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The eigenvalue problem for infinite systems of linear equations

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1. *Introduction.* Given an infinite system of linear equations

$$\sum_{j=1}^{\infty} a_{ij}x_j = 0 \quad (i = 1, 2, 3, \ldots),$$

where the a_{ij} depend on a parameter λ, the eigenvalue problem is to determine values of λ for which x_j $(j = 1, 2, \ldots)$ are not all zero. This problem (Taylor (3) and Vaughan (4)) can arise in the vibration of rectangular plates. Little theoretical work, however, appears to have been done concerning the existence and determination of the eigenvalues. The usual procedure (see (3) and (4)) is to consider a truncated or reduced system of N equations and find the values of λ for which the determinant of the $N \times N$ matrix $[a_{ij}]$ vanishes. If a particular λ tends to a constant value as N is increased then this value is assumed to be an eigenvalue. The question therefore arises as to what happens if no limit exists. Can we assert that there are no eigenvalues? By constructing an appropriate example we show that the non-existence of a limit does not imply the non-existence of eigenvalues. In order to construct our example we first establish a result concerning the Legendre polynomials.

2. *A result on Legendre polynomials.* If $P_n(x)$ is the Legendre polynomial of degree n then

$$\sum_{n=0}^{\infty} \frac{P_n(\cos c)\sin \tfrac{1}{2}(2n+1)c}{(2n+1)^2 - (2k\pi/c)^2} = 0,$$

where $0 < c < \pi$, $k = 1, 2, 3, \ldots$ and $2k\pi/c$ is not an integer. Subsequently for $k\pi/c$ we shall write p. Since $P_n(\cos c)$ is $O(n^{\frac{1}{2}})$ when n is large, this result is equivalent to

$$\sum_{n=0}^{\infty} \frac{P_n(\cos c)\sin \tfrac{1}{2}(2n+1)c}{2n+1-2p} = \sum_{n=0}^{\infty} \frac{P_n(\cos c)\sin \tfrac{1}{2}(2n+1)c}{2n+1+2p}.$$

It is therefore sufficient to show that these two series have the common sum

$$\frac{(-1)^k}{\sqrt{2}} \int_0^{\frac{1}{2}c} \frac{\cos 2p\theta \, d\theta}{(\cos 2\theta - \cos c)^{\frac{1}{2}}}.$$

Taking the complex variable $z = re^{i\theta}$ $(0 \leqslant \theta < 2\pi)$ as the parameter in the generating function for the Legendre polynomials we have, for $|z| < 1$,

$$\frac{z^{2p}}{(1 - 2\cos c z^2 + z^4)^{\frac{1}{2}}} = \sum_{n=0}^{\infty} P_n(\cos c)\, z^{2n+2p}. \tag{2·1}$$

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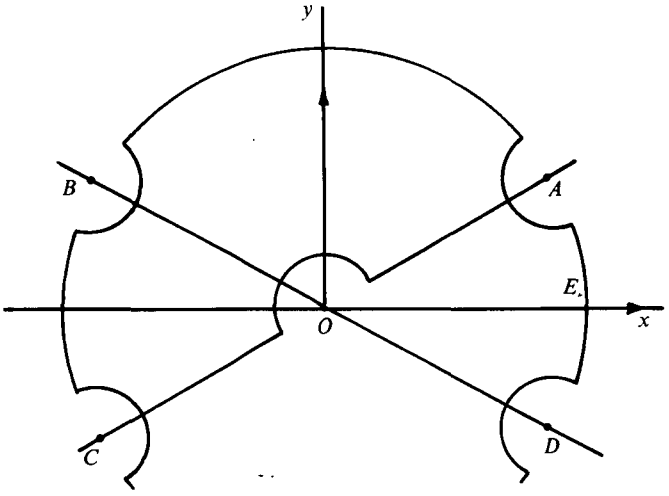


