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On Perron's method of summation

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1. *Introduction.* Let $\lambda > 0$ and let

$$\phi_\lambda(x) = a_0 + \sum_{\nu=0}^{\infty} a_\nu \frac{x}{x+\lambda}\frac{x+1}{x+\lambda+1}\cdots\frac{x+\nu-1}{x+\lambda+\nu-1}$$
$$= \sum_{\nu=0}^{\infty} a_\nu \frac{\Gamma(x+\lambda)\,\Gamma(x+\nu)}{\Gamma(x)\,\Gamma(x+\lambda+\nu)}. \tag{1}$$

If $\lim_{x\to\infty} \phi_\lambda(x) = s$, we say that the series $\sum_{\nu=0}^{\infty} a_\nu$ is summable (P,λ) to s.

This method of summation was introduced by Perron (1), who proved that the method is regular and that the Fourier series of a bounded function is summable $(P,2)$ at every point of continuity.

Szasz (2) has also discussed this method of summation. He proved that (P,λ) includes (C,λ), that, if the series (1) converges for all positive x, it also includes the Abel summability method (A) and that the (P,λ) transform of the series

$$\sum_{\nu=1}^{\infty} \frac{\sin \nu t}{\nu} = \frac{\pi - t}{2} \quad (0 < t < \pi)$$

does not present a Gibbs phenomenon for any $\lambda > 0$.

The summability method (P,λ) is similar to a special case of other summability methods which have been considered recently. It can be shown as follows.

If

$$s_\nu = o(\nu^\lambda), \tag{2}$$

then we see easily by partial summation that (1) is equivalent to

$$\phi_\lambda(x) = \lambda \sum_{\nu=0}^{\infty} \frac{x(x+1)\ldots(x+\nu-1)}{(x+\lambda)(x+\lambda+1)\ldots(x+\lambda+\nu)}\, s_\nu, \tag{3}$$

in the sense that if either of the series (1), (3) converges, then the other also converges, and has the same sum. Now for any given $x > 0$, the convergence of (1) is equivalent to the convergence of

$$\sum_{\nu=1}^{\infty} \frac{a_\nu}{\nu^\lambda}, \tag{4}$$

and it is easy to prove that the convergence of (4) implies (2). On the other hand, the convergence of (3) is equivalent to the convergence of

$$\sum_{\nu=1}^{\infty} \frac{s_\nu}{\nu^{\lambda+1}} \tag{5}$$

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which does not imply (2). Thus (3) is slightly more general than (1) in that, whenever (1) converges, (3) converges to the same sum, but that the converse is not true.

Given any sequence $\{\mu_n\}$, the quasi-Hausdorff transform (H^*, μ_n) is defined by

$$t_n = \sum_{\nu=n}^{\infty} \binom{\nu}{n} (\Delta^{\nu-n}\mu_n)\, s_\nu$$

(see Chapter XI of (3)). A method introduced by Ramanujan (5) is obtained by taking the matrix of the (H^*, μ_n) transformation and moving the elements of 'row n' n places to the left. Changing Ramanujan's notation slightly, I will denote this transformation by (S, μ_n). Thus the (S, μ_n) transformation is given by

$$t_n = \sum_{\nu=0}^{\infty} \binom{\nu+n}{n} (\Delta^{\nu}\mu_n)\, s_\nu.$$

The work of Ramanujan shows that there is a close connexion between the regularity of the (H, μ_n), (H^*, μ_{n+1}) and (S, μ_{n+1}) methods. It is familiar that in the special case

$$\mu_n = 1 \Big/ \binom{n+\alpha}{n} \tag{6}$$

(H, μ_n) reduces to (C, α). Thus it is reasonable to regard the (S, μ_{n+1}) method with μ_n given by (6) as in a certain sense corresponding to (C, α). Ishiguro (6) has given the name (D, α) to this method. For some of its properties, see Kuttner (7).

Now Borwein (8) has considered a generalization of the Cesàro transform, which he called (C, α, β); this is the (H, μ_n) transform with

$$\mu_n = \binom{n+\beta}{n} \Big/ \binom{n+\alpha+\beta}{n}. \tag{7}$$

Clearly, this reduces to (C, α) in the special case $\beta = 0$. By analogy with this, it would be natural to describe the special case of (S, μ_{n+1}) in which μ_n is given by (7) as (D, α, β). A straightforward calculation shows that the (D, α, β) transformation is given by

$$t_n = \frac{(\beta+1)(\beta+2)\dots(\beta+n+1)}{n} \sum_{\nu=0}^{\infty} \frac{\alpha(\alpha+1)\dots(\alpha+\nu-1)\dots(\nu+1)(\nu+2)\dots(\nu+n)}{(\alpha+\beta+1)(\alpha+\beta+2)\dots(\alpha+\beta+n+1+\nu)}\, s_n.$$

