\pi(m-1,0) + \frac{1}{m+1}\pi(m,1). \tag{2.6}$$

In (2.3) when $k = 0$ we interpret $\pi(-1,r)$ as 0. The boundary condition at $k = m$ in (2.4) can be replaced by the artificial boundary condition

$$\frac{m+1}{m+r+1}\pi(m+1,r) = \rho\ \pi(m,r-1). \tag{2.7}$$

This is obtained by extending (2.3) to hold also at $k = m$ and comparing this with (2.4).

We note that the total number $N_1 + N_2$ behaves as the number of customers in the $M/M/1-PS$ queue, which is well known to follow a geometric distribution. Thus, we

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have

$$\sum_{k+r=N} \pi(k,r) = (1-\rho)\rho^N,\ N \geqslant 0. \tag{2.8}$$

This identity will provide a useful check on the calculations that follow.

3 Summary

As the analysis shall become quite involved and technical, we collect here some of the main results.

We take $m \to \infty$, first assuming that $\rho < 1$. We use the scaled variables $x = k/m$ and $y = r/m$, so that x is the fraction of primary spaces that are utilised. The main approximation we obtain is then given by

$$\pi(k,r) \sim K(x,y)e^{m\phi(x,y)};\ \ 0 < x \leqslant 1,\ \ y > 0, \tag{3.1}$$

where

$$\phi(x,y) = x\phi_x + y\phi_y + \log\rho - \log(s+1), \tag{3.2}$$

$$\phi_x = \log\left[\frac{s+1-\rho-\rho s\, e^{(1-\rho)t}}{s+1-\rho-s\, e^{(1-\rho)t}}\right], \tag{3.3}$$

$$\phi_y = \log\left[\frac{\rho(s+1-\rho-\rho s\, e^{(1-\rho)t})}{\rho(s+1-\rho)-\rho s\, e^{(1-\rho)t}+(1-\rho)(s+1-\rho)e^t}\right], \tag{3.4}$$

and (x,y) are related to (s,t) via the mapping

$$x = x(s,t) = \frac{1}{(s+1)(1-\rho)^2}[s+1-\rho-s\, e^{(1-\rho)t}] \times [(s+1-\rho)\, e^{-(1-\rho)t} - s\rho^2 - \rho(1-\rho)se^{-t}], \tag{3.5}$$

$$y = y(s,t) = \frac{s}{(s+1)(1-\rho)}[(1-\rho)(s+1-\rho) - \rho s\, e^{-\rho t} + \rho(s+1-\rho)\, e^{-t}]. \tag{3.6}$$

Then $K(x,y)$ is given by

$$K(x,y) = \sqrt{\frac{(1-\rho)(s+1-\rho)^3[s+1-\rho-s\rho\, e^{(1-\rho)t}]^3}{(s+1-\rho-s\, e^{(1-\rho)t})[\rho(s+1-\rho)-\rho s\, e^{(1-\rho)t}+(1-\rho)(s+1-\rho)e^t]}} \times \frac{1}{(s+1)^2}\sqrt{\frac{e^{\rho t}}{|J|}}, \tag{3.7}$$

where $J = x_t y_s - x_s y_t$ is the Jacobian of the above transformation. This approximation to $\pi(k,r)$ is explicit in terms of (s,t) but implicit in terms of (x,y). However, for $x \approx 1$ it becomes much more explicit, with

$$\pi(k,r) \sim (1-\rho)\left(\frac{y+1-\rho}{y+1}\right)(y+1)^{k-m}\rho^{m(y+1)}. \tag{3.8}$$

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The above holds for $y > 0$ and $k = m - O(1)$, which corresponds to all but a few occupied primary spaces.

Different expansions must be constructed in various boundary and corner regions of the strip $\{(x, y) : 0 \leqslant x \leqslant 1, \quad y \geqslant 0\}$. For $k = O(1)$ $(x = O(m^{-1}))$, which means that only a few primary spaces are occupied,

$$\pi(k, r) \sim m^{k+1/2} e^{m\phi(0,y)} S_0(y) \frac{y^k}{k!} \left[1 + \rho - e^{\phi_y(0,y)}\right]^k, \tag{3.9}$$

where

$$\phi(0, y) = y \log \rho + \frac{y}{1-\rho} \log \left[\frac{y + \sqrt{(y-1+\rho)^2 + 4y} - (1-\rho)}{y + \sqrt{(y-1+\rho)^2 + 4y} + (1-\rho)}\right]$$
$$- \log \left[y + \sqrt{(y-1+\rho)^2 + 4y} + 1 + \rho\right] + \log 2 + \log \rho, \tag{3.10}$$

$$\phi_y(0, y) = \log \rho + \frac{1}{1-\rho} \log \left[\frac{y + \sqrt{(y-1+\rho)^2 + 4y} - (1-\rho)}{y + \sqrt{(y-1+\rho)^2 + 4y} + (1-\rho)}\right], \tag{3.11}$$

$$S_0(y) = \frac{\sqrt{2\pi}(1-\rho)(s_* + 1 - \rho)^{3/2} \sqrt{s_*}}{(s_* + 1)\sqrt{(s_* + 1)^2 - \rho}}, \tag{3.12}$$

$$s_*(y) = \frac{1}{2}\left[y - 1 + \rho + \sqrt{(y-1+\rho)^2 + 4y}\right]. \tag{3.13}$$

For $r = O(1)$ and $0 < x < 1$, which correspond to having a few secondary spaces occupied and a fraction of the primary ones, we obtain

$$\pi(k, r) \sim \frac{(1-\rho)^2 \rho^{m+r} x^{r/(1-\rho)}}{[1 - \rho + \rho x^{1/(1-\rho)}]^{r+1}}, \quad r \geqslant 1, \tag{3.14}$$

$$\pi(k, 0) - (1-\rho)\rho^k \sim -\frac{(1-\rho)\rho^{m+1} x^{1/(1-\rho)}}{1 - \rho + \rho x^{1/(1-\rho)}}. \tag{3.15}$$

Here we wrote the result for $r = 0$ so as to estimate the deviation of $\pi(k, 0)$ from the geometric distribution $(1-\rho)\rho^k = (1-\rho)\rho^{mx}$.

Near the corner $(x, y) = (0, 0)$ we use the original discrete variables (k, r) to find that

$$\pi(k, r) \sim \rho^{m+r} m^{-r/(1-\rho)} (1-\rho)^{2-r/(1-\rho)} \Gamma\left(1 + \frac{r}{1-\rho}\right)$$
$$\times \frac{1}{2\pi i} \oint \frac{z^{-k-1}(1 - \rho z)^{\rho r/(1-\rho)-1}}{(1-z)^{1+r/(1-\rho)}} dz, \quad r \geqslant 1 \tag{3.16}$$

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and

$$\pi(k,0)-(1-\rho)\rho^k \sim -\rho^m m^{-1/(1-\rho)} \sum_{j=0}^{k-1} \left(\frac{1-\rho^{k-j}}{1-\rho}\right) \frac{P(j,1)}{j+1}, \tag{3.17}$$

$$P(k,1)=\rho(1-\rho)^{2-1/(1-\rho)}\Gamma\left(\frac{2-\rho}{1-\rho}\right) \times \frac{1}{2\pi i}\oint \frac{z^{-k-1}(1-\rho z)^{\rho/(1-\rho)-1}}{(1-z)^{1+1/(1-\rho)}}\,dz. \tag{3.18}$$

Here the integrals are over a small loop about $z=0$, and these contour integrals may be expressed in terms of hypergeometric functions. Near the other corner $(x,y)=(1,0)$ we use the variables $n=m-k$ and r, and obtain

$$\pi(k,r)\sim(1-\rho)^2\rho^{m+r}, \quad r\geqslant 1, \tag{3.19}$$

$$\pi(k,0)\sim(1-\rho)(\rho^{-n}-\rho)\rho^m, \quad r=0. \tag{3.20}$$

We note that for $\rho<1$ and $m\to\infty$ most of the probability mass occurs in the range $k=O(1)$ and $r=0$, and $\pi(k,r)$ is exponentially small in all of the other ranges.

Defining the marginal distribution by

$$\mathscr{M}(k)=\sum_{r=0}^{\infty}\pi(k,r), \quad 0\leqslant k\leqslant m, \tag{3.21}$$

and

$$\mathscr{N}(r)=\sum_{k=0}^{m}\pi(k,r), \quad r\geqslant 0, \tag{3.22}$$

we can easily obtain their expansions from the results for $\pi(k,r)$. For the distribution of the number of occupied primary spaces we have

$$\mathscr{M}(k)-(1-\rho)\rho^k\sim\frac{\rho^{m+1}x^{\rho/(1-\rho)}}{(1-\rho)m}, \quad 0<x\leqslant 1, \tag{3.23}$$

$$\mathscr{M}(k)-(1-\rho)\rho^k\sim\rho^m m^{-1/(1-\rho)}[P(k,0)+P(k,1)], \quad k=O(1), \tag{3.24}$$

where $P(k,0)$ is given by (4.74). For the distribution of the number of occupied secondary spaces we obtain

$$\mathscr{N}(r)\sim(1-\rho)\rho^{m(y+1)}\left(\frac{y+1-\rho}{y}\right), \quad y>0, \tag{3.25}$$

$$\mathscr{N}(r)\sim m(1-\rho)^3\rho^{m+r}\int_0^1\frac{u^{r-\rho}}{(1-\rho+\rho u)^{r+1}}\,du, \quad r\geqslant 1, \tag{3.26}$$

$$1-\mathscr{N}(0)\sim m(1-\rho)^2\rho^{m+1}\int_0^1\frac{u^{1-\rho}}{1-\rho+\rho u}\,du. \tag{3.27}$$

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In particular the mean number of occupied secondary spaces is

$$\sum_{r=1}^{\infty} r\mathcal{N}(r) \sim \left(\frac{1-\rho}{2-\rho}\right) m\rho^{m+1}. \tag{3.28}$$

Finally, we consider the double limit where $m \to \infty$ and $\rho \uparrow 1$. We introduce the parameter $a = m(1-\rho) = O(1)$. On the (x, y) scale we find that

$$\pi(k, r) \sim m^{-1}\mathcal{K}(x, y)e^{m\Psi(x,y)} \; ; \; 0 < x \leqslant 1, \quad y > 0, \tag{3.29}$$

where

$$\Psi = x\Psi_x + y\Psi_y - \log(s+1), \tag{3.30}$$

with

$$\Psi_x = \log\left(\frac{1+s-st}{1-st}\right), \tag{3.31}$$

$$\Psi_y = \log\left(\frac{1+s-st}{1-st+se^t}\right), \tag{3.32}$$

$$x(s,t) = \frac{1}{s+1}(1-st)(1+2s-st-se^{-t}), \tag{3.33}$$

$$y(s,t) = \frac{s}{s+1}[s+(1-st)e^{-t}], \tag{3.34}$$

and

$$\mathcal{K} = \frac{as^{3/2}(s-st+1)e^{t/2}}{\sqrt{s(s+1)(1-st)(se^t-st+1)}\sqrt{2s(s+2)-[s(s+2)t^2-2t+s(s+2)-1]e^{-t}}} \times \exp\left[ast - \frac{as^2t^2}{2(s+1)} - a(s+1)\right]. \tag{3.35}$$

For $k = m - O(1)$ ($x = 1 - O(m^{-1})$) the expression simplifies to

$$\pi(k, r) \sim \frac{1}{m}\frac{ay}{y+1}(y+1)^{k-m}e^{-a(y+1)}, \quad y > 0, \tag{3.36}$$

which shows that the total probability mass in the range $k = m - O(1)$ and $y > 0$ is asymptotically e^{-a}. The remaining mass occurs in the range $r = 0$ and $0 < x < 1$, as the results below show that $\sum_{k=0}^{m} \pi(k, 0) \sim 1 - e^{-a}$.

From (3.36) we can also infer the conditional limit laws

$$Prob\left[N_1 = m - n | N_2 = my\right] \sim y(y+1)^{-n-1}, \quad y > 0,$$

$$Prob\left[N_2 = my | N_1 = m - n\right] \sim \frac{y(y+1)^{-n-1}e^{-ay}}{\mathcal{C}_n},$$

with

$$\mathcal{C}_n = \int_0^{\infty} \frac{ue^{-au}}{(u+1)^{n+1}}\, du.$$

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For $k = O(1)$ ($x = O(m^{-1})$) we find that

$$\pi(k,r) \sim m^{k-1/2}\mathscr{S}_0(y)\frac{y^k}{k!}\left[2 - e^{\Psi_y(0,y)}\right]^k e^{m\Psi(0,y)}, \tag{3.37}$$

where

$$\Psi(0,y) = -\frac{y+\sqrt{y^2+4y}}{y+\sqrt{y^2+4y}+2} + \log\left(\frac{2}{y+\sqrt{y^2+4y}+2}\right), \tag{3.38}$$

$$\Psi_y(0,y) = -\frac{2}{y+\sqrt{y^2+4y}}, \tag{3.39}$$

$$\mathscr{S}_0(y) = \frac{a\sqrt{2\pi}\left(y+\sqrt{y^2+4y}\right)^{3/2}}{\left(y+\sqrt{y^2+4y}+2\right)\left(y+\sqrt{y^2+4y}+4\right)^{1/2}} \times \exp\left[-a\left(\frac{y+\sqrt{y^2+4y}}{2} + \frac{1}{y+\sqrt{y^2+4y}+2}\right)\right]. \tag{3.40}$$

If $y > 0$, the analysis and results for $1-\rho = O(m^{-1})$ are similar to the case $\rho < 1$. However, this is not the case if $y = o(1)$. We give below various results for $r = O(1)$ ($y = O(m^{-1})$) and $r = O(\sqrt{m})$ ($y = O(m^{-1/2})$). First, for $r = O(1)$ and $0 < x < 1$ we have

$$\pi(k,r) \sim \frac{a}{2}\, m^{r/2-1} r^{-r/2} x^{r/2+1/4} e^{-a(x+1)/2} e^{-2\sqrt{mr}(1-\sqrt{x})}, \quad r \geqslant 1, \tag{3.41}$$

and

$$\pi(k,0) - (1-\rho)\rho^k \sim -\frac{a}{2}\frac{x^{3/4}}{\sqrt{m}} e^{-a(x+1)/2} e^{-2\sqrt{m}(1-\sqrt{x})}. \tag{3.42}$$

Also note that with this scaling of ρ

$$(1-\rho)\rho^k = \frac{ae^{-ax}}{m}\left[1 - \frac{a^2 x}{2m} + O(m^{-2})\right]. \tag{3.43}$$

When $k, r = O(1)$ we have, for $r \geqslant 1$,

$$\pi(k,r) \sim a\sqrt{\pi}\, m^{-5/4} r^{3/4} e^{-(a+r)/2} e^{-2\sqrt{mr}} \frac{1}{2\pi i}\oint \frac{z^{-k-1}}{(1-z)^{r+2}}\, e^{r/(1-z)}\, dz, \tag{3.44}$$

and

$$\pi(k,0) - (1-\rho)\rho^k \sim -a\sqrt{\pi} e^{-(a+1)/2} m^{-5/4} e^{-2\sqrt{m}}\, \frac{1}{2\pi i}\oint \frac{z^{-k} e^{1/(1-z)}}{(1-z)^3}\, dz. \tag{3.45}$$

The contour integral may be expressed in terms of a confluent hypergeometric function.