Fig. 1

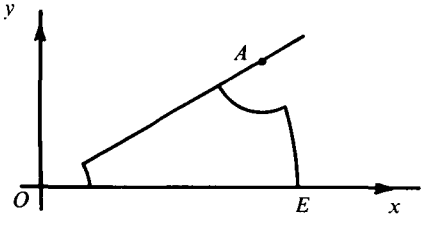


Fig. 2

In (2·1) we define z^{2p}, which has a branch point at $z = 0$, to be $r^p e^{ip\theta} (0 \leqslant \theta < \pi)$. For the function $1 - 2\cos c\, z^2 + z^4$ we write w and take, as is customary in the definition of the Legendre polynomials, $w^{\frac{1}{2}}$ to be that branch of the square root which has the value $+1$ at $z = 0$. We observe that the points $e^{i\frac{1}{2}c}$, $e^{i(\pi - \frac{1}{2}c)}$, $e^{i(\pi + \frac{1}{2}c)}$ and $e^{i(2\pi - \frac{1}{2}c)}$ are branch points of $w^{\frac{1}{2}}$. They are denoted in Fig. 1 by A, B, C and D respectively, while E is the point $z = 1$. The function $w^{\frac{1}{2}}$ is analytic within and on the contour formed by indenting, as shown in Fig. 1, the circle $|z| = 1$ at the points A, B, C and D. In particular we note that, along this contour between E and C, $w^{\frac{1}{2}}$ is given by

$$w^{\frac{1}{2}} = \begin{cases} \sqrt{2}(\cos 2\theta - \cos c)^{\frac{1}{2}} e^{i\theta} & (0 < \theta < \tfrac{1}{2}c), \\ \sqrt{2}(\cos c - \cos 2\theta)^{\frac{1}{2}} e^{i(\theta - \frac{1}{2}\pi)} & (\tfrac{1}{2}c < \theta < \pi - \tfrac{1}{2}c), \\ \sqrt{2}(\cos 2\theta - \cos c)^{\frac{1}{2}} e^{i(\theta - \pi)} & (\pi - \tfrac{1}{2}c < \theta < \pi + \tfrac{1}{2}c). \end{cases} \tag{2·2}$$

We now consider the integral of the function $z^{2p} w^{-\frac{1}{2}}$ round the contour $OEAO$ indented, as shown in Fig. 2, at the points O and A. The radii of the indents are ϵ_1 and ϵ_2 respectively. Since the function is analytic within and on the indented contour, the value of the integral is zero. Further, the integrals round the indentations tend to zero as ϵ_1, ϵ_2 both tend to zero. Hence we have

$$\int_0^1 \frac{x^{2p}\, dx}{(1 - 2\cos c\, x^2 + x^4)^{\frac{1}{2}}} + \frac{1}{\sqrt{2}} \int_0^{\frac{1}{2}c} \frac{i\, e^{i2p\theta}\, d\theta}{(\cos 2\theta - \cos c)^{\frac{1}{2}}} = \lim_{\epsilon_1, \epsilon_2 \to 0} \int_{\epsilon_1 e^{i\frac{1}{2}c}}^{(1-\epsilon_2) e^{i\frac{1}{2}c}} \frac{z^{2p}\, dz}{(1 - 2\cos c\, z^2 + z^4)^{\frac{1}{2}}}, \tag{2·3}$$

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the path of integration in the third integral of (2·3) being the straight line OA. Now we can integrate both sides of (2·1) along OA from $\epsilon_1 e^{i\frac{1}{2}c}$ to $(1-\epsilon_2)e^{\frac{1}{2}ic}$. Performing this integration we deduce, with the help of (2·3), that

$$\lim_{\epsilon_1,\epsilon_2\to 0}\sum_{n=0}^{\infty}\frac{(-1)^k P_n(\cos c)}{2n+2p+1}e^{i\frac{1}{2}(2n+1)c}\{(1-\epsilon_2)^{2n+2p+1}-\epsilon_1^{2n+2p+1}\}$$
$$=\int_0^1\frac{x^{2p}\,dx}{(1-2\cos c\,x^2+x^4)^{\frac{1}{2}}}+\frac{1}{\sqrt{2}}\int_0^{\frac{1}{2}c}\frac{i\,e^{i2p\theta}\,d\theta}{(\cos 2\theta-\cos c)^{\frac{1}{2}}}. \quad (2\cdot4)$$

