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# REFINED TESTS FOR SPATIAL CORRELATION

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We consider testing the null hypothesis of no spatial correlation against the alternative of pure first order spatial autoregression. A test statistic based on the least squares estimate has good first-order asymptotic properties, but these may not be relevant in small- or moderate-sized samples, especially as (depending on properties of the spatial weight matrix) the usual parametric rate of convergence may not be attained. We thus develop tests with more accurate size properties, by means of Edgeworth expansions and the bootstrap. Although the least squares estimate is inconsistent for the correlation parameter, we show that under quite general conditions its probability limit has the correct sign, and that least squares testing is consistent; we also establish asymptotic local power properties. The finite-sample performance of our tests is compared with others in Monte Carlo simulations.

## 1. INTRODUCTION

The modeling and analysis of spatially correlated data can pose significant complications and difficulties. Correlation across spatial data is typically a possibility, due to competition, spillovers, aggregation, and other circumstances, and might be anticipated in observable variables or in the unobserved disturbances in an econometric model, or both. In, for example, a linear regression model with exogenous regressors, if only the regressors are spatially correlated the usual rules for large sample inference (based on least squares) are unaffected. However, if also the disturbances are spatially correlated then though the least squares estimate (LSE) of the regression coefficients is likely to retain its consistency, its asymptotic variance matrix reflects the correlation. This matrix needs to be consistently estimated in order to carry out statistical inference, and its estimation (whether parametric or nonparametric) offers greater challenges than when time series data are involved, due to the lack of ordering in spatial data, as well as possible irregular spacing or lack of reliable information on locations. In addition the LSE is rendered asymptotically inefficient by spatial correlation, and developing generalized least squares estimates is similarly beset by ambiguities.

A sensible first step in data analysis is therefore to investigate whether or not there is evidence of spatial correlation, by carrying out a statistical test of the null hypothesis of no spatial correlation. A number of such tests have been developed,

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see e.g., Burridge (1980), Cliff and Ord (1981), Lee and Yu (2012), Kelejian and Prucha (2001), Li, Calder, and Cressie (2007), Martellosio (2012), Moran (1950), and Pinkse (2004). A number of them have been directed against the (first-order) spatial autoregression (SAR). For simplicity we stress the case of zero mean observable data, and will also allow for an unknown intercept, but our work can be extended to test for lack of spatial correlation in unobservable disturbances in more general models, such as regressions. Given the $n \times 1$ vector of observations $y = (y_1, \ldots, y_n)'$, the prime denoting transposition, the SAR model is

$$y = \lambda W y + \epsilon, \tag{1.1}$$

where $\epsilon = (\epsilon_1, \ldots, \epsilon_n)'$ consists of unobservable, uncorrelated random variables with zero mean and unknown variance $\sigma^2$, $\lambda$ is an unknown scalar, and $W$ is an $n \times n$ user-specified "weight" matrix, having $(i, j)$-th element $w_{ij}$, where $w_{ii} = 0$ for all $i$ and (in order to identify $\lambda$) normalization restrictions may be applied. Such restrictions imply that in general each element $w_{ij}$ changes as $n$increases, implying that $W$, and thus $y$, form triangular arrays (i.e., $W = W_n = (w_{ijn})$, $y = y_n = (y_{in})$) but we suppress reference to the $n$ subscript. The element $w_{ij}$ can be regarded as a (scaled) inverse economic distance between locations $i$ and $j$, where symmetry of $W$ is not necessarily imposed. Thus knowledge of actual locations is not required, extending the applicability of the model beyond situations when they are known, and entailing simpler modeling and theory than is typically possible when one attempts to incorporate locations of irregularly spaced geographical observations.

The null hypothesis of interest is

$$H_0 : \lambda = 0, \tag{1.2}$$

whence the $y_i$ are uncorrelated (and homoscedastic). An obvious statistic for testing (1.2) is the statistic based on the LSE $\hat{\lambda}$ of $\lambda$, which is given by

$$\hat{\lambda} = \frac{y'Wy}{y'W'Wy}. \tag{1.3}$$

Due to the dependence between right-hand side observables and disturbances in (1.1), $\hat{\lambda}$ is inconsistent for $\lambda$, as discussed by Lee (2002). However, $\hat{\lambda}$ does converge in probability to zero when $\lambda = 0$, so a test for (1.2) based on $\hat{\lambda}$ might be expected to be asymptotically valid. In particular, under (1.1), (1.2), and regularity conditions a central limit theorem for independent nonidentically distributed random variables gives

$$\left[ tr\left(WW'\right) / \left\{ tr\left(W^2 + WW'\right) \right\}^{1/2} \right] \hat{\lambda} \to_d \mathcal{N}(0, 1), \tag{1.4}$$

as $n \to \infty$. Since the square-bracketed norming factor can be directly computed, asymptotically valid tests against one-sided ($\lambda > 0$ or $\lambda < 0$) or two-sided ($\lambda \neq 0$) hypotheses are readily carried out.

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The accuracy of such tests is dependent on the magnitude of $n$, and the normal approximation might not be expected to be good for smallish $n$. Moreover, under conditions described later and as shown by Lee (2004) for the Gaussian maximum likelihood estimate (MLE) of $\lambda$, the rate of convergence in (1.4) can be less than the usual parametric rate $n^{1/2}$, depending on the assumptions imposed on $W$ as $n$ increases. In particular if $w_{ij} = O(1/h)$ is imposed, where the positive sequence $h = h_n$ can increase no faster than $n$, the rate is $(n/h)^{1/2}$, which increases more slowly than $n^{1/2}$ unless $h$ remains bounded. This outcome renders the usefulness of tests based on first-order asymptotics more dubious than in standard parametric situations.

Cliff and Ord (1971) noted that the limit distributions of tests for spatial independence can be inaccurate, and proposed an *ad hoc* correction. Higher-order asymptotic expansions can offer theoretically justifiable improvements in finite samples. Bao and Ullah (2007) derived the second-order bias and mean square error of the Gaussian MLE of $\lambda$ using a Nagar-type expansion, and Bao (2013) gave extensions to models with exogenous regressors and nonnormal disturbances, but neither reference studied test statistics. Various refinements of the Moran I/LM statistics have been presented by Cliff and Ord (1981), Terui and Kikuchi (1994), Robinson (2008), Baltagi and Yang (2013), and Robinson and Rossi (2014). Validity of Edgeworth expansions for the distribution of statistics in models involving SAR(1) processes has been established by Jin and Lee (2012) and Yang (2013). Earlier, in a quite general setting of irregularly spaced spatial observations, García-Soidán (1996) studied the validity of Edgeworth expansions for studentized and unstudentized estimates of a scalar parameter, extending work Götze and Hipp (1983) for mixing time series to coverage processes defined on an expanding subset of $\Re^k$, under the assumption of exponentially decaying correlations.

Here we develop tests derived from Edgeworth expansion of the cumulative distribution function (cdf) of $\hat{\lambda}$, which unlike the MLE is advantageously explicitly defined. The cdf of $\hat{\lambda}$ can be computed by simulation or other numerical techniques, while bootstrap tests can be employed to match our higher-order improvements, but our analytical approach sheds light on theoretical features and the order of magnitude of corrections. This seems especially relevant for the SAR model (1.1) because asymptotic theory depends on $h$ as well as $n$, and explains for example why tests against two-sided alternatives can be more accurate than one-sided tests. Our refined tests are justified under the assumption of normality on the $\epsilon_i$, but in this situation it would also be possible to develop exact similar tests, based on ratios of quadratic forms and having certain optimality properties, but the same is true in many settings in which Edgeworth refinements have been developed. Our tests are asymptotically similar, and the consistency and asymptotic efficiency properties established later in the paper do not rely on normality.

Formal Edgeworth expansions are established in the following section for both $\hat{\lambda}$ and for the LSE of an extended model that includes an unknown intercept.

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In Section 3 we deduce corrected critical values and corrected (asymptotically normal) test statistics and their properties against both one- and two-sided alternatives. Section 4 compares finite sample performance of our tests with bootstrap and Lagrange multiplier (LM) tests, and also compares Edgeworth approximations to the cdf with numerical calculations. The simple test based on (1.4) generally performs worse than the others, but Section 5 shows that under quite general conditions it is consistent against fixed alternatives, indeed the asymptotically biased $\hat{\lambda}$ actually exaggerates spatial dependence, while against local alternatives the left side of (1.4) has the same limit distribution as both the LM statistic and the Wald statistic based on the MLE, so it is efficient. Some final comments are offered in Section 6. Proofs are left to the Appendix.

## 2. EDGEWORTH EXPANSIONS FOR THE LEAST SQUARES ESTIMATE

The present section develops a (third-order) formal Edgeworth expansion for $\hat{\lambda}$ (1.3) under the null hypothesis of no spatial correlation (1.2). We introduce some assumptions.

**Assumption 1.** The $\epsilon_i$ are independent normal random variables with mean zero and unknown variance $\sigma^2$.

Normality is an unnecessarily strong condition for the first-order result (1.4), but it provides some motivation for stressing a quadratic form objective function and is familiar in higher-order asymptotic theory. Edgeworth expansions and resulting test statistics are otherwise complicated by the presence of cumulants of $\epsilon_i$. Assumption 1 implies that under (1.2) the $y_i$ are spatially independent.

For a real matrix $A$, let $||A||$ be the spectral norm of $A$ (i.e., the square root of the largest eigenvalue of $A'A$) and let $||A||_\infty$ be the maximum absolute row sums norm of $A$ (i.e., $||A||_\infty = \max_i \sum_j |a_{ij}|$, in which $a_{ij}$ is the $(i, j)$th element of $A$ and $i$ and $j$ vary respectively across all rows and columns of $A$). Let $K$ be a finite generic constant.

**Assumption 2.**

(i) For all $n$, $w_{ii} = 0, i = 1, \ldots, n$.

(ii) For all sufficiently large $n$, $W$ is uniformly bounded in row and column sums in absolute value, i.e., $||W||_\infty + ||W'||_\infty \leq K$.

(iii) For all sufficiently large $n$, uniformly in $i, j = 1, \ldots, n$, $w_{ij} = O(1/h)$, *where* $h = h_n$ is a positive sequence bounded away from zero for all $n$ such that $h/n \to 0$ as $n \to \infty$.

Parts (i) and (ii) of Assumption 2 are standard conditions on $W$ imposed in the literature. In particular, (ii) was introduced by Kelejian and Prucha (1998) to keep spatial correlation manageable. If $W$ is symmetric with nonnegative elements and row normalized, such that $\Sigma_{j=1}^n w_{ij} = 1$ for all $i$, then Assumption 2(ii) is

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automatically satisfied. Part (iii) covers two cases which have rather different implications for our results: either $h$ is bounded (when in (1.4) $\hat{\lambda}$ enjoys a parametric $n^{1/2}$ rate of convergence), or $h$ is divergent (when $\hat{\lambda}$ has a slower than parametric, $(n/h)^{1/2}$, rate).

By way of illustration consider (see Case, 1991),

$$W_n = I_r \otimes B_m, \quad B_m = \frac{1}{m-1}(l_m l_m' - I_m), \tag{2.1}$$

where $I_m$ is the $m \times m$ identity matrix, $l_m$ is the $m \times 1$ vector of 1's, and $\otimes$ denotes Kronecker product. Here $W$ is symmetric with nonnegative elements and row normalized, $n = mr$. Parts (i) and (ii) of Assumption 2 are satisfied, and $h \sim m$, where "$\sim$" throughout indicates that the ratio of left and right sides converges to a finite, nonzero constant. Thus in the bounded $h$ case only $r \to \infty$ as $n \to \infty$, whereas in the divergent $h$ case $m \to \infty$ and $r \to \infty$.