As a special case, we see that $(D, 1, \lambda-1)$ is given by

$$t_n = \lambda \sum_{\nu=0}^{\infty} \frac{(n+1)(n+2)\dots(n+\nu)}{(\lambda+n+1)(\lambda+n+2)\dots(\lambda+n+\nu+1)}\, s_\nu. \tag{8}$$

Now if $\phi_\lambda(x)$ is given by (3), and t_n by (8), then

$$\phi_\lambda(n+1) = t_n.$$

Thus (P, λ) differs from $(D, 1, \lambda-1)$ only in that, first, we make $x \to \infty$ through all values instead of integer values only, and, secondly, we take the slightly less general series-to-function form (1) instead of the sequence-to-function form (3).

Let

$$\psi_\lambda(x) = \sum_{\nu=0}^{\infty} a_\nu \left(\frac{x}{x+\nu}\right)^{\lambda}$$

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and let us denote by (P', λ) the summability method given by $\lim_{x\to\infty} \psi_\lambda(x)$. The problem of investigating the relation between (P, λ) and (P', λ) in the general case appears to be difficult. However, the following remarks may be made. Since

$$\frac{\Gamma(x+\lambda)}{\Gamma(x)} \sim x^\lambda$$

it is clear that, if

$$\frac{\Gamma(x+\nu)}{\Gamma(x+\nu+\lambda)} = \frac{1}{(x+\nu)^\lambda} + R_\nu(x)$$

and

$$\sum_{\nu=0}^{\infty} a_\nu R_\nu(x) = o(x^{-\lambda}), \tag{9}$$

then

$$\phi_\lambda(x) = \psi_\lambda(x)\,(1+o(1)) + o(1). \tag{10}$$

Thus (P, λ) and (P', λ) are equivalent for those series which satisfy (9). Now it is easily verified that, for large x (uniformly in ν for $\nu \geqslant 0$)

$$R_\nu(x) - R_{\nu+1}(x) = O\left(\frac{1}{(x+\nu)^{\lambda+2}}\right).$$

We can therefore show by partial summation that (9) is satisfied in particular when

$$s_\nu = o(\nu). \tag{11}$$

Now the convergence of the series defining either $\phi_\lambda(x)$ or $\psi_\lambda(x)$ is equivalent to the convergence of (11); so we may restrict ourselves to the case in which (4) converges. As already remarked, this implies (2); and when $\lambda \leqslant 1$, (2) implies (11). Thus (P, λ) is equivalent to (P', λ) in the case $0 < \lambda \leqslant 1$.

The following theorems will be proved.

THEOREM 1. *Let $\{a_\nu\}$ be a sequence such that*

$$\limsup_{\nu\to\infty} |\nu a_\nu| < \infty.$$

Then

$$\lim_{n\to\infty,\, x\to\infty} \sup_{n/x\to\infty} |s_n - \phi_\lambda(x)| \leqslant A_q \limsup_{\nu\to\infty} |\nu a_\nu|, \tag{12}$$

where q is any positive constant and

$$A_q = \int_0^q \left(1 - \frac{1}{(1+u)^\lambda}\right)\frac{du}{u} + \int_q^\infty \frac{du}{u(1+u)^\lambda}.$$

Moreover, the constant A_q is the best in the following sense. There is a real sequence $\{a_\nu\}$ such that $0 < \limsup_{\nu\to\infty} |\nu a_\nu| < \infty$ and the members of the inequality (12) *are equal.*

THEOREM 2. *If $\sum_{\nu=0}^{\infty} a_\nu$ is summable (P, λ) to s, then a necessary and sufficient condition that it should converge to s is that*

$$a_1 + 2a_2 + \ldots + na_n = o(n). \tag{13}$$

THEOREM 3. *If $\sum_{\nu=0}^{\infty} a_\nu$ is summable (P, λ) to s and if*

$$\underline{\lim}\,(s_n - s_m) \geqslant 0 \tag{14}$$

when $n > m \to \infty$, $n/m \to 1$, then it converges to s.

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If $\chi(x)$ is of bounded variation in $0 \leqslant x \leqslant 1$, and $s(x)$ is Borel-measurable in every finite interval of $x \geqslant 0$, the function

$$t(x) = \int_0^1 s(xt)\, d\chi(t)$$

for $x \geqslant 0$ is called the Hausdorff transformation of $s(x)$ by means of $\chi(x)$.