The analysis near the corner $(x,y) = (1,0)$ is much more complicated in the case $1-\rho = O(m^{-1})$ than when $\rho < 1$. We have obtained some partial results in the corner range, and there are several nested corner layers that must be considered. First, when $1 - x = O(m^{-1/2})$ and $y = O(m^{-1/2})$ we let $k = m - \sqrt{m}\xi$ and $r = \sqrt{m}R$. We find that for

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$\xi, R > 0$,

$$\pi(k,r) \sim m^{-3/2}\Omega(\xi,R) = m^{-3/2}R^{-1}\mathscr{D}(\xi,R), \tag{3.46}$$

where $\mathscr{D}(\xi,R)$ is expressible in terms of $\mathscr{D}(0,R)$ via the integral

$$\begin{aligned}\mathscr{D}(\xi,R) = \frac{1}{2\sqrt{\pi}}\int_R^\infty \exp\left[-\frac{(\chi-R+\xi)^2}{4\log(\chi/R)}\right] \\ \times\left\{\frac{\mathscr{D}(0,\chi)}{\chi^2\sqrt{\log(\chi/R)}} + \frac{(\chi-R)\mathscr{D}(0,\chi)}{2\chi[\log(\chi/R)]^{3/2}} - \frac{\mathscr{D}_\chi(0,\chi)}{\chi\sqrt{\log(\chi/R)}}\right\} d\chi \\ + \frac{\xi}{2\sqrt{\pi}}\int_R^\infty \frac{\mathscr{D}(0,\chi)}{2\chi[\log(\chi/R)]^{3/2}}\exp\left[-\frac{(\chi-R+\xi)^2}{4\log(\chi/R)}\right] d\chi,\end{aligned} \tag{3.47}$$

and $\mathscr{D}(0,R)$ satisfies the integral equation

$$\begin{aligned}\mathscr{D}(0,R) = \frac{1}{\sqrt{\pi}}\int_R^\infty \left\{\frac{\mathscr{D}(0,\chi)}{\chi^2\sqrt{\log(\chi/R)}} + \frac{(\chi-R)\mathscr{D}(0,\chi)}{2\chi[\log(\chi/R)]^{3/2}} - \frac{\mathscr{D}_\chi(0,\chi)}{\chi\sqrt{\log(\chi/R)}}\right\} \\ \times \exp\left[-\frac{(\chi-R)^2}{4\log(\chi/R)}\right] d\chi.\end{aligned} \tag{3.48}$$

We establish several asymptotic results for $\mathscr{D}(\xi,R)$ in Section 5.3. In Section 5.4 we consider the scale $k = m - O(m^{3/4})$ and $r = O(\sqrt{m})$ and obtain explicit expressions for $\pi(k,r)$, but these simply follow by expanding the result on the (x,y) scale as $(x,y) \to (1,0)$ along certain parabolic paths. In Section 5.5 we consider $x \sim 1$ and $r = O(1)$, and find that with $k = m - \bar{\xi}(\log m)\sqrt{m}$ and $r \geqslant 1$

$$\pi(k,r) \sim \frac{1}{2}ae^{-a}m^{r/2-1}m^{-\bar{\xi}\sqrt{r}}r^{-r/2}, \quad \bar{\xi} > \sqrt{r}, \tag{3.49}$$

$$\pi(k,r) \sim \frac{ae^{-a}}{\sqrt{2\pi}}\frac{m^{-\bar{\xi}^2/2}}{m\sqrt{\log m}}\frac{\Gamma(r-\bar{\xi}^2)}{r!}\,\bar{\xi}^{2+\bar{\xi}^2}e^{-\bar{\xi}^2}, \quad 0 < \bar{\xi} < \sqrt{r}, \tag{3.50}$$

$$\begin{aligned}\pi(k,r) \sim \frac{ae^{-a}}{2\sqrt{2\pi}}m^{-r/2-1}r^{-r/2}e^{-\circledast\sqrt{r\log m}}\int_{-\infty}^{\circledast} e^{-u^2/2}du, \\ \circledast = (\bar{\xi}-\sqrt{r})\sqrt{\log m} = O(1).\end{aligned} \tag{3.51}$$

When $r = 0$, we obtain, with now $\circledast = (\bar{\xi}-1)\sqrt{\log m}$,

$$\pi(k,0) - (1-\rho)\rho^k \sim -\frac{1}{2}ae^{-a}m^{-\bar{\xi}-1/2}, \quad \bar{\xi} > 1, \tag{3.52}$$

$$\pi(k,0) - \frac{a}{m}e^{-ax} \sim \frac{ae^{-a}}{\sqrt{2\pi}}\frac{m^{-\bar{\xi}^2/2}}{m\sqrt{\log m}}\Gamma(1+\bar{\xi}^2)\Gamma(-\bar{\xi}^2)\bar{\xi}^{1+\bar{\xi}^2}, \quad 0 < \bar{\xi} < 1, \tag{3.53}$$

$$\pi(k,0) - \frac{a}{m}e^{-ax} \sim -\frac{ae^{-a}}{2\sqrt{2\pi}}m^{-3/2}e^{-\circledast\sqrt{\log m}}\int_{-\infty}^{\circledast} e^{-u^2/2}du, \quad \circledast = O(1). \tag{3.54}$$

Note that $x = 1 - \bar{\xi}(\log m)m^{-1/2}$ so that $e^{-ax} \sim e^{-a}\left[1 + a\bar{\xi}(\log m)/\sqrt{m}\right]$.

In Section 5.5 we shall also discuss the scales $k = m - O(\sqrt{m\log m})$, $k = m - O(\sqrt{m})$ and $k = m - O(1)$, but we have not been able to completely resolve all of the corner layer(s)

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near $(x, y) = (1, 0)$. Furthermore, the analysis also suggests that we may get different asymptotics for the limit $m \to \infty$ with $1 - \rho = O(m^{-1/2})$, which we do not consider here.

4 Asymptotic expansions for $m \to \infty$ with fixed $0 < \rho < 1$

In this section we examine the case when $m \to \infty$ and ρ is fixed $(0 < \rho < 1)$. We set

$$\delta = \frac{1}{m}, \quad x = \frac{k}{m} = \delta k, \quad y = \frac{r}{m} = \delta r, \tag{4.1}$$

and then let

$$\pi(k, r) = K(x, y; \delta) \exp\left[\frac{1}{\delta}\phi(x, y)\right]. \tag{4.2}$$

For $0 < x < 1$, and $y > 0$, (2.3) becomes

$$\begin{aligned}(1+\rho)(x+y+\delta)\, K(x, y; \delta) \exp\left[\frac{1}{\delta}\phi(x, y)\right] \\ = \rho(x+y+\delta)\, K(x-\delta, y; \delta) \exp\left[\frac{1}{\delta}\phi(x-\delta, y)\right] \\ + (x+\delta)\, K(x+\delta, y; \delta) \exp\left[\frac{1}{\delta}\phi(x+\delta, y)\right] \\ + (y+\delta)\, K(x, y+\delta; \delta) \exp\left[\frac{1}{\delta}\phi(x, y+\delta)\right].\end{aligned} \tag{4.3}$$

The boundary condition (2.7) at $x = 1$ $(k = m)$ becomes

$$\rho K(1, y-\delta; \delta) \exp\left[\frac{1}{\delta}\phi(1, y-\delta)\right] = \frac{1+\delta}{1+y+\delta}\, K(1+\delta, y; \delta) \exp\left[\frac{1}{\delta}\phi(1+\delta, y)\right]. \tag{4.4}$$

We assume that $K(x, y; \delta)$ has an asymptotic expansion in powers of δ, with

$$K(x, y; \delta) = K(x, y) + \delta K^{(1)}(x, y) + O(\delta^2). \tag{4.5}$$

To compute the leading term for $\pi(k, r)$ we must compute $\phi(x, y)$ and the leading term in (4.5). We divide (4.3) by $K(x, y) \exp[\phi(x, y)/\delta]$ and let $\delta \to 0$. Similarly we divide (4.4) by $K(1, y) \exp[\phi(1, y)/\delta]$ and let $\delta \to 0$. This leads to the following 'eikonal' equation for ϕ, for $0 < x < 1$ and $y > 0$,

$$(1+\rho)(x+y) - \rho(x+y)e^{-\phi_x} - xe^{\phi_x} - ye^{\phi_y} = 0, \tag{4.6}$$

and at $x = 1$ we have the boundary condition

$$(1+y)e^{-\phi_y} = \frac{e^{\phi_x}}{\rho}. \tag{4.7}$$

We solve (4.6) and (4.7) for $\phi(x, y)$ by using the method of characteristics. We write (4.6) as $F(x, y, \phi, \phi_x, \phi_y) = 0$, where

$$F \equiv (1+\rho)(x+y) - \rho(x+y)e^{-\phi_x} - xe^{\phi_x} - ye^{\phi_y}. \tag{4.8}$$

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The characteristic equations for this non-linear partial differential equations (PDE) are [9]

$$\dot{x} = \frac{dx}{dt} = \frac{\partial F}{\partial \phi_x} = \rho(x+y)e^{-\phi_x} - xe^{\phi_x}, \tag{4.9}$$

$$\dot{y} = \frac{dy}{dt} = \frac{\partial F}{\partial \phi_y} = -ye^{\phi_y}, \tag{4.10}$$

$$\dot{\phi} = \frac{d\phi}{dt} = \phi_x \dot{x} + \phi_y \dot{y}, \tag{4.11}$$

$$\dot{\phi_x} = -\frac{\partial F}{\partial x} = -(1+\rho) + \rho\, e^{-\phi_x} + e^{\phi_x}, \tag{4.12}$$

$$\dot{\phi_y} = -\frac{\partial F}{\partial y} = -(1+\rho) + \rho\, e^{-\phi_x} + e^{\phi_y}. \tag{4.13}$$

To solve this system we use rays starting from $(x, y) = (1, s)$ at $t = 0$. Thus the 'initial manifold' is $x = 1$, where the boundary condition in (4.7) applies, and all the rays start from $x = 1$ at $t = 0$. We view x and y as functions of s and t. We set $\phi_x \equiv \log \Upsilon(s, t)$ and, from (4.8), (4.9), (4.10) and (4.12), obtain

$$\frac{\dot{x} + \dot{y}}{x + y} = \frac{2\rho}{\Upsilon} - (1+\rho), \tag{4.14}$$

$$\frac{\partial \Upsilon}{\partial t} = (\Upsilon - \rho)(\Upsilon - 1). \tag{4.15}$$

Solving (4.15) gives

$$\Upsilon = \frac{C_0(s)e^{(\rho-1)t} - \rho}{C_0(s)e^{(\rho-1)t} - 1}, \tag{4.16}$$

and using (4.16) we solve (4.14) to obtain

$$x + y = C_1(s)[C_0(s)e^{(\rho-1)t} - \rho]^2 e^{(1-\rho)t}. \tag{4.17}$$

The functions $C_0(s)$ and $C_1(s)$ will be determined shortly.

From (4.16) and (4.17), (4.9) can be written as

$$\frac{\partial x}{\partial t} + \left[\frac{C_0(s)e^{(\rho-1)t} - \rho}{C_0(s)e^{(\rho-1)t} - 1}\right] x = \rho C_1(s)[C_0(s)e^{(\rho-1)t} - \rho][C_0(s) - e^{(1-\rho)t}]. \tag{4.18}$$

Solving (4.18) we obtain $x(s, t)$ as

$$x(s,t) = [C_0(s)C_1(s) - \rho^2 C_1(s)e^{(1-\rho)t} + C_2(s)e^{-\rho t}][C_0(s)e^{(\rho-1)t} - 1], \tag{4.19}$$

and then $y(s, t)$ follows from (4.17) as

$$y(s,t) = C_0(s)C_1(s)(1-\rho)^2 + C_2(s)[e^{-\rho t} - C_0(s)e^{-t}]. \tag{4.20}$$

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Applying the initial conditions $(x(s,0)=1,\ y(s,0)=s)$ we find that

$$C_1(s)=\frac{s+1}{[C_0(s)-\rho]^2}, \tag{4.21}$$

$$C_2(s)=\frac{1}{C_0(s)-1}-\frac{C_0(s)-\rho^2}{[C_0(s)-\rho]^2}(s+1). \tag{4.22}$$

To determine $C_0(s)$ we set $\phi_x=A(s)$ and $\phi_y=B(s)$ at $t=0$, and from (4.6) and (4.7) obtain the following equations along the initial manifold

$$(1+\rho)(1+s)=\rho(1+s)e^{-A}+e^{A}+se^{B}, \tag{4.23}$$

$$\rho(1+s)e^{-A}=e^{B}. \tag{4.24}$$

Solving (4.23) and (4.24) leads to

$$A(s)=\log(s+1),\quad B(s)=\log\rho.$$

Since $e^{\phi_y}=-\dot{y}/y$ from (4.10), $\phi_y(1,s)=\log[-\dot{y}(s,0)/s]=B(s)=\log\rho$. Thus we obtain $C_0(s)=(s+1-\rho)/s$ and then, from (4.19)–(4.22), obtain (3.5) and (3.6). Solving (4.12) and (4.13) then yields (3.3) and (3.4). Finally, we obtain $\phi(x,y)$ by integrating (4.11) with respect to t. Using integration by parts, we can write

$$\phi=\int(\phi_x\dot{x}+\phi_y\dot{y})\,dt=\phi_x x+\phi_y y-\int(\dot{\phi_x}x+\dot{\phi_y}y)\,dt. \tag{4.25}$$

The integrand in the last integral in (4.25) is $-F=0$. We can also check from (3.3) – (3.6) that indeed $\dot{\phi_x}/\dot{\phi_y}=-y/x$. Therefore, $\phi(x,y)=\phi_x x+\phi_y y+f(s)$, where $f(s)$ is some function of s. At $t=0$, $\phi(1,s)=\log(s+1)+s\log\rho+f(s)$ and since $y=s$ at $t=0$,

$$\frac{\partial\phi(1,s)}{\partial s}=\frac{1}{s+1}+\log\rho+f'(s)=\phi_y(1,s)=\log\rho.$$

Thus $f(s)=-\log(s+1)+constant$. We will later show by asymptotic matching that near the corner $(x,y)=(1,0)$ the solution $\pi(k,r)$ must be $O(\rho^m)=O\left(\exp[(\log\rho)/\delta]\right)$. Thus, $\phi(1,0)=\log\rho$ and we then obtain (3.2). The value of the *constant* can also be ultimately found by normalisation or by using (2.8). Note, however, that our expansion (4.2) applies only in the domain $0<x\leqslant 1,\ y>0$. We shall construct different expansions for $x\approx 0$ and $y\approx 0$, and also near the two corners $(x,y)=(0,0)$ and $(1,0)$.