Now define

$$t_{ij} = \frac{h}{n} tr(W^i W'^j), \quad i \geq 0, \quad j \geq 0, \quad i + j \geq 1, \tag{2.2}$$

$$t = \frac{h}{n} tr((WW')^2). \tag{2.3}$$

Under Assumption 2 all $t_{ij}$ in (2.2) and $t$ are $O(1)$ (because, for any real $A$ such that $||A||_\infty \leq K$, we have $tr(AW) = O(n/h)$ ). To ensure the leading terms of the expansion in the theorem below are well defined, we introduce

**Assumption 3.**

$$\underline{\lim}_{n \to \infty} (t_{20} + t_{11}) > 0. \tag{2.4}$$

By the Cauchy inequality, Assumption 3 implies $\underline{\lim}_{n \to \infty} t_{11} > 0$, and the two conditions are equivalent when $W$ is symmetric or when its elements are all nonnegative. Assumption 3 is automatically satisfied under (2.1). It follows from Assumptions 2 and 3 that in (1.4) the norming factor

$$\frac{tr(WW')}{(tr(W^2 + WW'))^{1/2}} = \frac{t_{11}}{(t_{20} + t_{11})^{1/2}} \left(\frac{n}{h}\right)^{1/2} \sim \left(\frac{n}{h}\right)^{1/2}. \tag{2.5}$$

Now define

$$a = \frac{t_{11}}{(t_{20} + t_{11})^{1/2}}, \quad b = \frac{t_{21}}{(t_{20} + t_{11})^{1/2} t_{11}}, \quad c = \frac{2t_{30} + 6t_{21}}{(t_{20} + t_{11})^{3/2}}, \tag{2.6}$$

$$d = \frac{t}{t_{11}^2}, \quad e = \frac{12(t_{31} + t_{22})}{(t_{20} + t_{11}) t_{11}}, \quad f = \frac{6t_{40} + 24t_{31} + 6t_{22} + 12t}{(t_{20} + t_{11})^2}, \quad g = \frac{1}{t_{20} + t_{11}} \tag{2.7}$$

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and

$$U(\zeta) = 2b\zeta^2 - \frac{c}{6}H_2(\zeta), \tag{2.8}$$

$$V(\zeta) = \frac{1}{6}(e-6bc)\zeta H_2(\zeta) - (d-6b^2)\zeta^3 - \frac{1}{24}fH_3(\zeta) + \frac{1}{3}bc\zeta^2 H_3(\zeta) - 2b^2\zeta^5, \tag{2.9}$$

where $H_j(\zeta)$ is the $j$th Hermite polynomial, such that

$$H_2(\zeta) = \zeta^2 - 1 \quad H_3(\zeta) = \zeta^3 - 3\zeta. \tag{2.10}$$

Thus $U(\zeta)$ is an even, generally nonhomogeneous, quadratic function of $\zeta$, while $V(\zeta)$ is an odd, generally nonhomogeneous, polynomial in $\zeta$ of degree 5.

Write $\Phi(\zeta) = Pr(Z \leq \zeta)$ for a standard normal random variable $Z$, and $\phi(\zeta)$ for the probability density function (pdf) of $Z$. Let $F(\zeta) = P\left((n/h)^{1/2}a\hat{\lambda} \leq \zeta\right)$.

THEOREM 1. *Let (1.1) and Assumptions 1–3 hold. Under $H_0$ in (1.2), for any real $\zeta$, $F(\zeta)$ admits the third order formal Edgeworth expansion*

$$F(\zeta) = \Phi(\zeta) + U(\zeta)\phi(\zeta)\left(\frac{h}{n}\right)^{1/2} + V(\zeta)\phi(\zeta)\frac{h}{n} + O\left(\left(\frac{h}{n}\right)^{3/2}\right), \tag{2.11}$$

*where*

$$U(\zeta) = O(1), \quad V(\zeta) = O(1), \tag{2.12}$$

*as $n \to \infty$.*

Generally, $U(\zeta)$ and $V(\zeta)$ are nonzero, whence there are leading correction terms of exact orders $(h/n)^{1/2}$ and $h/n$, and both terms are known functions of $\zeta$.

A corresponding result to Theorem 1 is available for the pure SAR model with unknown intercept, i.e.,

$$y = \mu l + \lambda Wy + \epsilon, \tag{2.13}$$

where $\mu$ is an unknown scalar and $l = l_n$. The LSE of $\lambda$ in (2.13) is

$$\tilde{\lambda} = \frac{y'W'Py}{y'W'PWy}, \tag{2.14}$$

where $P = I_n - ll'/n$. Under (1.2), the same kind of regularity conditions and the additional

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**Assumption 4.** For all $n$, $\Sigma_{j=1}^{n} w_{ij} = 1$, $i = 1, \ldots, n$,

$\tilde{\lambda}$ has the same first-order limit distribution as $\hat{\lambda}$, so (1.4) holds with $\hat{\lambda}$ replaced by $\tilde{\lambda}$. However, the second- and higher-order limit distributions differ. In case Assumption 4 is not satisfied also the first-order limit distribution of $\tilde{\lambda}$ under (1.2) differs from that of $\hat{\lambda}$ and, in particular, $\tilde{\lambda}$ converges to the true value at the standard $n^{1/2}$ rate whether $h$ is bounded or divergent as $n \to \infty$. Since the main goal of this paper is to provide refined tests when the rate of convergence might be slower than the parametric rate $n^{1/2}$, the case of model (2.13) when $W$ is not row-normalized is not considered here.

Define

$$\tilde{U}(\zeta) = U(\zeta) + g^{1/2} \tag{2.15}$$

and

$$\tilde{V}(\zeta) = V(\zeta) + \left\{ \frac{g}{2}(1+p) + 2bg^{1/2} - \frac{g^4}{2} \right\} \zeta - 2bg^2\zeta^3 + \frac{cg^{1/2}}{6} H_3(\zeta), \tag{2.16}$$

where

$$p = l'WW'l/n. \tag{2.17}$$

(When $W$ is symmetric Assumption 4 implies $p = 1$.) Let $\tilde{F}(\zeta) = P((n/h)^{1/2} a\tilde{\lambda} \leq \zeta)$.

THEOREM 2. *Let (2.13) and Assumptions 1–4 hold. Under $H_0$ in (1.2), for any real $\zeta$, $\tilde{F}(\zeta)$ admits the third order formal Edgeworth expansion*

$$\tilde{F}(\zeta) = \Phi(\zeta) + \tilde{U}(\zeta)\phi(\zeta)\left(\frac{h}{n}\right)^{1/2} + \tilde{V}(\zeta)\phi(\zeta)\frac{h}{n} + O\left(\left(\frac{h}{n}\right)^{3/2}\right), \tag{2.18}$$

*where*

$$\tilde{U}(\zeta) = O(1), \quad \tilde{V}(\zeta) = O(1), \tag{2.19}$$

*as $n \to \infty$.*

The second- and third-order correction terms are again generally nonzero, and of orders $(h/n)^{1/2}$ and $h/n$ respectively. Notice that $\tilde{U}(\zeta) > U(\zeta)$, so the second-order approximate distribution function (df) of $\tilde{\lambda}$ is greater than that of $\hat{\lambda}$. The Edgeworth approximation in (2.18) is unaffected by $\mu$ (and the approximations in both (2.11) and (2.18) are unaffected by $\sigma^2$). Consequently results can be similarly obtained when there is a more general linear regression component than in (2.13), at least when regressors are nonstochastic or strictly exogenous. Indeed, similar techniques will yield approximations with respect to the model

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$y - \mu l = \lambda W(y - \mu l) + \epsilon$, or more general linear regression models with SAR disturbances.

Theorems 1 and 2 continue to hold after replacing $\epsilon$ in (1.1) by $w = f(\epsilon)\epsilon$, for almost surely nonzero, scalar functions $f$ (so in general the elements of $w$ form a triangular array). For example, if $f(\epsilon) = \left(\sigma^2 n/\epsilon'\epsilon\right)^{1/2}$ the elements of $w$ have zero mean and variance $\sigma^2$ and are uncorrelated, but they are not independent, indeed having a singular distribution for each $n$ (as therefore do the observations $y_i$), being uniformly distributed on the $n$-sphere with radius $\left(\sigma^2 n\right)^{1/2}$.

## 3. IMPROVED TESTS FOR NO SPATIAL CORRELATION

We consider first tests of the null hypothesis (1.2) against the alternative

$$H_1 : \lambda > 0 \tag{3.1}$$

in the no-intercept model (1.1).

For $\alpha \in (0, 1)$ (for example $\alpha = 0.05$ or $\alpha = 0.01$) define the normal critical value $z_\alpha$ such that $1 - \alpha = \Phi(z_\alpha)$. Write $q = (n/h)^{1/2} a\hat{\lambda}$. On the basis of (1.4) a test that rejects (1.2) against (3.1) when

$$q > z_\alpha \tag{3.2}$$

has approximate size $\alpha$. Theorem 1 readily yields more accurate tests that are simple to calculate because the coefficients of $U(\zeta)$ and $V(\zeta)$ are known, $W$ being chosen by the practitioner.

Define the exact critical value $w_\alpha$ such that $1 - \alpha = F(w_\alpha)$, so a test that rejects when $q > w_\alpha$ has exact size $\alpha$. Also introduce the Edgeworth corrected critical value

$$u_\alpha = z_\alpha - \left(\frac{h}{n}\right)^{1/2} U(z_\alpha). \tag{3.3}$$

COROLLARY 1. *Let (1.1) and Assumptions 1–3 hold. Under $H_0$ in (1.2), as $n \to \infty$*

$$w_\alpha = z_\alpha + O\left(\left(\frac{h}{n}\right)^{1/2}\right) \tag{3.4}$$

$$= u_\alpha + O\left(\frac{h}{n}\right). \tag{3.5}$$

Corollary 1 follows immediately from Theorem 1. From Corollary 1, the test that rejects (1.2) against (3.1) when

$$q > u_\alpha \tag{3.6}$$

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is more accurate than (3.2). Of course when the alternative of interest is $\lambda < 0$, the same conclusion can be drawn for the tests which reject when $q < -z_\alpha$, $q < -u_\alpha$, respectively.

Instead of correcting critical values we can derive from Theorem 1 a corrected test statistic that can be compared with $z_\alpha$. Introduce the polynomial

$$G(\zeta) = \zeta + \left(\frac{h}{n}\right)^{1/2} U(\zeta) + \frac{h}{n}\frac{1}{3}\left(2b - \frac{c}{6}\right)^2 \zeta^3 \tag{3.7}$$

which has known coefficients (see Yanagihara and Yuan, 2005). Since $G(\zeta)$ has derivative $(1 + \zeta(2b - c/6)(h/n)^{1/2})^2 > 0$, it is monotonically increasing. Thus $F(\zeta) = P(G(q) \leq G(\zeta))$ and we invert the expansion in Theorem 1 to obtain

COROLLARY 2. *Let (1.1) and Assumptions 1–3 hold. Under $H_0$, as $n \to \infty$*

$$P(G(q) > z_\alpha) = \alpha + O\left(\frac{h}{n}\right). \tag{3.8}$$

Thus the test that rejects when

$$G(q) > z_\alpha \tag{3.9}$$

has size that differs from $\alpha$ by smaller order than the size of (3.2).

Still more accurate tests can be deduced from Theorem 1 by employing also the third-order correction factor $V(\zeta)$, but the above tests have the advantage of simplicity. The $V$ term, however, is especially relevant in deriving improved tests against the two-sided alternative hypothesis

$$H_0 : \lambda \neq 0. \tag{3.10}$$

Because $U(\zeta)$ is an even function it follows from Theorem 1 that

$$P(|q| \leq \zeta) = 2\Phi(\zeta) - 1 + 2\frac{h}{n}V(\zeta) + O\left(\left(\frac{h}{n}\right)^{3/2}\right). \tag{3.11}$$

Thence define the Edgeworth-corrected critical value for a two-sided test,

$$v_{\alpha/2} = z_{\alpha/2} - \frac{n}{h}V(z_{\alpha/2}), \tag{3.12}$$

noting that the approximate size-$\alpha$ two-sided test based on (1.4) rejects $H_0$ against (3.10) when

$$|q| > z_{\alpha/2}. \tag{3.13}$$

Also, define $s_{\alpha/2}$ such that $P(|q| \leq s_{\alpha/2}) = 1 - \alpha$.

