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CHARACTERISING FINITE SOLVABLE GROUPS THROUGH THE NILPOTENCY PROBABILITY

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Abstract

Given a finite group G, we denote by $\nu(G)$ the probability that two randomly chosen elements of G generate a nilpotent subgroup. We prove that if $\nu(G) > 1/12$, then G is solvable.

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1. Introduction

Given a finite group G, let $\mathcal{N}(G)$ be the set of ordered pairs $(g_1, g_2) \in G^2$ with the property that $\langle g_1, g_2 \rangle$ is a nilpotent subgroup of G. Then,

$$\nu(G) := \frac{|\mathcal{N}(G)|}{|G|^2}$$

expresses the probability that two randomly chosen elements of G generate a nilpotent subgroup. Guralnick and Wilson [3] proved that if $\nu(G) > \nu(\mathrm{Sym}(3)) = \frac{1}{2}$, then G is nilpotent. However, the following question, raised in [4], remains open: does $\nu(G) > 1/12$ imply that G is solvable? In this note, we provide an affirmative answer.

THEOREM 1.1. *Let G be a finite group. If $\nu(G) > 1/12$, then G is solvable.*

Since $\nu(\mathrm{Alt}(5)) = 1/12$, the previous result is sharp.

2. Proof of Theorem 1.1

Let us start this section with a couple of definitions.

DEFINITION 2.1. Let G be a finite group, N a normal subgroup of G and $g_1, g_2 \in G$. We define

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$$\mathcal{N}_{g_1,g_2}(G,N) := \{(n_1,n_2) \in N^2 \mid \langle n_1g_1, n_2g_2\rangle \text{ is nilpotent}\},$$
$$\nu_{g_1,g_2}(G,N) := \frac{|\mathcal{N}_{g_1,g_2}(G,N)|}{|N|^2}.$$

DEFINITION 2.2. Let S be a finite nonabelian simple group. We identify S with the subgroup of its inner automorphisms and we define

$$\tilde{\nu}(S) = \max_{a_1,a_2\in \operatorname{Aut} S} \nu_{a_1,a_2}(\operatorname{Aut} S, S).$$

PROPOSITION 2.3. *Assume that a finite group G contains a unique minimal normal subgroup, say N, and that N is nonabelian. Then,*

$$\nu(G) \leq \tilde{\nu}(S),$$

where S is a composition factor of N.

PROOF. Assume that $N = S^u$, with $u \in \mathbb{N}$. We may identify G with a subgroup of the wreath product $\operatorname{Aut} S \wr \operatorname{Sym}(u)$. In this identification, any element of G can be written in the form $(\alpha_1,\dots,\alpha_u)\rho$, with $\alpha_1,\dots,\alpha_u \in \operatorname{Aut} S$ and $\rho \in \operatorname{Sym}(u)$. Moreover, we identify S with the inner automorphism group $\operatorname{Inn}(S)$ and, therefore, we can write $N = \{(s_1,\dots,s_u) \mid s_1,\dots,s_u \in S\}$.

Now, consider $g_1, g_2 \in G$ given by

$$g_1 = (a_1,\dots,a_u)\sigma, \quad g_2 = (b_1,\dots,b_u)\tau.$$

Write

$$\sigma = \sigma_1\cdots\sigma_c, \quad \tau = \tau_1\cdots\tau_d,$$

as products of disjoint cyclic permutations, and assume that 1 belongs to the supports of σ_1 and τ_1. Suppose

$$\sigma_1 = (1, i_2,\dots,i_e), \quad \tau_1 = (1, j_2,\dots,j_f).$$

Let

$$X = \{(\alpha_1,\dots,\alpha_u)\rho \in G \mid \rho(1) = 1\}.$$

The map $\phi\colon X \to \operatorname{Aut} S$ sending $(\alpha_1,\dots,\alpha_u)\rho$ to α_1 is a group homomorphism. Now, let $n_1 = (s_1,\dots,s_u), n_2 = (t_1,\dots,t_u) \in N$. Then, $(n_1g_1)^e, (n_2g_2)^f \in X$ and

$$\phi((n_1g_1)^e) = s_1a_1s_{i_2}a_{i_2}\cdots s_{i_e}a_{i_e}, \quad \phi((n_2g_2)^f) = t_1b_1t_{j_2}b_{j_2}\cdots t_{j_f}b_{j_f}.$$

Let

$$\gamma_1 = a_1s_{i_2}a_{i_2}\cdots s_{i_e}a_{i_e}, \quad \gamma_2 = b_1t_{j_2}b_{j_2}\cdots t_{j_f}b_{j_f}.$$