Since $P_n(\cos c)$ is $O(n^{-\frac{1}{2}})$ for n large, we can apply Abel's theorem (see (2)) to the left-hand side of (2·4). Equating the imaginary parts of (2·4) we thus find

$$\sum_{n=0}^{\infty}\frac{P_n(\cos c)\sin\frac{1}{2}(2n+1)c}{2n+2p+1}=\frac{(-1)^k}{\sqrt{2}}\int_0^{\frac{1}{2}c}\frac{\cos 2p\theta\,d\theta}{(\cos 2\theta-\cos c)^{\frac{1}{2}}}.$$

Replacing p by $-p$ in (2·1) we have

$$\frac{z^{-2p}}{(1-2\cos c\,z^2+z^4)^{\frac{1}{2}}}=\sum_{n=0}^{\infty}P_n(\cos c)\,z^{2n-2p}$$

which is valid for $0<|z|<1$.

This relation can be rewritten in the alternative form

$$\frac{z^{-2p}}{(1-2\cos c\,z^2+z^4)^{\frac{1}{2}}}-\sum_{n=0}^{p'}P_n(\cos c)\,z^{2n-2p}=\sum_{n=p'+1}^{\infty}P_n(\cos c)\,z^{2n-2p}, \quad (2\cdot5)$$

where p' denotes the integral part of $p=k\pi/c$.

We note that the right-hand side of (2·5) tends to zero as $|z|\to 0$. Thus the left-hand side has a removable singularity at $z=0$. Hence assigning the value zero to the left-hand side when $z=0$ we may take (2·5) to hold for $0\leqslant|z|<1$.

Now the integral of the left-hand side of (2·5) round the closed contour formed by the semi-circle $|z|=1$, $\frac{1}{2}c<\theta<\frac{1}{2}c+\pi$, indented as in Fig. 1 at the points A, B and C, and the straight line CA indented at the origin is zero. The contributions to the integral arising from the indentations tend to zero as the radii of these indentations tend to zero. Proceeding to the limit we have

$$\int_C^O+\int_O^A\left\{\frac{z^{-2p}}{(1-2\cos cz^2+z^4)^{\frac{1}{2}}}-\sum_{n=0}^{p'}P_n(\cos c)\,z^{2n-2p}\right\}dz$$
$$=\int_\Gamma\left\{\sum_{n=0}^{p'}P_n(\cos c)\,z^{2n-2p}-\frac{z^{-2p}}{(1-2\cos c\,z^2+z^4)^{\frac{1}{2}}}\right\}dz, \quad (2\cdot6)$$

where Γ is the semi-circle ABC.

Using (2·2) we find, after some manipulation which involves a change of variable in some of the integrals, that

$$\int_\Gamma\left\{\sum_{n=0}^{p'}P_n(\cos c)\,z^{2n-2p}-\frac{z^{-2p}}{(1-2\cos c\,z^2+z^4)^{\frac{1}{2}}}\right\}dz$$
$$=\frac{1}{\sqrt{2}}\{1+e^{-i2p\pi}\}\left\{\sum_{n=0}^{p'}\frac{\sqrt{2}(-1)^{k+1}P_n(\cos c)}{2n-2p+1}e^{i\frac{1}{2}(2n+1)c}+\int_{\frac{1}{2}c}^{\frac{1}{2}\pi}\frac{\cos 2p\theta\,d\theta}{(\cos c-\cos 2\theta)^{\frac{1}{2}}}\right.$$
$$\left.+\tan p\pi\int_{\frac{1}{2}c}^{\frac{1}{2}\pi}\frac{\sin 2p\theta\,d\theta}{(\cos c-\cos 2\theta)^{\frac{1}{2}}}+(\tan p\pi+i)\int_0^{\frac{1}{2}c}\frac{\cos 2p\theta\,d\theta}{(\cos 2\theta-\cos c)^{\frac{1}{2}}}\right\}. \quad (2\cdot7)$$

