Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 1–23, *2026*
DOI:10.1017/prm.2026.10151

The optimal weight of vibrating string equations with the first two eigenvalues based on critical equations

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(Received 26 September 2025; revised 29 March 2026; accepted 12 April 2026)

The present paper is concerned with the optimal weight of vibrating string equations with the first two eigenvalues λ_1 and λ_2 being given. Applying the method of critical equations in $L^p[0,1]$ for $p>1$ and the inverse spectral theory of Sturm–Liouville problems with measure coefficients, we find that the optimal weight can be uniquely determined if and only if $\lambda_2 \geq 2\lambda_1$ provided that the weight is non-negative and symmetrical. As an application, we provide an estimation of the extremum for partial trace of the first two eigenvalues on a sphere in $L^1[0,1]$.

Keywords: critical equation; eigenvalue; extremum; optimal weight; Sturm–Liouville problem

AMS Subject Classifications: Primary 34B24; Secondary 34L15; 34L20

1. Introduction

Consider the eigenvalue problem of the vibrating string equation

$$-y'' = \lambda w y,\ \ y = y(x),\ \ x \in (0,1) \tag{1.1}$$

subject to the Dirichlet boundary condition

$$y(0) = 0 = y(1), \tag{1.2}$$

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where $y(x)$ represents the longitudinal displacement at location x on the string and the density $w(x)$ is a non-negative and integrable function on $[0,1]$. The natural vibration frequencies of the string are determined by the eigenvalues of (1.1) and (1.2). Classical spectral theory, cf. [5, 23], shows that for problems (1.1) and (1.2), the eigenvalues are countable, simple, and real, that is,

$$0 < \lambda_1(w) < \cdots < \lambda_n(w) \to \infty,\ n \to \infty, \tag{1.3}$$

where $\lambda_n(w)$ is the n-th eigenvalue. Let $0 < \lambda_1 < \cdots < \lambda_m$ be fixed for an integer $m \geq 1$ and $L^1([0,1],\mathbb{R})$ be the space of real-valued integrable functions. Define

$$\Omega(\lambda_1,\cdots,\lambda_m) = \big\{w \in L^1([0,1],\mathbb{R}) : w(x) \geq 0,\ \lambda_j(w) = \lambda_j,\ 1 \leq j \leq m\big\}. \tag{1.4}$$

We are interested in the infimum of the norm $\|w\|_1 := \int_0^1 |w(x)|\mathrm{d}x$ for the densities w in $\Omega(\lambda_1,\cdots,\lambda_m)$, specifically, we search for the quantity

$$E(\lambda_1,\cdots,\lambda_m) := \inf\{\|w\|_1 :\ w \in \Omega(\lambda_1,\cdots,\lambda_m)\}, \tag{1.5}$$

which is termed as the optimal inverse problem for weights [13].

For $m = 1$, problem (1.5) is equivalently the extremal problem of eigenvalues, which dates back to [10], see also [1, 6, 7]. A classical example is the Lyapunov inequality, and the optimal weight $E(\lambda_1) = 4/\lambda_1$ is attained uniquely by the Dirac distribution at $x = 1/2$, as could be found in [11], [17, (1.2)], and [2, p. 177]. There are several tools to study the extremal problems for eigenvalues with separated or coupled boundary conditions, including the Rayleigh–Ritz principle, the variational approach [3, 4], the generalized Lyapunov inequality [14], and Mercer's theorem [15].

When $m \geq 2$, there are few results reported on problem (1.5). The authors of [8] and [26] studied the optimal potentials in L^2 and L^1, respectively. We point out that the optimal inverse problems in L^p spaces should be managed distinctively for $p = 1$ and $p > 1$ since the latter are reflexive, thereby any bounded subsets of L^p for $p > 1$ are relatively compact. This property ensures that the critical point of problem (1.5) is attainable in such spaces. It is worth noting that the critical point (the optimal weight) of (1.5) may not belong to L^1, but a larger space, for instance, the measure space, see Section 2 of this paper. For example, the critical point of (1.5) for $m = 1$ is the Dirac distribution at $x = 1/2$.

In 2009, the method of critical equations in $L^p[0,1]$ for $p > 1$ was introduced in [20, 25] for Sturm–Liouville operators to study the extremal value problems of eigenvalues in a subset of $L^1[0,1]$. Later on, related critical equations were deduced to obtain the optimal lower or upper bounds for eigenvalue gaps of Sturm–Liouville operators in [18]. The critical equations were also established in [9] for elliptic operators. For the inverse problems, the method of critical equations is proved useful as well; for example, it has been adopted in [16] to obtain the optimal weight of vibrating string equations for $m = 1$.

Generally, it is necessary to solve a boundary value problem with nonlinear differential equations when studying the optimal inverse problems with the method of critical equations, see [20, 25] for $m = 1$ and [9] for $m \geq 2$. However, the nonlinearity inevitably results in difficulties in solving such boundary value problems directly, especially for the case $m \geq 2$.

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In the present paper, we are committed to the research of the optimal inverse problem $E(\lambda_1, \lambda_2)$, i.e., problem (1.5) for $m = 2$. The same problem has been studied in [15]. By using Mercer's theorem, it was proved in [15, Theorem 3.1] that if $w(1-x) = w(x) \geq 0, a.e.\ x \in (0,1)$ and $\lambda_2 \geq 2\lambda_1$, then

$$\int_0^1 w(x)\mathrm{d}x \geq \frac{4\lambda_2}{\lambda_1(\lambda_2 - \lambda_1)},\ w \in \Omega(\lambda_1, \lambda_2),$$

and the optimal weight can be attained by the Dirac distribution at two points in $(0,1)$. Almost simultaneously, the authors started pondering whether the condition $\lambda_2 \geq 2\lambda_1$ could be removed or not. On the one hand, they expected that the conclusion would always be true without the condition; on the other hand, they also realized that it might be indispensable. At the end of [15], several examples ensuring the validity of $\lambda_2 \geq 2\lambda_1$ were displayed and the necessity of the condition was becoming increasingly clear. In the present paper, we improve the result of [15] and derive the following theorem as the principle conclusion.

THEOREM 1.1. *Assume that $w(x) \geq 0$ is symmetric on $[0,1]$. Then the optimal inverse problem (1.5) for $m = 2$ attains its extremum uniquely by the Dirac distribution at two points in $(0,1)$ if and only if $\lambda_2 \geq 2\lambda_1$. Furthermore,*

$$E(\lambda_1, \lambda_2) = \frac{4\lambda_2}{\lambda_1(\lambda_2 - \lambda_1)}. \tag{1.6}$$

It is worth noting that the method adopted in [15] does not apply to the proof of necessity of [15, Theorem 3.1]. We will prove Theorem 1.1 with the method of critical equations. After establishing the critical equations, we do not solve the nonlinear boundary value problem directly but analyse the solution w_p, namely the critical point, in $L^p[0,1]$ to obtain some important properties of the weak limit w_0 of w_p as $p \to 1+$ in the measure space. Therefore, the necessary preliminary knowledge for the measure space and the measure differential equations are introduced in Section 2. As an application, we will apply Theorem 1.1 together with Mercer's theorem to seek the extremum of the partial trace of the first two eigenvalues on a sphere in $L^1[0,1]$; both lower and upper bounds are obtained. A similar dual problem for optimal potentials in L^2 space has been studied in [19].

Following this section, Section 2 is devoted to some notations and preliminary results of spectral theory of the vibrating string equation with positive weights in $L^1([0,1], \mathbb{R})$ and the measure space, respectively. In Section 3, we introduce the optimal inverse problem $E(\lambda_1, \lambda_2)$ in $L^p[0,1]$ for $p > 1$ and set up the correlative critical equations. We prove Theorem 1.1 in section 4. In Section 5, we apply Theorem 1.1 to study the extremal problem of the partial trace of the first two eigenvalues on a sphere with radius r in $L^1[0,1]$.

2. Preliminary knowledge of the Sturm–Liouville eigenvalue problems

In this section, we introduce some notations and preliminaries of the vibrating string equation with the Dirichlet boundary condition

$$-y''(x) = \lambda w(x) y(x), \quad y(0) = 0 = y(1). \tag{2.1}$$

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The weight w is considered in $L^1([0,1],\mathbb{R})$, or more generally, it can be interpreted as $\int_{[0,x]} w(s)\mathrm{d}s = \mu(x)$ with $\mu \in \mathcal{M}_0^+$, where the space $\mathcal{M}_0^+$ is defined later in this section.