We can plot the rays, which are given by (3.5) and (3.6) in parametric forms. Each fixed value of s corresponds to a particular ray parameterised by t. We sketch the rays in Figures 2 and 3. In Figure 2 we plot the rays for $s>0$ and t in the range $0\leqslant t\leqslant t_{\max}\equiv\log[1+(1-\rho)/s]/(1-\rho)$. At $t=0$ we have $x=1$ and at $t=t_{\max}(s)$ the rays reach $x=0$. We also note that, in view of (3.3), ϕ_x develops a singularity at $t=t_{\max}(s)$. However, using (3.5) and (3.6) we can continue the rays for $t>t_{\max}$, and this continuation is sketched in Figure 3. We see that each ray reaches a minimum value of x, where it has a cusp. The cusp occurs when $\dot{x}=\dot{y}=0$, which happens at $t=t_c(s)\equiv\log[(s+1-\rho)/(\rho s)]/(1-\rho)>t_{\max}(s)$. From (3.5) and (3.6) we also see that when $t=t_c,\ x+y=0$, so that the line $y=-x$ is the locus of the cusp points. For

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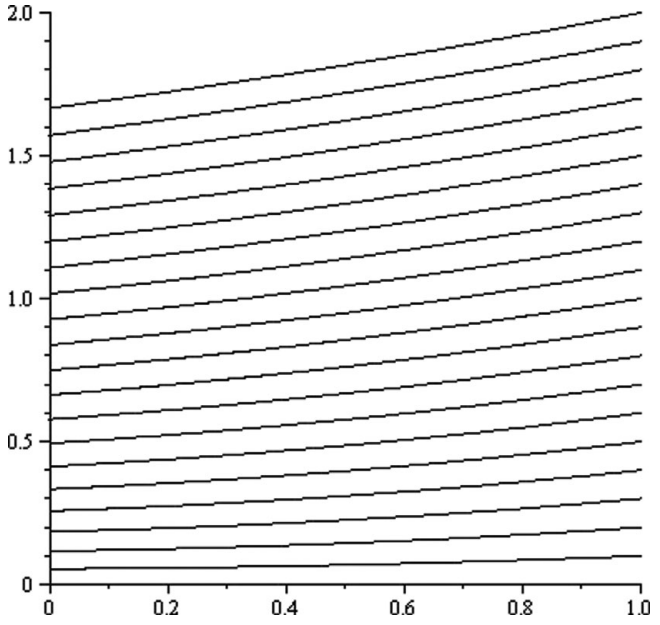


FIGURE 2. A sketch of the rays from $x = 1$.

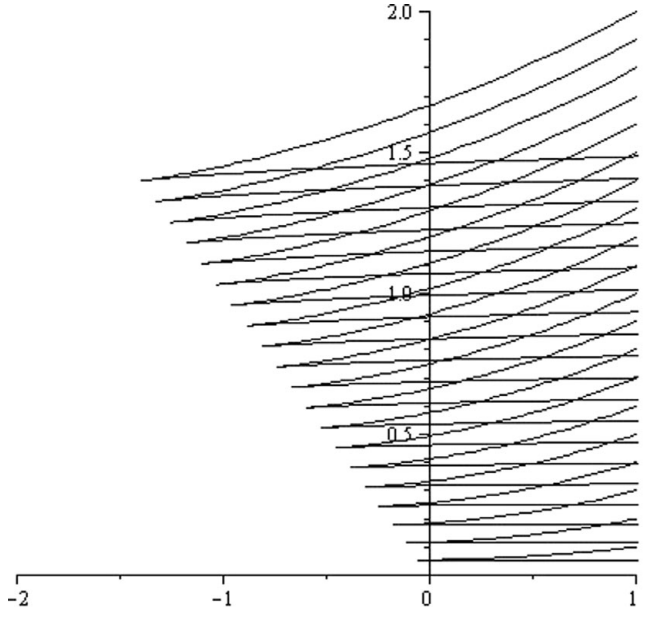


FIGURE 3. A sketch of the extended rays from $x = 1$.

$t > t_c$ we again have $\dot{x} > 0$ and the rays eventually re-enter the domain $x > 0$, and then x increases past $x = 1$. The full rays are sketched in Figure 3, up to the time they return to $x = 1$. The figure clearly shows the cusps and their locus. However, since the cusps occur outside of the domain of interest $\{0 \leqslant x \leqslant 1,\ y \geqslant 0\}$, they do not affect the solution very much, and we only consider the rays for $0 \leqslant t \leqslant t_{\max}$. The origin $(0, 0)$ is the only cusp

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point in the domain, and this would significantly affect the asymptotic structure of $\pi(k,r)$ for the scale $k,r = O(1)$. We analyse this range in Section 4.4.

We next calculate $K(x,y)$ in (4.5). Expanding (4.3) as $\delta \to 0$ and using the fact that ϕ satisfies (4.6) we obtain the following 'transport' equation:

$$\left[\left(\frac{x+y}{2}\phi_{xx}+1\right)\rho e^{-\phi_x}+\left(\frac{x}{2}\phi_{xx}+1\right)e^{\phi_x}+\left(\frac{y}{2}\phi_{yy}+1\right)e^{\phi_y}-(1+\rho)\right]K$$
$$=[\rho(x+y)e^{-\phi_x}-xe^{\phi_x}]K_x-ye^{\phi_y}K_y. \tag{4.26}$$

We note that the right-hand side is $\dot{x}K_x+\dot{y}K_y=\dot{K}$, which is the directional derivative of K along a ray. The factor that multiplies K in the left-hand side of (4.26) may be rewritten as

$$-\frac{1}{2}\frac{\partial}{\partial x}[\rho(x+y)e^{-\phi_x}-xe^{\phi_x}]+\frac{1}{2}\frac{\partial}{\partial y}(ye^{\phi_y})-(1+\rho)+\frac{1}{2}(e^{\phi_x}+e^{\phi_y})+\frac{3}{2}\rho e^{-\phi_x}$$
$$=-\frac{1}{2}\frac{\partial\dot{x}}{\partial x}-\frac{1}{2}\frac{\partial\dot{y}}{\partial y}-(1+\rho)+\frac{1}{2}(e^{\phi_x}+e^{\phi_y})+\frac{3}{2}\rho e^{-\phi_x}. \tag{4.27}$$

Defining $J(s,t)$ to be $x_ty_s-x_sy_t$, which is the Jacobian of the transformation from (x,y) to (s,t) coordinates, we find that

$$\frac{\partial\dot{x}}{\partial x}=\frac{\partial x_t}{\partial x}=x_{tt}t_x+x_{ts}s_x=\frac{x_{tt}y_s-x_{ts}y_t}{J}, \tag{4.28}$$

$$\frac{\partial\dot{y}}{\partial y}=\frac{\partial y_t}{\partial y}=y_{tt}t_y+y_{ts}s_y=\frac{y_{ts}x_t-y_{tt}x_s}{J}, \tag{4.29}$$

and

$$\frac{\partial J}{\partial t}=\dot{J}=x_{tt}y_s+y_{ts}x_t-x_{ts}y_t-y_{tt}x_s. \tag{4.30}$$

Thus, from (4.12), (4.13) and (4.27)–(4.30), we can write (4.26) as

$$\frac{\dot{K}}{K}=-\frac{1}{2}\frac{\dot{J}}{J}+\frac{1}{2}(\dot{\phi_x}+\dot{\phi_y})+\frac{1}{2}\rho e^{-\phi_x}. \tag{4.31}$$

Integrating (4.31) using (3.3) for ϕ_x we obtain

$$K(x,y)=K_0(s)\left[\frac{|(s+1-\rho)e^{-(1-\rho)t}-s\rho|}{|J|}\right]^{1/2}\exp\left(\frac{\phi_x+\phi_y+t}{2}\right), \tag{4.32}$$

where $K_0(s)$ is a function of s.

To determine $K_0(s)$ we use the boundary condition at $x=1$. From (4.4) we obtained (4.7) to leading order in δ, and at the next order we obtain

$$\rho e^{-\phi_y}\left[\frac{(y+1)}{2}\phi_{yy}K(1,s)+K(1,s)-(y+1)K_y(1,s)\right]$$
$$=e^{\phi_x}\left[\left(\frac{\phi_{xx}}{2}+1\right)K(1,s)+K_x(1,s)\right]. \tag{4.33}$$

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Since $\phi_{xx}(1,s) = s/(\rho - s - 1)$ and $\phi_{yy}(1,s) = 0$, (4.33) becomes

$$(s+1)[K_x(1,s) + K_y(1,s)] + \frac{s(2\rho - 1 - s)}{2(\rho - s - 1)} K(1,s) = 0. \tag{4.34}$$

Using the relations

$$\begin{aligned} K_x &= K_t t_x + K_s s_x = K_t \frac{y_s}{J} - K_s \frac{y_t}{J} = \frac{K_t + \rho s K_s}{\rho - s - 1}, \\ K_y &= K_t t_y + K_s s_y = -K_t \frac{x_s}{J} + K_s \frac{x_t}{J} = K_s, \\ J(s,0) &= \rho - 1 - s, \end{aligned}$$

(4.34) becomes

$$\frac{K_t}{K} + (\rho - 1)(s+1)\frac{K_s}{K} + \frac{\rho s}{1+s} - \frac{s}{2} = 0, \quad t = 0. \tag{4.35}$$

Evaluating (4.26) at $t = 0$ yields

$$\frac{K_t}{K} = \frac{s(\rho + s + 1)}{2(\rho - s - 1)} + \frac{\rho}{s+1} + s \tag{4.36}$$

and thus, from (4.35) and (4.36), we obtain the following ordinary differential equation for K at $t = 0$ with respect to s:

$$\frac{K_s}{K} = \frac{1}{s + 1 - \rho} - \frac{1}{s+1}. \tag{4.37}$$

Integrating (4.37) gives

$$K(1,s) = C^* \left(1 - \frac{\rho}{s+1}\right), \tag{4.38}$$

where C^* is a constant. Equating (4.32) with (4.38) at $t = 0$ we obtain $K_0(s)$ as

$$K_0(s) = C^* \frac{(s + 1 - \rho)^{3/2}}{(s+1)^2} \frac{1}{\sqrt{\rho(1-\rho)}}.$$

To obtain C^* we use the fact that $N_1 + N_2$ follows the geometric distribution in (2.8). Expanding ϕ and K near $x = 1$, which corresponds to $t = 0$, the approximation $\pi(k,r) \sim K(x,y)\exp[\phi(x,y)/\delta]$ simplifies to

$$\pi(k,r) \sim C^* \left(1 - \frac{\rho}{y+1}\right) \exp\left(\frac{y+1}{\delta} \log \rho\right) \exp\left[\frac{x-1}{\delta} \log(y+1)\right]. \tag{4.39}$$

Using (4.39) in (2.8) we find after some calculation that $C^* = 1 - \rho$ and thus obtain (3.7). We also obtain $\pi(k,r)$ near $x = 1$ from (4.39) as (3.8), which applies for $k = m - O(1)$ and $y > 0$.

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4.1 Boundary layer near $x = 0$

We observed from (3.3) that ϕ_x has a logarithmic singularity as $x \to 0$. We consider the scale $k = O(1)$ and $y > 0$, which corresponds to $x = O(\delta)$ and $r = O(\delta^{-1}) = O(m)$. We set

$$\pi(k,r) = \delta^{\nu-k} \exp\left[\frac{1}{\delta}\phi(0,y)\right] S_k(y;\delta),$$

and rewrite (2.3) as

$$\begin{aligned}(1+\rho)[y+\delta(k+1)]\, e^{\phi(0,y)/\delta} S_k(y;\delta) &= \delta\rho[y+\delta(k+1)] e^{\phi(0,y)/\delta} S_{k-1}(y;\delta) \\ &+ (k+1)\, e^{\phi(0,y)/\delta} S_{k+1}(y;\delta) + (y+\delta)\, e^{\phi(0,y+\delta)/\delta} S_k(y+\delta;\delta)\end{aligned} \tag{4.40}$$

for $k > 0$ and $y > 0$. The boundary condition at $k = 0$ is

$$(1+\rho)(y+\delta)\, e^{\phi(0,y)/\delta} S_0(y;\delta) = e^{\phi(0,y)/\delta} S_1(y;\delta) + (y+\delta) e^{\phi(0,y+\delta)/\delta} S_0(y+\delta;\delta). \tag{4.41}$$

We expand $S_k(y;\delta)$ in the form

$$S_k(y;\delta) = S_k(y) + \delta S_k^{(1)}(y) + O(\delta^2).$$

In (4.40) and (4.41) we let $\delta \to 0$ and obtain the following equation for the leading term $S_k(y)$

$$\left[1+\rho-e^{\phi_y(0,y)}\right] y = (k+1)\,\frac{S_{k+1}(y)}{S_k(y)}, \quad k \geqslant 0. \tag{4.42}$$

The general solution to (4.42) is

$$S_k(y) = S_0(y)\frac{y^k}{k!}\left[1+\rho-e^{\phi_y(0,y)}\right]^k,$$

and thus

$$\pi(k,r) \sim \delta^{\nu-k} e^{\phi(0,y)/\delta} S_0(y)\frac{y^k}{k!}\left[1+\rho-e^{\phi_y(0,y)}\right]^k. \tag{4.43}$$

It remains to determine the constant ν and the function $S_0(y)$.

