J. Inst. Math. Jussieu (2026), 1–11
doi:10.1017/S1474748026102175

7-LOCATED LOCALLY 5-LARGE COMPLEXES ARE ASPHERICAL

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(Received 5 January 2026 UTC; revised 19 July 2026 UTC; accepted 21 September 2026 UTC)

Abstract We prove that 7-located locally 5-large simplicial complexes are aspherical.

1. Introduction

A simplicial complex is *flag* if every set of vertices pairwise connected by edges spans a simplex. For $k \geq 5$, a simplicial complex is *k-large* if it is flag and has no induced cycles of length $4 \leq n < k$. A simplicial complex is *locally k-large* if each of its vertex links is k-large. This notion was introduced by Januszkiewicz and Świątkowski [8], and independently by Haglund [5], as a simplicial analogue of a locally CAT(0) (i.e. nonpositively curved) cube complex. They showed that such complexes are abundant in any dimension, and come with interesting automorphism groups. A cornerstone feature is that for $k \geq 6$ they are aspherical. The 1-skeleta of simply connected locally 6-large simplicial complexes were studied earlier in graph theory under the name of bridged graphs; see [1] for a survey.

The boundary of the icosahedron is locally 5-large, so in order to obtain asphericity under this weaker condition, Osajda introduced an extra hypothesis of m-location [12] (we will give the definition in a moment). 7-located locally 5-large simplicial complexes include many 3-manifolds, as well as all locally weakly systolic complexes [6], which were

Keywords and phrases: systolic complex; combinatorial non-positive curvature

Mathematics Subject Classification: Primary 20F65
Secondary 20F67; 55P20

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studied earlier in [4,11]. The properties of m-located complexes were investigated in [6,12]. A related condition was introduced in [9].

Maybe the most prominent example of a 7-located locally 5-large simplicial complex is the triangulation of the hyperbolic space $\mathbf{H}^4$ where each of the vertex links is isomorphic to the boundary of the 600-cell. The symmetry group of that triangulation is the Coxeter group with Coxeter–Dynkin diagram the linear graph of length 4 with consecutive labels 5333. We are interested in this triangulation since the associated Artin group is one of the smallest Artin groups for which the $K(\pi,1)$ conjecture, asking for the contractibility of the associated Artin complex, is still open.

In this paper, we prove the following related result.

Main Theorem. Every 7-located locally 5-large simplicial complex is aspherical.

2. Location

Let X be a flag simplicial complex. Note that we do not assume X to be locally finite or finite-dimensional.

Definition 2.1. A *k-wheel* W in X is an induced subcomplex isomorphic to the cone over the k-cycle. We write $W=(v_0,v_1v_2\cdots v_k)$, where the *centre* v_0 is the cone vertex and $v_1,\dots,v_k$ are the consecutive vertices of the *boundary* cycle.

A pair $W=(W_1,W_2)$ of wheels, with $W_1=(v_0,v_1\cdots v_k),W_2=(w_0,w_1\cdots w_\ell)$, is a (k,ℓ)-*dwheel* if

- $v_k=w_0$,
- $w_\ell=v_0$,
- $v_{k-1}=w_{\ell-1}$, and
- v_1 either equals w_1 or is a neighbour of w_1.

The *boundary* ∂W of the dwheel W is the cycle $v_1\cdots v_{k-1}w_{\ell-2}\cdots w_1$. (If $v_1=w_1$, then we discard the redundant w_1.) If $v_1=w_1$, then we say that W is *planar*.

Definition 2.2. Let $m\geq 4$. X is *m-located* if for every dwheel $W=(W_1,W_2)$ with $|\partial W|\leq m$, all the vertices of $W_1\cup W_2$ have a common neighbour in X.

Here vertices $v\neq w$ are *neighbours* (or *neighbouring*) in X if they span an edge of X.

Example 2.3. Let X be the simplicial complex that is the triangulation of the hyperbolic space $\mathbf{H}^4$ into hyperbolic simplices with all dihedral angles $\frac{2\pi}{5}$. Each of the vertex links of X is isomorphic to the boundary $\mathbf{S}^3_{600}$ of the 600-cell, which is 5-large. Note that the vertex links of $\mathbf{S}^3_{600}$ are isomorphic to the boundary $\mathbf{S}^2_{20}$ of the icosahedron. Since each induced 5-cycle in $\mathbf{S}^3_{600}$ and $\mathbf{S}^2_{20}$ is the boundary of a 5-wheel, each 5-wheel in X can be extended to the join of the 5-cycle and a triangle Δ. Furthermore, each induced 6-cycle in $\mathbf{S}^3_{600}$ is the boundary of a 6-wheel, and so each 6-wheel in X can be extended to the join of the 6-cycle and an edge e. Hence X does not contain a planar (5,6)-dwheel (W_1,W_2) with $W_1\cup W_2$ without a common neighbour, since otherwise appropriate Δ and e are disjoint and so Δ,e and $v_1=w_1$ span a simplex of dimension 5 in X, which is a contradiction. The (5,5)-dwheels are excluded similarly, which implies that X is 7-located.