Let $u(x;\lambda_n)$ be the eigenfunction of (2.1) corresponding to the n-th eigenvalue $\lambda_n = \lambda_n(w)$ satisfying

$$\int_0^1 |u(x;\lambda_n)|^2 w(x)\mathrm{d}x = 1. \tag{2.2}$$

Since λ_n is simple, after fixing the sign of $u'(0;\lambda_n)$, such an eigenfunction $u(x;\lambda_n)$ is unique, and note that it has exactly $n-1$ zero points in (0,1), cf. [5, 23]. Therefore, throughout this paper, we call an eigenfunction $u(x;\lambda_n)$ of a simple eigenvalue λ_n is *normalized* if $u(x;\lambda_n)$ satisfies (2.2) and $u'(0;\lambda_n) > 0$. For problem (2.1), if $w \in L^1[0,1]$ and $w(x) > 0$ a.e. $x \in [0,1]$, then every eigenvalue is simple, and hence the corresponding normalized eigenfunction is well defined. For other boundary conditions, the normalized eigenfunction can be defined similarly.

The following two lemmas state the continuity and differentiability of eigenvalues with respect to the boundary points and the weights, respectively.

LEMMA 2.1. ([23, p. 54, Theorem 3.5.1; p. 119, Theorem 5.6.1]) *Let $\lambda_n(b)$ be the n-th eigenvalue of*

$$-y'' = \lambda w y, \quad y = y(x),\ x \in [0,b],\ y(0) = 0 = y'(b),\ b \in (0,1). \tag{2.3}$$

where $w \in L^1([0,1],\mathbb{R})$ and $w(x) > 0$ a.e. $x \in [0,1]$. Then

(i) *$\lambda_n(b)$ is a continuous function of b;*

(ii) *If w is continuous at b, then $\lambda_n(b)$ is differentiable with respect to b and*

$$\lambda_n'(b) = -\lambda_n(b) w(b) |u_n|^2(b), \tag{2.4}$$

where $u_n(x)$ is the normalized eigenfunction corresponding to $\lambda_n(b)$.

LEMMA 2.2. ([23, p. 56]) *Let $w_0 \in L^1([0,1],\mathbb{R})$ and $w_0(x) > 0$ a.e. $x \in [0,1]$. Then there exists a neighbourhood $U(w_0)$ of w_0 such that for every $n \geq 1$,*

$$\lambda_n : U \to \mathbb{R},\ w \in U \mapsto \lambda_n(w) \in \mathbb{R}$$

is differentiable at w_0 and its Fréchet derivative is given by

$$\frac{\partial \lambda_n(w)}{\partial w}\Big|_{w=w_0} \cdot h = -\lambda_n(w_0) \int_0^1 u_n^2(x) h(x)\mathrm{d}x,\ h \in L^1([0,1],\mathbb{R}),$$

where $u_n(x)$ is the normalized eigenfunction corresponding to $\lambda_n(w_0)$.

In order to introduce the spectral theory of (2.1) with measure coefficients, we need to list some basic knowledge of the measure space, see [3, 21, 22] for more details. For a real function $\mu :\ [0,1] \to \mathbb{R}$, the total variation of μ on $[0,1]$ is defined

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as

$$\|\mu\|_V = \sup\left\{\sum_{i=0}^{n-1} |\mu(x_{i+1}) - \mu(x_i)| : \ 0 = x_0 < \cdots < x_n = 1, n \in \mathbb{N}\right\}.$$

The space of measures on $[0,1]$ is defined as

$$\mathcal{M}_0 = \{\mu : [0,1] \to \mathbb{R} : \mu(0+) = 0, \mu(x+) = \mu(x), \forall x \in (0,1), \|\mu\|_V < \infty\},$$

where $\mu(x+) := \lim_{s \downarrow x} \mu(s),\ x \in [0,1)$ represents the right limit, and the 'density function' ρ is introduced with $\int_{[0,x]} \rho(s)\mathrm{d}s = \mu(x)$. For example, $f \in L^1[0,1]$ is the density function of the absolutely continuous measure

$$\mu_f(x) := \int_{[0,x]} f(s)\mathrm{d}s,\ x \in [0,1], \tag{2.5}$$

and the Dirac-Delta function $\delta(x-a)$ at point $a \in (0,1)$ is the density function of the Heaviside function $H_a(x)$, where

$$H_a(x) = \begin{cases} 0, & x \in [0,a), \\ 1, & x \in [a,1]. \end{cases} \tag{2.6}$$

In the measure space $\mathcal{M}_0$, besides the usual topology induced by the norm $\|\cdot\|_V$, there also exists the following weak* topology, denoted by w^*.

Definition 1. *Let $\mu_n,\ \mu_0 \in \mathcal{M}_0,\ n \in \mathbb{N}$. μ_n is said to converge weakly* to μ_0, denoted as $\mu_n \xrightarrow{w^*} \mu_0$ in $(\mathcal{M}_0, w^*)$, if*

$$\lim_{n\to\infty} \int_{[0,1]} u(t)\mathrm{d}\mu_n(t) = \int_{[0,1]} u(t)\mathrm{d}\mu_0(t),\ \forall u \in C[0,1].$$

An important property of a subset of $(\mathcal{M}_0, w^*)$ is proposed below by the Banach–Alaoglu theorem.

Lemma 2.3. *A subset $B \subset (\mathcal{M}_0, w^*)$ is relatively sequentially compact if and only if B is norm-bounded in $(\mathcal{M}_0, \|\cdot\|_V)$.*

Remark 1. (i) Based on the properties of bounded variation functions and the density of the space of absolutely continuous functions in $(L^1[0,1], \|\cdot\|_1)$, it follows that $L^1[0,1]$ is dense in $(\mathcal{M}_0, w^*)$, that is, for $\forall \mu \in \mathcal{M}_0$, there exists $f_n \in L^1[0,1]$ such that $\mu_{f_n} \xrightarrow{w^*} \mu$, we also write $f_n \xrightarrow{w^*} \mathrm{d}\mu$ without any confusion.
(ii) If $\mu_n \xrightarrow{w^*} \mu_0$ in $(\mathcal{M}_0, w^*)$ and $f_n \to f$ in $C[0,1]$, then

$$\lim_{n\to\infty} \int_{[0,1]} f_n(t)\mathrm{d}\mu_n(t) = \int_{[0,1]} f(t)\mathrm{d}\mu_0(t).$$

(iii) We call $\mu \in \mathcal{M}_0$ a positive measure, denoted by $\mu \in \mathcal{M}_0^+$, if for all $f \in C[0,1]$ and $f(x) \geq 0$, it holds that $\int_0^1 f(x)\mathrm{d}\mu(x) \geq 0$. It is easy to verify that if $\mu_n \in \mathcal{M}_0^+$ and $\mu_n \xrightarrow{w^*} \mu_0$ in $(\mathcal{M}_0, w^*)$, then $\mu_0 \in \mathcal{M}_0^+$.

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For $\mu \in \mathcal{M}_0^+$, consider the eigenvalue problem

$$-\mathrm{d}\dot{y}(x) = \lambda y \mathrm{d}\mu(x),\ x \in (0,1),\ y(0) = 0 = y(1). \tag{2.7}$$

In the present paper, we refer to a specific case of the measure coefficient, explicitly, the density function of $\mu(x)$ is a *Dirac distribution at m points* on $(0,1)$,

$$\rho(x) = \sum_{i=1}^{m} r_i \delta(x - x_i),\ r_i > 0, x_i \in (0,1),\ 1 \le i \le m,\ m \in \mathbb{N}, \tag{2.8}$$

and then problem (2.7) can be written in the form of (2.1) as

$$-y''(x) = \lambda \rho(x) y,\ x \in (0,1),\ y(0) = 0 = y(1). \tag{2.9}$$

Compared with (2.1), (2.9) can be particularly a model of the fixed-end vibrating string with m lumped masses weighting r_i attached to it at locations $x = x_j$, $1 \le i \le m$. Usually, in order to avoid the complexity caused by the Dirac distribution, the boundary problem (2.9) can be interpreted in the sense of

$$\begin{cases} -y''(x) = 0,\ x \ne x_i,\ 1 \le i \le m,\ y(0) = 0 = y(1); \\ y'(x_i - 0) - y'(x_i + 0) = \lambda r_i y(x_i),\ 1 \le i \le m. \end{cases}$$

The following result indicates that the finiteness of the number of eigenvalues for (2.1) implies that the weight must be a Dirac distribution at finite points in $(0,1)$.