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COROLLARY 3. *Let (1.1) and Assumptions 1–3 hold. Under $H_0$, as $n \to \infty$*

$$s_{\alpha/2} = z_{\alpha/2} + O\left(\frac{h}{n}\right) \tag{3.14}$$

$$= \upsilon_{\alpha/2} + O\left(\left(\frac{h}{n}\right)^{3/2}\right). \tag{3.15}$$

Thus rejecting (1.2) against (3.10) when

$$|q| > \upsilon_{\alpha/2} \tag{3.16}$$

rather than (3.13) reduces the error to $O((h/n)^{3/2})$. In fact, Theorem 1 can be established to fourth-order, with fourth-order term that is even in $\zeta$, and error $O((h/n)^2)$, so the error in (3.15) can be improved to $O((h/n)^2)$.

As with the one-sided alternative (3.1), a corrected test statistic that can be compared with $z_{\alpha/2}$ can be derived from Theorem 1. Define (Yanagihara and Yuan, 2005)

$$L(\zeta) = \zeta + \frac{h}{n}V(\zeta) + \left(\frac{h}{n}\right)^2 \times \frac{1}{4}\left(L_1^2\zeta + \frac{L_2^2\zeta^5}{5} + \frac{L_3^2\zeta^9}{9} + \frac{2}{3}L_1L_2\zeta^3 + \frac{2}{5}L_1L_3\zeta^5 + \frac{2}{7}L_2L_3\zeta^7\right), \tag{3.17}$$

where $L_1 = -\frac{1}{6}(e - 6bc) + \frac{1}{8}f$, $L_2 = \frac{1}{2}(e - 6bc) - 3(d - 6b^2) - \frac{1}{8}f - 3bc$, and $L_3 = \frac{5}{3}bc - 10b^2$, so $L(\zeta)$ is a degree-7 polynomial in $\zeta$ with known coefficients. It is readily checked that $V(\zeta)$ has derivative $L_1 + L_2\zeta^2 + L_3\zeta^4$, where $L(\zeta)$ has derivative $(1 + (h/n)(L_1 + L_2\zeta^2 + L_3\zeta^4)/2)^2 > 0$ and is thus monotonically increasing. Therefore, from (3.11), we obtain

COROLLARY 4. *Let (1.1) and Assumptions 1–3 hold. Under $H_0$, as $n \to \infty$*

$$P(L(|q|) > z_{\alpha/2}) = \alpha + O\left(\left(\frac{h}{n}\right)^{3/2}\right). \tag{3.18}$$

The transformation in (3.17) and Corollary 4 follow from (3.11) using a minor modification of Theorem 2 of Yanagihara and Yuan (2005). From the latter result, we conclude that the test that rejects $H_0$ against (3.10) when

$$L(|q|) > z_{\alpha/2} \tag{3.19}$$

has size which is closer to $\alpha$ than (3.13).

Improved tests can be similarly derived from Theorem 2 for the intercept model in (2.13). We first consider tests of $H_0$ in (1.2) against (3.1). Let $\tilde{q} = (n/h)^{1/2}a\tilde{\lambda}$.

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A standard test based on first order asymptotic theory rejects (1.2) against (3.1) at approximate level $\alpha$ when

$$\tilde{q} > z_\alpha. \tag{3.20}$$

Define the exact and Edgeworth-corrected critical values $\tilde{w}_\alpha$, such that $1-\alpha = \tilde{F}(\tilde{w}_\alpha)$, and $\tilde{u}_\alpha = z_\alpha - \tilde{U}(z_\alpha)(h/n)^{1/2} = u_\alpha - g^{1/2}(h/n)^{1/2}$, respectively.

Similar to Corollaries 1 and 2, from Theorem 2 we deduce

COROLLARY 5. *Let (2.13) and Assumptions 1–4 hold. Under $H_0$ in (1.2), as $n \to \infty$*

$$\tilde{w}_\alpha = z_\alpha + O\left(\left(\frac{h}{n}\right)^{1/2}\right) \tag{3.21}$$

$$= \tilde{u}_\alpha + O\left(\frac{h}{n}\right). \tag{3.22}$$

Notice that $\tilde{u}_\alpha < u_\alpha$ for any $\alpha$, so that the second-order corrected critical value is lower for the intercept model.

Let

$$\tilde{G}(\zeta) = \zeta + \left(\frac{h}{n}\right)^{1/2} \tilde{U}(\zeta) + \frac{h}{n}\frac{1}{3}\left(2b - \frac{c}{6}\right)^2 \zeta^3 = G(\zeta) + \left(\frac{h}{n}\right)^{1/2} g^{1/2}. \tag{3.23}$$

COROLLARY 6. *Let (2.13) and Assumptions 1–4 hold. Under $H_0$ in (1.2), as $n \to \infty$*

$$P(\tilde{G}(\tilde{q}) > z_\alpha) = \alpha + O\left(\frac{h}{n}\right). \tag{3.24}$$

Thus, tests that reject (1.2) against (3.1) when either

$$\tilde{q} > \tilde{u}_\alpha \tag{3.25}$$

or

$$\tilde{G}(\tilde{q}) > z_\alpha \tag{3.26}$$

are more accurate than (3.20).

Also, from Theorem 2 improved tests of (1.2) against (3.10) can be deduced. From (2.18), since $\tilde{U}(\zeta)$ is an even function we obtain

$$P(|\tilde{q}| \le \zeta) = 2\Phi(\zeta) - 1 + 2\frac{h}{n}\tilde{V}(\zeta) + O\left(\left(\frac{h}{n}\right)^{3/2}\right). \tag{3.27}$$

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Define $\tilde{s}_{\alpha/2}$ such that $P(|\tilde{q}| \leq \tilde{s}_{\alpha/2}) = 1 - \alpha$ and $\tilde{v}_{\alpha/2} = z_{\alpha/2} - (n/h)\tilde{V}(z_{\alpha/2})$. A standard, approximate size $\alpha$, two-sided test rejects (1.2) against (3.10) when

$$|\tilde{q}| > z_{\alpha/2}. \tag{3.28}$$

From (3.27) we deduce

COROLLARY 7. *Let (2.13) and Assumptions 1–4 hold. Under $H_0$, as $n \to \infty$*

$$\tilde{s}_{\alpha/2} = z_{\alpha/2} + O\left(\frac{h}{n}\right) \tag{3.29}$$

$$= \tilde{v}_{\alpha/2} + O\left(\left(\frac{h}{n}\right)^{3/2}\right). \tag{3.30}$$

Finally, define

$$\tilde{L}(\zeta) = \zeta + \frac{h}{n}\tilde{V}(\zeta) + \left(\frac{h}{n}\right)^2 \frac{1}{4} \times \left(\tilde{L}_1^2\zeta + \frac{\tilde{L}_2^2\zeta^5}{5} + \frac{L_3^2\zeta^9}{9} + \frac{2}{3}\tilde{L}_1\tilde{L}_2\zeta^3 + \frac{2}{5}\tilde{L}_1 L_3\zeta^5 + \frac{2}{7}\tilde{L}_2 L_3\zeta^7\right), \tag{3.31}$$

where $\tilde{L}_1 = L_1 + \frac{g}{2}(1+p) + 2bg^{1/2} - \frac{g^4}{2} - \frac{cg^{1/2}}{2}$, $\tilde{L}_2 = L_2 - 6bg^{1/2} + \frac{cg^{1/2}}{2}$.

COROLLARY 8. *Let (2.13) and Assumptions 1–4 hold. Under $H_0$, as $n \to \infty$*

$$P(\tilde{L}(|\tilde{q}|) > z_{\alpha/2}) = \alpha + O\left(\left(\frac{h}{n}\right)^{3/2}\right). \tag{3.32}$$

From Corollaries 7 and 8, we conclude that the tests that reject $H_0$ against (3.10) when either

$$|\tilde{q}| > \tilde{v}_{\alpha/2} \tag{3.33}$$

or

$$\tilde{L}(|\tilde{q}|) > z_{\alpha/2} \tag{3.34}$$

have sizes closer to $\alpha$ than that obtained from (3.28).

## 4. MONTE CARLO COMPARISON OF FINITE SAMPLE PERFORMANCE

In this section we report and discuss a Monte Carlo investigation of the finite sample performance of the tests derived in Section 3 and of bootstrap tests, given that in many circumstances the bootstrap is known to achieve a first-order Edgeworth correction (see e.g., Singh, 1981). We focus here on the no-intercept

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model (1.1), corresponding results for the intercept model (2.13) can be found in Robinson and Rossi (2013a). For the no-intercept model (1.1) the bootstrap test is as follows (e.g., Paparoditis and Politis, 2005). We construct 199 $n \times 1$ vectors $\epsilon_j^*$ for $j = 1, \ldots, 199$, where each $\epsilon_j^*$ is obtained by resampling with replacement from $y_i - \sum_1^n y_i/n$, $i = 1, \ldots, n$. The bootstrap test statistic is $q_j^* = (n/h)^{1/2} a \epsilon_j^{*'} W_j'^* / \epsilon_j^{*'} W_j'^*$, $j = 1, \ldots, 199$, its $(1-\alpha)$th percentile being $u_\alpha^*$ which solves $\sum_{j=1}^{199} 1(q_j^* \leq u_\alpha^*)/199 \leq 1-\alpha$, where $1(.)$ indicates the indicator function. We reject (1.2) against the one-sided alternative (3.1) when

$$q > u_\alpha^*. \tag{4.1}$$

Defining the $(1-\alpha)$th percentile of $|q_j^*|$ as the value $v_\alpha^*$ solving $\sum_{j=1}^{199} 1(|q_j^*| \leq v_\alpha^*)/199 \leq 1-\alpha$, we reject (1.2) against the two-sided alternative (3.10) if

$$|q| > v_\alpha^*. \tag{4.2}$$

We also compare our tests with ones based on the (signed) square root of the LM statistic and its mean-variance corrected version (Moran, 1950; Cliff and Ord, 1981), defined respectively as

$$L = \frac{n}{(tr(WW'^2))^{1/2}} \frac{y'MWMy}{y'My} \tag{4.3}$$

and

$$CL = \gamma^{-1/2} \left( \frac{y'MWMy}{y'My} - \beta \right), \tag{4.4}$$

where

$$\begin{aligned} \beta &= -\frac{n}{n-k} tr((I-M)W), \\ \gamma &= \frac{n^2 tr(MWMW + MWMW')}{(n-k)(n-k+2)} - \frac{2n^2 (tr((I-M)W))^2}{(n-k)^2(n-k+2)}, \end{aligned} \tag{4.5}$$

in which we take $k = 0$, $M = I$ for the no-intercept model (1.1) and $k = 1$, $M = P$ for the intercept model (2.13). For the respective statistics, we reject $H_0$ (1.2) against (3.1) when

$$L > z_\alpha, \tag{4.6}$$

and when

$$CL > z_\alpha, \tag{4.7}$$

while we reject $H_0$ (1.2) against (3.10) when

$$|L| > z_{\alpha/2}, \tag{4.8}$$

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and when

$$|CL| > z_{\alpha/2}. \tag{4.9}$$

In the simulations we set $\sigma^2 = 1$ in Assumption 1 and choose $W$ as in (2.1), for various $m$ and $r$. Recalling that orders of magnitudes in Theorems 1 and 2 are affected by whether $h$ diverges or remains bounded as $n \to \infty$, we represent both cases by different choices of $m \sim h$. We choose $(m, r) = (8, 5)$, $(12, 8)$, $(18, 11)$, $(28, 14)$, i.e., $n = 40, 96, 198, 392$ to represent "divergent" $h$, and $(m, r) = (5, 8)$, $(5, 20)$, $(5, 40)$, $(5, 80)$, i.e., $n = 40, 100, 200, 400$ to represent "bounded" $h$. For each of these combinations we compute $\hat{\lambda}$ across $1,000$ replications. In all tests $\alpha = 0.05$.

Empirical sizes are displayed in Tables 1–4, in which "normal" , "Edgeworth" , "transformation" , "bootstrap" , "L", and "corrected L" refer respectively to tests using the standard normal approximation, Edgeworth-corrected critical values, Edgeworth-corrected test statistic, bootstrap critical values, LM statistic, and LM corrected statistic, and the respective abbreviations N, E, T, B, L, and CL will be extensively used in the text.