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2.1. Disc diagrams

A *disc diagram in* X is a simplicial map $f\colon D \to X$ from a simplicial disc D that is *nondegenerate*, i.e. does not send any edge to a vertex. The *boundary (cycle)* of f is the closed path $f_{|\partial D}$, which we identify with its image $f(\partial D)$ whenever it is embedded. A disc diagram $f\colon D \to X$ is *minimal* if it has minimal area (i.e. the number of triangles in D) among all the disc diagrams in X with the same boundary cycle. We say that f is *reduced* if it is locally injective at $D \setminus D^0$. The following is a well-known variation of a result by Van Kampen.

Lemma 2.4 [10, Lem 2.16 and 2.17]. *Any homotopically trivial cycle embedded in* X^1 *is the boundary cycle of a disc diagram in* X*. Any minimal disc diagram in* X *is reduced.*

Lemma 2.5 [6, Thm B]. *If* X *is* 7*-located and locally* 5*-large, then so is* D *for each minimal disc diagram* $D \to X$*. In other words,* D *has no*

- *interior vertices of valence* 3 *or* 4*, or*
- *neighbouring interior vertices with valences* 5 *and* 5 *or* 6*.*

Corollary 2.6. *Each* D *as in Lemma 2.5*

(i) *is* 5*-large, and*

(ii) *each of its induced* 5*-cycles is the boundary of a* 5*-wheel.*

Proof. Let $D' \subseteq D$ be a simplicial disc with boundary ω a 4-cycle or a 5-cycle. Following the proof of [6, Cor 4.7], we metrise each 5-wheel of D' as the regular Euclidean pentagon P of side length 1, and each remaining triangle of D' as the regular Euclidean triangle of side length 1. Consequently, all triangles have area $\geq \frac{1}{5}\mathrm{Area}(P) > 0.34$ (which is larger than in [6, Cor 4.7], where 4-wheels were allowed). Since D' is CAT(0) [6, Thm 4.6], by [3, III.2.17], the area of D' is $\leq \frac{|\omega|^2}{4\pi}$. Consequently, the number of triangles of D' is $\leq \left\lfloor \frac{|\omega|^2}{1.36\pi} \right\rfloor$, which equals 3 for $|\omega| = 4$, and equals 5 for $|\omega| = 5$. Thus ω is not induced or it is the boundary of a 5-wheel, as desired. □

Remark 2.7. The κ' method from the proof of Proposition 3.2 can be used to give an alternative proof of Corollary 2.6, since in a minimal counterexample to Corollary 2.6, each valence 5 interior vertex has nonpositive κ'.

3. Lunar diagrams

We metrise the 1-skeleton of each simplicial complex with the path metric, where all the edges have length 1.

Definition 3.1. Let D be a simplicial disc and let x,v be vertices of ∂D. We say that D is *lunar (between* x *and* v*)* if the two distinct injective paths γ_1 and γ_2 in ∂D from x to v are geodesics in D^1.

More generally, let X be a simplicial complex and let x and v be distinct vertices of X. Suppose that γ_1 and γ_2 are geodesics from x to v in X^1 that are disjoint except at the endpoints. A minimal disc diagram $D \to X$ with boundary $\gamma_1 \cup \gamma_2$ is a *lunar disc*

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diagram (between x and v). Note that if $D \to X$ is a lunar disc diagram, then D is a lunar simplicial disc.

Let D be a lunar simplicial disc. Then for a vertex u of ∂D, an interior vertex of D of valence 5 is *u-exposed* if it is a neighbour of both neighbours of u in ∂D.

Let D be a lunar simplicial disc between x and v that is 7-located and locally 5-large. By Corollary 2.6(i), if z is u-exposed, and the neighbours of u in ∂D are denoted u_1, u_2, then the cycle $u_1 u u_2 z$ has a diagonal. (Note that if $u \neq x, v$, then z is a neighbour of u.) Consequently, for each u there is at most one u-exposed vertex.