LEMMA 2.4. (cf. [24]) *Consider problem (2.1) with $\int_{[0,x]} w(s)ds = \mu(x)$, $\mu \in \mathcal{M}_0^+$. Then there exist exactly m eigenvalues if and only if w is a Dirac distribution at m points in $(0,1)$.*

REMARK 2. Especially, if ρ is a Dirac distribution at two points in $(0,1)$, being symmetric with respect to $x = 1/2$, that is

$$\rho(x) = r\delta(x - a) + r\delta[x - (1 - a)],\ r > 0,\ 0 < a < 1/2,$$

then (2.9) has exactly two eigenvalues λ_1 and λ_2, and notice that (λ_1, λ_2) and (r, a) are determined uniquely by each other,

$$\lambda_1 = \frac{1}{ra},\ \lambda_2 = \frac{1}{ra(1-2a)}, \tag{2.10}$$

and

$$r = \frac{2\lambda_2}{\lambda_1(\lambda_2 - \lambda_1)},\ a = \frac{\lambda_2 - \lambda_1}{2\lambda_2}. \tag{2.11}$$

Furthermore, the first two eigenfunctions of problem (2.9) are respectively

$$u(x;\lambda_1) = \begin{cases} \frac{\sqrt{2r}}{2ra}x, & x \in [0,a), \\ \frac{\sqrt{2r}}{2r}, & x \in [a, 1-a), \\ \frac{\sqrt{2r}}{2ra}(1-x), & x \in [1-a, 1], \end{cases} \tag{2.12}$$

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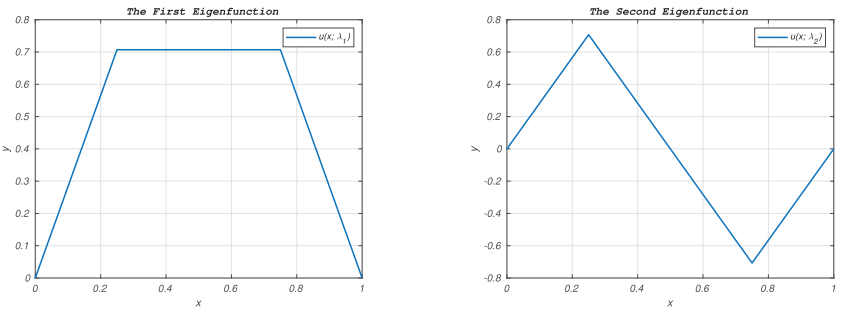


Figure 1. $u(x;\lambda_1)$ and $u(x;\lambda_2)$ for r=1 and a=1/4.

and

$$u(x;\lambda_2)=\begin{cases}\frac{\sqrt{2r}}{2ra}x, & x\in[0,a),\\ \frac{\sqrt{2r}}{2ra-r}(x-\frac{1}{2}), & x\in[a,1-a),\\ \frac{\sqrt{2r}}{2ra}(x-1), & x\in[1-a,1].\end{cases}\tag{2.13}$$

Assuming $r=1$ and $a=1/4$, then $\lambda_1=4$, $\lambda_2=8$, their eigenfunctions are plotted in Figure 1 for an intuitive presentation. Evidently, it can be found that $u(x;\lambda_1)$ of (2.9) is inverted-bathtub-shaped with no zero point in (0,1), and $u(x;\lambda_2)$ has exactly one zero point (1/2,0).

It is well known that eigenvalues are continuous on w with respect to the topology in $L^1([0,1])$, see [23]. Actually, results in [12, 24] show that eigenvalues possess even 'stronger' continuity.

LEMMA 2.5. (cf. [24, Theorem 4.1]) *Let $\lambda_k(w)$ be the k-th eigenvalue of (2.1) and ϕ_{kw} be the corresponding normalized eigenfunction for all $k\geq 1$. If $0\leq w_n\in L^1([0,1],\mathbb{R}), n\geq 1$ and $w_n\xrightarrow{w^*}\rho$, where ρ is a Dirac distribution at m points in $(0,1)$, then*

$$\lambda_k(w_n)\to\lambda_k(\rho),\ \ \phi_{kw_n}\xrightarrow{\|.\|_C}\phi_{k\rho}, 1\leq k\leq m$$

and for $k\geq m+1$, $\lambda_k(w_n)\to\infty$ as $n\to\infty$.

Lemma 2.1 and Lemma 2.5 show that eigenvalues are continuous on weights and end points independently, then the following lemma gives the continuity of them jointly in $(\mathcal{M}_0,w^*)\times\mathbb{R}$.

LEMMA 2.6. *Let $\lambda_k(w,b)$ be the kth eigenvalue of*

$$-y''=\lambda wy,\quad y=y(x),\ x\in[0,b],\ y(0)=0=y'(b),\ b\in(0,1).\tag{2.14}$$

If $w_n\in C[0,b]$, $w_n\xrightarrow{w^*}\rho$ and $b_n\to b$ as $n\to\infty$, where ρ is a Dirac distribution at m points in $(0,b)$, then $\lambda_k(w_n,b_n)\to\lambda_k(\rho,b)$ as $n\to\infty$ for $1\leq k\leq m$.

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Proof. Let $u(x,t)$ be the associated normalized eigenfunction of (2.14) with b replaced by t. It follows from (ii) of Lemma 2.1 that

$$\lambda'(w,t) := \frac{\partial \lambda(w,t)}{\partial t} = -\lambda(w,t)w(t)|u|^2(t,t),$$

and hence,

$$\lambda(w,b_n) - \lambda(w,b) = \int_{b_n}^{b} \lambda(w,t)w(t)|u|^2(t,t)dt. \tag{2.15}$$

Whereas, since $u(x,t)$ satisfies (2.14) on $[0,t]$, write $\lambda(w,t)$ as λ, then

$$\begin{aligned} u(x,t) &= \lambda \int_0^x \int_s^t w(\tau)u(\tau;t)d\tau ds = \lambda \left(\int_0^x (x-\tau)wu + x\int_x^t wu \right) \\ &= \lambda \left(x\int_0^t w(s)u(s,t)ds - \int_0^x sw(s)u(s,t)ds \right), \ x \in [0,t], \end{aligned}$$

and hence $u(t,t) = \lambda \int_0^t (t-s)w(s)u(s,t)ds$, and

$$\begin{aligned} |u(t,t)|^2 &\le t^2|\lambda|^2 \left(\int_0^t w(s)u(s,t)ds \right)^2 \\ &\le t^2|\lambda|^2 \int_0^t w(s)ds \int_0^t w(s)u^2(s,t)ds \\ &= t^2|\lambda|^2 \int_0^t w(s)ds \le b^2|\lambda(w,t)|^2\|w\|_1, \end{aligned} \tag{2.16}$$

where $\|w\|_1 = \int_0^b w$. Since $\lambda(w,t)$ is continuous on t by (i) of Lemma 2.1, we have $|\lambda(w,t)| \le M$ for $t \in [b/2, b]$, and hence, (2.15) together with (2.16) yields that

$$|\lambda(w,b_n) - \lambda(w,b)| \le b^2M^3\|w\|_1 \int_{b_n}^{b} w(t)dt. \tag{2.17}$$

Since $\{\|w_n\|_1 : \ n \ge 1\}$ is bounded by Lemma 2.3 due to the weak convergence of w_n, say $\|w_n\|_1 \le C$, we have from (2.17) that for all $n \ge 1$,

$$|\lambda(w_n,b_n) - \lambda(w_n,b)| \le Cb^2M^3 \int_{b_n}^{b} w_n(t)dt. \tag{2.18}$$

Note that $w_n \xrightarrow{w^*} \rho$ and ρ is the Dirac distribution at m points in $(0,b)$, which implies that $\int_{b_n}^{b} w_n(t)dt \to 0$ as $n \to \infty$. As a result, 2.17 yields that

$$\lambda_k(w_n,b_n) - \lambda_k(w_n,b) \to 0, \ n \to \infty, \tag{2.19}$$

and on the other hand, Lemma 2.5 ensures that for $1 \le k \le m$,

$$\lambda_k(w_n,b) - \lambda_k(\rho,b) \to 0, \ n \to \infty.$$

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Then we obtain that as $n \to \infty$,

$$\lambda_k(w_n, b_n) - \lambda_k(\rho, b) = \lambda_k(w_n, b_n) - \lambda_k(w_n, b) + \lambda_k(w_n, b) - \lambda_k(\rho, b) \to 0.$$

□

REMARK 3. Checking the proof of Lemma 2.6, it can be detected that for any positive measure $\rho \in \mathcal{M}_0^+$, the conclusion is also valid if $\mathrm{d}\rho(x)$ does not contain the density $\delta(x-b)$. In fact, such a restriction is also necessary. See the following example.

EXAMPLE 2.7. Take $b = 1, b_n = 1 - 1/(2n)$. Let

$$w_n(x) = \begin{cases} 0, & x \in [0, 1 - 1/n], \\ n & x \in (1 - 1/n, 1]. \end{cases}$$

Clearly $w_n \xrightarrow{w^*} w_0 = \delta(x-1)$ as $n \to \infty$. The eigenvalue problem

$$-y'' = \lambda w_0 y, \ \ y(0) = 0 = y'(1)$$

is equivalent to

$$-y'' = 0, \ \ y(0) = 0 = y'(1-0) - \lambda y(1)$$

since $y'(1-0) - y'(1) = \lambda y(1)$, which means that $\lambda_1(w_0) = 1$ with the eigenfunction $\psi(x) = x$. Whereas, the first eigenvalue $\lambda_1(w_n)$ of

$$-y'' = \lambda w_n y, \ \ y(0) = 0 = y'(b_n)$$

satisfies

$$x \tan x = \frac{1}{2(n-1)}, \ \ x = \frac{\sqrt{n\lambda_1(w_n)}}{2n}.$$

A calculation shows that $\lambda_1(w_n) \to 2$, as $n \to \infty$.