Tables 1 and 2 cover one-sided tests (3.2), (3.6), (3.9), (4.1), (4.6), (4.7) in the no-intercept model (1.1), when $h$ is respectively "divergent" and "bounded".

**TABLE 1.** Empirical sizes (nominal $\alpha = 0.05$) of tests of $H_0$ (1.2) against $H_1$ (3.1) in no-intercept model (1.1) when $h$ is "divergent"

| | $m = 8$ $r = 5$ | $m = 12$ $r = 8$ | $m = 18$ $r = 11$ | $m = 28$ $r = 14$ |
|---|---|---|---|---|
| Normal | 0 | 0 | 0.001 | 0.001 |
| Edgeworth | 0.004 | 0.008 | 0.010 | 0.016 |
| Transformation | 0.036 | 0.038 | 0.040 | 0.047 |
| Bootstrap | 0.048 | 0.055 | 0.038 | 0.042 |
| L | 0.078 | 0.072 | 0.064 | 0.074 |
| Corrected L | 0.077 | 0.066 | 0.061 | 0.067 |

**TABLE 2.** Empirical sizes (nominal $\alpha = 0.05$) of tests of $H_0$ (1.2) against $H_1$ (3.1) in no-intercept model (1.1) when $h$ is "bounded"

| | $m = 5$ $r = 8$ | $m = 5$ $r = 20$ | $m = 5$ $r = 40$ | $m = 5$ $r = 80$ |
|---|---|---|---|---|
| Normal | 0.001 | 0.001 | 0.001 | 0.011 |
| Edgeworth | 0.001 | 0.025 | 0.028 | 0.034 |
| Transformation | 0.042 | 0.045 | 0.043 | 0.052 |
| Bootstrap | 0.059 | 0.052 | 0.043 | 0.046 |
| L | 0.069 | 0.060 | 0.064 | 0.055 |
| Corrected L | 0.065 | 0.061 | 0.060 | 0.53 |

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Test N is drastically undersized for each $n$ in both tables. The sizes for E are somewhat better, and improve as $n$ increases, in particular, for "divergent" $h$ the discrepancy between empirical and nominal sizes is 18% lower relative to N, on average across sample size. Both T and B perform well for all $n$. Indeed, on average, when $h$ is "divergent" empirical sizes for T and B are 80% and 86%, respectively, closer to 0.05 than those for N, with a similar pattern in Table 2. From Table 2, the average improvements offered by E, T, and B over N are about 41%, 89%, and 88%, respectively. Overall, T and B perform best among the tests based on LSE. Tables 1 and 2 are consistent with Theorem 1 in which $F$ converges to $\Phi$ at rate $n^{1/2}$ when $h$ is bounded, but only at rate $(n/h)^{1/2}$ when $h$ is divergent. Indeed, for N, when $h$ is "bounded" , the difference between empirical and nominal size decreases by 20% as $n$ increases from $n = 40$ to $n = 400$, while this difference only decreases by 2% in case $h$ is "divergent" as $n$ increases from $n = 40$ to $n = 392$. From Tables 1 and 2, L and CL drastically outperform both N and E, but on average sizes for T are, respectively, 53% and 52% closer to 5% than those of L and CL when $h$ is "divergent" . The latter figures are 54% and 51% when $h$ is "bounded".

Tables 3 and 4 cover two-sided tests for the no-intercept model (1.1), namely (3.13), (3.16), (3.19), (4.2), (4.8), and (4.9). Again, N is very poor, though

**TABLE 3.** Empirical sizes (nominal $\alpha = 0.05$) of tests of $H_0$ (1.2) against $H_1$ (3.10) in no-intercept model (1.1) when $h$ is "divergent"

| | $m = 8$ $r = 5$ | $m = 12$ $r = 8$ | $m = 18$ $r = 11$ | $m = 28$ $r = 14$ |
|---|---|---|---|---|
| Normal | 0.132 | 0.130 | 0.126 | 0.106 |
| Edgeworth | 0.062 | 0.058 | 0.060 | 0.057 |
| Transformation | 0.105 | 0.088 | 0.073 | 0.060 |
| Bootstrap | 0.064 | 0.048 | 0.046 | 0.045 |
| L | 0.033 | 0.036 | 0.038 | 0.033 |
| Corrected L | 0.036 | 0.040 | 0.042 | 0.044 |

**TABLE 4.** Empirical sizes (nominal $\alpha = 0.05$) of tests of $H_0$ (1.2) against $H_1$ (3.10) in no-intercept model (1.1) when $h$ is "bounded"

| | $m = 5$ $r = 8$ | $m = 5$ $r = 20$ | $m = 5$ $r = 40$ | $m = 5$ $r = 80$ |
|---|---|---|---|---|
| Normal | 0.096 | 0.078 | 0.068 | 0.061 |
| Edgeworth | 0.062 | 0.051 | 0.049 | 0.052 |
| Transformation | 0.055 | 0.025 | 0.042 | 0.052 |
| Bootstrap | 0.044 | 0.053 | 0.047 | 0.059 |
| L | 0.032 | 0.043 | 0.044 | 0.044 |
| Corrected L | 0.039 | 0.044 | 0.042 | 0.045 |

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contrary to the one-sided test case the problem is now oversizing, and E, T, and B all offer notable improvements. Instead, both L and CL appear to be undersized. When $h$ is "divergent" the difference between empirical and nominal sizes is reduced respectively on average across sample sizes by 87%, 59%, and 91% for E, T, and B relative to N, and by 86%, 59%, and 69% when $h$ is "bounded". In case $h$ is "divergent" , L and CL are closer to the nominal 0.05 than both T and N. However, on average E outperforms L by 37%, while CL and E are comparable. When $h$ is "bounded" , again L and CL perform better than N and T, but E offers a significant improvement over both L and CL of 67% and 55%, respectively. From Robinson and Rossi (2013), the pattern of results for the intercept model (2.13) is similar to that displayed in Tables 1–4.

Monte Carlo results for our tests of (1.2) against a two-sided alternative can be compared with the LM-based Edgeworth-corrected tests derived in Robinson and Rossi (2014). Sizes for E and T in Tables 3 and 4 are relatively satisfactory also in light of those in Robinson and Rossi (2014). In particular, for "bounded" $h$ the deviation of empirical sizes for both E and T from the nominal 5% appears to be similar to that for their Edgeworth-corrected tests, while for "divergent" $h$ only E has a similar performance to theirs, T only offering a modest improvement over N. In general the Edgeworth-corrected tests in Robinson and Rossi (2014) are undersized, while from Tables 3 and 4 it is clear that the problem here is oversizing.

To illustrate the effect of the transformation $G(.)$ used in Section 3, in Figure 1 we plot the histograms with 100 bins of $q$ and $G(q)$ obtained from 1,000 replications when $m = 28$ and $r = 14$. The figure suggests that the density of $q$ is very skewed to the left and that most of the skewness is removed by the transformations, as in Hall (1992).

In Tables 5 and 6 we assess power against a fixed alternative, i.e.,

$$H_1 : \lambda = \bar{\lambda} > 0 \tag{4.10}$$

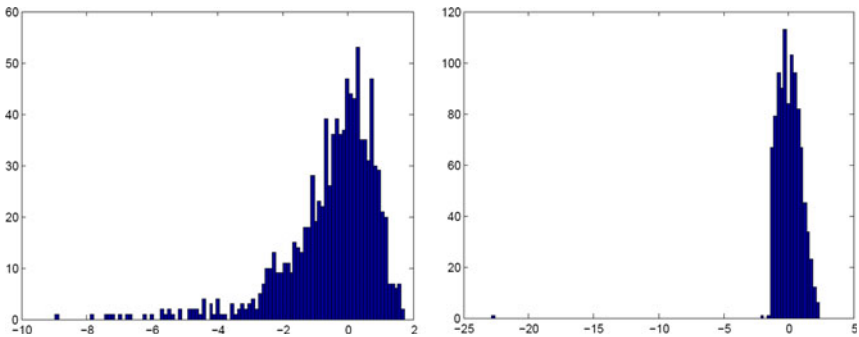



**FIGURE 1.** Histograms of $q$ (left picture) and $G(q)$ (right picture) for 1,000 replications, $m = 28$, $r = 14$.

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**TABLE 5.** Empirical powers of tests of $H_0$ (1.2) against $H_1$ (4.12), with nominal size $\alpha = 0.05$ in no-intercept model (1.1) when $h$ is "divergent"

| | $\bar{\lambda}$ | $m = 8$ $r = 5$ | $m = 12$ $r = 8$ | $m = 18$ $r = 11$ | $m = 28$ $r = 14$ |
|---|---|---|---|---|---|
| | 0.1 | 0 | 0 | 0.005 | 0.009 |
| Normal | 0.5 | 0 | 0.335 | 0.673 | 0.854 |
| | 0.8 | 0.257 | 0.994 | 1 | 1 |
| | 0.1 | 0.001 | 0.008 | 0.013 | 0.019 |
| Edgeworth | 0.5 | 0.200 | 0.562 | 0.764 | 0.904 |
| | 0.8 | 0.957 | 0.998 | 1 | 1 |
| | 0.1 | 0.059 | 0.087 | 0.129 | 0.130 |
| Transformation | 0.5 | 0.680 | 0.854 | 0.924 | 0.958 |
| | 0.8 | 0.986 | 0.999 | 1 | 1 |
| | 0.1 | 0.085 | 0.130 | 0.165 | 0.168 |
| Bootstrap | 0.5 | 0.738 | 0.886 | 0.928 | 0.973 |
| | 0.8 | 0.990 | 1 | 1 | 1 |
| | 0.1 | 0.109 | 0.137 | 0.168 | 0.183 |
| L | 0.5 | 0.758 | 0.885 | 0.940 | 0.978 |
| | 0.8 | 0.998 | 1 | 1 | 1 |
| | 0.1 | 0.124 | 0.148 | 0.165 | 0.174 |
| Corrected L | 0.5 | 0.775 | 0.884 | 0.938 | 0.960 |
| | 0.8 | 0.995 | 0.998 | 1 | 1 |

when $h$ is "divergent" and "bounded" respectively. These are non-size-corrected tests. Except for the smallest sample size when $h$ is "divergent", even N performs well for the largest $\bar{\lambda} = 0.8$, as do all other tests in all settings. N also does comparably well to E, T, and B when $h$ is bounded and $\bar{\lambda} = 0.5$. But overall N is outperformed by the other tests, with T and B offering the greatest power among the LSE-based tests. Tables 5 and 6 suggest that T is slightly outperformed by L and CL for almost all sample sizes considered, but the opposite holds true for the intercept model in (2.13), as reported in Robinson and Rossi (2013).

A direct comparison with empirical powers of corrected-tests in Robinson and Rossi (2014) is not possible because here we are focusing on tests of $H_0$ against a one-sided alternative, (4.10), while in their Monte Carlo study empirical powers of tests of $H_0$ against $H_1 : \lambda = \bar{\lambda} \neq 0$ are reported. From Robinson and Rossi (2014), however, it is clear that Edgeworth-corrected tests display similar power to that of the standard $\chi^2$ test. Thus, we might expect that one-sided Edgeworth-corrected tests based on L would have similar power to that of L reported in Tables 5 and 6.