If $(s_1,t_1) \notin \mathcal{N}_{\gamma_1,\gamma_2}(\operatorname{Aut} S, S)$, then

$$\phi(\langle (n_1g_1)^e, (n_2g_2)^f\rangle) = \langle s_1\gamma_1, t_1\gamma_2\rangle$$

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is not nilpotent. In particular, $\langle n_1 g_1, n_2 g_2 \rangle$ is not nilpotent. Thus, given s_i, t_j for $i \neq 1$ and $j \neq 1$, there are at least

$$|S|^2(1 - \nu_{\gamma_1,\gamma_2}(\operatorname{Aut} S, S)) \geq |S|^2(1 - \tilde{\nu}(S))$$

choices for s_1 and t_1 such that $\langle n_1 g_1, n_2 g_2 \rangle$ is not nilpotent. It follows that there are at least $|S|^{2u}(1 - \tilde{\nu}(S))$ pairs $(n_1, n_2) \in N^2$ such that $\langle n_1 g_1, n_2 g_2 \rangle$ is not nilpotent. This is equivalent to saying that $\nu_{g_1,g_2}(G, N) \leq \tilde{\nu}(S)$ for every $(g_1, g_2) \in G^2$. Hence

$$\begin{aligned} \nu(G)|G|^2 &= \sum_{(Ng_1, Ng_2) \in (G/N)^2} \nu_{g_1,g_2}(G, N) \\ &\leq \sum_{(Ng_1, Ng_2) \in (G/N)^2} |N|^2 \tilde{\nu}(S) \\ &\leq |G|^2 \tilde{\nu}(S) \end{aligned}$$

and, consequently, $\nu(G) \leq \tilde{\nu}(S)$. □

PROPOSITION 2.4. *If S is a finite nonabelian simple group, then $\tilde{\nu}(S) \leq \tilde{\nu}(\mathrm{Alt}(5)) = 1/12$, with equality only if $S = \mathrm{Alt}(5)$.*

PROOF. Given $a_1, a_2 \in \operatorname{Aut} S$, let $\mathcal{P}_{a_1,a_2}(S) = \{(s_1, s_2) \in S^2 \mid S \leq \langle s_1 a_1, s_2 a_2 \rangle\}$. We define

$$\pi_{a_1,a_2}(S) := \frac{|\mathcal{P}_{a_1,a_2}(S)|}{|S|^2}, \quad \tilde{\pi}(S) = \min_{a_1,a_2 \in \operatorname{Aut} S} \pi_{a_1,a_2}(S).$$

Clearly, if $(s_1, s_2) \in \mathcal{P}_{a_1,a_2}(S)$, then $\langle s_1 a_1, s_2 a_2 \rangle$ is not nilpotent and, therefore, $(s_1, s_2) \notin \mathcal{N}_{a_1,a_2}(\operatorname{Aut} S, S)$. Thus,

$$\pi_{a_1,a_2}(S) \leq 1 - \nu_{a_1,a_2}(\operatorname{Aut}(S), S)$$

for every $a_1, a_2 \in \operatorname{Aut} S$ and, therefore,

$$\tilde{\nu}(S) \leq 1 - \tilde{\pi}(S).$$

Results of Dixon [1], Kantor and Lubotzky [5], and Liebeck and Shalev [6] establish that $\tilde{\pi}(S) \to 1$ as $|S| \to \infty$. Explicit lower bounds for $\tilde{\pi}(S)$ are given in [7], where, in particular, all nonabelian simple groups S with $\tilde{\pi}(S) \leq 9/10$ are listed. Since $1 - 9/10 = 1/10 > 1/12$, this is not enough for our purposes, but more detailed information can be found in [8]. In particular, combining [8, Table 5.1, Theorem 6.01 and Lemma 7.1.1], it turns out that if S is not an alternating group, then either $\tilde{\pi}(S) > 11/12$ (and, consequently, $\tilde{\nu}(S) < 1/12$) or S belongs to the family $\mathcal{S}$ of the following groups:

$$\begin{aligned} &\mathrm{PSL}(2, 7), \mathrm{PSL}(2, 8), \mathrm{PSL}(2, 11), \mathrm{PSL}(2, 13), \mathrm{PSL}(3, 3), \\ &\mathrm{PSL}(3, 4), \mathrm{PSp}(6, 2), \mathrm{PSU}(4, 2), \mathrm{M}_{11}, \mathrm{M}_{12}. \end{aligned}$$

The GAP Library of Tables of Marks [9] contains the tables of marks of all almost simple groups whose socle belongs to $\mathcal{S}$ and, using them, the exact value of $\tilde{\nu}(S)$, reported in Table 1, can be easily obtained (see Section 3 for more details).

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TABLE 1. $\tilde{\nu}(S)$ for $S \in \mathcal{S}$.