3. The extremal problems and critical equations

In this section, we first introduce the optimal inverse problem E_p in $L^p[0,1]$ for $p > 1$ associated with the original problem (1.5). Then, we give some properties of critical points and set up the critical equations of E_p.

We start with the positive and symmetric weights. Here, w is called *symmetric* on $[0,1]$ if $w(1-x) = w(x)$, *a.e.* $x \in [0,1]$. For simplicity, w is replaced by $\lambda_1 w$ and problem (1.5) reads

$$\begin{aligned} &\Omega(\lambda) = \left\{ w \in L^1([0,1], \mathbb{R}) : \lambda_1(w) = 1, \ \lambda_2(w) = \lambda := \lambda_2/\lambda_1 \right\}; \\ &E(\lambda) = \inf \left\{ \int_0^1 w : \ w \in \Omega(\lambda) \right\}. \end{aligned} \tag{3.1}$$

That is, $m = 2$ and we always suppose that $\lambda_1(w) = 1$ and $\lambda_2(w) = \lambda$ in this section. Besides, the infimum $E(\lambda)$ is attained by μ_0, or $\mathrm{d}\mu_0$, meaning that $\mu_0 \in \mathcal{M}_0^+$ and $\|\mu_0\|_V \le \|w\|_1, \ \forall w \in \Omega(\lambda)$.

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3.1. The associated optimal inverse problems in $L^P[0,1]$

We propose the optimal inverse problems in $L^p[0,1]$ for $p > 1$ associated with problem (3.1). Define

$$\Omega_p(\lambda) = \{w \in L^p[0,1] : 0 \le w \in \Omega(\lambda) \ is \ symmetric\},\ p > 1, \tag{3.2}$$

and consider the optimal inverse problem

$$E_p(\lambda) = \inf\left\{\int_0^1 w^p :\ w \in \Omega_p(\lambda)\right\}. \tag{3.3}$$

The existence of the *critical point*, say w_p, in $\Omega_p(\lambda)$ of E_p is guaranteed by the results of Eberlein-Smulian theorem since $L^p[0,1]$ is a reflexive space for $p > 1$. For more details, see [9].

The next lemma indicates that $\{\|w_p\|_p :\ p > 1]\}$ is bounded below.

LEMMA 3.1. *Let w_p be a critical point of E_p for $p > 1$. Then*

$$\|w_p\|_p = \left(\int_0^1 |w_p|^p\right)^{1/p}$$

is bounded below and $\|w_p\|_p > 4$ for $p > 1$.

Proof. Since $L^q[0,1] \subset L^p[0,1]$ if $1 < p < q$, we have from the definition of w_p that $\int_0^1 w_p^p \le \int_0^1 w_q^p$, which together with the *Hölder* inequality yields that

$$\|w_p\|_p = \left(\int_0^1 w_p^p\right)^{1/p} \le \left(\int_0^1 w_q^p\right)^{1/p} \le \left(\int_0^1 w_q^q\right)^{1/q} = \|w_q\|_q. \tag{3.4}$$

3.4 means that $\|w_p\|_p$ is nondecreasing for $p > 1$, and hence $\|w_p\|_p$ is bounded for $p > 1$. Moreover, the classical Lyapunov inequality gives that $\int_0^1 w_p > 4$, and hence we arrive at

$$\left(\int_0^1 w_p^p\right)^{1/p} > \int_0^1 w_p > 4,\ \forall p > 1. \tag{3.5}$$

□

As is proved, the boundedness of $\|w_p\|_p$ holds for $p > 1$. However, in the following sections, p is always deemed fully close to 1 from the right side. Thus, the conclusion of Lemma 3.1 may actually be utilized on a finite interval, say, for $p \in (1, 2]$.

If problem (3.1) attains its extremal value uniquely by a Dirac distribution at two points, then one can verify that this optimal weight must be w_0, see Remark 2, where

$$w_0 = r\delta(x - x_0) + r\delta[x - (1 - x_0)],\ r = \frac{2\lambda}{(\lambda - 1)},\ x_0 = \frac{\lambda - 1}{2\lambda}. \tag{3.6}$$

Although whether the critical point w_p is uniquely determined for each $p > 1$ remains unclear, we affirm that the critical point w_p always has weak limitation as $p \to 1+$ provided that the infimum $E(\lambda)$ is attained uniquely.

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THEOREM 3.2. *Let w_p be a critical point of (3.3). If the infimum $E(\lambda)$ is attained uniquely by w_0 defined in (3.6), then $w_p \xrightarrow{w^*} w_0$ as $p \to 1+$.*

Proof. We prove the theorem in three steps.

Step 1. There exists w^*-convergent subsequence of w_p as $p \to 1+$. Since for $w \in L^1 := L^1([0,1], \mathbb{R})$, the absolute continuous measure

$$\mu_w(x) := \int_0^x w(s)\mathrm{d}s, \quad x \in [0,1]$$

satisfies $\|\mu_w\|_V = \|w\|_{L^1}$, we have from Lemma 3.1 that $\{\|\mu_{w_p}\|_V : p \in (1,2]\}$ is bounded. Therefore, Lemma 2.3 yields that μ_{w_p} has w^*-convergent subsequence, say, $\mu_{w_p} \xrightarrow{w^*} \mu_\rho$ for some subsequence of $p \to 1+$ and $\mu_\rho \in \mathcal{M}_0^+$ by (iii) of Remark 1. Taking $f \equiv 1$, we have from $\int_0^1 w_p f \to \int_{[0,1]} \rho f$ that

$$\int_0^1 w_p(x)\mathrm{d}x \to \int_{[0,1]} \rho(x)\mathrm{d}x := \|\mu_\rho\|_V \tag{3.7}$$

for some subsequence of $p \to 1+$.

Step 2. For all $w \in \Omega(\lambda)$, we claim that

$$\int_0^1 w(x)\mathrm{d}x \geq \lim_{p\to 1+} \int_0^1 w_p = \int_{[0,1]} \rho(x)\mathrm{d}x,$$

where $\mu_{w_p} \xrightarrow{w^*} \mu_\rho$ for some subsequence of $p \to 1+$. In fact, for $w \in \Omega(\lambda)$, set $A_p = \{x \in [0,1] : \ |w(x)| \leq 1/(p-1)\}$ and introduce

$$\omega_p(x) = \chi(A_p)(x)w(x), \ x \in [0,1],$$

where $\chi(A_p)$ is the characteristic function on A_p. Clearly, $\omega_p \to w$, $p \to 1+$ in $L^1[0,1]$. Notice that $\omega_p \in L^p[0,1]$ for $p > 1$ and w_p is a critical point of E_p, then

$$\left(\int_0^1 \omega_p^p(x)\mathrm{d}x\right)^{1/p} \geq \left(\int_0^1 w_p^p(x)\mathrm{d}x\right)^{1/p} \geq \int_0^1 w_p(x)\mathrm{d}x \to \int_{[0,1]} \rho(x)\mathrm{d}x. \tag{3.8}$$

Because

$$\begin{aligned}\int_0^1 \left|\omega_p^p(x) - w(x)\right| \mathrm{d}x &\leq \int_0^1 |\omega_p^p(x) - \omega_p(x)|\mathrm{d}x + \int_0^1 |\omega_p(x) - w(x)|\mathrm{d}x \\ &= \int_0^1 \omega_p(x)|\omega_p^{p-1}(x) - 1|\mathrm{d}x + \int_0^1 |\omega_p(x) - w(x)|\mathrm{d}x\end{aligned} \tag{3.9}$$

By the definition of $\omega_p(x)$ and $x^x \to 1$ as $x \to 0+$, we have

$$\omega_p(x)|\omega_p^{p-1}(x) - 1| \leq \omega_p(x)\left|\left(\frac{1}{p-1}\right)^{p-1} - 1\right| \leq \omega_p(x) \leq w(x), \ x \in [0,1]$$

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for sufficiently small $p-1>0$. As a result, the dominated convergence theorem ensures that

$$\int_0^1 \omega_p(x)|\omega_p^{p-1}(x)-1|\mathrm{d}x \to 0,\ p\to 1+,$$

and hence we have from (3.9) that

$$\int_0^1 \omega_p^p(x)\mathrm{d}x \to \int_0^1 w(x)\mathrm{d}x,\ p\to 1+.$$

Letting $p\to 1+$ in (3.8), we have

$$\int_0^1 w(x)\mathrm{d}x \geq \int_{[0,1]} \rho(x)\mathrm{d}x,\ \forall w\in\Omega(\lambda). \tag{3.10}$$