Proposition 3.2. *Let D be a lunar simplicial disc between x and v and let v_1, v_2 be the neighbours of v in ∂D. If D is 7-located and locally 5-large, then*

(i) *either v_1 and v_2 are neighbours and have a common neighbour closer to x than v_1, v_2, or*

(ii) *v_1 and v_2 are not neighbours, and there is a v-exposed neighbour v' of v_1, v, v_2, whose remaining neighbours v_1', v_2' are closer to x than v_1, v_2.*

In the proof, we need the following. We are using the path metric d on D^1, the path metric d' on D'^1 and the path metric d'' on D''^1.

Lemma 3.3. *Let D be a lunar simplicial disc between x and v and let v_1, v_2 be the neighbours of v in ∂D. If D is 7-located and locally 5-large, then there is a lunar simplicial disc $D' \subseteq D$ between z and v such that*

(1) *the path $v_1 v v_2$ lies in $\partial D'$,*

(2) *the function $d(\cdot, x) - d'(\cdot, z)$ is constant on all the vertices of D' at d'-distance ≤ 2 from v,*

(3) *each vertex on $\partial D'$ has valence at least 4, except possibly for z, v, v_1 or v_2,*

(4) *each interior vertex of D' of valence 5 that is a neighbour of ≥ 3 vertices of $\partial D'$ is u-exposed in D' with $u \in \{z, v, v_1, v_2\}$.*

Remark 3.4. By (1) and (2), if D' satisfies Proposition 3.2(i) or (ii), then so does D.

In the proof of Lemma 3.3, we will use the following.

Lemma 3.5. *Let $D' \subseteq D$ be connected simplicial complexes with vertices $z' \in D', z \in D$. Let $I \subset D'$ be a subcomplex, and let C be a connected component of $D' \setminus I$ that is equal to a connected component of $D \setminus I$, with $z, z' \notin C$. If the function $d(\cdot, z) - d'(\cdot, z')$ is constant on all the vertices of I, then it is constant on all the vertices of $I \cup C$.*

Proof. Denote the constant value of $d(\cdot, z) - d'(\cdot, z')$ on the vertices of I by K. Let $\overline{C}$ be the closure of C in D, and let d_C denote the path metric on the 1-skeleton of $\overline{C}$. For each vertex w of C, each path in D^1 from w to z decomposes into a (nontrivial) path in $\overline{C}^1$ from w to some $i \in I$ and a (possibly trivial) path in D^1 from i to z. Thus we have

$$d(w, z) = \min_{i \in I} \big(d_C(w, i) + d(i, z) \big).$$

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Analogously,

$$d'(w,z') = \min_{i\in I}\big(d_C(w,i) + d'(i,z')\big).$$

Since the last terms always differ by K, we have $d(w,z) = d'(w,z') + K$, as desired. □

Lemma 3.6. *Let D be a lunar simplicial disc between z and v, and let $D' \subseteq D$ be a simplicial disc with $\partial D'$ a concatenation of paths γ'_1,γ'_2 from z' to v. Suppose that for $j = 1,2$ there is a vertex u_j on γ'_j with $d(u_j,z) - d'(u_j,z') = d(v,z) - d'(v,z')$ and such that γ'_j is a concatenation $\alpha_j\beta_j$ of the subpath $\alpha_j = vu_j$ of the geodesic γ_j, and a geodesic β_j of D'^1. Then each γ'_j is a geodesic in D' and so D' is a lunar simplicial disc between z' and v.*

Proof. By the hypothesis, for each j we have

$$d'(v,z') = d(v,z) - d(u_j,z) + d'(u_j,z') = |\alpha_j| + |\beta_j|,$$

and so $\alpha_j\beta_j$ is a geodesic, as desired. □

Proof of Lemma 3.3. Let $D' \subseteq D$ be a lunar simplicial disc of minimal area between z and v satisfying (1) and (2), among all the vertices z of D. We denote by γ'_1,γ'_2 the geodesics in $\partial D'$ from z to v.