We next asymptotically match (4.43) to the ray expansion $K(x,y)\exp[\phi(x,y)/\delta]$ in an intermediate limit where $x \to 0$ but $k = x/\delta \to \infty$. Using Stirling's formula for $k!$, the expansion in (4.43) for $k \to \infty$ becomes

$$\begin{aligned}\pi(k,r) &\sim \delta^{\nu} e^{\phi(0,y)/\delta} S_0(y)\frac{e^k}{\sqrt{2\pi k}} \exp(-k\log k - k\log\delta)(1+\rho-e^{\phi_y(0,y)})^k y^k \\ &= \delta^{\nu+1/2}\frac{S_0(y)}{\sqrt{2\pi x}} \exp\left[\frac{1}{\delta}\left\{\phi(0,y) + x + x\log\left(\frac{y}{x}\right) + x\log\left[1+\rho-e^{\phi_y(0,y)}\right]\right\}\right].\end{aligned} \tag{4.44}$$

Here we rewrote the result in terms of x.

We expand the ray solution as $x \to 0$. Along $x = 0$ we can explicitly invert the transformation from (x,y) to ray coordinates. When $x = 0$ we have $t = t_{\max}(s)$ and $s + 1 - \rho - se^{(1-\rho)t} = 0$. Then from (3.6) we find that s and y are related by

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$y = s(s+1-\rho)/(s+1)$. We thus define $s_* = s_*(y)$ by (3.13). Thus, a ray that starts from $(x,y) = (1, s_*)$ hits the y-axis at the point $(0,y)$. Then we use (3.2) and (3.4) and evaluate these expressions at $t = t_{\max}$ and $s = s_*$ to obtain explicit expressions (3.10) and (3.11) for $\phi(0,y)$ and $\phi_y(0,y)$. We note that $\phi_y(0,y) = \log\rho - t_{\max}(s_*)$. Also, from (3.3) and (3.5) we obtain

$$\phi_x = -\log x + \log\left(\frac{\left[s+1-\rho-\rho s\, e^{(1-\rho)t}\right]\left[(s+1-\rho)\, e^{-(1-\rho)t} - s\rho^2 - \rho(1-\rho)se^{-t}\right]}{(s+1)(1-\rho)^2}\right), \tag{4.45}$$

which is exact for all (x,y) and indicates the logarithmic singularity in ϕ_x as $x \to 0$. By expanding (4.45) for $s \to s_*$ and $t \to t_{\max}$ we obtain

$$\begin{aligned}\phi_x &= -\log x + \log\left[\frac{s_*(s_*+1-\rho)(1+\rho-\rho e^{-t})}{(s_*+1)}\right] + o(1)\\ &= -\log x + \log y + \log\left[1+\rho-e^{\phi_y(0,y)}\right] + o(1),\end{aligned} \tag{4.46}$$

which integrates to

$$\phi(x,y) = \phi(0,y) - x\log x + x + x\log y + x\log\left[1+\rho-e^{\phi_y(0,y)}\right] + o(x). \tag{4.47}$$

The above precisely agrees with the exponential terms in (4.44). Thus the expansions will match asymptotically, provided that $v = -1/2$ and

$$S_0(y) = \lim_{x\to 0}\left[\sqrt{2\pi x}\, K(x,y)\right].$$

By expanding $J = x_t y_s - x_s y_t$ and then K in (3.7) for $s \to s_*$ and $t \to t_{\max}$, we obtain after some calculation the expression in (3.12) for $S_0(y)$.

4.2 Boundary layer near $y = 0$

We consider small values of y, and return to the original discrete variable $r = y/\delta$, thus setting

$$\pi(k,r) = \Pi_r(x;\delta) \quad r \geqslant 1, \tag{4.48}$$

and

$$\pi(k,0) = (1-\rho)\rho^k + \Pi_0(x;\delta). \tag{4.49}$$

We assume for now that $0 < x < 1$, and will treat the corner regions, where $x \approx 0$ or $x \approx 1$, separately. Also, we expect that for large m and $\rho < 1$, the secondary servers will rarely be needed. Thus we expect that $\pi(k,r)$ will be mostly concentrated along $\pi(k,0)$, with this function being roughly geometric in ρ. Thus we wrote $\pi(k,0)$ in a form so as to estimate its deviation from a geometric distribution. We also note that $(1-\rho)\rho^k I_{\{r=0\}}$ is an exact solution to (2.3), and also satisfies the corner condition (2.5) at $(k,r) = (0,0)$. It fails to satisfy the full problem in (2.1) only because of the boundary condition at $k = m$.

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With (4.48) and (4.49), (2.3) becomes

$$(1+\rho)\,\Pi_r(x;\delta) = \rho\,\Pi_r(x-\delta;\delta) + \frac{x+\delta}{x+\delta(r+1)}\,\Pi_r(x+\delta;\delta) + \frac{\delta(r+1)}{x+\delta(r+1)}\,\Pi_{r+1}(x;\delta), \quad r>0. \tag{4.50}$$

$$(1+\rho)\,\Pi_0(x;\delta) = \rho\,\Pi_0(x-\delta;\delta) + \Pi_0(x+\delta;\delta) + \frac{\delta}{x+\delta}\,\Pi_1(x;\delta), \quad r=0. \tag{4.51}$$

We expand $\Pi_r(x;\delta)$ as

$$\Pi_r(x;\delta) = \Pi_r(x) + \delta\Pi_r^{(1)}(x) + O(\delta^2).$$

From (4.50) and (4.51) we obtain to leading order the following equations

$$x(1-\rho)\Pi_r' - r\Pi_r + (r+1)\Pi_{r+1} = 0, \tag{4.52}$$

$$x(1-\rho)\Pi_0' + \Pi_1 = 0. \tag{4.53}$$

Equations (4.52) and (4.53) express the 'current' value $\Pi_r(x)$ in terms of the 'future' value $\Pi_{r+1}(x)$. However, for sufficiently large r we can use the ray expansion to compute $\pi(k,r)$ asymptotically. Expanding $K(x,y)\exp[m\phi(x,y)]$ for $y\to 0$ and $0<x<1$ leads to

$$\begin{aligned}\pi(k,r) &= K(x,0)\exp\{m[\phi(x,0)+y\phi_y(x,0)+O(y^2)]\}\\ &\sim \frac{(1-\rho)^2}{1-\rho+\rho e^{-t}}\,\rho^m[e^{\phi_y(x,0)}]^r\\ &= \frac{(1-\rho)^2\rho^{m+r}(e^{-t})^r}{[1-\rho+\rho e^{-t}]^{r+1}}.\end{aligned} \tag{4.54}$$

Here we used the fact that $y=0$ implies that $s=0$, and thus (cf. (3.5)) $x(0,t)=e^{(\rho-1)t}$ and $e^{-t}=x^{1/(1-\rho)}$. The ray that starts from the corner $(1,0)$ is the line segment $y=0$, $0\leqslant x\leqslant 1$ and this ray reaches $(x,0)$ when $t=-(\log x)/(1-\rho)$. Therefore, from (4.54), as $y\to 0$ $\pi(k,r)$ becomes (3.14). We can easily check that (3.14) satisfies (4.52). Thus, we have the leading term $\Pi_r(x)$ for $r\geqslant 1$. To obtain $\Pi_0(x)$ we solve (4.53) with $\Pi_1(x)$ computed from the right-hand side of (3.14) with $r=1$. We thus obtain

$$\Pi_0(x) = \rho^m\left[\tilde{C} + \frac{(1-\rho)^2}{1-\rho+\rho x^{1/(1-\rho)}}\right],$$

where $\tilde{C}$ is a constant. As $\pi(0,0)=1-\rho$, we expect that $\Pi_0(0)=\rho^m[\tilde{C}+(1-\rho)]=0$, which implies that $\tilde{C}=-(1-\rho)$. Therefore, for $r=0$ we have (3.15). In Section 4.4 we will consider $\pi(k,r)$ for r and $k=O(1)$, and will give a more precise argument, based on asymptotic matching, to show that $\Pi_0(0)=0$.

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4.3 Corner layer near $(x, y) = (1, 0)$

We consider (x, y) near the corner $(1, 0)$. We again use the discrete variable r and consider $k = m - O(1)$ (or $1 - x = O(\delta)$). Thus we define n and L by

$$n = m - k, \quad \pi(k, r) = L(n, r; m).$$

The main balance equation (2.3), written in terms of (n, r), becomes

$$(1+\rho)L(n,r;m) = \rho L(n+1,r;m) + \frac{m-n+1}{m-n+1+r}L(n-1,r;m) + \frac{r+1}{m-n+1+r}L(n,r+1;m). \tag{4.55}$$

Here we used $\pi(k \pm 1, r) = L(n \mp 1, r; m)$. The artificial boundary condition (2.7) becomes

$$\frac{m+1}{m+r+1}L(-1,r;m) = \rho\, L(0,r-1;m), \quad r \geqslant 2, \tag{4.56}$$

and the corner condition in (2.6) is

$$(1+\rho)L(0,0;m) = \rho L(1,0;m) + \frac{1}{m+1}L(0,1;m). \tag{4.57}$$

By requiring (4.55) to also hold at $n = 0$, thus defining $L(-1, r; m)$, we can use (4.56) (or (2.7)) instead of (2.3). Then (4.57) may be replaced by

$$L(-1,0;m) = 0. \tag{4.58}$$

The condition at the other corner $(0, 0)$ corresponds asymptotically to $n = \infty$ and will play no role.

We expand L for $m \to \infty$ (or $\delta \to 0$) with

$$L(n,r;m) = L(n,r) + \frac{1}{m}L^{(1)}(n,r) + O(m^{-2}).$$

To leading order we obtain from (4.55)

$$(1+\rho)L(n,r) = \rho L(n+1,r) + L(n-1,r), \ r \geqslant 0, \tag{4.59}$$

and (4.56) and (4.58) lead to

$$L(-1,r) = \rho\, L(0,r-1), \quad r \geqslant 2, \tag{4.60}$$

$$L(-1,0) = 0. \tag{4.61}$$

We can infer the solution of (4.59)–(4.61) by expanding the ray solution as $(x, y) \to (1, 0)$. This can be obtained by simply letting $x \to 1$ in (3.14), which suggests that

$$L(n,r) = (1-\rho)^2\rho^{m+r}, \quad r \geqslant 1, \tag{4.62}$$

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which is independent of n. We can easily verify that (4.62) satisfies (4.59) and (4.60). To obtain $L(n,0)$ we let $x \to 1$ in (3.15) and recall that $k = m - n$. Hence,

$$L(n,0) = (1-\rho)(\rho^{-n} - \rho)\rho^m. \tag{4.63}$$

4.4 Corner layer near $(x,y) = (0,0)$

We consider near $(x,y) = (0,0)$ and go back to the original discrete variable (k,r), with $k, r = O(1)$. We also set

$$\pi(k,r) = \rho^m m^{-r/(1-\rho)} P(k,r;m), \quad r > 0, \tag{4.64}$$

and

$$\pi(k,0) = (1-\rho)\rho^k + \rho^m m^{-1/(1-\rho)} P(k,0;m), \quad r = 0. \tag{4.65}$$

We also require that this corner expansion match with the expansion in Section 4.2, in an intermediate limit where $k \to \infty$ but $x = k/m \to 0$. By expanding (3.14) and (3.15) as $x \to 0$, we see that $x^{r/(1-\rho)}$ becomes $O(m^{-r/(1-\rho)})$ on the k-scale. We thus scaled $\pi(k,r)$ to be $O(\rho^m m^{-r/(1-\rho)})$ expecting that $P(k,r;m)$ will be $O(1)$. Also note that $\pi(k,1)$ and $\pi(k,0) - (1-\rho)\rho^k$ are scaled to be of the same order in m. Then the balance equation (2.3) becomes

$$\begin{aligned}(1+\rho)P(k,r;m) &= \rho P(k-1,r;m) + \frac{k+1}{k+r+1} P(k+1,r;m) \\ &\quad + \frac{r+1}{k+r+1} m^{-1/(1-\rho)} P(k,r+1;m), \quad k > 0, \quad r > 0, \end{aligned} \tag{4.66}$$

$$\begin{aligned}(1+\rho)P(k,0;m) &= \rho P(k-1,0;m) + P(k+1,0;m) \\ &\quad + \frac{1}{k+1} P(k,1;m), \quad k > 0, \; r = 0. \end{aligned} \tag{4.67}$$

Expanding $P(k,r;m)$ as $P(k,r;m) = P(k,r) + o(1)$ we obtain from (4.66) and (4.67) the following equations for the leading term

$$(1+\rho)P(k,r) = \rho P(k-1,r) + \frac{k+1}{k+r+1} P(k+1,r), \quad r > 0, \tag{4.68}$$

$$(1+\rho)P(k,0) = \rho P(k-1,0) + P(k+1,0) + \frac{1}{k+1} P(k,1), \quad k > 0, \; r = 0. \tag{4.69}$$

Here we define $P(-1,r) = 0$ so that (4.68) holds for $k \geqslant 0$. However, we must now consider the corner condition (2.5). Since $\pi(0,0) = 1 - \rho$, (4.65) implies that $P(0,0) = 0$ and thus (2.5) asymptotically becomes

$$0 = P(1,0) + P(0,1). \tag{4.70}$$

This is also consistent with (4.69) at $k = 0$. We shall first analyse (4.68) for $r > 0$, and then solve (4.69) for $P(k,0)$.