Step 3. We claim that $w_p \xrightarrow{w^*} w_0$ as $p\to 1+$. Since $w_p \xrightarrow{w^*} \rho$ as $p\to 1+$, it follows from Lemma 2.5 that

$$\lambda_1(\rho)=1,\ \lambda_2(\rho)=\lambda. \tag{3.11}$$

Consequently, Step 1 and Step 2 give that $\mu_\rho\in\mathcal{M}_0^+$ and

$$\|\mu_\rho\|_V \leq \inf\left\{\int_0^1 w:\ w\in\Omega(\lambda)\right\}.$$

Thus, by the assumption in Theorem 3.2, we know that $\rho=w_0$, more specifically, $w_p \xrightarrow{w^*} w_0$ for all subsequences of $p\to 1+$. □

3.2. The critical equations in $L^p[0,1]$

In this subsection, we set up the critical equations of problem E_p for $p>1$. Define

$$F(w,c_1,c_2) := \int_0^1 w^p(x)\mathrm{d}x + c_1[\lambda_1(w)-1] + c_2[\lambda_2(w)-\lambda], \tag{3.12}$$

here c_1 and c_2 are the Lagrange multipliers for optimizing $\int_0^1 w^p(x)\mathrm{d}x$ with the constraints $\lambda_1(w)=1$ and $\lambda_2(w)=\lambda$. Applying Pontryagin's maximum principle, we know that the critical point w_p satisfies

$$\begin{cases} \dfrac{\partial F}{\partial w}(w_p,\hat{c_{1p}},\hat{c_{2p}})=0; \\ \dfrac{\partial F}{\partial c_j}(w_p,\hat{c_{1p}},\hat{c_{2p}})=0,\ j=1,2, \end{cases} \tag{3.13}$$

for some constants $\hat{c_{1p}}$ and $\hat{c_{2p}}$. To proceed, we adopt the results of Lemma 2.2 and derive that

$$\begin{cases} pw_p^{p-1}(x)-\hat{c_{1p}}\lambda_1(w_p)u_{1p}^2(x)-\hat{c_{2p}}\lambda_2(w_p)u_{2p}^2(x)=0; \\ \lambda_1(w_p)=1,\ \lambda_2(w_p)=\lambda, \end{cases} \tag{3.14}$$

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where u_{1p} and u_{2p} are respectively the first and second normalized eigenfunctions of (2.1) given that $w = w_p$ and the first two eigenvalues are restricted. Rewriting (3.14) as

$$pw_p^{p-1}(x) = c_{1p}u_{1p}^2(x) + c_{2p}u_{2p}^2(x), \tag{3.15}$$

where $c_{1p} = c_{\hat{1}p}$, $c_{2p} = \lambda c_{\hat{2}p}$ such that $|c_{1p}| + |c_{2p}| \neq 0$, we obtain the following nonlinear boundary value problem

$$\begin{cases} -u_{1p}''(x) = w_p u_{1p}(x), \ u_{1p}(0) = 0 = u_{1p}(1); \\ -u_{2p}''(x) = \lambda w_p u_{2p}(x), \ u_{2p}(0) = 0 = u_{2p}(1); \\ pw_p^{p-1}(x) = c_{1p}u_{1p}^2(x) + c_{2p}u_{2p}^2(x). \end{cases} \tag{3.16}$$

(3.16) is namely the *critical equations* associated with the optimization problem E_p. The sign of c_{1p} and c_{2p} in (3.16) is determined in the following theorem.

THEOREM 3.3. *Let w_p be a critical point of (3.3). If the infimum $E(\lambda)$ is attained uniquely by w_0 defined in (3.6), then w_p satisfies the critical equation (3.16) with $c_{1p}, c_{2p} \geq 0$ for sufficiently small $p - 1 > 0$.*

Proof. Since u_{2p} has one zero point in $(0, 1)$ and $u_{1p}(x) > 0$ on $(0, 1)$, one sees from (3.15) that $c_{1p} \geq 0$ for all $p > 1$. We will now prove that $c_{2p} \geq 0$ for sufficiently small $p - 1 > 0$.

Suppose on the contrary that there exists a subsequence of $p \to 1+$ such that $c_{2p} < 0$. Then it follows from (3.16) that

$$w_p^{p-1}(x) = \phi_{1p}^2(x) - \phi_{2p}^2(x), \tag{3.17}$$

with $\phi_{1p} = \sqrt{c_{1p}/p}\, u_{1p}$ and $\phi_{2p} = \sqrt{-c_{2p}/p} u_{2p}$ satisfying

$$\begin{cases} -\phi_{1p}''(x) = w_p \phi_{1p}(x), \ \phi_{1p}(0) = 0 = \phi_{1p}(1), \\ -\phi_{2p}''(x) = \lambda w_p \phi_{2p}(x), \ \phi_{2p}(0) = 0 = \phi_{2p}(1). \end{cases} \tag{3.18}$$

Based on the property of u_{1p} and u_{2p}, immediately we have

$$\phi_{1p}'\left(\frac{1}{2}\right) = 0 = \phi_{2p}\left(\frac{1}{2}\right) \tag{3.19}$$

since the symmetry of w_p implies that $\phi_{1p}(1-x) = \phi_{1p}(x)$, $\phi_{2p}(1-x) = -\phi_{2p}(x)$ on $[0, 1]$. Multiplying both sides of the first and second equations in (3.18) by $\lambda\phi_{1p}'(x)$ and $\phi_{2p}'(x)$, respectively, and integrating their difference over $[0, x]$, we have

$$\lambda\phi_{1p}'^2(x) - \phi_{2p}'^2(x) + \lambda\frac{p-1}{p}w_p^p(x) \equiv C_p := \lambda\phi_{1p}'^2(0) - \phi_{2p}'^2(0), \tag{3.20}$$

where the equality (3.17) is used. And then, multiplying both sides of the first and second equations in (3.18) by $\lambda\phi_{1p}(x)$ and $\phi_{2p}(x)$, respectively, and integrating

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them on $[0,1]$, we have

$$\lambda\int_0^1 \phi_{1p}'^2 = \lambda\int_0^1 w_p\phi_{1p}^2,\ \int_0^1 \phi_{2p}'^2 = \lambda\int_0^1 w_p\phi_{2p}^2.$$

These equalities together with (3.17) yield that

$$\int_0^1 \left[\lambda\phi_{1p}'^2 - \phi_{2p}'^2\right] = \lambda\int_0^1 w_p(\phi_{1p}^2 - \phi_{2p}^2) = \lambda\int_0^1 w_p^p. \tag{3.21}$$

Integrating two sides of (3.20) on $[0,1]$ and using (3.21), we have

$$C_p = \lambda\left(1+\frac{p-1}{p}\right)\int_0^1 w_p^p = \lambda\frac{2p-1}{p}\int_0^1 w_p^p > 0. \tag{3.22}$$

The boundedness of $\{\|w_p\|_p :\ p\in(1,2]\}$ in Lemma 3.1 implies that C_p is bounded for $p\in(1,2]$. Set

$$\eta_p := \max\{x :\ u_{2p}'(x) = 0,\ x\in(0,1/2)\}. \tag{3.23}$$

Then $\eta_p < 1/2$ since $u_{2p}'(1/2) < 0$. This also means that $\phi_{2p}'(x)\le 0$ on $(\eta_p, 1/2]$. Moreover, (3.20) gives that

$$\lambda\phi_{1p}'^2(\eta_p) + \lambda\frac{p-1}{p}w_p^p(\eta_p) = C_p,$$

and hence $\lambda\phi_{1p}'^2(\eta_p)\le C_p$. Since $\phi_{1p}''(x)\le 0$ on $(0,1/2)$, one sees that

$$\lambda\phi_{1p}'^2(x)\le C_p,\ x\in[\eta_p,1/2],\ \forall p>1. \tag{3.24}$$

In order to obtain a contradiction to the assumption that $c_{2p}<0$ for some sequence of $p\to 1+$, we carry out the remaining proof in five steps.

Step 1. We assert that $\eta_0 = \overline{\lim_{p\to 1+}}\eta_p < 1/2$. Observe that $\phi_{2p}(x)$ is the first eigenfunction of

$$-y''(x) = \lambda w_p(x)y(x),\ y(0) = 0 = y'(\eta_p),$$

on $[0,\eta_p]$ with the corresponding eigenvalue $\lambda(w_p,\eta_p)$. If $\eta_p\to 1/2$ for some subsequence of $p\to 1+$, then it follows from Lemma 2.6 that $\lambda(w_p,\eta_p)\to\lambda(w_0,1/2)=\lambda$ jointly, which implies that eigenvalue problems

$$-y'' = \gamma w_0 y,\ y(0) = 0 = y\left(\frac{1}{2}\right)$$

and

$$-y'' = \gamma w_0 y,\ y(0) = 0 = y'\left(\frac{1}{2}\right)$$

have the same first eigenvalue λ, where γ denotes the spectral parameter. Nevertheless, one may find that this cannot be true by calculation, which leads to a contradiction.