To verify (3), suppose without loss of generality that $z_0 \in \gamma'_1 \setminus \{z,v,v_1\}$ has valence ≤ 3, which then equals 3, since γ'_1 is a geodesic in D'^1. Let z_1,z_2,z_3 be the consecutive neighbours of z_0 in D', where z_1 separates z_0 from v on γ'_1. If $z_2 \notin \gamma'_2$, then let D'' be the lunar simplicial disc between z and v obtained from D' by removing the *open star* of z_0, which is the union of z_0 and all the simplices of D' containing z_0 in their closure. Then D'' still satisfies (1) and (2), which contradicts the minimality hypothesis. If $z_2 \in \gamma'_2$, then let D'' be the simplicial disc that is the closure of the connected component of v in $D' \setminus z_1z_2$. Observe that $d'(z,z_1) = d'(z,z_2) + 1$, and so by Lemma 3.6 applied to $D'' \subset D'$ with $z' = u_2 = z_2, u_1 = z_1$, we have that D'' is a lunar simplicial disc between z_2 and v. Applying Lemma 3.5 to $D'' \subset D', I = z_1z_2$ and $C = D'' \setminus I$, we obtain that $d'(\cdot,z) - d''(\cdot,z')$ is constant on the vertices of D''. Thus $d(\cdot,x) - d''(\cdot,z') = \big(d(\cdot,x) - d'(\cdot,z)\big) + \big(d'(\cdot,z) - d''(\cdot,z')\big)$ is constant on the vertices of D'' at d''-distance ≤ 2 from v. Hence D'' still satisfies (1) and (2), which contradicts the minimality hypothesis.

To verify (4), let z_0 be an interior vertex of D' of valence 5 that is a neighbour of $m \geq 3$ vertices of $\partial D'$.

If $m = 5$, then by (3), z_0 is v-exposed in D'. The same holds for $m = 4$, unless z_0 has exactly two neighbours on each γ'_i, which are distinct from z,v, and consecutive by Corollary 2.6(i). We will discuss this possibility below, together with the case $m = 3$. Namely, if $m = 3$, then z_0 is u-exposed in D' with $u \in \{z,v,v_1,v_2\}$, unless it has, say, two consecutive neighbours z_1,z_2 on γ'_1 and a neighbour z_3 on γ'_2, all of which are distinct from z,v. We can assume $d'(z_1,z) = d'(z_2,z) + 1$.

In each of the cases below, we will prove that z_0 is u-exposed in D', with $u \in \{z,v,v_1,v_2\}$, or we will reach a contradiction by finding a properly contained lunar simplicial disc $D'' \subsetneq D'$ between a vertex z' of D' and v satisfying (1) and (2) and hence contradicting the minimality hypothesis. In fact, property (1) will always follow directly from the

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construction, so we will not be mentioning it. The simplicial disc D'' will always be a lunar simplicial disc between z' and v by Lemma 3.6 applied to $D'' \subsetneq D'$ and $u_j \in I \cap \gamma'_j$, as before. The path $\beta_j = u_j z' \subset \partial D''$ will always have length 0,1 or 2 and it will be clear that it is a geodesic, as desired. Thus we will only be focusing on justifying (2) in each case. We will be applying Lemma 3.5 with C the connected component of $D'' \setminus I$ containing v.

Consider the *top* and *bottom* connected components of D' minus the path $z_1 z_0 z_3$ containing v and z, respectively. Each of the two neighbours of z_0 distinct from z_1, z_2, z_3 is *top* (resp. *bottom*) if it lies in the top (resp. bottom) component.

If z_0 has no top neighbours, and $d'(z_1,z) \neq d'(z_3,z)$, then let D'' be the lunar simplicial disc between $z' = z_1$ or z_3 and v that is the closure of the connected component C of v in $D' \setminus z_1 z_3$. Applying Lemma 3.5 to $D'' \subsetneq D'$ and $I = z_1 z_3$ contradicts the minimality hypothesis. If z_0 has no top neighbours, and $d'(z_1,z) = d'(z_3,z)$, then let D'' be the lunar simplicial disc between $z' = z_0$ and v that is the closure of the connected component of v in D' minus the path $z_1 z_0 z_3$. Applying Lemma 3.5 to $D'' \subsetneq D'$ and $I = z_1 z_3$, we have that $d'(\cdot,z) - d''(\cdot,z')$ is constant on all the vertices of D'' except possibly for z_0. However, if z_0 is not v-exposed in D', then $d''(v,z_0) > 2$, and so D'' still satisfies (2).