S	$\tilde{\nu}(S)$
PSL(2, 7)	$\frac{3}{56} \approx 0.0536$
PSL(2, 8)	$\frac{1}{56} \approx 0.0179$
PSL(2, 11)	$\frac{2}{165} \approx 0.0121$
PSL(2, 13)	$\frac{3}{364} \approx 0.0082$
PSL(3, 3)	$\frac{1}{234} \approx 0.0043$
PSL(3, 4)	$\frac{5}{4032} \approx 0.0032$
PSU(4, 2)	$\frac{67}{23760} \approx 0.0026$
PSp(6, 2)	$\frac{1}{4536} \approx 0.0002$
M_{11}	$\frac{1}{440} \approx 0.0023$
M_{12}	$\frac{7}{11880} \approx 0.0006$

Now, assume that $S = \mathrm{Alt}(n)$. Using GAP and, in particular, the library of tables of marks, it can be checked that

$$\tilde{\nu}(\mathrm{Alt}(5)) = \frac{1}{12}, \quad \tilde{\nu}(\mathrm{Alt}(6)) = \frac{1}{36}, \quad \tilde{\nu}(\mathrm{Alt}(7)) = \frac{1}{210}, \quad \tilde{\nu}(\mathrm{Alt}(8)) = \frac{19}{9720}, \quad \tilde{\nu}(\mathrm{Alt}(9)) = \frac{1}{2160}.$$

Now, suppose $n \geq 10$ and, for $1 \leq i \leq n$, let

$$X_i = \{s \in \mathrm{Sym}(n) \mid s(i) = i\} \cong \mathrm{Sym}(n-1), \quad Y_i = X_i \cap \mathrm{Alt}(n) \cong \mathrm{Alt}(n-1).$$

Consider $a_1, a_2 \in \mathrm{Sym}(n)$ and let $Y_{a_1,a_2} = \langle a_1, a_2, \mathrm{Alt}(n)\rangle$. Moreover, let

$$\Omega_0 = \{(s_1, s_2) \in (\mathrm{Alt}(n))^2 \mid \langle s_1 a_1, s_2 a_2\rangle = Y_{a_1,a_2}\}$$

and, for $1 \leq i \leq n$, let

$$\Omega_i = \{(s_1, s_2) \in (\mathrm{Alt}(n))^2 \mid \langle s_1 a_1, s_2 a_2\rangle = Y_{a_1,a_2} \cap X_i\}.$$

Notice that $\Omega_0, \Omega_1, \ldots, \Omega_n$ are disjoint subsets of $(\mathrm{Alt}(n))^2$ and

$$\Omega_0 \cup \Omega_1 \cup \cdots \cup \Omega_n \subseteq (\mathrm{Alt}(n))^2 \setminus \mathcal{N}_{a_1,a_2}(\mathrm{Sym}(n), \mathrm{Alt}(n)).$$

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It follows that

$$\nu_{a_1,a_2}(\mathrm{Sym}(n), \mathrm{Alt}(n)) \leq 1 - \sum_{0\leq i\leq n} \frac{|\Omega_i|}{|\mathrm{Alt}(n)|^2}.$$

For $1 \leq i \leq n$, there exist $s_1, s_2 \in \mathrm{Alt}(n)$ such that $\langle s_1a_1, s_2a_2\rangle \leq X_i$ and

$$|\Omega_i| = |Y_i|^2\pi_{s_1a_1,s_2a_2}(X_i, Y_i) \geq |\mathrm{Alt}(n-1)|^2\tilde{\pi}(\mathrm{Alt}(n-1)).$$

Moreover,

$$|\Omega_0| \geq |\mathrm{Alt}(n)|^2\tilde{\pi}(\mathrm{Alt}(n)).$$

Hence,

$$\begin{aligned}\nu_{a_1,a_2}(\mathrm{Sym}(n), \mathrm{Alt}(n)) &\leq 1 - \tilde{\pi}(\mathrm{Alt}(n)) - n\left(\frac{|\mathrm{Alt}(n-1)|^2\tilde{\pi}(\mathrm{Alt}(n-1))}{|\mathrm{Alt}(n)|^2}\right)\\ &= 1 - \tilde{\pi}(\mathrm{Alt}(n)) - \frac{\tilde{\pi}(\mathrm{Alt}(n-1))}{n}.\end{aligned}$$

It follows from [7] that if $m \geq 9$, then $\tilde{\pi}(\mathrm{Alt}(m)) \geq \tilde{\pi}(\mathrm{Alt}(9)) = 15403/18144$ and, therefore, for $n \geq 10$,

$$\begin{aligned}\tilde{\nu}(\mathrm{Alt}(n)) &\leq 1 - \tilde{\pi}(\mathrm{Alt}(n)) - \frac{\tilde{\pi}(\mathrm{Alt}(n-1))}{n}\\ &\leq 1 - \frac{15403}{18144}\left(1 + \frac{1}{10}\right) = \frac{12007}{181440} \approx 0.0622.\end{aligned}$$