For $r > 0$ we use the generating function $\xi(z) = \xi(z;r) \equiv \sum_{k=0}^{\infty} P(k,r) z^k$. Multiplying (4.68) by z^k and summing over $k \geqslant 0$ gives

$$\left[-\rho z^2 + (1+\rho)z - 1\right] \xi'(z) = [\rho(r+2)z - (1+\rho)(r+1)]\, \xi(z). \tag{4.71}$$

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The general solution to (4.71) is

$$\xi(z;r) = \xi_*(r)(1-z)^{-1-r/(1-\rho)}\left(\frac{\rho}{1-\rho z}\right)^{1-\rho r/(1-\rho)},$$

where $\xi_*(r)$ is a function of r. In particular,

$$\xi(0;r) = \xi_*(r)\rho^{1-\rho r/(1-\rho)} = P(0,r).$$

Inverting the generating function, we can write for $r > 0$,

$$P(k,r) = \xi_*(r)\frac{1}{2\pi i}\oint \frac{z^{-k-1}}{(1-z)^{1+r/(1-\rho)}}\left(\frac{\rho}{1-\rho z}\right)^{1-\rho r/(1-\rho)} dz, \tag{4.72}$$

where the integral is a complex contour integral along a small loop around $z = 0$. We obtain $\xi_*(r)$ by matching. Letting $x \to 0$ ($x = k/m$) in (3.14), and invoking the asymptotic matching condition shows that

$$P(k,r) \sim (1-\rho)\left(\frac{\rho}{1-\rho}\right)^r k^{r/(1-\rho)}, \quad k \to \infty. \tag{4.73}$$

Expanding (4.72) as $k \to \infty$ and noting that the singularity at $z = 1$ is closest to the origin, we find that (4.73) holds, provided that

$$\xi_*(r) = \frac{(1-\rho)^2}{\rho}\left(\frac{\rho}{1-\rho}\right)^{r/(1-\rho)}\Gamma\left(1+\frac{r}{1-\rho}\right).$$

It remains to compute $P(k,0)$, by solving (4.69) and (4.70). We use the generating function

$$\xi_0(z) \equiv \sum_{k=0}^{\infty} P(k,0)z^k,$$

multiply (4.69) by z^k, and sum over $k \geqslant 0$, to obtain

$$\xi_0(z) = \frac{1}{(1+\rho)z - \rho z^2 - 1}\sum_{k=0}^{\infty}\frac{z^{k+1}}{k+1}P(k,1),$$

and thus

$$P(k,0) = -\frac{1}{2\pi i}\oint \frac{z^{-k-1}}{(1-z)(1-\rho z)}\left[\sum_{j=0}^{\infty}\frac{z^{j+1}}{j+1}P(j,1)\right] dz. \tag{4.74}$$

Evaluating the contour integral leads to (3.17).

5 Asymptotic expansions for $m \to \infty$ with $\rho \uparrow 1$

In this section we study the heavy traffic case, in which $\rho \uparrow 1$, with $m \to \infty$. We introduce the parameter a, with

$$\rho = 1 - \frac{a}{m} = 1 - a\delta, \quad a = O(1). \tag{5.1}$$

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We shall again analyse (2.3) for different ranges of (k,r) (or (x,y)). For some of the ranges the analysis will closely parallel that of Section 4, which had $\rho < 1$. However, when $y \approx 0$ the analysis is much different, and we will show that the scalings $r = O(\sqrt{m})$ $(y = O(\sqrt{\delta}))$ and $r = O(1)$ $(y = O(\delta))$ will lead to very different types of asymptotics.

We first consider $0 < x \leqslant 1$ and $y > 0$. Using a ray expansion in the form (3.29), we ultimately obtain (3.30)–(3.35). We omit the details as the analysis parallels that in Section 4. For $k = O(1)$ we then construct a boundary layer correction near $x = 0$ to the ray expansion, and this gives (3.37)–(3.40). This analysis is similar to that in Section 4.1.

5.1 Boundary layer near $y = 0$

We consider small values of y and use the original discrete variable $r = y/\delta$. We set, for $0 < x < 1$,

$$\pi(k,r) = m^{\nu_1 + r/2} e^{-\sqrt{mr}\mathscr{H}(x)} \mathscr{Q}_r(x;\delta) \quad r \geqslant 1, \tag{5.2}$$

and

$$\pi(k,0) = (1-\rho)\rho^k + m^{\nu_1}\sqrt{m} e^{-\sqrt{m}\mathscr{H}(x)} \mathscr{Q}_0(x;\delta), \quad r = 0, \tag{5.3}$$

where $\mathscr{H}(x) > 0$. Omitting the details, we find that $\mathscr{H}(x)$ and $\mathscr{Q}_r$ satisfy the ordinary differential equations

$$[\mathscr{H}'(x)]^2 = \frac{1}{x}, \tag{5.4}$$

$$2\mathscr{H}'(x)\mathscr{Q}_r'(x) = \left[\frac{r}{x}\mathscr{H}'(x) - a\mathscr{H}'(x) - \mathscr{H}''(x)\right]\mathscr{Q}_r(x), \tag{5.5}$$

whose solutions are

$$\mathscr{H}(x) = 2\left(1 - \sqrt{x}\right); \quad \mathscr{Q}_r(x) = \mathscr{Q}_*(r) x^{r/2 + 1/4} e^{-ax/2}, \; r > 0. \tag{5.6}$$

By matching to the ray expansion (3.29), which corresponds to taking $s \to 0$, $t \to \infty$ with st fixed, we ultimately find that

$$\nu_1 = -1, \quad \mathscr{Q}_*(r) = \frac{a}{2} e^{-a/2} r^{-r/2}, \quad \mathscr{Q}_0(x) = -\mathscr{Q}_1(x),$$

which leads to (3.41) and (3.42).

5.2 Corner layer near $(x,y) = (0,0)$

We consider (x,y) near $(0,0)$ and go back to the original discrete variables (k,r), with $k, r = O(1)$. We also set

$$\pi(k,r) = m^{-5/4} e^{-2\sqrt{mr}} \mathscr{T}(k,r;\, m), \quad r > 0, \tag{5.7}$$

$$\pi(k,0) = (1-\rho)\rho^k + m^{-5/4} e^{-2\sqrt{m}} \mathscr{T}(k,0;\, m), \quad r = 0, \tag{5.8}$$

and note that $\mathscr{T}(0,0;\, m) = 0$. By analysing the difference equations for $\mathscr{T}(k,r;\, m)$ and using asymptotic matching to (3.29), (3.37), (3.41) and (3.42), we eventually obtain (3.44) and (3.45).

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5.3 Analysis near the corner $(x, y) = (1, 0)$: $1 - x, y = O(m^{-1/2})$

We examine the problem for $x = 1 - O(\sqrt{\delta})$ and $y = O(\sqrt{\delta})$, which corresponds to $k = m - O(\sqrt{m})$ and $r = O(\sqrt{m})$. We let

$$x = 1 - \xi\sqrt{\delta}, \quad y = R\sqrt{\delta}, \tag{5.9}$$

and set

$$\pi(k, r) = \delta^{3/2}\Omega(\xi, R;\ \delta) \sim \delta^{3/2}\Omega(\xi, R). \tag{5.10}$$

The scaling in (5.9) can be inferred by expanding (3.36) as $x = 1 - \xi\sqrt{\delta} \to 1$ and $y = R\sqrt{\delta} \to 0$. Then, from (2.3) and (2.7) we obtain the limiting PDE and a boundary condition

$$\Omega + \Omega_{\xi\xi} + R(\Omega_\xi + \Omega_R) = 0; \quad \xi,\ R > 0, \tag{5.11}$$

$$R\,\Omega + \Omega_\xi - \Omega_R = 0; \quad \xi = 0,\ R > 0. \tag{5.12}$$

We must thus solve a parabolic PDE in the quarter plane, subject to an oblique derivative boundary condition along $\xi = 0$. While we were not able to solve this problem exactly, we shall use asymptotic matching to infer various properties of $\Omega(\xi, R)$ as ξ and/or R become(s) large, or if $R \to 0$. We shall also obtain an integral equation for the boundary values $\Omega(0, R)$. We comment that the PDE in (5.11) is not separable due to the term $R\Omega_\xi$. When considering the analogous infinite server model (see [11]) Knessl obtained, in a certain heavy traffic limit, a problem very similar to (5.11) and (5.12). However, there the term $R\Omega_\xi$ was replaced by $\xi\Omega_\xi$ so that the PDE was separable.

By matching to the ray expansion (3.29) we can infer the behaviour of $\Omega(\xi, R)$ when ξ and R are simultaneously large, with $0 < \xi/R < \infty$. Defining W by

$$We^W = \frac{1}{2\sqrt{e}}\left(\frac{\xi}{R} - 1\right),$$

we find that

$$\Omega(\xi, R) \sim ae^{-a}\frac{(\xi - R)^2}{4R\sqrt{2(W+1)}W^2}\exp\left[-(\xi - R)^2\left(\frac{W + 1/2}{4W^2}\right)\right]. \tag{5.13}$$

For $\xi/R \gg 1$, (5.13) simplifies to

$$\Omega(\xi, R) \sim \left(\frac{ae^{-a}}{4\sqrt{2}}\right)\frac{\xi^2}{R}(\log(\xi/R))^{-5/2}\exp\left[-\frac{(\xi - R)^2}{4\log(\xi/R)}\right], \tag{5.14}$$

while for $\xi \to 0$, $R \to \infty$ with ξR fixed, we obtain, by analysing (5.11) and (5.12) asymptotically,

$$\Omega(\xi, R) = ae^{-a}e^{-\xi R}\left[R + \left(\frac{1}{R} + \xi - \frac{R\xi^2}{2}\right) + \left(-\frac{6}{R^3} - \frac{\xi}{R^2} - \frac{\xi^2}{2R} - \frac{\xi^3}{6} + \frac{R\xi^4}{8}\right) + O(R^{-5})\right]. \tag{5.15}$$

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Next we analyse (5.11) and (5.12) using a Laplace transform. We set $\Omega(\xi,R)=\mathscr{D}(\xi,R)/R$. Then we rewrite (5.11) and (5.12) as

$$\mathscr{D}_{\xi\xi}+R(\mathscr{D}_{\xi}+\mathscr{D}_{R})=0;\quad \xi,\ R>0, \tag{5.16}$$

$$\mathscr{D}_{\xi}-\mathscr{D}_{R}+\left(\frac{1}{R}+R\right)\mathscr{D}=0;\quad \xi=0,\quad R>0. \tag{5.17}$$

We assume that $\mathscr{D}\to 0$ as $R\to 0$ and use a double Laplace transform, with

$$\mathscr{U}(\alpha,\beta)\equiv\int_0^{\infty}\int_0^{\infty}\mathscr{D}(\xi,R)e^{-\alpha\xi}e^{-\beta R}\,d\xi\,dR. \tag{5.18}$$

Taking the Laplace transform of (5.16) gives

$$\alpha^2\mathscr{U}-(\alpha+\beta)\mathscr{U}_{\beta}-\mathscr{U}=\int_0^{\infty}e^{-\beta R}\left\{\alpha\mathscr{D}(0,R)+R\left[\frac{d}{dR}\left(\frac{\mathscr{D}(0,R)}{R}\right)\right]\right\}dR. \tag{5.19}$$

Here we also used (5.17) to eliminate $\mathscr{D}_{\xi}(0,R)$. Integrating by parts on the right-hand side and solving (5.19) we obtain

$$\begin{aligned}\mathscr{U}(\alpha,\beta)&=\frac{1}{\alpha+\beta}\int_{-\alpha}^{\beta}\int_0^{\infty}\left(\frac{\alpha+\beta}{\alpha+z}\right)^{\alpha^2}\left(\frac{1}{R}-\alpha-z\right)\mathscr{D}(0,R)e^{-zR}\,dR\,dz\\&=\int_{-1}^{0}\int_0^{\infty}\left(\frac{1}{1+\theta}\right)^{-\alpha^2}\left[\frac{1}{\chi}-(\alpha+\beta)(1+\theta)\right]\mathscr{D}(0,\chi)e^{-\beta\chi}e^{-(\alpha+\beta)\chi\theta}\,d\chi\,d\theta.\end{aligned} \tag{5.20}$$

Inverting the double transform in (5.18) we get

$$\begin{aligned}\mathscr{D}(\xi,R)&=\frac{1}{2\sqrt{\pi}}\int_R^{\infty}\frac{(\chi-R+\xi)^2}{4[\log(\chi/R)]^{5/2}}\frac{\mathscr{D}(0,\chi)}{\chi^2}\exp\left[-\frac{(\chi-R+\xi)^2}{4\log(\chi/R)}\right]d\chi\\&\quad-\frac{1}{2\sqrt{\pi}}\int_R^{\infty}\frac{d}{d\chi}\left\{\frac{\mathscr{D}(0,\chi)}{\chi}\exp\left[-\frac{(\chi-R+\xi)^2}{4\log(\chi/R)}\right]\right\}\frac{d\chi}{\sqrt{\log(\chi/R)}},\end{aligned} \tag{5.21}$$

which is equivalent to (3.47). Integrating the second integral by parts, we obtain an alternate expression for $\mathscr{D}(\xi,R)$ as

$$\begin{aligned}\mathscr{D}(\xi,R)&=\frac{1}{2\sqrt{\pi}}\int_R^{\infty}\left\{\frac{(\chi-R+\xi)^2}{4[\log(\chi/R)]^{5/2}}-\frac{1}{2[\log(\chi/R)]^{3/2}}\right\}\frac{\mathscr{D}(0,\chi)}{\chi^2}\\&\quad\times\exp\left[-\frac{(\chi-R+\xi)^2}{4\log(\chi/R)}\right]d\chi.\end{aligned} \tag{5.22}$$

Next we derive an integral equation for $\mathscr{D}(0,R)$ by letting $\xi\to 0$ in (3.47). If $\xi=0$, the second integral in (3.47) has a non-integrable singularity at the boundary $\chi=R$ due to the factor $[\log(\chi/R)]^{-3/2}$. If we set $\chi=Re^{v}$, the second integral becomes

$$\begin{aligned}&\frac{1}{2\sqrt{\pi}}\int_0^{\infty}\frac{\xi\mathscr{D}(0,Re^{v})}{2v^{3/2}}\exp\left[-\frac{R^2(e^{v}-1+\xi/R)^2}{4v}\right]dv\\&\quad=\frac{1}{4\sqrt{\pi}}\int_0^{\infty}\frac{\mathscr{D}(0,Re^{w\xi^2})}{w^{3/2}}\exp\left[-\frac{R^2(e^{w\xi^2}-1+\xi/R)^2}{4w\xi^2}\right]dw,\end{aligned} \tag{5.23}$$