If z_0 has no bottom neighbours, and $d'(z_2,z) \neq d'(z_3,z)$, then let D'' be the lunar simplicial disc between $z' = z_2$ or z_3 and v that is the closure of the connected component of v in $D' \setminus z_2 z_3$. Applying Lemma 3.5 to $D'' \subsetneq D'$ and $I = z_2 z_3$ contradicts the minimality hypothesis. If z_0 has no bottom neighbours, and $d'(z_2,z) = d'(z_3,z)$, then let z' be the common neighbour of z_2, z_3 distinct from z_0. Let D'' be the lunar simplicial disc between z' and v that is the closure of the connected component of v in D' minus the path $z_2 z' z_3$. Applying Lemma 3.5 to $D'' \subset D'$ and $I = z_2 z_3$, we again obtain that D'' satisfies (2). Note that $D'' \neq D'$, unless z_0 is z-exposed in D'.

Thus from now on we can assume that z_0 has exactly one top neighbour z_t and one bottom neighbour z_b (one of which might lie in $\partial D'$ if $m = 4$). Let $k = d'(z_0,z)$. By the triangle inequality, we have $d'(z_1,z) = k+1$ or k, and $d'(z_3,z) = k+1, k$ or $k-1$.

Case 1: $d'(z_1,z) = k+1$.

(a) $d'(z_3,z) = k+1$. Let D'' be the lunar simplicial disc between $z' = z_0$ and v that is the closure of the connected component of v in D' minus the path $I = z_1 z_0 z_3$. We can then apply Lemma 3.5 to $D'' \subsetneq D'$ and I as above.

(b) $d'(z_3,z) = k$. Then $d'(z_b,z) = k-1$. We take $z' = z_b$ and D'' to be the closure of the connected component of v in D' minus the path $I = z_2 z_b z_3$. By Lemma 3.5, the function $d'(\cdot,z) - d''(\cdot,z')$ is again constant on all the vertices of D''. Note that $D'' \neq D'$, unless z_0 is z-exposed in D'.

(c) $d'(z_3,z) = k-1$. We take $z' = z_3$ and D'' to be the closure of the connected component of v in D' minus the path $I = z_1 z_0 z_3$.

Case 2: $d'(z_1,z) = k$.

(a) $d'(z_3,z) = k+1$. We take $z' = z_2$ and D'' to be the closure of the connected component of v in D' minus the path $I = z_2 z_0 z_3$.

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(b) $d'(z_3,z)=k$. Let z' be the common neighbour of z_2 and z_b distinct from z_0. Let D'' be the lunar simplicial disc between z' and v that is the closure of the connected component of v in D' minus the path $z_2z'z_bz_3$. Let $I\subset D''$ be the path $z_2z_0z_3$. Note that $d'(\cdot,z)-d''(\cdot,z')\equiv k-2$ on I. By Lemma 3.5, the function $d'(\cdot,z)-d''(\cdot,z')$ is constant on all the vertices of D'' except possibly at z' and z_b. Note that $d''(z',v)>2$. If $d''(z_b,v)=2$, then $d'(z,v)=k+1$ and so v is a common neighbour of z_1 and z_3. Thus z_0 is v-exposed in D'. Note that $D''\neq D'$, unless z_0 is z-exposed in D'.

(c) $d'(z_3,z)=k-1$. We take $z'=z_b$ and D'' to be the closure of the connected component of v in D' minus the path $z_2z_bz_3$. Let I be the path $z_2z_1z_tz_3$. Note that $d'(\cdot,z)-d''(\cdot,z')\equiv k-2$ on I. By Lemma 3.5, the function $d'(\cdot,z)-d''(\cdot,z')$ is constant on all the vertices of D'' except possibly at z' and z_0. Note that $d''(z',v)>2$. If $d''(z_0,v)=2$, then $d'(z,v)=k+1$ and so v is a neighbour of z_1 and connected by the path vv_2z_3 of length 2 with z_3. By Corollary 2.6(ii) applied to the 5-cycle $z_0z_1vv_2z_3$, we have that either $z_t=v_2$, and so z_0 is v-exposed, or this 5-cycle is the boundary of a 5-wheel with centre z_t, and so both z_t,z_0 have valence 5, which is a contradiction. Again $D''\neq D'$, unless z_0 is z-exposed in D'. □