□

PROOF OF THEOREM 1.1. Assume that G is a nonsolvable finite group. Then, G admits a nonabelian chief factor X/Y. In particular, $G/C_G(X/Y)$ has a unique minimal normal subgroup and this minimal normal subgroup is isomorphic to X/Y. It can be easily proved that $\nu(G) \leq \nu(G/N)$ for every normal subgroup N of G (see, for example, [4, Lemma 2.1]). This implies that $\nu(G) \leq \nu(G/C_G(X/Y))$. Moreover, by Proposition 2.3, $\nu(G/C_G(X/Y)) \leq \tilde{\nu}(S)$, where S is a composition factor of X/Y. Since, by Proposition 2.4, $\tilde{\nu}(S) \leq 1/12$, we conclude

$$\nu(G) \leq \nu(G/C_G(X/Y)) \leq \tilde{\nu}(S) \leq \frac{1}{12}.$$

□

3. Details on the computation of $\tilde{\nu}(S)$

To compute the exact value of $\tilde{\nu}(S)$, with $S \in \mathcal{S}$, the following observation is useful.

LEMMA 3.1. *Let N be a normal subgroup of a finite group G and let Ω be the set of pairs $(g_1, g_2) \in G^2$ such that $\langle g_1, g_2\rangle N = G$. Then, the value of $\nu_{g_1,g_2}(G, N)$ is the same for every choice of $(g_1, g_2) \in \Omega$.*

PROOF. Denote by $\mathcal{H}$ the set of 2-generated subgroups H of G that are nilpotent and such that $G = HN$. Assume that $(g_1, g_2) \in \Omega$ and, for every $H \in \mathcal{H}$, let $\Delta(g_1, g_2, H)$ be the set of (n_1, n_2) such that $H = \langle n_1g_1, n_2g_2\rangle$. Then, $\mathcal{N}_{g_1,g_2}(G, N)$ is the disjoint union of the subsets $\Delta(g_1, g_2, H)$, with H ranging over $\mathcal{H}$.

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TABLE 2. $\tau(G,S)$ when $\mathrm{soc}(G) = \mathrm{PSL}(3,4)$.

G	S	$S.2_1$	$S.2_2$	$S.2_3$	$S.2^2$	$S.6$
$\tau(G,S)$	$\frac{5}{4032}$	$\frac{13}{4032}$	$\frac{19}{6720}$	$\frac{5}{4032}$	$\frac{13}{4032}$	$\frac{1}{2520}$

Fix $H \in \mathcal{H}$. Since $HN = G$, there exist $n_1, n_2 \in N$ such that $g_1 = n_1 h_1$ and $g_2 = n_2 h_2$, and therefore, $(n_1^{-1}, n_2^{-1}) \in \Delta(g_1, g_2, H)$. Moreover,

$$\Delta(g_1, g_2, H) = \{(m_1, m_2) \in (N \cap H)^2 \mid \langle (m_1 n_1)(n_1^{-1} g_1), (m_2 n_2)(n_2^{-1} g_2) \rangle = H\}.$$

It follows from the main result in [2] that $|\Delta(g_1, g_2, H)|/|N \cap H|^2$ does not depend on the choice of (g_1, g_2) and coincides with the conditional probability that two randomly chosen elements of H generate H given that they generate H modulo $H \cap N$. Consequently, the cardinality of the sets $\Delta(g_1, g_2, H)$, and hence also that of their disjoint union, does not depend on the choice of (g_1, g_2). □

Given $S \in \mathcal{S}$, for every almost simple group G with $\mathrm{soc}(G) = S$, we use the table of marks of G to determine the nilpotent 2-generated subgroups of G that supplement $\mathrm{soc}(G)$ and, for each such subgroup, we compute the number of ordered pairs that generate it. Let $\tau(G,S)$ be the sum of the numbers thus obtained, divided by $|S|^2$ and by the number of pairs that generate G/S. The resulting number coincides with $\nu_{g_1,g_2}(G,S)$, where $\langle g_1, g_2 \rangle S = G$. Finally, $\tilde{\nu}(S)$ is the maximum of the values $\tau(G,S)$ as G ranges over the almost simple groups with socle S.

The case which requires more attention is when $S = \mathrm{PSL}(3,4)$. In this case, $\mathrm{Out}\, S \cong C_2 \times \mathrm{Sym}(3)$ and there are 6 different subgroups $S \leq G \leq \mathrm{Aut}\, S$ such that G/S is nilpotent. The values of $\tau(G,S)$ are as in Table 2.

In particular, $\tilde{\nu}(\mathrm{PSL}(3,4)) = 13/4032$.

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