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where $v = w\xi^2$. Then letting $\xi \downarrow 0$ in (5.23) we obtain

$$\frac{\mathscr{D}(0,R)}{4\sqrt{\pi}}\int_0^\infty w^{-3/2}e^{-1/(4w)}dw = \frac{\mathscr{D}(0,R)}{2\sqrt{\pi}}\int_0^\infty u^{-1/2}e^{-u}du = \frac{\mathscr{D}(0,R)}{2}. \tag{5.24}$$

Therefore, from (3.47) and (5.24) we obtain (3.48). Since $\mathscr{D}(0,R) = R\Omega(0,R)$, we have the following integral equation for $\Omega(0,R)$,

$$R\Omega(0,R) = \frac{1}{\sqrt{\pi}}\int_R^\infty \frac{1}{\sqrt{\log(\chi/R)}}\exp\left[-\frac{(\chi-R)^2}{4\log(\chi/R)}\right] \times \left[\frac{\chi-R}{2\log(\chi/R)}\,\Omega(0,\chi) - \Omega_\chi(0,\chi)\right]d\chi. \tag{5.25}$$

5.4 Analysis near the corner $(x,y) = (1,0)$: $x = 1 - O(m^{-1/4})$

Next we examine the behaviour of $\pi(k,r)$ in a region that is further away from the boundary of $x = 1$ than the ξ scale. In this range the asymptotics of $\pi(k,r)$ can be obtained by expanding the ray solution $\mathscr{P}(x,y) \sim \delta\mathscr{K}\exp(\Psi/\delta)$ as $(x,y) \to (0,0)$. We set

$$1 - x = \delta^{1/4}\eta, \quad y = R\sqrt{\delta},$$

and expand the ray solution $\mathscr{P}(x,y)$ as $s \to 0$ and $t \to \infty$. On the (η,R) scale the ray expansion simplifies considerably, and with this simplified form we will be able to relate this scale to the (ξ,R) scale in Section 5.3, as well as the (x,r) scale in Section 5.1. We obtain, after a lengthy calculation

$$\pi(k,r) \sim \frac{\delta a e^{-a}e^{T_0/2}}{\sqrt{1+e^{-T_0}}}\left(4e^{T_0} + 2 + \eta\sqrt{\frac{1+e^{-T_0}}{R}}\right)^{-1/2}\exp\left[\frac{\Psi^{(0)}}{\sqrt{\delta}} + \frac{\Psi^{(1)}}{\delta^{1/4}} + \Psi^{(2)}\right], \tag{5.26}$$

where

$$\Psi^{(0)} = -\frac{\eta}{2}\sqrt{\frac{R}{1+e^{-T_0}}} - R\log\left(1+e^{T_0}\right) + \frac{1}{2}\left(\frac{R}{1+e^{-T_0}}\right), \tag{5.27}$$

$$\Psi^{(1)} = -\frac{\eta^2}{4}\sqrt{\frac{R}{1+e^{-T_0}}} - \frac{\eta}{2}\left(\frac{R}{1+e^{-T_0}}\right) + R\sqrt{\frac{R}{1+e^{-T_0}}} - \frac{3}{4}\left(\frac{R}{1+e^{-T_0}}\right)^{3/2}, \tag{5.28}$$

and

$$\Psi^{(2)} = -\left(4e^{T_0} + 2 + \eta\sqrt{\frac{1+e^{-T_0}}{R}}\right)^{-1}\left[\frac{5}{32}\eta^4 + \eta^3\left(\frac{e^{T_0}}{2} + \frac{3}{4}\right)\sqrt{\frac{R}{1+e^{-T_0}}} + \eta^2\frac{R}{1+e^{-T_0}}\left(e^{T_0} + \frac{55}{48} - \frac{3}{4}e^{-T_0}\right) - \eta\left(\frac{R}{1+e^{-T_0}}\right)^{3/2}\left(\frac{2}{3}e^{T_0} + \frac{4}{3} + \frac{5}{2}e^{-T_0}\right) - \left(\frac{R}{1+e^{-T_0}}\right)^2\left(e^{T_0} + \frac{55}{32} + \frac{9}{4}e^{-T_0} - \frac{1}{2}e^{-2T_0}\right)\right]. \tag{5.29}$$

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Here T_0 is implicitly defined by

$$\eta\sqrt{\frac{1+e^{-T_0}}{R}} = 2T_0 - \frac{1}{2}\log\delta - \log R + \log\left(1+e^{-T_0}\right) - 1. \tag{5.30}$$

Note that T_0 depends on η and R, and also weakly upon m, due to the term $-\log\delta = \log m$.

We examine the asymptotic behaviour of (5.26) as $R \to 0$ with a fixed $\eta > 0$ (corresponding to $T_0 \to \infty$), which should match with (3.41) when it is expanded as $r \to \infty$. Letting $T_0 \to \infty$ (and $R \to 0$) in (5.30) we obtain

$$T_0 \sim \frac{1}{4}\log\delta + \frac{1}{2}\log R + \frac{1}{2}\left(\frac{\eta}{\sqrt{R}} + 1\right), \tag{5.31}$$

which when used in (5.26) gives, using $R = r/\sqrt{m}$,

$$\pi(k,r) \sim \frac{ae^{-a}}{2}\, m^{r/2-1} r^{-r/2} \exp\left(-\eta\sqrt{r}m^{1/4} - \frac{\eta^2\sqrt{r}}{4}\right). \tag{5.32}$$

Expanding (3.41) as $x \to 1$ and letting $x = 1 - \eta m^{-1/4}$ also lead to (5.32), which verifies the matching between the (η, R) and (x, r) scales.

We have also verified the matching between (5.26) and the expansion in Section 5.3 by considering the limit $T_0 \to -\infty$, $S_0 = \sqrt{R/(1+e^{-T_0})} \to 0$ with $S_0 T_0 \ll 1$.

5.5 Analysis near the corner $(x, y) = (1, 0)$: $r = O(1)$

We consider $r = O(1)$, which corresponds to there being only a few occupied secondary spaces, and $x \sim 1$. We shall discuss the scales $k = m - O(1)$, $k = m - O(\sqrt{m})$, $k = m - O(\sqrt{m\log m})$ and $k = m - O(\sqrt{m}\log m) = m - \sqrt{m}(\log m)\bar{\xi}$. To get an idea of the forms of the expansions for $\pi(k,r)$ on these scales, we first expand the results of Section 5.4, which apply on the (η, R) scale, in the limit $\eta \to 0$, $R \to 0$, and then rewrite the results in terms of $\bar{\xi}$ and r. We note that

$$1 - x = \delta^{1/4}\eta = \delta^{1/2}\xi = -\delta^{1/2}(\log\delta)\,\bar{\xi}, \quad R = \delta^{1/2} r, \tag{5.33}$$

which relates the variables x, η, $\bar{\xi}$ and ξ.

We rewrite (5.30), which defines T_0 as

$$\frac{\eta}{\sqrt{R}} = \frac{\xi}{\sqrt{r}} = -\frac{\bar{\xi}\log\delta}{\sqrt{r}} = \frac{1}{\sqrt{1+e^{-T_0}}}\left[2T_0 - \log\delta - \log r + \log\left(1+e^{-T_0}\right) - 1\right]. \tag{5.34}$$

Thus, if $T_0 = O(1)$,

$$\frac{\bar{\xi}}{\sqrt{r}} \sim \frac{1}{\sqrt{1+e^{-T_0}}}, \quad T_0 \sim -\log\left(\frac{r}{\bar{\xi}^2} - 1\right). \tag{5.35}$$

There is a singularity in this approximation to T_0 when $r = \bar{\xi}^2$, and this will lead to a transition in the asymptotics along $r = \bar{\xi}^2$. We also note that***

$$T_0 = 0 \text{ if } r = 2\bar{\xi}^2, \quad T_0 < 0 \text{ if } r > 2\bar{\xi}^2, \quad T_0 > 0 \text{ if } \bar{\xi}^2 < r < 2\bar{\xi}^2,$$

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and (5.34) implies that if $r < \bar{\xi}^2$, $T_0 \to \infty$ with

$$T_0 \sim -\frac{\log \delta}{2}\left(\frac{\bar{\xi}}{\sqrt{r}} - 1\right) = \frac{\log m}{2}\left(\frac{\bar{\xi}}{\sqrt{r}} - 1\right). \tag{5.36}$$

We first consider the case $r < \bar{\xi}^2$. Letting $\eta = m^{-1/4}(\log m)\bar{\xi}$, $R = m^{-1/2}r$ and $T_0 \to \infty$ in (5.26) yields

$$\begin{aligned}\pi(k,r) &\sim \delta \frac{ae^{-a}}{2} \exp\left[-\bar{\xi}\sqrt{r}\log m + \frac{r}{2}\log m - \frac{r\log r}{2}\right] \\ &= \frac{ae^{-a}}{2} m^{r/2-\bar{\xi}\sqrt{r}-1} r^{-r/2}, \quad \bar{\xi} > \sqrt{r}. \end{aligned} \tag{5.37}$$

We note that the expression in (5.37) is algebraically small in m, and becomes $O(m^{-r/2-1})$ when $\bar{\xi} = \sqrt{r}$. It is easy to see that (5.37) is precisely the expansion of (5.32) as $\eta \to 0$ ($\eta = m^{-1/4}(\log m)\,\bar{\xi}$) and thus also matches the boundary layer solution (3.41), when it is expanded as $x \to 1$.

When $r > \bar{\xi}^2$ we refine (5.35) by setting

$$T_0 = -\log\left(\frac{r}{\bar{\xi}^2} - 1\right) + \frac{T_1}{\log m}. \tag{5.38}$$

Using (5.38) in (5.34) we obtain

$$T_1 \sim \frac{4\log(r/\bar{\xi} - \bar{\xi}) + 2}{1 - \bar{\xi}^2/r}, \tag{5.39}$$

and thus

$$T_0 \sim -\log\left(\frac{r}{\bar{\xi}^2} - 1\right) + \frac{4\log(r/\bar{\xi} - \bar{\xi}) + 2}{(1 - \bar{\xi}^2/r)\log m}. \tag{5.40}$$

Using (5.40) in (5.26) leads to

$$\pi(k,r) \sim ae^{-a} \frac{m^{-\bar{\xi}^2/2}}{m\sqrt{\log m}} \frac{\bar{\xi}^2}{\sqrt{r(r-\bar{\xi}^2)}} \left(\frac{\bar{\xi}}{r - \bar{\xi}^2}\right)^{\bar{\xi}^2} \left(\frac{r - \bar{\xi}^2}{r}\right)^r, \quad \bar{\xi} < \sqrt{r}. \tag{5.41}$$

Next we examine the behaviour of (5.26) in the transition region where $\bar{\xi} \sim \sqrt{r}$. We set

$$T_0 = \log\log m + \log \Lambda, \tag{5.42}$$

and

$$\bar{\xi} = \sqrt{r} + \frac{\kappa}{\log m} \tag{5.43}$$

for Λ, $\kappa \ll \log m$. Then we obtain from (5.34)

$$\kappa \sim 2\sqrt{r}\log\log m, \tag{5.44}$$

and defining $\bar{\xi}^*$ by

$$\bar{\xi} = \sqrt{r} + 2\sqrt{r}\left(\frac{\log\log m}{\log m}\right) + \frac{\bar{\xi}^*}{\log m} \tag{5.45}$$

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we find that (5.34) becomes

$$\frac{\bar{\xi}^*}{\sqrt{r}} = 2\log \Lambda - \frac{1}{2\Lambda} - \log r - 1 + o(1). \tag{5.46}$$

This defines Λ implicitly in terms of $\bar{\xi}^*$ and r. Then on the $(\bar{\xi}^*, r)$ scale we obtain from (5.26)

$$\pi(k,r) \sim a e^{-a} \sqrt{\frac{\Lambda}{4\Lambda+1}} m^{-r/2-1} (\log m)^{-2r} r^{-r/2} e^{-\bar{\xi}^*\sqrt{r}}. \tag{5.47}$$

Next we examine $\pi(k,r)$ on the $(\bar{\xi}, r)$ scale by analysing the balance equation (2.3). We set

$$\pi \sim \frac{1}{m\sqrt{\log m}} \bar{K}(\bar{\xi}, r) \exp[f(\bar{\xi}) \log m], \tag{5.48}$$

and rewrite (2.3) as

$$\begin{aligned}
&\left(2 - \frac{a}{m}\right) \bar{K}(\bar{\xi}, r) e^{f(\bar{\xi})\log m} \\
&= \left(1 - \frac{a}{m}\right) \bar{K}\left(\bar{\xi} + \frac{1}{\sqrt{m}\log m}, r\right) e^{f(\bar{\xi}+1/(\sqrt{m}\log m))\log m} \\
&\quad + \frac{m - \bar{\xi}\sqrt{m}\log m + 1}{m - \bar{\xi}\sqrt{m}\log m + r + 1} \bar{K}\left(\bar{\xi} - \frac{1}{\sqrt{m}\log m}, r\right) e^{f(\bar{\xi}-1/(\sqrt{m}\log m))\log m} \\
&\quad + \frac{r+1}{m - \bar{\xi}\sqrt{m}\log m + r + 1} \bar{K}(\bar{\xi}, r+1) e^{f(\bar{\xi})\log m}.
\end{aligned} \tag{5.49}$$

From (5.49) to leading order ($O(1/m)$) we obtain the limiting equation

$$-\bar{K}(\bar{\xi}, r)[f'(\bar{\xi})]^2 + r\bar{K}(\bar{\xi}, r) - (r+1)\bar{K}(\bar{\xi}, r+1) = 0. \tag{5.50}$$

By matching (5.48) to (5.41), when $r > \bar{\xi}^2$ we must have $f(\bar{\xi}) = -\bar{\xi}^2/2$. Using this in (5.50) we obtain

$$\frac{\bar{K}(\bar{\xi}, r+1)}{\bar{K}(\bar{\xi}, r)} = \frac{r - \bar{\xi}^2}{r+1}. \tag{5.51}$$