Proof of Proposition 3.2. By Remark 3.4 and Lemma 3.3, we can assume that $D=D'$ and that it satisfies Lemma 3.3(3,4). For any interior vertex w of D', let $\kappa(w)=6$ minus the valence of w. For w in $\partial D'$, let $\kappa(w)=4$ minus the valence of w. By the combinatorial Gauss–Bonnet theorem (see e.g. [10, Thm 4.6]), the sum of all $\kappa(w)$ equals 6. For each interior vertex w of valence 5, let $\kappa'(w)=\kappa(w)-\frac{N(w)}{3}$, where $N(w)$ is the number of the interior neighbours of w (all of which have valence ≥ 7 by Lemma 2.5). For each interior vertex w of valence ≥ 7, let $\kappa'(w)=\kappa(w)+\frac{N_5(w)}{3}$, where $N_5(w)$ is the number of the interior neighbours of w of valence 5. We let $\kappa'(w)=\kappa(w)$ for the remaining w. Note that the sum of $N(w)$ over interior w of valence 5 is equal to the sum of $N_5(w)$ over interior w of valence ≥ 7, since in each of these sums we count the number of pairs of interior neighbours of valence 5 and ≥ 7. Thus the sum of all $\kappa'(w)$ equals 6 as well.

Claim 1. κ' is nonpositive except possibly at

- x,v, where it is ≤ 2,
- v_i, where it is ≤ 1,
- u-exposed vertices, for $u\in\{x,v,v_1,v_2\}$, where it is ≤ 1. It is <1 if the u-exposed vertex has at least one interior neighbour.

Indeed, if an interior vertex w of valence 5 is not u-exposed for $u\in\{x,v,v_1,v_2\}$, then by (4), we have $N\geq 3$, and so $\kappa'(w)=\kappa(w)-\frac{N}{3}\leq 1-1$. On the other hand, if an interior vertex w has valence 7, then by Lemma 2.5, we have $N_5\leq 3$ and so $\kappa'(w)=\kappa(w)+\frac{N_5}{3}\leq -1+1$, and if it has valence $k\geq 8$, then $N_5\leq\frac{k}{2}$ and so $\kappa'(w)=\kappa(w)+\frac{N_5}{3}\leq 6-k+\frac{k}{6}=6-\frac{5k}{6}<0$. This justifies the Claim.

If there is a v-exposed vertex, then we call it v'. If such a vertex does not exist, but there are v_j-exposed vertices, then we call them v'_j. Note that if v' exists, then we do not define v'_j, because if a v_j-exposed vertex exists, then it equals v'. If there is an x-exposed vertex, then we call it x'. By the Claim, the sum of κ' over all the vertices except at v,v',v_j,v'_j is at most 3. Consequently, the sum of κ' at v,v',v_j,v'_j is at least 3.

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To verify (i) or (ii), it suffices to check that

(i) $\kappa'(v) = 2$ and $\kappa(v_j) = 1$ for some j, or

(ii) v' exists and $\kappa'(v) = \kappa'(v_1) = \kappa'(v_2) = 1$.

We will justify that the only remaining possibilities, up to a symmetry, are:

(a) $\kappa'(x) = \kappa'(v) = 2, \kappa'(x') = \kappa'(v') = 1, \kappa'(v_1) = \kappa'(v_2) = 0$,

(b) $\kappa'(x) = 2, \kappa'(v) = \kappa'(v_1) = \kappa'(x') = \kappa'(v') = 1, \kappa'(v_2) = 0$,

(c) $\kappa'(x) = 2, \kappa'(v) = \kappa'(v_1) = \kappa'(v_2) = \kappa'(x') = 1$, and there are no v', v'_j,

(d) $\kappa'(x) = 2, \kappa'(v_1) = \kappa'(v_2) = \kappa'(v'_1) = \kappa'(x') = 1, \kappa'(v) = 0$, and there are no v', v'_2,

(e) $\kappa'(x) = 2, \kappa'(v_1) = \kappa'(v_2) = \kappa'(v'_1) = \kappa'(v'_2) = \kappa'(x') = 1, \kappa'(v) = -1$.

Indeed, consider first the case where v' exists. If $\kappa'(v) = 2$, then $\kappa'(v_1), \kappa'(v_2)$ are nonpositive. This brings us to case (a). If $\kappa'(v) = 1$, then we have (ii), or, say, we have $\kappa'(v_2) \leq 0$. This brings us to case (b).

Second, suppose that there is no v'. Note that if one of the v'_j exists, then $\kappa'(v) \leq 0$. If both v'_j exist, then they are not neighbours by Lemma 2.5, and so $\kappa'(v) \leq -1$. This brings us to (i) or cases (c–e).

However, in all cases (a–e), by (3), the vertex x' has at least one interior neighbour, which contradicts $\kappa'(x') = 1$. □

4. Contractibility

Here, as before, we do not assume our simplicial complexes to be locally finite or finite-dimensional.