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**TABLE 6.** Empirical powers of tests of $H_0$ (1.2) against $H_1$ (4.12), with nominal size $\alpha = 0.05$ in no-intercept model (1.1) when $h$ is "bounded"

| | $\bar{\lambda}$ | $m=5$ $r=8$ | $m=5$ $r=20$ | $m=5$ $r=40$ | $m=5$ $r=80$ |
|---|---|---|---|---|---|
| | 0.1 | 0.010 | 0.083 | 0.187 | 0.363 |
| Normal | 0.5 | 0.551 | 0.988 | 1 | 1 |
| | 0.8 | 0.999 | 1 | 1 | 1 |
| | 0.1 | 0.016 | 0.095 | 0.200 | 0.375 |
| Edgeworth | 0.5 | 0.676 | 0.992 | 1 | 1 |
| | 0.8 | 1 | 1 | 1 | 1 |
| | 0.1 | 0.122 | 0.172 | 0.280 | 0.420 |
| Transformation | 0.5 | 0.858 | 0.993 | 1 | 1 |
| | 0.8 | 1 | 1 | 1 | 1 |
| | 0.1 | 0.127 | 0.204 | 0.291 | 0.441 |
| Bootstrap | 0.5 | 0.872 | 0.991 | 1 | 1 |
| | 0.8 | 1 | 1 | 1 | 1 |
| | 0.1 | 0.135 | 0.194 | 0.293 | 0.442 |
| L | 0.5 | 0.873 | 0.993 | 1 | 1 |
| | 0.8 | 1 | 1 | 1 | 1 |
| | 0.1 | 0.146 | 0.190 | 0.277 | 0.454 |
| Corrected L | 0.5 | 0.894 | 0.995 | 1 | 1 |
| | 0.8 | 0.997 | 1 | 1 | 1 |

Finally we calculate numerically $F(\zeta)$ for various $\zeta$ by means of Imhof's (1961) procedure and find that Theorem 1 approximation works fairly well. Numerical algorithms do have limitations; Lu and King (2002) surveyed the numerical evaluation of the cdf of normal quadratic forms. Let

$$F_3(\zeta) = \Phi(\zeta) + U(\zeta)\phi(\zeta)\left(\frac{h}{n}\right)^{1/2} + V(\zeta)\phi(\zeta)\frac{h}{n} \tag{4.11}$$

be the third-order Edgeworth corrected cdf of $q$. Tables 7 and 8 compare $F_3(\zeta)$ for $\zeta$ representing one- and two-sided 5% normal critical values with numerical calculations for "divergent" and "bounded" $h$, respectively, and in both Tables "Edgeworth" refers to (4.11), while "exact" refers to the numerical procedure.

In Tables 7 and 8, the "exact" results confirm that $F(\zeta)$ is heavily skewed to the left, all values being above the normal cdf ones, for all sample sizes considered. Although skewness decreases with increasing $n$, $F(\zeta)$ converges quite slowly, especially for "divergent" $h$, confirming the theory. For very small $n$, (4.11) returns some values that slightly exceed 1, but the problem disappears as $n$ increases. Otherwise, the agreement between Edgeworth-corrected and exact values in the

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**TABLE 7.** Comparison between (4.13) and "exact" cdf for various values of $\zeta$ for the no-intercept model (1.1) when $h$ is "divergent"

| $\zeta$ | | $m=8$ $r=5$ | $m=12$ $r=8$ | $m=18$ $r=11$ | $m=28$ $r=14$ |
|---|---|---|---|---|---|
| 1.96 | Edgeworth | 0.986 | 0.992 | 0.995 | 0.996 |
| | exact | 0.999 | 0.999 | 0.999 | 0.999 |
| 1.645 | Edgeworth | 0.002 | 1 | 1 | 1 |
| | exact | 0.999 | 0.999 | 0.999 | 0.998 |
| −1.645 | Edgeworth | 0.208 | 0.181 | 0.164 | 0.153 |
| | exact | 0.180 | 0.161 | 0.149 | 0.140 |
| −1.96 | Edgeworth | 0.185 | 0.154 | 0.136 | 0.124 |
| | exact | 0.145 | 0.127 | 0.111 | 0.107 |

**TABLE 8.** Comparison between (4.13) and "exact" cdf for various values of $\zeta$ for the no-intercept model (1.1) when $h$ is "bounded"

| $\zeta$ | | $m=5$ $r=8$ | $m=5$ $r=20$ | $m=5$ $r=40$ | $m=5$ $r=80$ |
|---|---|---|---|---|---|
| 1.96 | Edgeworth | 1 | 0.998 | 0.994 | 0.989 |
| | exact | 0.999 | 0.998 | 0.994 | 0.990 |
| 1.645 | Edgeworth | 1 | 0.989 | 0.979 | 0.971 |
| | exact | 0.999 | 0.988 | 0.979 | 0.971 |
| 1.645 | Edgeworth | 0.142 | 0.104 | 0.086 | 0.075 |
| | exact | 0.132 | 0.100 | 0.085 | 0.074 |
| −1.96 | Edgeworth | 0.112 | 0.073 | 0.056 | 0.046 |
| | exact | 0.097 | 0.069 | 0.054 | 0.045 |

lower tail leaves something to be desired, especially for the intercept model (2.13) as reported in Robinson and Rossi (2013), but it improves quickly as $n$ increases. In the upper tail the agreement is instead very satisfactory, for both "divergent" and "bounded" $h$.

## 5. CONSISTENCY AND LOCAL POWER OF LSE-BASED TESTING

As previously remarked, $\hat{\lambda}$ and $\tilde{\lambda}$ are inconsistent when $\lambda$ is nonzero. Therefore, if it should be the case that $p\lim_{n\to\infty}\hat{\lambda} < \lambda \ \ (> \lambda)$ when $\lambda > 0$ ($\lambda < 0$), it might be feared that for some $\lambda \quad p\lim_{n\to\infty}\hat{\lambda} = 0$ as $n \to \infty$, with the same possibility for $\tilde{\lambda}$. Then the standard and corrected tests would be inconsistent. The following

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theorem shows that under fairly general conditions the direction of inconsistency of $\hat{\lambda}$ and $\tilde{\lambda}$ follows the sign of $\lambda$, so that the tests are actually consistent against fixed alternatives. We relax Assumption 1 to:

**Assumption 1.** $'$ The $\epsilon_i$ are independently and identically distributed with zero mean, variance $\sigma^2$, and finite fourth moment.

THEOREM 3. *Let Assumptions 1' and 2 (i) hold, let*

$$\underline{\lim}_{n\to\infty} t_{11} > 0 \tag{5.1}$$

*with h as defined in Assumption 2 (iii), and let*

$$\overline{\lim}_{n\to\infty} \|W\| \leq 1 \tag{5.2}$$

*and*

$$\lim_{n\to\infty} \frac{tr\left(W^2\right)}{tr\left(W'W\right)} \to \tau > 0. \tag{5.3}$$

*If also (a) W has nonnegative elements for all sufficiently large n, then for all* $\lambda \in [0, 1)$,

$$p\lim_{n\to\infty} \widetilde{\lambda} = p\lim_{n\to\infty} \widehat{\lambda} = \lambda(1+\xi(\lambda)), \quad (1-\lambda)^2\tau \leq \xi(\lambda) \leq \frac{(1+\lambda)^2}{1-\lambda}, \tag{5.4}$$

*or if (b) W is similar to a symmetric matrix for all sufficiently large n, then for all* $\lambda \in (-1, 1)$

$$p\lim_{n\to\infty} \widetilde{\lambda} = p\lim_{n\to\infty} \widehat{\lambda} = \lambda(1+\xi(\lambda)), \quad \frac{(1-|\lambda|)^2}{1+|\lambda|}\tau \leq \xi(\lambda) \leq \frac{(1+|\lambda|)^2}{1-|\lambda|}. \tag{5.5}$$

The proof is in the Appendix. Both sets of conditions on $W$ are clearly satisfied in the case (2.1). An example of a nonsymmetric $W$ for which (b) holds arises when an initially symmetric matrix is row-normalized. Condition (5.3) holds automatically if $W$ is symmetric, when $\tau = 1$. Note that we do not necessarily require that $\lim_{n\to\infty} t_{11}$ exists, and that as $|\lambda| \uparrow 1$ the upper bounds for $\xi(\lambda)$ tend to infinity while the lower bounds tend to zero, whereas as $\lambda \to 0$ both bounds under condition (b) tend to 1 while under (a) the upper and lower bounds respectively tend to 1 and $\tau \leq 1$. But most significantly, for each $|\lambda| \in (0, 1)$ the bounds are finite and positive, so the asymptotic bias $\lambda\xi(\lambda)$ of both $\widehat{\lambda}$ and $\widetilde{\lambda}$ is finite and shares the sign of $\lambda$, indeed $\widehat{\lambda}$ and $\widetilde{\lambda}$ tend to exaggerate the spatial correlation. We stress that (a) rules out any negative $\lambda$, $w_{ij}$, in which situation we have been unable to obtain a suitable lower bound under simple conditions, but $\lambda > 0$ seems the case of main practical relevance and negative $\lambda$, $w_{ij}$ are covered under (b). Recalling that $q = (n/h)^{1/2}a\hat{\lambda}$, $a = t_{11}^{1/2}(t_{20}/t_{11}+1)^{-1/2}$, it follows from (5.1)

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and (5.3) and Theorem 3 that when $\lambda > 0$, $q \to_p \infty$ as $n \to \infty$, so for (1.1) $P(q > z_\alpha) \to 1$, $P(q > u_\alpha) \to 1$, and $P(G(q) > z_\alpha) \to 1$ as $n \to \infty$, while for (2.13) $P(\tilde{q} > z_\alpha) \to 1$, $P(\tilde{q} > \tilde{u}_\alpha) \to 1$, and $P(\tilde{G}(\tilde{q}) > z_\alpha) \to 1$ as $n \to \infty$, with similar results when $\lambda < 0$ under condition (b), and obvious implications for two-sided tests.

We now consider behavior in the presence of local alternatives, namely

$$\lambda = \lambda_n = \delta \left(\frac{h}{n}\right)^{1/2}, \delta \neq 0. \tag{5.6}$$

THEOREM 4. *Let Assumptions 1' , 2, (5.2), and (5.3) hold, and let*

$$\omega = \lim_{n\to\infty} t_{11} \tag{5.7}$$

*exist and be positive. Then as* $n \to \infty$,

$$\frac{tr(WW')}{\left(tr(W^2 + WW')\right)^{1/2}} \hat{\lambda} \to_d \mathcal{N}(\omega^{1/2}(1+\tau)^{1/2}\delta, 1), \tag{5.8}$$

$$\frac{tr(WW')}{\left(tr(W^2 + WW')\right)^{1/2}} \tilde{\lambda} \to_d \mathcal{N}(\omega^{1/2}(1+\tau)^{1/2}\delta, 1). \tag{5.9}$$

Since $\omega(1+\tau)$ is the asymptotic information, it follows that the limit distributions in (5.8) and (5.9) are the same as those for the Wald statistic based on the MLE of $\lambda$ and for the LM statistic, so tests based on the normal approximation for $\hat{\lambda}$ and $\tilde{\lambda}$ are efficient.

## 6. FINAL COMMENTS

We have developed tests for lack of spatial correlation based on the LSE of the correlation parameter $\lambda$ in pure SAR and SAR with intercept, that have improved higher-order properties, and compared their finite-sample performance with other tests via Monte Carlo simulations. Although the LSE is inconsistent, we have shown under quite general conditions that LSE-based tests are consistent against fixed alternatives and locally efficient, to add to their computational appeal. Our methods can straightforwardly extend to derive improved LSE-based tests in models involving regressors.