Solving (5.51) gives

$$\bar{K}(\bar{\xi}, r) = \left[\prod_{j=1}^{r-1} (j - \bar{\xi}^2)\right] \frac{\bar{K}(\bar{\xi}, 1)}{r!} = \frac{\Gamma(r - \bar{\xi}^2)}{\Gamma(1 - \bar{\xi}^2)} \frac{\bar{K}(\bar{\xi}, 1)}{r!}, \tag{5.52}$$

and thus

$$\pi(k,r) \sim \frac{m^{-\bar{\xi}^2/2}}{m\sqrt{\log m}} \frac{\Gamma(r - \bar{\xi}^2)}{r!} \bar{H}(\bar{\xi}), \quad \bar{H}(\bar{\xi}) = \frac{\bar{K}(\bar{\xi}, 1)}{\Gamma(1 - \bar{\xi}^2)}. \tag{5.53}$$

We see that as $\bar{\xi} \uparrow \sqrt{r}$, the factor $\Gamma(r - \bar{\xi}^2)$ becomes singular, which indicates a transition in the asymptotics. Analysis of the range $\bar{\xi} \sim \sqrt{r}$ will also determine $\bar{H}(\bar{\xi})$ in (5.53), and thus $\bar{K}(\bar{\xi}, 1)$. To study the transition we scale $\bar{\xi} - \sqrt{r} = O[(\log m)^{-1/2}]$ with

$$\bar{\xi} = \sqrt{r} + \frac{\circledast}{\sqrt{\log m}}, \quad \pi(k,r) = \mathscr{H}_r(\circledast; m).$$

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Then (2.3) becomes

$$\left(2-\frac{a}{m}\right)\ \mathscr{H}_r(\circledast;m)=\left[1-\frac{r}{m}+O\left(\frac{\log m}{m\sqrt{m}}\right)\right]\ \mathscr{H}_r\left(\circledast-\frac{1}{\sqrt{m\log m}};m\right)$$
$$+\left[\frac{r+1}{m}+O\left(\frac{\log m}{m\sqrt{m}}\right)\right]\ \mathscr{H}_{r+1}(\circledast;m)+\left(1-\frac{a}{m}\right)\mathscr{H}_r\left(\circledast+\frac{1}{\sqrt{m\log m}};m\right),\quad(5.54)$$

which we further approximate by

$$(r+1)\mathscr{H}_{r+1}(\circledast;m)-r\mathscr{H}_r(\circledast;m)+\frac{1}{\log m}\mathscr{H}_r''(\circledast;m)+o\left(\frac{1}{\log m}\right)=0.\quad(5.55)$$

Before analysing (5.55) we derive matching conditions for $\mathscr{H}_r(\circledast;m)$ as $\circledast\to\pm\infty$. On the $(\bar{\xi},r)$ scale (5.37) applies for $\bar{\xi}>\sqrt{r}$. This can also be obtained by analysing (2.3) on the $(\bar{\xi},r)$ scale for $\bar{\xi}>\sqrt{r}$ and using asymptotic matching. In either case we ultimately conclude that (3.41) remains valid for $x\to 1$ as long as $\bar{\xi}=(1-x)\sqrt{m}/\log m>\sqrt{r}$. By setting $\bar{\xi}=\sqrt{r}+\circledast/\sqrt{\log m}$ in (5.37) we obtain

$$\frac{ae^{-a}}{2}m^{-r/2-1}r^{-r/2}e^{-\circledast\sqrt{r\log m}}\,[1+o(1)]$$

so that as $\circledast\to+\infty$ the matching condition for $\mathscr{H}_r(\circledast;m)$ is

$$\mathscr{H}_r(\circledast;m)\sim\frac{ae^{-a}}{2}m^{-r/2-1}r^{-r/2}e^{-\circledast\sqrt{r\log m}},\quad\circledast\to+\infty.\quad(5.56)$$

By examining (5.53) as $\bar{\xi}\to\sqrt{r}$ and using

$$\Gamma(r-\bar{\xi}^2)\sim\frac{1}{\sqrt{r}-\bar{\xi}}\cdot\frac{1}{\sqrt{r}+\bar{\xi}}=-\frac{\sqrt{\log m}}{\circledast}\left(2\sqrt{r}+\frac{\circledast}{\sqrt{\log m}}\right)^{-1}\sim-\frac{\sqrt{\log m}}{2\circledast\sqrt{r}}\quad(5.57)$$

we obtain from (5.53)

$$\mathscr{H}_r(\circledast;m)\sim m^{-r/2-1}e^{-\circledast\sqrt{r\log m}}\frac{e^{-\circledast^2/2}}{-2\circledast}\frac{\bar{H}(\sqrt{r})}{r!\sqrt{r}},\quad\circledast\to-\infty.\quad(5.58)$$

In view of the matching conditions that we set

$$\mathscr{H}_r(\circledast;m)=m^{-r/2-1}e^{-\circledast\sqrt{r\log m}}[\hbar_r(\circledast)+o(1)].\quad(5.59)$$

With (5.59) we see that

$$\mathscr{H}_r''(\circledast;m)-(\log m)r\mathscr{H}_r'(\circledast;m)\sim m^{-r/2-1}e^{-\circledast\sqrt{r\log m}}(-2)\sqrt{r\log m}\,\hbar_r(\circledast).\quad(5.60)$$

Also, since $\bar{\xi}\sim\sqrt{r}$, we have $\bar{\xi}<\sqrt{r+1}$ and thus (5.53) may be used to approximate $\mathscr{H}_{r+1}(\circledast;m)$ in (5.55), as

$$\mathscr{H}_{r+1}(\circledast;m)\sim\frac{m^{-r/2-1}}{\sqrt{\log m}}e^{-\circledast\sqrt{r\log m}}e^{-\circledast^2/2}\frac{\bar{H}(\sqrt{r})}{(r+1)!}.\quad(5.61)$$

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Using (5.60) and (5.61) in (5.55) we obtain to leading order

$$2\sqrt{r}\hbar_r'(\circledast) = \frac{e^{-\circledast^2/2}}{r!}\bar{H}(\sqrt{r}). \tag{5.62}$$

The solution to (5.62) that decays as $\circledast \to -\infty$ is

$$\hbar_r(\circledast) = \frac{\bar{H}(\sqrt{r})}{2r!\sqrt{r}}\int_{-\infty}^{\circledast} e^{-u^2/2}du. \tag{5.63}$$

To determine $\bar{H}(\sqrt{r})$ we use the matching condition in (5.56). Letting $\circledast \to \infty$, we evaluate the integral in (5.63) as $\sqrt{2\pi}$ and rewrite (5.59), for $\circledast \to \infty$, as

$$\mathscr{H}_r(\circledast;m) \sim m^{-r/2-1}e^{-\circledast\sqrt{r\log m}}\frac{\bar{H}(\sqrt{r})}{r!\sqrt{r}}\sqrt{\frac{\pi}{2}}. \tag{5.64}$$

Comparing (5.56) with (5.64) we obtain

$$\bar{H}(\sqrt{r}) = \frac{ae^{-a}}{\sqrt{2\pi}}r^{-r/2}\sqrt{r}\,\Gamma(r+1), \tag{5.65}$$

so that

$$\bar{H}(z) = \frac{ae^{-a}}{\sqrt{2\pi}}z^{-z^2}z\Gamma(1+z^2). \tag{5.66}$$

Thus on the $(\circledast, r)$ transition scale we have

$$\pi(k,r) = \mathscr{H}_r(\circledast;m) \sim \frac{ae^{-a}}{2\sqrt{2\pi}}m^{-r/2-1}r^{-r/2}e^{-\circledast\sqrt{r\log m}}\int_{-\infty}^{\circledast} e^{-u^2/2}du, \quad r \geqslant 1, \tag{5.67}$$

and when $\bar{\xi} < \sqrt{r}$, from (5.53) and (5.66), we obtain

$$\pi(k,r) \sim \frac{ae^{-a}}{\sqrt{2\pi}}\frac{m^{-\bar{\xi}^2/2}}{m\sqrt{\log m}}\frac{\Gamma(1+\bar{\xi}^2)\Gamma(r-\bar{\xi}^2)}{r!}\,\bar{\xi}^{-\bar{\xi}^2}\bar{\xi}, \quad r \geqslant 1. \tag{5.68}$$

We note that letting $\bar{\xi}, r \to \infty$ in (5.68) and using Stirling's formula for the Gamma functions lead to (5.41). This verifies the matching between the $(\bar{\xi}, r)$ scale and the ray expansion on the (x, y) scale.

We next examine $r = 0$. For $\bar{\xi} > 1$ we can obtain the expansion of $\pi(k,0)$ as a limiting case of (3.42), which leads to

$$\pi(k,0) - (1-\rho)\rho^k \sim -\frac{ae^{-a}}{2}m^{-1/2-\bar{\xi}}, \quad \bar{\xi} > 1. \tag{5.69}$$

For $0 < \bar{\xi} < 1$ we find that (5.50) still holds at $r = 0$, if we write

$$\pi(k,0) - (1-\rho)\rho^k \sim \frac{m^{-\bar{\xi}^2/2}}{m\sqrt{\log m}}\bar{K}(\bar{\xi},0).$$

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Hence,

$$\pi(k,0)-(1-\rho)\rho^k \sim \frac{ae^{-a}}{\sqrt{2\pi}}\frac{m^{-\bar{\xi}^2/2}}{m\sqrt{\log m}}\bar{\xi}^{1-\bar{\xi}^2}\Gamma(1+\bar{\xi}^2)\Gamma(-\bar{\xi}^2),\quad 0<\bar{\xi}<1. \tag{5.70}$$

We note that $\pi(k,0)-(1-\rho)\rho^k$ is negative because $\Gamma(-\bar{\xi}^2)<0$. There is a transition in the asymptotics when $\bar{\xi}=1$, and (5.70) also breaks down as $\bar{\xi}\to 0$, due to the singularities of $\Gamma(-\bar{\xi}^2)$. We also note that on the $\bar{\xi}$ scale

$$(1-\rho)\rho^k=\frac{ae^{-a}}{m}\left[1+\frac{a\log m}{\sqrt{m}}\bar{\xi}+O\left(\frac{\log^2 m}{m}\right)\right]=O(m^{-1}), \tag{5.71}$$

so that the geometric part of $\pi(k,0)$ is still asymptotically dominant for $\bar{\xi}>0$.

To study the transition range $\bar{\xi}\sim 1$ we let $\pi(k,0)-(1-\rho)\rho^k\sim \mathscr{H}_0(\circledast;m)$ where now $\circledast=\sqrt{\log m}(\bar{\xi}-1)$. Note that the transition point is the same for $\pi(k,0)$ and $\pi(k,1)$. By examining the matching condition to (5.69) as $\circledast\to+\infty$ and to (5.70) as $\circledast\to-\infty$, we again set

$$\mathscr{H}_0(\circledast;m)\sim m^{-3/2}e^{-\circledast\sqrt{\log m}}\hbar_0(\circledast), \tag{5.72}$$

and from (5.55) with $r=0$ we find that $(\log m)^{-1}\mathscr{H}_0''(\circledast;m)\sim-\mathscr{H}_1(\circledast;m)$ so that $\hbar_0(\circledast)=-\hbar_1(\circledast)$ and hence

$$\pi(k,0)-(1-\rho)\rho^k\sim-\frac{ae^{-a}}{2\sqrt{2\pi}}m^{-3/2}e^{-\circledast\sqrt{\log m}}\int_{-\infty}^{\circledast}e^{-u^2/2}du,\quad \circledast=\sqrt{\log m}(\bar{\xi}-1). \tag{5.73}$$

Now we consider scales that have $\bar{\xi}\to 0$, so that $m-k=o(\sqrt{m}\log m)$. If we define $\tilde{\xi}$ by

$$m-k=\sqrt{m\log m}\,\tilde{\xi},\quad \tilde{\xi}=\sqrt{\log m}\,\bar{\xi},$$

then the factor $m^{-\bar{\xi}^2/2}=\exp(-\bar{\xi}^2\log m/2)=e^{-\tilde{\xi}^2/2}$ becomes $O(1)$ and (5.53) simplifies to

$$\pi(k,r)\sim\frac{1}{m\log m}\frac{ae^{-a}}{\sqrt{2\pi}}\frac{\tilde{\xi}}{r}e^{-\tilde{\xi}^2/2},\quad r\geqslant 1. \tag{5.74}$$

Here we used $\Gamma(r-\bar{\xi}^2)\sim(r-1)!$ and $\bar{H}(\bar{\xi})\sim ae^{-a}\bar{\xi}/\sqrt{2\pi}$ as $\bar{\xi}\to 0$. We have verified, after a lengthy calculation which we omit that (5.74) can also be obtained by expanding $m^{-3/2}\Omega(\xi,R)$ for $\xi\to\infty$, $R\to 0$ with $\xi=O(\sqrt{-\log R})$, using the integral in (3.47) and our knowledge of $\mathscr{D}(0,R)$ (or $\Omega(0,R)$) as $R\to\infty$. If we expand $\pi(k,0)$ on the $\bar{\xi}$ scale for $\bar{\xi}\to 0$, we obtain from (5.70)

$$\pi(k,0)\sim ae^{-a}\left[\frac{1}{m}-\frac{1}{m}\frac{e^{-\tilde{\xi}^2/2}}{\sqrt{2\pi}\tilde{\xi}}\right] \tag{5.75}$$

so that on the $\tilde{\xi}$ scale the geometric part of $\pi(k,0)$ no longer dominates the right-hand side of (5.70). We thus re-examine the balance equation (2.3) along $r=0$, i.e.

$$\left(2-\frac{a}{m}\right)\pi(k,0)=\left(1-\frac{a}{m}\right)\pi(k-1,0)+\pi(k+1,0)+\frac{1}{k+1}\pi(k,1). \tag{5.76}$$

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On the $\tilde{\xi}$ scale we let $\pi(k,0) \sim m^{-1}\mathcal{G}(\tilde{\xi})$ and use (5.74) to approximate $\pi(k,1)$. Then (5.76) asymptotically becomes (after we multiply by $m^2 \log m$)

$$-\mathcal{G}''(\tilde{\xi}) \sim \frac{ae^{-a}}{\sqrt{2\pi}}\,\tilde{\xi} e^{-\tilde{\xi}^2/2}. \tag{5.77}$$

The solution of (5.77) that matches, as $\tilde{\xi} \to \infty$, to (5.70) is given by

$$\mathcal{G}(\tilde{\xi}) = ae^{-a}\left[1 - \frac{1}{\sqrt{2\pi}}\int_{\tilde{\xi}}^{\infty} e^{-u^2/2}du\right] \tag{5.78}$$

so that on the $\tilde{\xi}$ scale with $r = 0$ we have

$$\pi(k,0) \sim \frac{ae^{-a}}{m}\left[1 - \frac{1}{\sqrt{2\pi}}\int_{\tilde{\xi}}^{\infty} e^{-u^2/2}du\right]. \tag{5.79}$$

For $\tilde{\xi} \to \infty$ (5.79) reduces to (5.75). This shows also that on the $\tilde{\xi}$ scale ($k = m - O(\sqrt{m \log m})$) $\pi(k,r)$ ($r \geqslant 1$) is smaller than $\pi(k,0)$ by a factor of $(\log m)^{-1}$.