Lemma 4.1. *Suppose that K is a nonempty flag simplicial complex*

(1) *of diameter ≤ 2,*

(2) *5-large, and*

(3) *such that any induced 5-cycle is the boundary of a 5-wheel of K.*

Then K is contractible.

In the proof, we will use the following.

Lemma 4.2 [7, Lem 8.11]. *Let K be as in Lemma 4.1. Then for any pair of simplices of K with vertex sets A_1, A_2, there is a vertex a of K that is a neighbour of or equal to each of the elements of $A_1 \cup A_2$.*

Consequently, both $A_1 \cup \{a\}$ and $A_2 \cup \{a\}$ span a simplex.

Proof of Lemma 4.1. By the Whitehead theorem, it suffices to show that any finite subcomplex K' of K is contained in a contractible subcomplex K'' of K. We consider all the subsets $V_0, \ldots, V_n$ of the vertex set of K' that span a simplex of K. Let M_0 be the simplex spanned by V_0. Using Lemma 4.2, we inductively construct simplices $M_1, \ldots, M_n$ so that $M_i \supseteq M_{i-1}$ and M_i contains a vertex a_i such that $V_i \cup \{a_i\}$ spans a simplex. Let

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K'' be the span of the union of K' and M_n. Note that K'' is flag and each maximal simplex of K'' intersects M_n. Then K'' is contractible (which follows essentially from gradually collapsing K'' to M_n; see e.g. [7, Lem 8.13]). □

A subcomplex C of a simplicial complex K is *3-convex* if it is a connected induced subcomplex of K and for every path abc with vertices a,c in C at distance 2 in the 1-skeleton K^1, we have that b also belongs to C.

Remark 4.3. Let K be as in Lemma 4.1. If C is a 3-convex subcomplex of K, then K also satisfies the hypotheses of Lemma 4.1.

4.1. Downward links

In the entire subsection, we assume that X is a simply connected, 7-located (in particular, flag) and locally 5-large simplicial complex.

We fix a basepoint vertex x of X. The *ball* $B_n(x)$ (resp. the *sphere* $S_n(x)$) is the subcomplex of X spanned by all the vertices at distance $\leq n$ (resp. $= n$) from x in X^1. Let $n > 0$ and let σ be a simplex contained in $S_n(x)$. The *link* of σ is the intersection of the links of all the vertices of σ, treated as subcomplexes of X. The intersection $K(\sigma)$ of the link of σ with $S_{n-1}(x)$ is the *downward link* of σ. We will prove in Proposition 4.7 that $K(\sigma)$ is nonempty, which is obvious only in the case where σ is a vertex.

Our goal is to show that downward links satisfy the hypotheses of Lemma 4.1, and so they are contractible. Note that they are flag, since they are induced subcomplexes of the flag simplicial complex X. To start with, let $\sigma = v$ be a vertex. Note that $K(v)$ satisfies Lemma 4.1(2) since X is locally 5-large.

From Proposition 3.2, it follows that $K(v)$ satisfies Lemma 4.1(1), and more generally:

Corollary 4.4. *Let v_1,v_2 be vertices of $K(v)$.*

(i) *If v_1 and v_2 are neighbours, then $K(v_1)$ intersects $K(v_2)$.*

(ii) *If v_1 and v_2 are not neighbours, then they have a common neighbour v' in $K(v)$ such that there is an edge $v_1'v_2'$ with v_1' in $K(v_1v')$ and v_2' in $K(v_2v')$.*

Proposition 4.5. *Any induced 5-cycle in $K(v)$ is the boundary of a 5-wheel in $K(v)$.*

Proof. Let $\gamma = v_2w_2w_1v_1u$ be an induced 5-cycle in $K(v)$. Let v',v_1',v_2' be as in Corollary 4.4(ii). Since $K(v)$ is 5-large, u and v' are either equal or neighbours. If v' is a neighbour of w_1 or w_2, then, since $K(v)$ is 5-large, we have that (v',γ) is the required 5-wheel. Otherwise, $W_1 = (v,v_2w_2w_1v_1v')$ is a 5-wheel. Since $W_2 = (v',v_2v_2'v_1'v_1v)$ is also a 5-wheel, (W_1,W_2) is a (5,5)-dwheel, and so all the vertices of $W_1 \cup W_2$ have a common neighbour y in X. Since y is a neighbour of both v and v_1', we have that y is a vertex of $K(v)$. Since $K(v)$ is 5-large, considering the cycle v_1uv_2y, we obtain that y is also a neighbour of u. Thus (y,γ) is the required 5-wheel. □

Corollary 4.6. *Each $K(v)$ satisfies the hypotheses of Lemma 4.1, and so it is contractible.*

Proposition 4.7. *Let $n > 0$, and let σ be a simplex of $S_n(x)$. Then $K(\sigma)$ is nonempty.*

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Proof. Suppose first that σ is an edge of $S_n(x)$ with vertices v_1 and v_2. We may obtain a new complex X' by artificially adding to X a vertex v and a triangle vv_1v_2. This does not affect local 5-largeness or 7-location, so the proposition follows from Corollary 4.4(i) applied to v in X'.