It is known (3) *that an $(H, \chi(x))$ transformation is regular if and only if*

$$\chi(0) = \chi(0+), \quad \chi(1) - \chi(0) = 1.$$

We define the (P, λ) limit of $s(u)$ as the limit as $w \to \infty$ (if it exists) of

$$\lambda w^\lambda \int_0^\infty \frac{s(u)}{(w+u)^{\lambda+1}}\, du.$$

THEOREM 4. *If $s(x)$ is bounded in every finite interval of $x \geqslant 0$ and summable (P, λ) to s, and if $t(x)$ is any regular $(H, \chi(x))$ transform of $s(x)$, then for all positive values of w*

$$\int_0^\infty t(u) \frac{w^\lambda}{(w+u)^{\lambda+1}}\, du = \int_0^1 \left\{ \int_0^\infty s(u) \frac{(wt)^\lambda}{(wt+u)^{\lambda+1}}\, du \right\} d\chi(t) \tag{15}$$

and further $t(x)$ is summable (P, λ) to s.

2. *Lemmas.* For the proofs of these theorems we require the following lemmas.

LEMMA 1. *Suppose $\{s_\nu\}$ is any bounded (real or complex) sequence. Let $\{c_\nu(x)\}$ be a sequence of functions defined for $0 < x < +\infty$ and satisfying*

$$\lim_{x \to \infty} c_\nu(x) = 0 \tag{16}$$

for $\nu = 0, 1, \ldots,$

$$\limsup_{x \to \infty} \sum_{\nu=0}^{\infty} |c_\nu(x)| = M < +\infty.$$

Then we have

$$\limsup_{x \to \infty} \sum_{\nu=0}^{\infty} c_\nu(x) s_\nu \leqslant M \limsup_{x \to \infty} |s_\nu|. \tag{17}$$

Moreover, M is the best constant in the following sense, there exists a bounded sequence $\{s_\nu\}$, $0 < \limsup_{\nu \to \infty} |s_\nu| < +\infty$, such that the members of inequality (17) *are equal.*

This lemma is due to Agnew (4).

LEMMA 2. *Let*

$$\tau(x) = \sum_{\nu=0}^{\infty} c_\nu(x) s_\nu \quad (x \geqslant 0).$$

Suppose that the following conditions are satisfied:

(i) $c_\nu(x) \geqslant 0$, $c_\nu(x) \to 0$ $(x \to \infty)$, $\sum_{\nu=0}^{\infty} c_\nu(x) = 1$.

(ii) *The coefficients c_ν have, in addition to* (i), *the properties*

$$\sum_{\nu=1}^{M} c_\nu(x) \to 0$$

if $M > x \to \infty$, $x/m \to \infty$;

$$\sum_{\nu=N}^{\infty} c_\nu(x) \log \frac{\nu}{N} \to 0,$$

if $N > x \to \infty$, $N/x \to \infty$.

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(iii) *If* $s(t) = s_\nu$ *for* $\nu \leqslant t < \nu+1$ *then*

$$\underline{\lim}\,\{s(t)-s(u)\} \geqslant 0$$

when $t > u \to \infty$, $t/u \to 1$.

(iv) $\tau(x)$ *is bounded.*

Then s_ν *is bounded.*

This is a special case of Theorem 238 in (3).

LEMMA 3. *If the Fourier transform of* $g(x)$ *never vanishes,* $f(x)$ *is real, bounded, and satisfies the condition*

$$\underline{\lim}\,(f(y)-f(x)) \geqslant 0$$

when $y > x \to \infty$, $y/x \to 1$, *and*

$$\frac{1}{x}\int_0^\infty g\left(\frac{t}{x}\right) f(t)\,dt \to l\int_0^\infty g(t)\,dt,$$

then $f(x) \to l$ *as* $x \to \infty$.

This is Theorem 233 in (3).

3. *Proof of Theorem* 1. The Tauberian condition imposed in this theorem can easily be shown to imply that

$$\phi_\lambda(x) - \psi_\lambda(x) \to 0$$

as $x \to \infty$; so that in proving this theorem we may consider (P', λ) in place of (P, λ).

We have

$$s_n - \psi_\lambda(x) = \sum_1^n a_\nu \left\{1-\left(\frac{x}{x+\nu}\right)^\lambda\right\} - \sum_{n+1}^\infty a_\nu \left(\frac{x}{x+\nu}\right)^\lambda.$$

Let

$$c_0(x) = 0,$$

$$c_\nu(x) = \frac{1}{\nu}\left\{1-\left(\frac{x}{x+\nu}\right)^\lambda\right\} \quad \text{for} \quad 1 \leqslant \nu \leqslant n,$$

$$= -\frac{1}{\nu}\left(\frac{x}{x+\nu}\right)^\lambda \quad \text{for} \quad \nu > n.$$

Since

$$\lim_{n\to\infty,\, x\to\infty,\, n/x\to q} c_\nu(x) = \frac{1}{\nu}\left\{1-\left(\frac{x}{x+\nu}\right)^\lambda\right\} = 0$$

for every fixed ν, condition (16) is satisfied.