Next we discuss the problem on the (n,r) scale, where $k = m - n$ and $n = O(1)$. Setting $\pi(k,r) = \mathcal{L}(n,r;\ m)$ we write (2.3) and (2.4) as

$$\left(2 - \frac{a}{m}\right)\mathcal{L}(n,r;\ m) = \left(1 - \frac{a}{m}\right)\mathcal{L}(n+1,r;\ m) + \frac{m-n+1}{m+r-n+1}\mathcal{L}(n-1,r;\ m) + \frac{r+1}{m+r-n+1}\mathcal{L}(n,r+1;\ m), \quad n \geqslant 1, \tag{5.80}$$

and

$$\left(2 - \frac{a}{m}\right)\mathcal{L}(0,r;\ m) = \left(1 - \frac{a}{m}\right)\mathcal{L}(1,r;\ m) + \frac{r+1}{m+r+1}\mathcal{L}(0,r+1;m) + \left(1 - \frac{a}{m}\right)\mathcal{L}(0,r-1;\ m), \quad r \geqslant 1. \tag{5.81}$$

The corner condition (2.6) becomes

$$\left(2 - \frac{1}{m}\right)\mathcal{L}(0,0;\ m) = \left(1 - \frac{1}{m}\right)\mathcal{L}(1,0;\ m) + \frac{1}{m+1}\mathcal{L}(0,1;\ m). \tag{5.82}$$

We expand $\mathcal{L}$ as

$$\mathcal{L}(n,r;\ m) = \frac{1}{m^{3/2}\sqrt{\log m}}\left[\mathcal{L}(n,r) + \frac{1}{\log m}\mathcal{L}^{(1)}(n,r) + O(\log^{-2} m)\right]. \tag{5.83}$$

The scale factor $m^{-3/2}(\log m)^{-1/2}$ must be included in view of matching considerations, which we discuss shortly. Using (5.83) in (5.80)–(5.82) and solving for $\mathcal{L}$ and $\mathcal{L}^{(1)}$ leads

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to

$$\mathscr{L}(n,r;\ m) \sim \frac{1}{m^{3/2}\sqrt{\log m}} \left\{ \mathscr{V}(0) + \sum_{l=1}^{r} \mathscr{W}(l) + n\mathscr{W}(r) + \frac{1}{\log m} \left[\mathscr{V}^{(1)}(0) + \sum_{l=1}^{r} \mathscr{W}^{(1)}(l) + n\mathscr{W}^{(1)}(r) \right] \right\} \tag{5.84}$$

for $r \geqslant 1$, and

$$\mathscr{L}(n,0;\ m) \sim \frac{n+1}{m^{3/2}\sqrt{\log m}} \left[\mathscr{V}(0) + \frac{1}{\log m} \mathscr{V}^{(1)}(0) \right]. \tag{5.85}$$

Here $\mathscr{V}(0)$ and $\mathscr{W}(r)$ must be determined by asymptotic matching, as they cannot be determined by only analysing the (n,r) scale.

Now we try to match (5.84) and (5.85) to (5.74) and (5.79), noting that $n = \tilde{\xi}\sqrt{m \log m}$. On the $\tilde{\xi}$ scale $\mathscr{L}(n,0)$ in (5.85) becomes $O(m^{-1})$, which is of the same order as (5.79), but the expansions cannot match because (5.85) will be linear in n (or $\tilde{\xi}$), while (5.79) does not vanish as $\tilde{\xi} \to 0$. Problems also arise in matching the n and $\tilde{\xi}$ scales for $r \geqslant 1$. For a fixed r and $n \to \infty$, the leading term in (5.84) becomes $m^{-3/2}(\log m)^{-1/2} n\mathscr{W}(r) = m^{-1}\tilde{\xi}\mathscr{W}(r)$, which is larger than (5.74) by a factor of $\log m$. Thus, we must set $\mathscr{W}(r) = 0$. We can match (5.74) to the correction term in (5.84) by setting $\mathscr{W}^{(1)}(r) = ae^{-a}/(\sqrt{2\pi}r)$. Then we would have, for $r \geqslant 1$,

$$\mathscr{L}(n,r;\ m) \sim \frac{1}{m^{3/2}\sqrt{\log m}} \left\{ \mathscr{V}(0) + \frac{1}{\log m} \left[\left(\frac{n}{r} + \sum_{l=1}^{r} \frac{1}{l} \right) \frac{ae^{-a}}{\sqrt{2\pi}} + \mathscr{V}^{(1)}(0) \right] \right\}, \tag{5.86}$$

and

$$\mathscr{L}(n,0;\ m) \sim \frac{n+1}{m^{3/2}\sqrt{\log m}} \left[\mathscr{V}(0) + \frac{1}{\log m} \mathscr{V}^{(1)}(0) \right]. \tag{5.87}$$

We have verified by numerical computations that the order of magnitude of $\pi(k,r)$ on the (n,r) scale does seem to be $O(m^{-3/2}(\log m)^{-1/2})$, that $\pi(k,0)$ is approximately proportional to $n+1$, and that $\pi(k,r)$ is approximately constant for $r \geqslant 1$ and $k = m - O(1)$. However, the problems with the matching suggest that there is yet another scale in the problem, which corresponds to $n \to \infty$ and $\tilde{\xi} \to 0$. We have not been able to identify this new scale.

To summarise this subsection, we obtained results for $\pi(k,r)$ on the $(\bar{\xi},r)$ scale, treating separately the cases $\bar{\xi} > \sqrt{r}$, $\bar{\xi} \sim \sqrt{r}$ and $\bar{\xi} < \sqrt{r}$, for $r \geqslant 1$. For $r = 0$ we gave results for $\bar{\xi} > 1$, $\bar{\xi} \sim 1$ and $\bar{\xi} < 1$. For $\tilde{\xi} = \bar{\xi}\sqrt{\log m} = O(1)$ we obtained the simplified result in (5.74) for $\pi(k,r)$ for $r \geqslant 1$. For $\pi(k,0)$, (5.79) applies on the $\tilde{\xi}$ scale. On the (n,r) scale we obtained (5.86) for $r \geqslant 1$ and (5.87) for $r = 0$. However, there is still a 'gap' in the asymptotics between the n and $\tilde{\xi}$ scale, a gap which we have not been able to fill.

6 Numerical studies

We assess the accuracy of some of the asymptotic formulas that we obtained.

In Tables 1 and 2 we test our asymptotic results for $m \to \infty$ with a fixed $\rho < 1$. Table 1 has $\pi(0,1)$, where (3.16) applies with $k = 0$ and $r = 1$, $\pi(m/2,1)$, where (3.14) applies with $x = 1/2$ and $r = 1$, and $\pi(m,0)$, where (3.20) applies with $n = 0$. We consider $\rho = 0.5$ and

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Table 1. *Numerical comparisons for $r = 0, 1$.*
$\rho = 0.5$; $(k,r) = (0,1),\ (m/2,1),\ (m,0)$; $10 \leqslant m \leqslant 30$.

(k,r)	m	Exact	Asymptotic
$(0,1)$	10	5.83×10^{-6}	9.76×10^{-6}
	20	1.78×10^{-9}	2.38×10^{-9}
	30	8.43×10^{-13}	1.03×10^{-12}
$(m/2,1)$	10	8.57×10^{-5}	7.81×10^{-5}
	20	7.92×10^{-8}	7.62×10^{-8}
	30	7.61×10^{-11}	7.45×10^{-11}
$(m,0)$	10	2.71×10^{-4}	2.44×10^{-4}
	20	2.51×10^{-7}	2.38×10^{-7}
	30	2.41×10^{-10}	2.32×10^{-10}

Table 2. *Numerical comparisons for $r = m$.*
$\rho = 0.5$; $(k,r) = (0,m),\ (m/2,m),\ (m,m)$; $10 \leqslant m \leqslant 30$.

(k,r)	m	Exact	Asymptotic
$(0,m)$	10	7.59×10^{-13}	7.40×10^{-13}
	20	3.64×10^{-25}	3.59×10^{-25}
	30	1.52×10^{-37}	1.51×10^{-37}
$(m/2,m)$	10	4.86×10^{-9}	4.81×10^{-9}
	20	4.93×10^{-17}	4.90×10^{-17}
	30	5.02×10^{-25}	5.00×10^{-25}
(m,m)	10	3.67×10^{-7}	3.57×10^{-7}
	20	3.45×10^{-13}	3.41×10^{-13}
	30	3.28×10^{-19}	3.25×10^{-19}

increase m from 10 to 30. The agreement is not particularly good for $(k,r) = (0,1)$ but does improve significantly as m increases. For $(k,r) = (m/2,1)$ (boundary layer near $y = 0$ in Section 4.2) and $(k,r) = (m,0)$ (corner layer in Section 4.3) the agreement is good even for $m = 10$, with errors of at most 10%. In Table 2, we consider $(k,r) = (0,m)$, $(m/2,m)$, (m,m), so that $y = 1$ and $x = 0,\ 1/2,\ 1$. The asymptotic formulas that apply are now (3.9) (with (3.10)–(3.13)), (3.1) (with (3.2)– (3.7)) and (3.8) (with $x = y = 1$). We again consider $\rho = 0.5$ and $m = 10,\ 20,\ 30$. Now we obtain generally excellent agreement, with the worst error in Table 2 being about 3%, which occurs for $k = 0$ and $r = m = 10$.

These comparisons show that the asymptotics agree reasonably well with the exact numerical values of $\pi(k,r)$. The comparisons also clearly demonstrate the necessity of analysing separately the different ranges of (k,r), when $m \to \infty$.

7 Discussion

While the preceding analysis was quite technical, we can infer from it several simple qualitative insights into the behaviour of the model, some of which are quite surprising.

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We first consider the case $\rho < 1$. Now secondary spaces will rarely be needed, and $\pi(k,r)$ will be uniformly exponentially small for $r \geqslant 1$. Expression (3.1) applies both for $r = my = O(m)$ and $r = O(1)$ (reducing then to (3.14)), and gives an accurate measure of this smallness. The ray method shows how the joint scaled process $m^{-1}(N_1(t), N_2(t)) \equiv (X(t), Y(t))$ attains a value (x_0, y_0) with $0 \leqslant x_0 < 1$ and $y_0 > 0$. The most likely state of the scaled process is $(0,0)$, and to achieve (x_0, y_0) it must first travel to $(1,0)$, which means that all of the primary spaces will become occupied. From $(1,0)$ it will then move to $(1, y_*)$ and finally to (x_0, y_0) along one of the rays sketched in Figure 2. The y_* value is determined by tracing backward the ray from (x_0, y_0). Since these rays have positive slope, we necessarily have $y_* > y_0$. This means that the number of occupied secondary spaces will first exceed the target value of y_0, while all or almost all primary spaces will remain occupied. Then both X and Y will decrease along a ray to reach (x_0, y_0), meaning that some primary and some secondary spaces will become empty. The overshoot to y_* is perhaps counter-intuitive.

When $\rho < 1$ we also showed that the mean number of occupied secondary spaces is of the order $O(m\rho^m)$ (cf. (3.28)), while the number of occupied primary spaces is roughly geometric (cf. (3.23)) in ρ, with a deviation that is of order $O(\rho^m/m)$, and this deviation depends on the fraction x of occupied primary spaces.

In the 'heavy traffic' case, where $1 - \rho = a/m = O(m^{-1})$ we found that there is a probability $1 - e^{-a}$ that there are zero secondary spaces and only a few primary ones are occupied. With the remaining probability e^{-a} all but a few of the m primary spaces are occupied, and the number of occupied secondary spaces varies weakly and exponentially with r, as $e^{-ay} = e^{-ar/m}$ (cf. (3.36)). Thus, there are two distinct scales, $r = 0$ with $k = O(1)$ and $r = O(m)$ with $k = m - O(1)$ that carry comparable mass. When $\rho = 1 - a/m$ the probabilistic interpretation of the ray expansion (3.29) is similar to that when $\rho < 1$, except that it is easier for the scaled process to go from $(0,0)$ to $(1,0)$ and then to $(1, y_*)$ because $\pi(k,r)$ is no longer exponentially small along these segments. The rays themselves do not exhibit any discontinuous change as $\rho \uparrow 1$, except in their behaviour near the corner $(x,y) = (1,0)$. Whereas $\phi = \phi(x,y;\rho)$ in (3.1) (for $\rho < 1$) is perfectly smooth as $(x,y) \to (1,0)$, the function $\Psi(x,y) = \phi(x,y;1)$ in (3.29) has very singular behaviour near $(1,0)$, and this leads to the non-uniformities in the asymptotics that were treated in Section 5. These non-uniformities lead to new scales that correspond to situations where the number of occupied primary spaces is close to the maximum value m, and there are either $O(1)$ or $O(\sqrt{m})$ occupied secondary spaces.

From a viewpoint of purely asymptotic methods, the analysis of this model led to several novel features. First, there is the continuum of cusps formed by the rays (cf. Figure 3) and the corresponding corner layer near $(x,y) = (0,0)$, which we resolved using hypergeometric function. Second, there is the transition layer that we found for $r = O(1)$ and $k = m - \sqrt{m}(\log m)\bar{\xi}$, as the location of this layer has $\bar{\xi} = \sqrt{r}$ for $r \geqslant 1$, so its location is different for each r, and increasing r to $r+1$ shifts the transition layer by a significant $(O(\sqrt{m}\log m))$ amount.

Finally, we mention that by using the asymptotic results for the joint distribution $\pi(k,r)$, we should be able to compute other performance measures for this storage model, such as the index of the largest occupied space, and the number of wasted spaces. The latter would presumably make contact with the rough estimate in [3] that we discussed in the

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Introduction. This would, however, require more analysis, and perhaps a more thorough understanding of the corner region(s) near $(x, y) = (1, 0)$, when $\rho \sim 1$.

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