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Step 2. We declare that for every $x \in [\eta_0, 1/2)$, $w_p(x)$ is bounded as $p \to 1+$. Otherwise, there exists $\tilde{x} \in [\eta_0, 1/2)$ such that $w_p(\tilde{x}) \to \infty$ for some subsequence of $p \to 1+$. (3.17) yields that

$$\frac{1}{2}(w_p^{p-1}(x))' = \phi_{1p}'(x)\phi_{1p}(x) - \phi_{2p}'(x)\phi_{2p}(x).$$

Considering the sign of each term obtained previously, we find that $(w_p^{p-1}(x))' \geq 0$, that is, $w_p'(x) \geq 0$ on $[\eta_0, 1/2]$, and hence

$$\int_0^1 w_p(t)\mathrm{d}t \geq \int_{\tilde{x}}^{1/2} w_p(t)\mathrm{d}t \geq w_p(\tilde{x})\left(\frac{1}{2} - \tilde{x}\right) \to \infty$$

for some subsequence of $p \to 1+$, which contradicts the boundedness of $\|w_p\|_p$ proved in Lemma 3.1.

Step 3. We claim that $\lambda\phi_{1p}'^2(x) - C_p \to 0$ as $p \to 1+$ for $x \in [\eta_0, 1/2)$. If not, by (3.24), we know that there exists $\tilde{x} \in [\eta_0, 1/2)$ and $\varepsilon_0 > 0$ such that

$$\lambda\phi_{1p}'^2(\tilde{x}) \leq C_p - \varepsilon_0 \tag{3.25}$$

for some subsequence of $p \to 1+$. Taking $x = \tilde{x}$ in (3.20) and letting $p \to 1+$ in

$$\lambda\phi_{1p}'^2(\tilde{x}) - \phi_{2p}'^2(\tilde{x}) + \lambda\frac{p-1}{p}w_p^p(\tilde{x}) = C_p,$$

we encounter a contradiction between (3.25) and the conclusion in Step 2.

Step 4. We show that $\phi_{1p}(1/2)$ is bounded as $p \to 1+$. Otherwise, assume that $\phi_{1p}(1/2) \to \infty$ for some subsequence of $p \to 1+$. On the one hand, since $\phi_{1p}'^2(x) \leq C_p/\lambda$ on $[\eta_p, 1/2)$ as is shown in (3.24) and C_p is bounded for $p \in (1, 2]$ in (3.22), we find that $\phi_{1p}(\eta_0) \to \infty$ for some subsequence of $p \to 1+$. On the other hand, it follows from (3.20), Step 2 and Step 3 that $\phi_{2p}'^2(x)$ is bounded on $[\eta_0, 1/2]$ as $p \to 1+$, and so is $\phi_{2p}'(x)$. As $\phi_{2p}(1/2) = 0$ and for some $\xi \in (\eta_0, 1/2)$

$$\phi_{2p}\left(\frac{1}{2}\right) - \phi_{2p}(\eta_0) = \phi_{2p}'(\xi)\left(\frac{1}{2} - \eta_0\right),$$

we know that $\phi_{2p}(\eta_0)$ is bounded as $p \to 1+$. Finally, by (3.17), we derive that

$$w_p^{p-1}(\eta_0) = \phi_{1p}^2(\eta_0) - \phi_{2p}^2(\eta_0) \to \infty$$

for some subsequence of $p \to 1+$, and hence $w_p(\eta_0) \to \infty$ accordingly, which contradicts the conclusion in Step 2.

Step 5. Set $\alpha_p = \phi_{1p}'(0),\ \beta_p = \phi_{2p}'(0)$. Since

$$\phi_{1p}\left(\frac{1}{2}\right) = \frac{\alpha_p}{2} - \int_0^{1/2}\left(\frac{1}{2} - t\right)w_p(t)\phi_{1p}(t)dt,$$

we have

$$|\alpha_p| \leq \left|\phi_{1p}\left(\frac{1}{2}\right)\right|\left(2 + \int_0^{1/2}|w_p(t)dt|\right),$$

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and hence α_p is bounded as $p \to 1+$ by the boundedness of $\phi_{1p}(1/2)$ and $\|w_p\|$. As discussed earlier, C_p is bounded, and therefore by (3.20), $\beta_p^2 = \lambda\alpha_p^2 - C_p$ is surely bounded as $p \to 1+$.

Subsequently, set $\alpha_p \to \alpha$ and $\beta_p \to \beta$ for some subsequence of $p \to 1+$. Since $w_p(x) \xrightarrow{w^*} w_\delta$, it follows from Lemma 2.5 that as $p \to 1+$,

$$\phi_{1p} \to \phi_1, \ \phi_{2p} \to \phi_2$$

uniformly on $[0, 1/2]$, where ϕ_1 and ϕ_2 are respectively the first and second eigenfunctions of $-y'' = \mu w_0 y, \ y(0) = 0 = y(1)$ such that $\phi_1'(0) = \alpha$ and $\phi_2'(0) = \beta$, namely,

$$\phi_1(x) = \begin{cases} \alpha x, & x \in [0, x_0], \\ \alpha x_0, & x \in [x_0, 1/2], \end{cases}$$
$$\phi_2(x) = \begin{cases} \beta x, & x \in [0, x_0], \\ \beta(\lambda - 1)(\frac{1}{2} - x), & x \in [x_0, 1/2], \end{cases}$$

where $x_0 = \frac{\lambda - 1}{2\lambda}$. Since $w_p^{p-1}(x) = \phi_{1p}^2(x) - \phi_{2p}^2(x)$ and ϕ_{1p}, ϕ_{2p} converge to ϕ_1, ϕ_2 uniformly on $[0, 1/2]$, respectively, applying the expression of ϕ_1 and ϕ_2 in (3.26), one may verify that $w_p^{p-1}(x_1) < w_p^{p-1}(x_2)$, or equivalently, $w_p(x_1) < w_p(x_2)$ for $0 \le x_1 < x_2 \le 1/2$ and p sufficiently close to the right of 1. Moreover, $w_p(x) \xrightarrow{w^*} w_0$ implies that $w_p(x)$ is unbounded in every neighbourhood $U(x_0)$ of x_0 as $p \to 1+$, it follows that $w_p(x) \to \infty$ as $p \to 1+$ on $(x_0, 1/2]$, which clearly contradicts the conclusion in Step 2. Hence, Theorem 3.3 is proved. □

From now on (3.15) takes the form

$$w_p^{p-1}(x) = \phi_{1p}^2(x) + \phi_{2p}^2(x), \tag{3.27}$$

where $\phi_{1p} = \sqrt{c_{1p}/p}u_{1p}, \ \phi_{2p} = \sqrt{c_{2p}/p}u_{2p}$, and they satisfy (3.18). Then

$$\lambda\phi_{1p}''(x)\phi_{1p}'(x) - \phi_{2p}''(x)\phi_{2p}'(x) + \lambda w_p(\phi_{1p}(x)\phi_{1p}'(x) - \phi_{2p}(x)\phi_{2p}'(x)) = 0. \tag{3.28}$$

Integrating (3.28) on $[0, x]$, we have

$$\lambda\phi_{1p}'^2(x) + \phi_{2p}'^2(x) + \lambda\frac{p-1}{p}w_p^p(x) \equiv C_p := \lambda\phi_{1p}'^2(0) + \phi_{2p}'^2(0). \tag{3.29}$$

We call (3.29) *the first integral* associated with the critical equation (3.27). Similarly as before, we have from (3.28) that

$$\lambda\int_0^1 \phi_{1p}'^2 = \lambda\int_0^1 w_p\phi_{1p}^2, \ \int_0^1 \phi_{2p}'^2 = \lambda\int_0^1 w_p\phi_{2p}^2, \tag{3.30}$$

and hence

$$\int_0^1 \left[\lambda\phi_{1p}'^2 + \phi_{2p}'^2\right] = \lambda\int_0^1 \left[w_p(\phi_{1p}^2 + \phi_{2p}^2)\right]. \tag{3.31}$$

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Integrating (3.29) on $[0,1]$ and employing (3.31), we have

$$C_p = \lambda\left(1+\frac{p-1}{p}\right)\int_0^1 w_p^p(x) = \lambda\frac{2p-1}{p}\int_0^1 w_p^p(x). \tag{3.32}$$

Evidently, it can be seen from (3.32) that C_p is bounded for $p \in (1,2]$ due to the boundedness of $\|w_p\|$ by Lemma 3.1.

4. The proof of Theorem 1.1

In this section, we will use the critical equation (3.27) and the first integral (3.29) to prove Theorem 1.1. We begin with the following lemma in consideration of the boundedness of C_p in (3.32).