Let

$$f(u) = \frac{1}{u}\left(1-\frac{1}{(1+u)^\lambda}\right),$$

$$g(u) = \frac{1}{u(1+u)^\lambda}.$$

Then

$$\sum_{\nu=0}^\infty |c_\nu(x)| = \frac{1}{x}\sum_{\nu=1}^n f\left(\frac{\nu}{x}\right) + \frac{1}{x}\sum_{\nu=n+1}^\infty g\left(\frac{\nu}{x}\right).$$

As $n \to \infty$, $x \to \infty$, $n/x \to q$, this sum tends to

$$\int_0^q f(u)\,du + \int_q^\infty g(u)\,du.$$

Theorem 1 follows from Lemma 1.

4. *Proof of Theorem* 2. Clearly the condition (13) is necessary.

Let $v_0 = 0$, $v_n = a_1 + 2a_2 + \ldots + na_n$ $(n > 0)$. Then $v_n = o(n)$. By partial summation we have

$$s_n = a_0 + \sum_{k=1}^n \frac{v_k - v_{k-1}}{k} = a_0 + \sum_{k=1}^n \frac{v_k}{k(k+1)} + \frac{v_n}{n+1}.$$

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Since the last term tends to zero the series

$$\sum_{k=0}^{\infty} a_k \quad \text{and} \quad a_0 + \sum_{k=1}^{\infty} \frac{v_k}{k(k+1)}$$

are equiconvergent. Let t_n and $g_\lambda(x)$ be the partial sum and Perron sum of the second series. We have

$$s_n - t_n \to 0 \quad (n \to \infty),$$

$$\phi_\lambda(x) - g_\lambda(x) \to 0 \quad (x \to \infty).$$

Since the terms of the second series are $o(1/n)$, it follows from Theorem 1 that $t_n - g_\lambda(x) \to 0$ as $x = n \to \infty$ and hence $s_n - \psi_\lambda(x) \to 0$ as $x = n \to \infty$. (13) is also sufficient.

The proof of Theorem 2 does not really involve the specific properties of (P, λ) at all. The argument shows that if A is any regular summability method for which

$$a_n = o(1/n) \tag{18}$$

is a Tauberian condition (that is to say, which is such that (18), together with summability A, implies convergence), then

$$a_1 + 2a_2 + \ldots + na_n = o(n)$$

is also a Tauberian condition (in an analogous sense).

5. *Proof of Theorem* 3. By taking

$$c_\nu(x) = \frac{\lambda x(x+1)\ldots(x+\nu-1)}{(x+\lambda)(x+\lambda+1)\ldots(x+\lambda+\nu)}$$

we can easily verify that the conditions of Lemma 2 are satisfied. Hence (P, λ) summability of $\sum_{\nu=1}^{\infty} a_\nu$ and (14) imply that s_ν is bounded.

It follows that (11) holds, and hence that we may replace (P, λ) by $(P' \lambda)$.

The Fourier transform of

$$g(t) = \lambda \left(\frac{1}{1+t}\right)^{\lambda+1}$$

is

$$\lambda \int_0^\infty \frac{t^{-ix}}{(1+t)^{\lambda+1}}\, dt,$$

which never vanishes (since the gamma function never vanishes). Let $f(x) = s_\nu$ $(\nu \leqslant x < \nu+1)$. Then the conditions of Lemma 3 are satisfied. Hence $\sum_{\nu=0}^{\infty} a_\nu$ converges to s.

6. *Proof of Theorem* 4. Let

$$s_1(x) = s(x), \quad s_2(x) = 0 \quad \text{for} \quad 0 \leqslant x \leqslant 1,$$

$$s_1(x) = 0, \quad s_2(x) = s(x) \quad \text{for} \quad x > 1,$$

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and let $t_1(x)$ and $t_2(x)$ be the regular Hausdorff transform of $s_1(x)$ and $s_2(x)$. Then, by Fubini's theorem, since $s_1(x)$ is bounded

$$\begin{aligned}\int_0^1 \left\{\int_0^\infty s_1(u)\frac{(Wt)^\lambda}{(Wt+u)^{\lambda+1}}\,du\right\}d\chi(t) &= \int_0^1 \left\{\int_0^\infty s_1(tu)\frac{W^\lambda}{(W+u)^{\lambda+1}}\,du\right\}d\chi(t)\\ &= \int_0^\infty t_1(u)\frac{W^\lambda}{(W+u)^{\lambda+1}}\,du. \end{aligned} \tag{19}$$