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# APPENDIX

**Proof of Theorem 1.**

Under $H_0$, $\hat{\lambda} = \epsilon' W' \epsilon / \epsilon' W' W \epsilon$ and thus $P(\hat{\lambda} \leq x) = P(\varsigma \leq 0)$, where $\varsigma = \epsilon' (C + C')\epsilon/2$, $C = W' - xW'W$ and $x$ is any real number. We proceed much as in, e.g., Phillips (1977). Under Assumption 1, the characteristic function (cf) of $\varsigma$ is

$$
\begin{aligned}
E\left(e^{\frac{it}{2}\epsilon'(C+C')\epsilon}\right) &= \frac{1}{(2\pi)^{n/2}\sigma^n}\int_{\Re^n} e^{\frac{it}{2}\xi'(C+C')\xi} e^{-\frac{\xi'\xi}{2\sigma^2}} d\xi \\
&= \frac{1}{(2\pi)^{n/2}\sigma^n}\int_{\Re^n} e^{-\frac{1}{2\sigma^2}\xi'\left(I - it\sigma^2(C+C')\right)\xi} d\xi \\
&= det\left(I - it\sigma^2\left(C+C'\right)\right)^{-1/2} = \prod_{j=1}^{n}\left(1 - it\sigma^2\eta_j\right)^{-1/2}, \qquad \textbf{(A.1)}
\end{aligned}
$$

where the $\eta_j$ are eigenvalues of $C + C'$ and $det(A)$ denotes the determinant of a generic square matrix $A$. From (A.1) the cumulant generating function (cgf) of $\varsigma$ is

$$
\begin{aligned}
\psi(t) &= -\frac{1}{2}\sum_{j=1}^{n} ln\left(1 - it\sigma^2\eta_j\right) = \frac{1}{2}\sum_{j=1}^{n}\sum_{s=1}^{\infty}\frac{\left(it\sigma^2\eta_j\right)^s}{s} \\
&= \frac{1}{2}\sum_{s=1}^{\infty}\frac{\left(it\sigma^2\right)^s}{s}\sum_{j=1}^{n}\eta_j^s = \frac{1}{2}\sum_{s=1}^{\infty}\frac{\left(it\sigma^2\right)^s}{s} tr\left(\left(C+C'\right)^s\right). \qquad \textbf{(A.2)}
\end{aligned}
$$

Denoting by $\kappa_s$ the $s$-th cumulant of $\varsigma$, from (A.2)

$$\kappa_1 = \sigma^2 tr(C), \qquad \textbf{(A.3)}$$

$$\kappa_2 = \frac{\sigma^4}{2} tr\left(\left(C+C'\right)^2\right), \qquad \textbf{(A.4)}$$

$$\kappa_s = \frac{\sigma^{2s} s!}{2}\frac{tr\left(\left(C+C'\right)^s\right)}{s}, s > 2. \qquad \textbf{(A.5)}$$

Let $\varsigma^c = (\varsigma - \kappa_1)/\kappa_2^{1/2}$. The cgf of $\varsigma^c$ is

$$\psi^c(t) = -\frac{1}{2}t^2 + \sum_{s=3}^{\infty}\frac{\kappa_s^c (it)^s}{s!}, \qquad \textbf{(A.6)}$$

where

$$\kappa_s^c = \frac{\kappa_s}{\kappa_2^{s/2}}, \qquad \textbf{(A.7)}$$

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so the cf of $\varsigma^c$ is

$$E\left(e^{it\varsigma^c}\right) = e^{-\frac{1}{2}t^2}\exp\left\{\sum_{s=3}^{\infty}\frac{\kappa_s^c(it)^s}{s!}\right\}$$
$$= e^{-\frac{1}{2}t^2}\left\{1+\sum_{s=3}^{\infty}\frac{\kappa_s^c(it)^s}{s!}+\frac{1}{2!}\left(\sum_{s=3}^{\infty}\frac{\kappa_s^c(it)^s}{s!}\right)^2+\frac{1}{3!}\left(\sum_{s=3}^{\infty}\frac{\kappa_s^c(it)^s}{s!}\right)^3+\cdots\right\}$$
$$= e^{-\frac{1}{2}t^2}\left\{1+\frac{\kappa_3^c(it)^3}{3!}+\frac{\kappa_4^c(it)^4}{4!}+\frac{\kappa_5^c(it)^5}{5!}+\left\{\frac{\kappa_6^c}{6!}+\frac{(\kappa_3^c)^2}{(3!)^2}\right\}(it)^6+\cdots\right\}. \quad \textbf{(A.8)}$$

Thus by Fourier inversion, formally

$$P\left(\varsigma^c \le z\right) = \int_{-\infty}^{z}\phi(z)dz+\frac{\kappa_3^c}{3!}\int_{-\infty}^{z}H_3(z)\phi(z)dz+\frac{\kappa_4^c}{4!}\int_{-\infty}^{z}H_4(z)\phi(z)dz+\cdots. \quad \textbf{(A.9)}$$

Collecting the above results,

$$P\left(\hat{\lambda}\le x\right) = P\left(\varsigma \le 0\right) = P\left(\varsigma^c\kappa_2^{1/2}+\kappa_1\le 0\right) = P\left(\varsigma^c \le -\kappa_1^c\right)$$
$$= \Phi\left(-\kappa_1^c\right)-\frac{\kappa_3^c}{3!}\Phi^{(3)}\left(-\kappa_1^c\right)+\frac{\kappa_4^c}{4!}\Phi^{(4)}\left(-\kappa_1'\right)+\cdots. \quad \textbf{(A.10)}$$

From (A.3), (A.4), and (A.7),

$$\kappa_1^c = \frac{tr(C)}{\left(\frac{1}{2}tr\left(\left(C+C'\right)^2\right)\right)^{1/2}}. \quad \textbf{(A.11)}$$

The numerator of $\kappa_1^c$ is

$$tr\left(W\right)-xtr\left(WW'\right) = -xtr\left(WW'\right) = -\frac{n}{h}xt_{11}, \quad \textbf{(A.12)}$$

while its denominator is

$$\left(\frac{1}{2}tr\left(C+C'\right)^2\right)^{1/2} = \left(tr(W^2)+tr\left(WW'\right)-4xtr\left(W^2W'\right)+2x^2tr\left(\left(WW'\right)^2\right)\right)^{1/2}$$
$$= \left(\frac{n}{h}\right)^{1/2}\left(t_{20}+t_{11}-4xt_{21}+2x^2t\right)^{1/2}. \quad \textbf{(A.13)}$$

Thus

$$\kappa_1^c = \frac{-xt_{11}(n/h)^{1/2}}{\left(t_{20}+t_{11}-4xt_{21}+2x^2t\right)^{1/2}} = \frac{-xt_{11}(n/h)^{1/2}}{(t_{20}+t_{11})^{1/2}\left(1-\frac{4xt_{21}-2x^2t}{(t_{20}+t_{11})}\right)^{1/2}}. \quad \textbf{(A.14)}$$

Choose

$$x = \left(\frac{h}{n}\right)^{1/2}\frac{(t_{20}+t_{11})^{1/2}}{t_{11}}\zeta = \left(\frac{h}{n}\right)^{1/2}a^{-1}\zeta, \quad \textbf{(A.15)}$$

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where $a$ was defined in (2.6). By Taylor expansion

$$\kappa_1^c = -\zeta\left(1-\frac{4xt_{21}-2x^2t}{(t_{20}+t_{11})}\right)^{-1/2} = -\zeta - 2\left(\frac{h}{n}\right)^{1/2}\frac{t_{21}}{t_{11}(t_{20}+t_{11})^{1/2}}\zeta^2$$
$$+\frac{h}{n}\frac{t}{t_{11}^2}\zeta^3 - 6\frac{h}{n}\left(\frac{t_{21}}{(t_{20}+t_{11})^{1/2}t_{11}}\right)^2\zeta^3 + O\left(\left(\frac{h}{n}\right)^{3/2}\right)$$
$$= -\zeta - 2\left(\frac{h}{n}\right)^{1/2}b\zeta^2 + \frac{h}{n}d\zeta^3 - 6\frac{h}{n}b^2\zeta^3 + O\left(\left(\frac{h}{n}\right)^{3/2}\right), \tag{A.16}$$

where $b$ and $d$ were defined in (2.6) and (2.7). Then by Taylor expansion and using

$$(-d/dx)^j\Phi(x) = -H_{j-1}(x)\phi(x), \tag{A.17}$$

we have

$$\Phi(-\kappa_1^c) = \Phi\left(\zeta + 2\left(\frac{h}{n}\right)^{1/2}b\zeta^2 - \frac{h}{n}d\zeta^3 + 6\frac{h}{n}b^2\zeta^3 + O\left(\left(\frac{h}{n}\right)^{3/2}\right)\right)$$
$$= \Phi(\zeta) + \left(2\left(\frac{h}{n}\right)^{1/2}b\zeta^2 - \frac{h}{n}d\zeta^3 + 6\frac{h}{n}b^2\zeta^3\right)\phi(\zeta)$$
$$+2\frac{h}{n}b^2\zeta^4\Phi^{(2)}(\zeta) + O\left(\left(\frac{h}{n}\right)^{3/2}\right)$$
$$= \Phi(\zeta) + 2\left(\frac{h}{n}\right)^{1/2}b\zeta^2\phi(\zeta)$$
$$+\frac{h}{n}\left(-d\zeta^3 + b^2\left(6\zeta^3 - 2\zeta^4H_1(\zeta)\right)\right)\phi(\zeta) + O\left(\left(\frac{h}{n}\right)^{3/2}\right)$$
$$= \Phi(\zeta) + 2\left(\frac{h}{n}\right)^{1/2}b\zeta^2\phi(\zeta) + \frac{h}{n}\left(-d\zeta^3 + b^2\left(6\zeta^3 - 2\zeta^5\right)\right)\phi(\zeta)$$
$$+O\left(\left(\frac{h}{n}\right)^{3/2}\right). \tag{A.18}$$

Similarly,

$$\Phi^{(3)}\left(-\kappa_1^c\right) = \Phi^{(3)}(\zeta) + 2\left(\frac{h}{h}\right)^{1/2}b\zeta^2\Phi^{(4)}(\zeta) + O\left(\frac{h}{n}\right)$$
$$= \left(H_2(\zeta) - 2\left(\frac{h}{h}\right)^{1/2}b\zeta^2H_3(\zeta)\right)\phi(\zeta) + O\left(\frac{h}{n}\right). \tag{A.19}$$

From (A.5), (A.7),

$$\kappa_3^c = \frac{tr\left(\left(C+C'\right)^3\right)}{\left(\frac{1}{2}tr\left(\left(C+C'\right)^2\right)\right)^{3/2}}.$$

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By standard algebra, for $x$ defined in (A.15),

$$\frac{1}{2}tr\left(\left(C+C'\right)^2\right) = \frac{n}{h}\left(t_{20}+t_{11}-4\left(\frac{h}{n}\right)^{1/2}\frac{(t_{20}+t_{11})^{1/2}t_{21}}{t_{11}}\zeta+O\left(\frac{h}{n}\right)\right)$$
$$= \frac{n}{h}(t_{20}+t_{11})-4\left(\frac{n}{h}\right)^{1/2}\frac{(t_{20}+t_{11})^{1/2}t_{21}}{t_{11}}\zeta+O(1), \tag{A.20}$$
$$tr\left(\left(C+C'\right)^3\right) = \frac{n}{h}\left(2t_{30}+6t_{21}-12\left(\frac{h}{n}\right)^{1/2}\frac{(t_{20}+t_{11})^{1/2}(t_{31}+t_{22})}{t_{11}}\zeta+O\left(\frac{h}{n}\right)\right)$$
$$= \frac{n}{h}(2t_{30}+6t_{21})-12\left(\frac{n}{h}\right)^{1/2}\frac{(t_{20}+t_{11})^{1/2}(t_{31}+t_{22})}{t_{11}}\zeta+O(1) \tag{A.21}$$

and thus

$$\kappa_3^c = \frac{\frac{n}{h}(2t_{30}+6t_{21})-12\left(\frac{n}{h}\right)^{1/2}(t_{20}+t_{11})^{1/2}(t_{31}+t_{22})t_{11}^{-1}\zeta+O(1)}{\left(\frac{n}{h}\right)^{3/2}(t_{20}+t_{11})^{3/2}\left(1-4\left(\frac{h}{n}\right)^{1/2}t_{21}t_{11}^{-1}(t_{20}+t_{11})^{-1/2}\zeta+O\left(\frac{h}{n}\right)\right)^{3/2}}$$
$$= \left(\left(\frac{h}{n}\right)^{1/2}\frac{2t_{30}+6t_{21}}{(t_{20}+t_{11})^{3/2}}-12\frac{h}{n}\frac{t_{31}+t_{22}}{t_{11}(t_{20}+t_{11})}\zeta+O\left(\left(\frac{h}{n}\right)^{3/2}\right)\right)$$
$$\times\left(1+6\left(\frac{h}{n}\right)^{1/2}\frac{t_{21}}{t_{11}(t_{20}+t_{11})^{1/2}}\zeta+O\left(\frac{h}{n}\right)\right)$$
$$= \left(\frac{h}{n}\right)^{1/2}\frac{2t_{30}+6t_{21}}{(t_{20}+t_{11})^{3/2}}-12\frac{h}{n}\frac{t_{31}+t_{22}}{t_{11}(t_{20}+t_{11})}\zeta$$
$$+\frac{h}{n}\frac{6(2t_{30}+6t_{21})t_{21}}{(t_{20}+t_{11})^2t_{11}}\zeta+O\left(\left(\frac{h}{n}\right)^{3/2}\right)$$
$$= \left(\frac{h}{n}\right)^{1/2}c-\frac{h}{n}(e-6bc)\zeta+O\left(\left(\frac{h}{n}\right)^{3/2}\right), \tag{A.22}$$

where $b$, $c$, and $e$ were defined in (2.6) and (2.7). Similarly,

$$3tr\left(\left(C+C'\right)^4\right) = \frac{n}{h}(6t_{40}+24t_{31}+12t+6t_{22})+O\left(\left(\frac{n}{h}\right)^{1/2}\right) \tag{A.23}$$

and thus

$$\kappa_4^c = \frac{h}{n}\frac{6t_{40}+24t_{31}+12t+6t_{22}}{(t_{20}+t_{11})^2}+O\left(\left(\frac{h}{n}\right)^{3/2}\right) = \frac{h}{n}f+O\left(\left(\frac{h}{n}\right)^{3/2}\right), \tag{A.24}$$

where $f$ was defined in (2.7).