LEMMA 4.1. *Suppose that all the assumptions in Theorem 3.3 are satisfied. Let ϕ_{1p} and ϕ_{2p} be defined as in (3.27). Then the set*

$$\{\phi_{1p}(x), \phi'_{1p}(x), \phi_{2p}(x), \phi'_{2p}(x) : x \in [0,1], p \in (1,2]\}$$

is bounded.

Proof. Since $C_p = \lambda\phi'^2_{1p}(0)+\phi'^2_{2p}(0)$ is bounded, both $\phi'_{1p}(0)$ and $\phi'_{2p}(0)$ are bounded for $p \in (1,2]$. Furthermore, we get from $\phi''_{1p}(x) \le 0$ and $\phi'_{1p}(x)(1/2-x) \ge 0$ that $\phi'_{1p}(x)$ is bounded on $[0,1]$ for $p \in (1,2]$. Therefore, $\phi_{1p}(x) = \int_0^x \phi'_{1p}(t)\mathrm{d}t$ gives that $\phi_{1p}(x)$ is bounded on $[0,1]$ for $p \in (1,2]$. Similarly, the boundedness of $\phi_{2p}(x)$ and $\phi'_{2p}(x)$ can be proved on $[0,1]$ for $p \in (1,2]$. □

Let ϕ_{1p} and ϕ_{2p} be defined as in (3.27). Lemma 4.1 implies that ϕ_{1p} and ϕ_{2p} respectively converge to ϕ_1 and ϕ_2 as $p \to 1+$ uniformly on $[0,1]$, where ϕ_1 and ϕ_2, as shown in (3.26), are the first and second eigenfunctions of $-y'' = \mu w_0 y,\ y(0) = 0 = y(1)$ such that $\phi'_1(0) = \alpha$ and $\phi'_2(0) = \beta$, with α and β given by

$$\phi'_{1p}(0) \to \alpha,\ \phi'_{2p}(0) \to \beta,\ p \to 1+. \tag{4.1}$$

LEMMA 4.2. *Suppose that all the assumptions in Theorem 3.3 are satisfied. Let w_p be a critical point of E_p, and $\phi_1(x)$, $\phi_2(x)$ be defined as in (3.26). Then the following properties hold.*

(i) $(\alpha^2+\beta^2)x_0^2 = 1$;
(ii) $\beta \neq 0$;
(iii) *There exists $\delta > 0$, s.t. $w_p(x) \to 0$ as $p \to 1+$ uniformly on $[1/2-\delta, 1/2+\delta]$.*

Proof. (i) Suppose on the contrary that there exists $\varepsilon_0 > 0$ such that

$$\left|(\alpha^2+\beta^2)x_0^2 - 1\right| \ge 2\varepsilon_0.$$

Without loss of generality, we assume that $(\alpha^2+\beta^2)x_0^2 \ge 1+2\varepsilon_0$. By the continuity of $\phi_1(x)$, $\phi_2(x)$ and the uniform convergence of $\phi_{jp}(j=1,2)$ on $[0,1]$ as $p \to 1+$,

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there exists $\delta > 0$ such that for sufficiently small $p - 1 > 0$,

$$\phi_{1p}^2(x) + \phi_{2p}^2(x) \geq 1 + \varepsilon_0. \quad x_0 - \delta < x < x_0 + \delta.$$

Then the critical equation (3.27) yields that $(w_p(x))^{p-1} \geq 1 + \varepsilon_0$ on $[x_0 - \delta, x_0 + \delta]$, and hence

$$\int_{x_0-\delta}^{x_0+\delta} w_p(x)\mathrm{d}x \geq 2\delta(1+\varepsilon_0)^{\frac{1}{p-1}} \to +\infty,\ p \to 1+,$$

which contradicts Lemma 3.1.

(ii) Note that (3.32) implies that

$$C_p = \lambda \frac{2p-1}{p} \int_0^1 w_p^p(x) \to 2r\lambda,\ r = \frac{2\lambda}{\lambda - 1} (see\ 3.6).$$

Whereas, the definition of C_p in (3.29) together with (4.1) gives that

$$C_p = \lambda\phi_{1p}'(0) + \phi_{1p}'(0) \to \lambda\alpha^2 + \beta^2,\ p \to 1+.$$

Then $2r\lambda = \lambda\alpha^2 + \beta^2$, and hence $\beta^2 > 0$ by $(\alpha^2 + \beta^2)x_0^2 = 1$ and $rx_0 = 1$.

(iii) Considering that $\beta \neq 0$, we have that

$$\alpha^2 x_0^2 < (\alpha^2 + \beta^2)x_0^2 = 1.$$

Set $\alpha^2 x_0^2 = 1 - 2\varepsilon_1$. By the continuity of $\phi_1^2(x) + \phi_2^2(x)$ at $x = 1/2$ and the uniform convergence of $\phi_{jp} (j = 1, 2)$, there exists $\delta > 0$ such that

$$\phi_{1p}^2(x) + \phi_{2p}^2(x) \leq 1 - \varepsilon_1,\ |x - 1/2| \leq \delta$$

for sufficiently small $p - 1 > 0$, and hence $w_p(x) \leq (1 - \varepsilon_1)^{\frac{1}{p-1}} \to 0,\ p \to 1+.$ □

Now, we proceed to prove the main result of this paper, Theorem 1.1, with the aid of Lemma 4.1 and Lemma 4.2.

The proof of Theorem 1.1. ⇒) Suppose that the problem (3.1) attains its optimal weight uniquely by a Dirac distribution at two points. Then a calculation gives that w_0 defined as in (3.6) is the unique optimal weight. Let w_p, ϕ_{1p}, ϕ_{2p} and ϕ_1, ϕ_2 be defined as before. Lemma 4.1 and Lemma 4.2 yield that $C_p = \lambda\phi_{1p}'^2(0) + \phi_{2p}'^2(0) \to \lambda\alpha^2 + \beta^2$ as $p \to 1+$. On the other hand, as $p \to 1+$, since $w_p \xrightarrow{w^*} w_0$, based on the uniform convergence of ϕ_{2p}, we obtain

$$\phi_{2p}'(0) - \phi_{2p}'(1/2) = \lambda \int_0^{1/2} w_p\phi_{2p} \to \lambda \int_{[0,1/2]} w_0\phi_2 = \lambda r\beta x_0 = \beta\lambda. \tag{4.2}$$

(4.2) together with $\phi_{2p}'(0) = \beta_p \to \beta$, see (4.1), guarantees that $\phi_{2p}'(1/2) \to \beta(1-\lambda)$ as $p \to 1+$.

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Set $x = 1/2$ in (3.29). We have

$$\phi_{2p}'^2(1/2) + \lambda\frac{p-1}{p}w_p^p(1/2) = C_p := \lambda\phi_{1p}'^2(0) + \phi_{2p}'^2(0) \tag{4.3}$$

since $\phi_{1p}'(1/2) = 0$. Letting $p \to 1+$ in (4.3), it follows from $w_p(1/2) \to 0$, (4.1) and (4.2) that

$$\lambda\alpha^2 + \beta^2 = \beta^2(\lambda-1)^2,$$

which means that $\lambda \geq 2$. A calculation gives that

$$\int_0^1 w_0 = 2r = \frac{4\lambda}{\lambda-1}.$$

Recover the original form, we have proven that $\lambda_2 \geq 2\lambda_1$ and

$$E(\lambda_1,\lambda_2) = \frac{4\lambda_2}{\lambda_1(\lambda_2-\lambda_1)}.$$

$\Leftarrow$) Suppose that $\lambda_2 \geq 2\lambda_1$. The optimal weight w_0 has been illustrated in Theorem 3.1 of [15]; we only need to prove that w_0 is unique herein. For the sake of completeness, we provide both the proof of existence and uniqueness since some inequalities in the proof of existence will be used later.

It follows from Corollary 3.2 in [15] that

$$\int_0^1 w(x)\mathrm{d}x \geq \frac{4\sigma_1^2}{\sigma_1-\sigma_2},\ \sigma_1 = \sum_{j=1}^{\infty}\frac{1}{\lambda_{2j-1}},\ \sigma_2 = \sum_{j=1}^{\infty}\frac{1}{\lambda_{2j}}. \tag{4.4}$$

Set $\sigma_1 = t/\lambda_1,\ \sigma_2 = s/\lambda_2$. Then $s,\ t \in [1,+\infty)$ and (4.4) gives

$$\int_0^1 w(x)\mathrm{d}x \geq \frac{4t^2\lambda_2}{\lambda_1(t\lambda_2 - s\lambda_1)}. \tag{4.5}$$

Since $\lambda_2 \geq 2\lambda_1$, we have

$$(t^2-t)\lambda_2 \geq 2(t^2-t)\lambda_1 = (t-1)^2\lambda_1 + (t^2-s)\lambda_1 + (s-1)\lambda_1 \geq (t^2-s)\lambda_1,$$

which means that

$$\frac{4t^2\lambda_2}{\lambda_1(t\lambda_2-s\lambda_1)} \geq \frac{4\lambda_2}{\lambda_1(\lambda_2-\lambda_1)}, \tag{4.6}$$

and hence

$$E(\lambda_1,\lambda_2) \geq \frac{4\lambda_2}{\lambda_1(\lambda_2-\lambda_1)}.$$

Let w_0 be defined as in (3.6). Then the '=' holds in the above equality.