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We now integrate the expressions on each side of (2·5) along CA indented at O, the integration commencing at the point $(1-\epsilon_3)\,e^{i(\pi+\frac{1}{2}c)}$, where ϵ_3 is the radius of the indent at C, and finishing at the point $(1-\epsilon_2)\,e^{i\frac{1}{2}c}$. As ϵ_1, ϵ_2 and ϵ_3 all tend to zero the integral of the left-hand side tends to a limit which is precisely the expression in the left-hand side of (2·6). Moreover, the value of this expression is known by virtue of (2·7). The limiting value of the integral of the right-hand side of (2·5) is given by

$$\lim_{\epsilon_2,\,\epsilon_3\to 0}\sum_{n=p'+1}^{\infty}\frac{(-1)^k P_n(\cos c)}{2n-2p+1}\,e^{i\frac{1}{2}(2n+1)c}\{(1-\epsilon_2)^{2n-2p+1}+e^{-i2p\pi}(1-\epsilon_3)^{2n-2p+1}\},$$

there being no contribution from the indent at O. Using Abel's theorem once more we see that the limit is equal to

$$(-1)^k(1+e^{-i2p\pi})\sum_{n=p'+1}^{\infty}\frac{P_n(\cos c)\,e^{i\frac{1}{2}(2n+1)c}}{2n-2p+1}.$$

Equating this to the right-hand side of (2·7) we deduce, after division by $1+e^{-i2p\pi}\neq 0$ and comparing imaginary parts, that

$$\sum_{n=0}^{\infty}\frac{P_n(\cos c)\sin\frac{1}{2}(2n+1)c}{2n-2p+1}=\frac{(-1)^k}{\sqrt{2}}\int_0^{\frac{1}{2}c}\frac{\cos 2p\theta\,d\theta}{(\cos 2\theta-\cos c)^{\frac{1}{2}}}.$$

Thus the result concerning Legendre polynomials is proved.

3. *The infinite system* $\sum_{j=1}^{\infty}\dfrac{x_j}{(2j-1)^2-\lambda^2 i^2}=0\quad(i=1,2,\ldots)$.

From the result on the Legendre polynomials we deduce that the infinite system has the non-zero solution

$$x_j=P_{j-1}(\cos c)\sin\tfrac{1}{2}(2j-1)c\quad(j=1,2,3,\ldots)$$

corresponding to the eigenvalue $\lambda=2\pi/c$, where $0<c<\pi$ and $2\pi/c$ is irrational.

We now consider a reduced system of N equations. It is known (see (1)) that Cauchy's double alternant

$$\det\frac{1}{a_i+b_j}\quad(i,j=1,2,\ldots,N)$$

has the value

$$\frac{\prod_{r=1}^{N-1}\left(\prod_{s=r+1}^{N}(a_s-a_r)\right)\prod_{r=1}^{N-1}\left(\prod_{s=r+1}^{N}(b_s-b_r)\right)}{\prod_{s=1}^{N}\left(\prod_{r=1}^{N}(a_r+b_s)\right)}.$$

Taking $a_i=-\lambda^2 i^2$ and $b_j=(2j-1)^2$ the determinant of the reduced system is found after manipulation to have the value

$$\frac{(-1)^{\frac{1}{2}N(N-1)}(2\lambda)^{N(N-1)}}{\prod_{s=1}^{N}\left(\prod_{r=1}^{N}((2s-1)^2-\lambda^2 r^2)\right)}\prod_{r=1}^{N-1}\frac{\{(N-r)!\}^2(N+r)!\,(N+r-1)!}{(2r)!\,(2r-1)!}.$$

This vanishes only when $\lambda=0$, which is a trivial case since the infinite system is then replaced by a single equation. Thus the result stated in the introduction is proved.

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