Substituting (A.15), (A.18), (A.19), (A.22), and (A.24) in (A.10) and rearranging using (2.8) and (2.9) complete the proof. ■

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**Proof of Theorem 2.**

Under $H_0$ and by Assumption 2(i), $\hat{\lambda} = \epsilon' W' P\epsilon / \epsilon' W' P W \epsilon$. Proceeding as before, $P(\tilde{\lambda} \leq x) = P(\varsigma \leq 0)$, which can be written as the right side of (A.10), with $\varsigma = \epsilon'(C + C')\epsilon/2$ and

$$C = W' P (I - xW). \tag{A.25}$$

Derivation of the cumulants $\kappa_j$ of $\varsigma$ is very similar to that in the proof of Theorem 1, and so is not described in detail. From (A.25), (2.2), and (2.17),

$$\kappa_1 = \sigma^2 tr(C) = -\sigma^2 \left(1 + x tr(W'W) - \frac{x}{n}(l'WW'l)\right) = -\sigma^2 \left(1 + x\frac{n}{h}t_{11} - xp\right). \tag{A.26}$$

Similarly, since

$$l' W^i W'^j l = O(n) \quad \text{for all} \quad i \geq 0, \quad j \geq 0, \tag{A.27}$$

$$\begin{aligned}
\kappa_2 &= \frac{\sigma^4}{2} tr\left(\left(C + C'\right)^2\right) \\
&= \sigma^4 \left( tr\left(W^2\right) + tr\left(W'W\right) - 1 - \frac{1}{n} l'W'Wl - 4x\left(tr\left(WW'W\right) + O(1)\right) \right. \\
&\qquad \left. + 2x^2 \left( tr\left(\left(W'W\right)^2\right) + O(1)\right)\right) \\
&= \sigma^4 \left(\frac{n}{h}(t_{20} + t_{11}) - 1 - p - 4x\left(\frac{n}{h}t_{21} + O(1)\right) + 2x^2\left(\frac{n}{h}t + O(1)\right)\right).
\end{aligned} \tag{A.28}$$

Proceeding as in the proof of Theorem 1, the first centered cumulant of $\varsigma$ is

$$\kappa_1^c = \frac{-x\frac{n}{h}t_{11} - 1 + xp}{\left(\frac{n}{h}(t_{20} + t_{11})\right)^{1/2}} \left(1 - \frac{1 + p + 4x\left(\frac{n}{h}t_{21} + O(1)\right) - 2x^2\left(\frac{n}{h}t + O(1)\right)}{\frac{n}{h}(t_{20} + t_{11})}\right)^{-1/2}. \tag{A.29}$$

Setting $x$ as in (A.15) and by Taylor expansion,

$$\begin{aligned}
\kappa_1^c &= -\left(\zeta + \frac{(h/n)^{1/2}}{(t_{20} + t_{11})^{1/2}} - \frac{h}{n}\frac{p}{t_{11}}\zeta\right) \\
&\quad \times \left(1 + \left(\frac{h}{n}\right)^{1/2} \frac{2t_{21}}{t_{11}(t_{20} + t_{11})^{1/2}}\zeta \right. \\
&\qquad \left. + \frac{h}{n}\left(\frac{1}{2(t_{20} + t_{11})} + \frac{1}{2}\frac{p}{t_{20} + t_{11}} - \frac{t}{t_{11}^2}\zeta^2 + \frac{6t_{21}^2}{t_{11}^2(t_{20} + t_{11})}\zeta^2\right)\right) + O\left(\left(\frac{h}{n}\right)^{3/2}\right) \\
&= -\left(\zeta + \left(\frac{h}{n}\right)^{1/2} g^{1/2} - \frac{h}{n}\frac{p}{t_{11}}\zeta\right)\left(1 + \left(\frac{h}{n}\right)^{1/2} 2b\zeta + \frac{h}{n}\left(\frac{g}{2} + \frac{g}{2}p - d\zeta^2 + 6b^2\zeta^2\right)\right) \\
&\quad + O\left(\left(\frac{h}{n}\right)^{3/2}\right) \\
&= -\zeta - \left(\frac{h}{n}\right)^{1/2}(2b\zeta^2 + g^{1/2}) - \frac{h}{n}\left(\frac{g}{2}\zeta + \frac{g}{2}p\zeta - d\zeta^3 + 6b^2\zeta^3 + 2bg^{1/2}\zeta\right) \\
&\quad + O\left(\left(\frac{h}{n}\right)^{3/2}\right),
\end{aligned} \tag{A.30}$$

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with $b$, $d$, $g$, and $p$ defined in (2.6), (2.7), and (2.17). Similarly, by standard algebra and using (A.27),

$$tr\left(\left(C+C'\right)^3\right)=\frac{n}{h}\left(2t_{30}+6t_{21}\right)-12\left(\frac{n}{h}\right)^{1/2}\frac{\left(t_{20}+t_{11}\right)^{1/2}\left(t_{31}+t_{22}\right)}{t_{11}}\zeta+O(1), \quad \textbf{(A.31)}$$

agreeing with the corresponding formula in the proof of Theorem 1, so that the third centered cumulant of $\varsigma$, $\kappa_3^c$, is (A.22), whereas the fourth centered cumulant of $\varsigma$, $\kappa_4^c$, is again (A.24).

Next,

$$\begin{aligned}\Phi(-\kappa_1^c)&=\Phi(\zeta)+\left(\frac{h}{n}\right)^{1/2}(2b\zeta^2+g^{1/2})\phi(\zeta)\\&\quad+\frac{h}{n}\left(\frac{g}{2}\zeta+\frac{g}{2}p\zeta-d\zeta^3+6b^2\zeta^3+2bg^{1/2}\zeta\right)\phi(\zeta)\\&\quad+\frac{1}{2}(2b\zeta^2+g^{1/2})^2\Phi^{(2)}(\zeta)+O\left(\left(\frac{h}{n}\right)^{3/2}\right)\\&=\Phi(\zeta)+\left(\frac{h}{n}\right)^{1/2}(2b\zeta+g^{1/2})\phi(\zeta)\\&\quad+\frac{h}{n}\left(\frac{g}{2}\zeta+\frac{g}{2}p\zeta-d\zeta^3+6b^2\zeta^3+2bg^{1/2}\zeta-\frac{1}{2}\left(2b\zeta^2+g^{1/2}\right)^2H_1(\zeta)\right)\phi(\zeta)\\&\quad+O\left(\left(\frac{h}{n}\right)^{3/2}\right)\end{aligned} \quad \textbf{(A.32)}$$

and

$$\begin{aligned}\Phi^{(3)}(-\kappa_1^c)&=\Phi^{(3)}(\zeta)+\left(\frac{h}{h}\right)^{1/2}(2b\zeta^2+g^{1/2})\Phi^{(4)}(\zeta)+O\left(\frac{h}{n}\right)\\&=\left(H_2(\zeta)-\left(\frac{h}{h}\right)^{1/2}(2b\zeta^2+g^{1/2})H_3(\zeta)\right)\phi(\zeta)+O\left(\frac{h}{n}\right).\end{aligned} \quad \textbf{(A.33)}$$

Substituting (A.15), (A.22), (A.24), (A.32), and (A.33) in the right side of (A.10) completes the proof. ■

**Proof of Theorem 3.**

Define $S(x)=I-xW$. We have

$$\begin{aligned}\tilde{\lambda}-\lambda&=\frac{\epsilon' PWS^{-1}(\lambda)\epsilon}{\epsilon' S^{-1\prime}(\lambda)W'PWS^{-1}(\lambda)\epsilon}\\&=\frac{\epsilon' WS^{-1}(\lambda)\epsilon-\epsilon' ll'WS^{-1}(\lambda)\epsilon/n}{\epsilon' S^{-1\prime}(\lambda)W'WS^{-1}(\lambda)\epsilon-\epsilon' S^{-1\prime}(\lambda)W'll'WS^{-1}(\lambda)\epsilon/n}.\end{aligned} \quad \textbf{(A.34)}$$

Now

$$\begin{aligned}E\left|\epsilon' ll'WS^{-1}(\lambda)\epsilon\right|&\le\left(E\left(\epsilon' l\right)^2E\left(l'WS^{-1}(\lambda)\epsilon\right)^2\right)^{1/2}\\&=\sigma^2n^{1/2}\left(l'WS^{-1}(\lambda)S^{-1\prime}(\lambda)W'l\right)^{1/2}\\&\le\sigma^2n\left\|S^{-1}(\lambda)\right\|\|W\|.\end{aligned} \quad \textbf{(A.35)}$$

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From (5.2),

$$\overline{\lim_{n\to\infty}}\left\|S^{-1}(\lambda)\right\| \leq \sum_{j=0}^{\infty}|\lambda|^j\,\overline{\lim_{n\to\infty}}\|W\|^j \leq (1-|\lambda|)^{-1}, \tag{A.36}$$

and thus (A.35)$= (1-|\lambda|)^{-1}\,O(n)$. Likewise

$$E\left(\epsilon' S^{-1\prime}(\lambda)W'll'WS^{-1}(\lambda)\epsilon\right) = \sigma^2 l'WS^{-1}(\lambda)S^{-1\prime}(\lambda)W'l \leq \sigma^2 n\left\|S^{-1}(\lambda)\right\|^2\|W\|^2 = (1-|\lambda|)^{-2}\,O(n). \tag{A.37}$$

Thus

$$\tilde{\lambda}-\lambda = \frac{(h/n)\epsilon'WS^{-1}(\lambda)\epsilon + (1-|\lambda|)^{-1}\,O_p(h/n)}{(h/n)\epsilon' S^{-1\prime}(\lambda)W'WS^{-1}(\lambda)\epsilon + (1-|\lambda|)^{-2}\,O_p(h/n)}. \tag{A.38}$$

The subsequent proof will show that the leading terms of both numerator and denominator are of larger order than the remainder terms, and so the latter may be ignored and it suffices to examine

$$\hat{\lambda}-\lambda = \frac{\epsilon'WS^{-1}(\lambda)\epsilon}{\epsilon' S^{-1\prime}(\lambda)W'WS^{-1}(\lambda)\epsilon}. \tag{A.39}$$