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It remains to prove that the optimal weight is attained uniquely by w_0. That is, if $\lambda_1(\hat{w}) = \lambda_1, \lambda_2(\hat{w}) = \lambda_2$ and

$$\int_0^1 \hat{w}\mathrm{d}x = \frac{4\lambda_2}{\lambda_1(\lambda_2 - \lambda_1)},$$

then $\hat{w} = w_0$. Let σ_1 and σ_2 be defined as (4.4) and set

$$\hat{t} = \lambda_1\sigma_1, \ \hat{s} = \lambda_2\sigma_2.$$

Then $\hat{t}, \hat{s} \in [1, \infty)$ and it follows from (4.5) that

$$\int_0^1 \hat{w}\mathrm{d}x \geq \frac{4\hat{t}^2\lambda_2}{\lambda_1(\hat{t}\lambda_2 - \hat{s}\lambda_1)}. \tag{4.7}$$

Therefore, (4.6) together with (4.7) means that

$$\frac{4\hat{t}^2\lambda_2}{\lambda_1(\hat{t}\lambda_2\hat{s} - \lambda_1)} = \frac{4\lambda_2}{\lambda_1(\lambda_2 - \lambda_1)}, \ \textit{or} \ \frac{\hat{t}^2}{\hat{t}\lambda_2 - \hat{s}\lambda_1} = \frac{1}{\lambda_1(\lambda_2 - \lambda_1)}. \tag{4.8}$$

This also means that

$$(\hat{t}^2 - \hat{t})\lambda_2 = (\hat{t}^2 - \hat{s})\lambda_1. \tag{4.9}$$

On the other hand, since $\lambda_2 \geq 2\lambda_1 > 0$ and $\hat{t}, \hat{s} \geq 1$, we have from

$$(\hat{t}^2 - \hat{s})\lambda_1 = (\hat{t}^2 - \hat{t})\lambda_2 \geq 2(\hat{t}^2 - \hat{t})\lambda_1 = (\hat{t} - 1)^2\lambda_1 + (\hat{t}^2 - \hat{s})\lambda_1 + (\hat{s} - 1)\lambda_1$$

that $\hat{t} = \hat{s} = 1$. This implies that (2.1) has exactly two eigenvalues, and hence Lemma 2.4 implies that the weight must be the Dirac distribution at two points and a calculation gives that the weight is w_0 as defined in (3.6). □

Remark 4. Theorem 1.1 presents a necessary and sufficient condition for optimal inverse problem (1.5) as $m = 2$, which improves the result in [15] by filling in the necessity of the conclusion. In particular, the necessity here signifies that for (1.5), if its extremum is attained uniquely by the Dirac distribution at two points in $(0, 1)$, then $\lambda_2 \geq 2\lambda_1$. As noted in Remark 1 of [15], it is expected that Corollary 3.4 therein could be proved without the extra condition $\lambda_2 \geq 2\lambda_1$. Based on the conclusion obtained in the present paper, if such a restriction is removed, then the optimal weight function would not be a Dirac distribution at two points on (0,1), but more complex. Intuitively, we conjecture that it may be a Dirac distribution at more than two points.

5. The extremal problem for partial trace

In this section, as an application, we study the extremal problem for partial trace of the first two eigenvalues associated with (2.1), i.e.,

$$-y''(x) = \lambda w(x)y(x), \quad y(0) = 0 = y(1), \tag{5.1}$$

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where $w \in B_r^+ \subset L^1([0,1],\mathbb{R})$,

$$B_r^+ = \{w \in L^1([0,1],\mathbb{R}) : \ \|w\|_1 = r, \ w(x) \geq 0, a.e. x \in [0,1]\}.$$

We search for the infimum of the set $V(r)$, where

$$V(r) = \{\lambda_1(w) + \lambda_2(w) : \ w \in B_r^+\}, \ r > 0, \tag{5.2}$$

with $\lambda_1(w)$ and $\lambda_2(w)$ being respectively the first and second eigenvalues of (5.1). Introduce

$$B_r^+(s) = \{w \in B_r^+ : \ \lambda_2(w) \geq 2\lambda_1(w), \ w(x) = w(1-x), x \in [0,1]\}. \tag{5.3}$$

Note that $B_r^+(s)$ is not empty by Remark 2 in Section 1 for $a \in [1/4, 1/2)$. Clearly, $B_r^+(s) \subset B_r^+$, and hence $\inf V(r) \leq \inf V(r,s)$, where

$$V(r,s) = \{\lambda_1(w) + \lambda_2(w) : \ w \in B_r^+(s)\}. \tag{5.4}$$

Fix $\Lambda > 0$, set

$$\begin{aligned} \Omega(\Lambda) =& \big\{w \in L^1([0,1],\mathbb{R}) : \lambda_1(w) + \lambda_2(w) = \Lambda, \ \lambda_2(w) \geq 2\lambda_1(w) \\ & 0 \leq w, \ w(x) = w(1-x), x \in [0,1]\big\}; \\ E(\Lambda) =& \inf\{\|w\|_1 : \ w \in \Omega(\Lambda)\}. \end{aligned} \tag{5.5}$$

Then the dual problem of $\inf V(r,s)$ is just problem $E(\Lambda)$ for weights. That is, if $r = E(\Lambda) = f(\Lambda)$, then $\Lambda = \inf V(r,s) = f^{-1}(r)$, where f^{-1} is the inverse function of f. Similar problems of $E(\Lambda)$ in L^2 space for optimal potential of

$$-y'' + qy = \lambda y, \ y(0) = 0 = y(1)$$

has been studied in [19, Theorems 1.1 and 1.2] and the existence and uniqueness of the optimal potential are established.

For given λ_1, λ_2 such that $0 < 2\lambda_1 \leq \lambda_2$ and $\lambda_1 + \lambda_2 = \Lambda$, we set $\lambda_1 = t\Lambda$ and $\lambda_2 = (1-t)\Lambda$ for $t \in (0, 1/3]$. See (1.6), the definition of $E(\lambda_1, \lambda_2))$ for $w \in \Omega(\Lambda)$ and $\lambda_1(w) = \lambda_1, \lambda_2(w) = \lambda_2$ reads

$$E(\lambda_1, \lambda_2) = \frac{4\lambda_2}{\lambda_1(\lambda_2 - \lambda_1)} = \frac{4}{\Lambda}\frac{1-t}{t(1-2t)} := H(t), \ t \in (0, 1/3].$$

A calculation gives that the minimal value of $H(t)$ on $(0, 1/3]$ is attained by

$$H_{min} = H(a) = \frac{4}{\Lambda}(1+\sqrt{2})^2, \ a = 1/(2+\sqrt{2}).$$

Therefore, $E(\Lambda) = H(a)$, which also means that

$$\inf V(r,s) = \frac{4}{r}(1+\sqrt{2})^2 \geq \inf V(r). \tag{5.6}$$

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(5.6) offers an upper bound of $\inf V(r)$. Now, we apply Mercer's theorem to find a lower bound. According to Lemma 2.4 in [16],

$$\int_0^1 G(x,x)w(x)\mathrm{d}x = \sum_{n=1}^{\infty} \frac{1}{\lambda_n(w)} \geq \frac{1}{\lambda_1} + \frac{1}{\lambda_2} = \frac{\lambda_1+\lambda_2}{\lambda_2\lambda_1},$$

where $G(x,t)$ is the Green function of $-y'' = 0$ with $y(0) = 0 = y(1)$, and hence $G(x,x) = x(1-x)$ on $[0,1]$. Set $\Lambda = \lambda_1 + \lambda_2$, $\lambda_1 = t\Lambda$, and $\lambda_2 = (1-t)\Lambda$ for $t \in (0,1/2)$. Then

$$r = \int_0^1 w(x)\mathrm{d}x \geq \frac{4}{\Lambda t(1-t)} > \frac{16}{\Lambda},\ t \in (0,1/2) \Rightarrow \Lambda > \frac{16}{r}.$$

In fact, the next theorem readily follows the above discussion.

THEOREM 5.1. *Consider the extremal problem for partial trace of the first two eigenvalues* (5.1) *and let* $V(r)$ *be defined as in* (5.2). *Then*

$$\inf V(r) \in \left(\frac{16}{r}, \frac{4}{r}(1+\sqrt{2})^2\right].$$

Acknowledgements.
This work was supported by the Natural Science Foundation of Shandong Province (ZR2024MA005) and the National Natural Science Foundation of China (Grants 12271299 and 12071254). The authors would like to express their sincere gratitude to the anonymous referees for their constructive criticism and helpful suggestions.

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