Next we prove that $M(K,t) = \int_{Kt}^\infty s_2(u)\frac{(Wt)^\lambda}{(Wt+u)^{\lambda+1}}\,du$

tends to zero uniformly in $0 \leqslant t \leqslant 1$. Write $A = \max(Kt, 1)$. Then

$$\begin{aligned} M(K,t) &= t^\lambda \int_A^\infty \frac{s_2(u)\,W^\lambda}{(W+u)^{\lambda+1}}\left(\frac{W+u}{Wt+u)}\right)^{\lambda+1}du\\ &= t^\lambda\left(\frac{W+A}{Wt+A}\right)^{\lambda+1}\int_A^\xi \frac{s_2(u)\,W^\lambda}{(W+u)^{\lambda+1}}\,du \end{aligned}$$

for some $\xi > A$ (by the second mean value theorem). Now, since $A \geqslant 1$,

$$\frac{W+A}{Wt+A} \leqslant \frac{W+A}{A} \leqslant W+1.$$

Thus
$$|M(K,t)| < (W+1)^{\lambda+1}t^\lambda\left|\int_A^\xi \frac{s_2(u)\,W^\lambda}{(W+u)^{\lambda+1}}\,du\right|. \tag{20}$$

Again using the fact that $A \geqslant 1$, we see that the integral in (20) is bounded for all K, t; and it is arbitrarily small for sufficiently large Kt. Hence, given any $\epsilon > 0$ we can make $|M(K,t)| < \epsilon$ by taking $t < t_0$ (say) (whatever K). Having chosen $t_0 > 0$ we can make Kt arbitrarily large uniformly in $t_0 \leqslant t \leqslant 1$ by taking K sufficiently large. Hence $M(K,t) \to 0$ as $K \to \infty$ uniformly for $0 \leqslant t \leqslant 1$.

Since $s_2(x)$ is bounded in every finite interval, it follows from Fubini's theorem that

$$\begin{aligned}&\int_0^1 \left\{\int_0^\infty s_2(u)\frac{(Wt)^\lambda}{(Wt+u)^{\lambda+1}}\,du\right\}d\chi(t)\\ &= \lim_{K\to\infty}\left\{\int_0^1\left[\int_0^{Kt} s_2(u)\frac{(Wt)^\lambda}{(Wt+u)^{\lambda+1}}\,du\right]d\chi(t)\right\}\\ &\quad + \lim_{K\to\infty}\left\{\int_0^1\left[\int_{Kt}^\infty s_2(u)\frac{(Wt)^\lambda}{(Wt+u)^{\lambda+1}}\,du\right]d\chi(t)\right\}\\ &= \lim_{K\to\infty}\int_0^1\left\{\int_0^K s_2(tu)\frac{W^\lambda}{(W+u)^{\lambda+1}}\,du\right\}d\chi(t)\\ &= \lim_{K\to\infty}\int_0^K t_2(u)\frac{W^\lambda}{(W+u)^{\lambda+1}}\,du\\ &= \int_0^\infty t_2(u)\frac{W^\lambda}{(W+u)^{\lambda+1}}\,du. \end{aligned}$$

This, together with (19), gives (15).

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REFERENCES

(1) PERRON, O. Beitrag zur Theorie divergenten reihen. *Math. Z.* **6** (1920), 286–310.
(2) SZASZ, O. On a summation method of O. Perron. *Math. Z.* **52** (1949–50), 631–636.
(3) HARDY, G. H. *Divergent series* (Oxford University Press).
(4) AGNEW, R. P. Abel transforms and partial sums of Tauberian series. *Ann. of Math.* **50** (1949), 110–117.
(5) RAMANUJAN, M. S. On Hausdorff and quasi-Hausdorff methods of summability. *Quart. J. Math. (Oxford)*, (2), **8** (1957), 197–213.
(6) ISHIGURO, K. On the summability methods of divergent series. *Acad. Roy. Belg. Cl. Sci. Mém. Coll. in* -8°, **35** (1965), 1–43.
(7) KUTTNER, B. On 'translated quasi-Cesàro' summability. *Proc. Cambridge Philos. Soc.* **62** (1966), 705–712.
(8) BORWEIN, D. Theorems on some methods of summability. *Quart. J. Math. (Oxford)*, (2), **36** (1958), 310–316.

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