Now

$$Var\left(\epsilon'WS^{-1}(\lambda)\epsilon\right) = \sigma^4 tr\left(WS^{-1}(\lambda)WS^{-1}(\lambda) + WS^{-1}(\lambda)S^{-1\prime}(\lambda)W'\right) + \kappa\sum_{i=1}^{n}u_i^2, \tag{A.40}$$

where $\kappa$ is the 4th cumulant of $\epsilon_i$ and $u_i$ is the $ith$ diagonal element of $U = WS^{-1}(\lambda)$. The first term on the right is bounded by $2\sigma^4 tr(WS^{-1}(\lambda)S^{-1\prime}(\lambda)W') \leq K\left\|S^{-1}(\lambda)\right\|^2 tr\left(W'W\right)$, using the Cauchy inequality and the inequality $tr(A'B'BA) \leq \|B\|^2\ tr(A'A)$. Denoting by $w_i'$ the $ith$ row of $W$ and $e_i$ the $n\times 1$ vector whose $ith$ element is 1 and remaining elements are 0, we have $u_i^2 = (w_i'S^{-1}(\lambda)e_i)^2 \leq \|w_i\|^2\left\|S^{-1}(\lambda)\right\|^2$ and so $\sum_{i=1}^n u_i^2 \leq K\left\|S^{-1}(\lambda)\right\|^2 tr\left(W'W\right)$ also. Thus (A.40) is bounded by $K\left\|S^{-1}(\lambda)\right\|^2 tr\left(W'W\right) = (1-|\lambda|)^{-2}\,O(n/h)$. Likewise

$$Var\left(\epsilon' S^{-1\prime}(\lambda)W'WS^{-1}(\lambda)\epsilon\right) = 2\sigma^4 tr\left(S^{-1\prime}(\lambda)W'WS^{-1}(\lambda)S^{-1\prime}(\lambda)W'WS^{-1}(\lambda)\right) + \kappa\sum_{i=1}^{n}v_i^2, \tag{A.41}$$

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where $v_i$ is the $ith$ diagonal element of $V = U'U$. Proceeding as before, the first term on the right is bounded by $K \left\| S^{-1}(\lambda) \right\|^4 \|W\|^2 tr\left(W'W\right)$. Since $v_i = e_i' V e_i \leq \|V\|$,

$$\begin{aligned}
\sum_{i=1}^{n} v_i^2 &\leq \|V\| \sum_{i=1}^{n} v_i \\
&= \|V\| \sum_{i=1}^{n} e_i' S^{-1\prime}(\lambda) \sum_{j=1}^{n} w_j w_j' S^{-1}(\lambda) e_i \\
&= \|V\| \sum_{j=1}^{n} w_j' S^{-1}(\lambda) \sum_{i=1}^{n} e_i e_i' S^{-1\prime}(\lambda) w_j \\
&= \|V\| \sum_{j=1}^{n} w_j' S^{-1}(\lambda) S^{-1\prime}(\lambda) w_j \\
&\leq \|V\| \left\| S^{-1}(\lambda) \right\|^2 \sum_{j=1}^{n} \|w_j\|^2 \\
&\leq \|W\|^2 \left\| S^{-1}(\lambda) \right\|^4 tr\left(W'W\right). \qquad \textbf{(A.42)}
\end{aligned}$$

It follows that (A.41) is bounded by $K \left\| S^{-1}(\lambda) \right\|^4 tr\left(W'W\right) = (1-|\lambda|)^{-4} O(n/h)$.

Thus

$$\hat{\lambda} - \lambda = \frac{tr\left(WS^{-1}(\lambda)\right) + (1-|\lambda|)^{-1} O_p\left((n/h)^{1/2}\right)}{tr\left(S^{-1\prime}(\lambda)W'WS^{-1}(\lambda)\right) + (1-|\lambda|)^{-2} O_p\left((n/h)^{1/2}\right)}. \qquad \textbf{(A.43)}$$

We obtain upper and lower bounds for the leading terms in denominator and numerator. First, as already found,

$$tr\left(S^{-1\prime}(\lambda)W'WS^{-1}(\lambda)\right) \leq (1-|\lambda|)^{-2} tr\left(W'W\right), \qquad \textbf{(A.44)}$$

whereas

$$tr\left(S^{-1\prime}(\lambda)W'WS^{-1}(\lambda)\right) \geq \|S(\lambda)\|^{-2} tr\left(W'W\right) \geq (1+|\lambda|)^{-2} tr\left(W'W\right), \qquad \textbf{(A.45)}$$

since

$$\|S(\lambda)\| \leq 1 + |\lambda|. \qquad \textbf{(A.46)}$$

Next note that

$$tr\left(WS^{-1}(\lambda)\right) = tr\left(W \sum_{j=0}^{\infty} \lambda^j W^j\right) = \lambda tr\left(W^2 \sum_{j=0}^{\infty} \lambda^j W^j\right) = \lambda tr\left(W^2 S^{-1}(\lambda)\right). \qquad \textbf{(A.47)}$$

Now under condition (a) all elements of $S^{-1}(\lambda)$ are nonnegative so $tr(W^2 S^{-1}(\lambda)) \geq 0$, and by the Cauchy inequality

$$\begin{aligned}
tr(W^2 S^{-1}(\lambda)) &\leq \left(tr\left(W'W\right) tr(WS^{-1}(\lambda)S^{-1\prime}(\lambda)W')\right)^{1/2} \\
&\leq \left\| S^{-1}(\lambda) \right\| tr\left(W'W\right) \leq (1-\lambda)^{-1} tr\left(W'W\right). \qquad \textbf{(A.48)}
\end{aligned}$$

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Under condition (b), $W$ is similar to a symmetric matrix $W^*$, so $W = Q^{-1}W^*Q$, for some nonsingular matrix $Q$. Thus with $S^*(\lambda) = I_n - \lambda W^*$, we have $S^{-1}(\lambda) = Q^{-1}S^{*-1}(\lambda)Q$, $tr(W^2S^{-1}(\lambda)) = tr(W^{*2}S^{*-1}(\lambda))$. The Cauchy inequality gives

$$\begin{aligned} tr(W^{*2}S^{*-1}(\lambda)) &\leq \left(tr\left(W^{*2}S^{*-2}(\lambda)\right) tr\left(W^{*2}\right)\right)^{1/2} \\ &\leq \left\|S^{*-1}(\lambda)\right\| tr\left(W^{*2}\right) = (1-|\lambda|)^{-1} tr\left(W^2\right) \\ &\leq (1-|\lambda|)^{-1} tr\left(W'W\right). \end{aligned} \tag{A.49}$$

As for lower bounds, under condition (a) all elements of $S(\lambda) - I_n$ are nonnegative, so

$$tr\left(W^2S^{-1}(\lambda)\right) \geq tr\left(W^2\right), \tag{A.50}$$

whereas under condition (b),

$$\begin{aligned} tr\left(W^{*2}S^{*-1}(\lambda)\right) &\geq \|S^*(\lambda)\|^{-2} tr\left(W^{*2}\right) \geq (1+|\lambda|)^{-1} tr\left(W^{*2}\right) \\ &= (1+|\lambda|)^{-1} tr\left(W^2\right). \end{aligned} \tag{A.51}$$

Thus under both sets of conditions

$$\begin{aligned} \hat{\lambda} - \lambda &= \frac{tr(WS^{-1}(\lambda))\left(1+(1-|\lambda|)^{-1}O_p\left((n/h)^{1/2}/tr(WS^{-1}(\lambda))\right)\right)}{tr\left(S^{-1\prime}(\lambda)W'WS^{-1}(\lambda)\right)\left(1+(1-|\lambda|)^{-2}O_p\left((n/h)^{1/2}/tr\left(S^{-1\prime}(\lambda)W'WS^{-1}(\lambda)\right)\right)\right)} \\ &= \frac{(h/n)tr(WS^{-1}(\lambda))\left(1+(1-|\lambda|)^{-1}O_p((h/n)^{1/2})\right)}{(h/n)tr\left(S^{-1\prime}(\lambda)W'WS^{-1}(\lambda)\right)\left(1+(1-|\lambda|)^{-2}O_p\left((h/n)^{1/2}\right)\right)} \\ &= \frac{tr(WS^{-1}(\lambda))}{tr\left(S^{-1\prime}(\lambda)W'WS^{-1}(\lambda)\right)}\left(1+\frac{(1-|\lambda|)^{-2}o_p(1)}{1+(1-|\lambda|)^{-2}o_p(1)}\right) \\ &= \frac{tr(WS^{-1}(\lambda))}{tr\left(S^{-1\prime}(\lambda)W'WS^{-1}(\lambda)\right)}\left(1+(1-|\lambda|)^{-2}o_p(1)\right). \end{aligned} \tag{A.52}$$

Further, from the previous calculations, under condition (a)

$$\frac{tr\left(W^2\right)}{(1-\lambda)^{-2}tr\left(W'W\right)} \leq \frac{tr\left(W^2S^{-1}(\lambda)\right)}{tr\left(S^{-1\prime}(\lambda)W'WS^{-1}(\lambda)\right)} \leq \frac{(1-\lambda)^{-1}tr\left(W'W\right)}{(1+\lambda)^{-2}tr\left(W'W\right)}, \tag{A.53}$$

and under condition (b)

$$\frac{(1+|\lambda|)^{-1}tr\left(W^2\right)}{(1-|\lambda|)^{-2}tr(W'W)} \leq \frac{tr\left(W^2S^{-1}(\lambda)\right)}{tr\left(S^{-1\prime}(\lambda)W'WS^{-1}(\lambda)\right)} \leq \frac{(1-|\lambda|)^{-1}tr\left(W'W\right)}{(1+|\lambda|)^{-2}tr\left(W'W\right)}. \tag{A.54}$$

For all sufficiently large $n$, the factors $tr\left(W'W\right)$ cancel in the upper bounds in both (A.53) and (A.54), since $\eta(n/h) \leq (n/h)\underline{\lim}_{n\to\infty}t_{11} \leq tr\left(W'W\right) \leq n\|W\|^2 \leq n$ for some $\eta > 0$, and application of (5.3) completes the proof. ■

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**Proof of Theorem 4.**

In view of previous calculations we give only the proof of (5.8). We have

$$\frac{tr(WW')}{\left(tr(W^2+WW')\right)^{1/2}}\hat{\lambda} = \left(\frac{n}{h}\right)^{1/2} a \frac{\epsilon' S^{-1\prime}(\lambda_n) W S^{-1}(\lambda_n)\epsilon}{\epsilon' S^{-1\prime}(\lambda_n) W' W S^{-1}(\lambda_n)\epsilon}. \tag{A.55}$$

This can be written

$$\begin{aligned}
&\left(\frac{n}{h}\right)^{1/2} a \frac{\epsilon'\left(I+\lambda_n W+\lambda_n^2 W^2 S^{-1}(\lambda_n)\right)' W \left(I+\lambda_n W+\lambda_n^2 W^2 S^{-1}(\lambda_n)\right)\epsilon}{\epsilon'\left(I+\lambda_n W+\lambda_n^2 W^2 S^{-1}(\lambda_n)\right)' W \left(I+\lambda_n W'W+\lambda_n^2 W^2 S^{-1}(\lambda_n)\right)\epsilon} \\
&= \left(\frac{n}{h}\right)^{1/2} a \frac{\epsilon' W\epsilon + \lambda_n \epsilon'\left(WW'+W^2\right)\epsilon + O_p(1)}{\epsilon' W'W\epsilon + \lambda_n \epsilon'\left(W'W'W+W'WW\right)\epsilon + O_p(1)} \\
&= a \frac{(h/n)^{1/2}\epsilon' W\epsilon + \delta h \epsilon'\left(WW'+W^2\right)\epsilon/n + O_p\left((h/n)^{1/2}\right)}{h\epsilon' W'W\epsilon/n + \lambda_n h\epsilon'\left(W'W'W+W'WW\right)\epsilon/n + O_p(h/n)} \\
&= (\omega^{1/2}(1+\tau)^{-1/2} + o(1)) \frac{(h/n)^{1/2}\epsilon' W\epsilon + \sigma^2\delta\omega(1+\tau) + o_p(1)}{\sigma^2\omega + o_p(1)} \\
&= \omega^{-1/2}(1+\tau)^{-1/2}\left(\frac{h}{n}\right)^{1/2}\frac{\epsilon' W\epsilon}{\sigma^2} + \delta\omega^{1/2}(1+\tau)^{1/2} + o_p(1),
\end{aligned} \tag{A.56}$$

from which (5.8) readily follows. ■

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