Now suppose $\dim(\sigma) > 1$. We fix two distinct vertices v and y of σ. Let σ' be the subsimplex of σ spanned by all the vertices except for y, and let $e = vy$. By induction, we have vertices v_1 in $K(\sigma')$ and v_2 in $K(e)$. If neither v_1 nor v_2 lies in $K(\sigma)$, then there is $u \neq v$ in σ' that is not a neighbour of v_2, and y is not a neighbour of v_1. By the 5-largeness of the link of v, the vertex v_1 is not a neighbour of v_2. Let v',v'_1,v'_2 be the vertices from Corollary 4.4(ii). Note that if v' is a neighbour of y, then by the 5-largeness of the link of v, it lies in $K(\sigma')$. Thus we can assume that v' is not a neighbour of y and so $W_1 = (v,v_2yuv_1v')$ is a 5-wheel. Since $W_2 = (v',v_2v'_2v'_1v_1v)$ is also a 5-wheel, (W_1,W_2) is a (5,5)-dwheel, and so all the vertices of $W_1 \cup W_2$ have a common neighbour z, which lies in $K(v)$. By the 5-largeness of the link of v, the vertex z lies in $K(\sigma')$, and hence also in $K(\sigma)$. □

Lemma 4.8. *Let $n > 0$, and let σ be a simplex of $S_n(x)$. Then for any vertex v of σ, the complex $K(\sigma)$ is a 3-convex subcomplex of $K(v)$.*

Before the proof, let us note that from Corollary 4.6, Proposition 4.7, Lemma 4.8, Remark 4.3 and Lemma 4.1, we deduce:

Corollary 4.9. *Each $K(\sigma)$ is contractible.*

Proof of Lemma 4.8. Let abc be a path in $K(v)$ with a,c in $K(\sigma)$ at distance 2 in $K(v)$. Let y be any vertex of σ distinct from v. Applying the 5-largeness of the link of v to the cycle $abcy$, we obtain that b is a neighbour of y. Since this holds for each y, we have that b belongs to $K(\sigma)$, as desired. The connectivity of $K(\sigma)$ now follows from the fact that $K(v)$ has diameter ≤ 2. □

Below, the *open star* of a simplex σ of a simplicial complex X is the union of the interior of the simplex σ and all the open simplices of X containing the interior of σ in their closure.

Proof of the Main Theorem. Let X be a 7-located locally 5-large simplicial complex (possibly not locally finite and infinite-dimensional). By passing to the universal cover of X, we can assume that X is simply connected. It suffices to prove that each $B_n(x)$ is contractible, which we do by induction on n. Note that each finite subcomplex of $B_n(x)$ is contained in the span of the union of a finite subcomplex of $S_n(x)$ and the entire $B_{n-1}(x)$. Hence, by the Whitehead theorem, to prove the contractibility of $B_n(x)$, it suffices to show that for each finite induced subcomplex A of $S_n(x)$, the span A_0 of $A \cup B_{n-1}(x)$ is homotopy equivalent to $B_{n-1}(x)$. To this end, we order the simplices of A in the order of nonincreasing dimension as $\sigma_1,\sigma_2,\ldots,\sigma_k$. Let A_i be the (not necessarily induced) subcomplex obtained from A_{i-1} by removing the open star of σ_i. Equivalently, A_{i-1} is the union of A_i and the join of $K(\sigma_i)$ with σ_i along the join of $K(\sigma_i)$

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with $\partial\sigma_i$. Since both of these joins are contractible, A_{i-1} and A_i are homotopy equivalent [2, Lem 10.3(i)]. Consequently, A_0 is homotopy equivalent to $A_k = B_{n-1}(x)$. □

Acknowledgements. We are grateful to the anonymous referee for pointing out several omissions in the original version of the article, and for suggesting how to improve the presentation. We also thank Jingyin Huang for pointing us to [2].

Funding. The first author is partially supported by NSF DMS-2402105. The second author is partially supported by NSERC.

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