Ergod. Th. & Dynam. Sys. (2005), **25**, 1935–1950
doi:10.1017/S0143385705000180
© *2005 Cambridge University Press*
Printed in the United Kingdom

Topological entropy in C^*-algebras associated with lambda-graph systems

KENGO MATSUMOTO

Department of Mathematical Sciences, Yokohama City University,
Seto 22-2, Kanazawa-ku, Yokohama 236-0027, Japan
(e-mail: kengo@yokohama-cu.ac.jp)

(*Received* 2 *November* 2004 *and accepted in revised form* 21 *April* 2005)

Abstract. A λ-graph system is a labeled Bratteli diagram with shift transformation. It is a generalization of finite labeled graphs and presents a subshift. The present author has previously constructed a C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with a λ-graph system $\mathfrak{L}$ that is a generalization of the Cuntz–Krieger algebras. In this paper, we introduce an entropic quantity for the λ-graph system $\mathfrak{L}$, called the volume entropy for $\mathfrak{L}$ and written as $h_{\mathrm{vol}}(\mathfrak{L})$. The volume entropy $h_{\mathrm{vol}}(\mathfrak{L})$ is invariant under shift equivalence of λ-graph systems, and yields a new topological conjugacy invariant of subshifts. We prove that Voiculescu's non-commutative topological entropy for the canonical completely positive map of the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is the volume entropy $h_{\mathrm{vol}}(\mathfrak{L})$.

0. *Introduction*

For a finite set Σ, a subshift (Λ, σ) is a topological dynamics defined by a closed shift-invariant subset Λ of the compact set $\Sigma^{\mathbb{Z}}$ of all bi-infinite sequences of Σ with shift transformation σ defined by $\sigma((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}$. The present author has previously introduced notions of a λ-graph system and a symbolic matrix system as presentations of subshifts [**Ma2**]. They are generalized notions of a λ-graph (= labeled graph) and a symbolic matrix for sofic subshifts. We henceforth denote by $\mathbb{Z}_+$ and by $\mathbb{N}$ the set of all non-negative integers and the set of all positive integers, respectively. A λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ consists of a vertex set $V = V_0 \cup V_1 \cup V_2 \cup \cdots$, an edge set $E = E_{0,1} \cup E_{1,2} \cup E_{2,3} \cup \cdots$, a labeling map $\lambda : E \to \Sigma$ and a surjective map $\iota(= \iota_{l,l+1}) : V_{l+1} \to V_l$ for each $l \in \mathbb{Z}_+$ with a certain compatible condition. A symbolic matrix system $(\mathcal{M}, I)$ over Σ consists of two sequences of rectangular matrices $(\mathcal{M}_{l,l+1}, I_{l,l+1}), l \in \mathbb{Z}_+$. The matrices $\mathcal{M}_{l,l+1}$ have their entries in formal sums of Σ and the matrices $I_{l,l+1}$ have their entries in $\{0, 1\}$. They satisfy the commutation relations: $I_{l,l+1}\mathcal{M}_{l+1,l+2} = \mathcal{M}_{l,l+1}I_{l+1,l+2}$ for $l \in \mathbb{Z}_+$. It is required that each row

https://doi.org/10.1017/S0143385705000180 Published online by Cambridge University Press

of $I_{l,l+1}$ has at least one 1 and each column of $I_{l,l+1}$ has exactly one 1. A λ-graph system naturally arises from a symbolic matrix system $(\mathcal{M}, I)$. The labeled edges from a vertex $v_i^l \in V_l$ to a vertex $v_j^{l+1} \in V_{l+1}$ are given by the (i, j)-component $\mathcal{M}_{l,l+1}(i, j)$ of $\mathcal{M}_{l,l+1}$. The map $\iota(= \iota_{l,l+1})$ is defined by $\iota_{l,l+1}(v_j^{l+1}) = v_i^l$ precisely if $I_{l,l+1}(i, j) = 1$. The λ-graph systems and the symbolic matrix systems are the same objects and give rise to subshifts by gathering label sequences appearing in the labeled Bratteli diagrams of the λ-graph systems. Conversely we have a canonical method to construct a λ-graph system and a symbolic matrix system from an arbitrary subshift [**Ma2**]. They are called the canonical λ-graph system and the canonical symbolic matrix system for subshift Λ and written as $\mathfrak{L}^\Lambda$ and $(\mathcal{M}^\Lambda, I^\Lambda)$, respectively.

In [**Ma3**], the present author introduced a unital nuclear C^*-algebra $\mathcal{O}_\mathfrak{L}$ from a λ-graph system $\mathfrak{L}$ and studied its structure. For a λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ over Σ, let $\{v_1^l, \ldots, v_{m(l)}^l\}$ be the vertex set V_l. The C^*-algebra $\mathcal{O}_\mathfrak{L}$ is generated by partial isometries S_α corresponding to the symbols $\alpha \in \Sigma$ and projections E_i^l corresponding to the vertices $v_i^l \in V_l, i = 1, \ldots, m(l), l \in \mathbb{Z}_+$. It is realized as a universal unique C^*-algebra subject to certain operator relations among $S_\alpha, \alpha \in \Sigma$ and $E_i^l, i = 1, \ldots, m(l), l \in \mathbb{Z}_+$ encoded by the structure of $\mathfrak{L}$. If $\mathfrak{L}$ satisfies condition (I) and is irreducible, the C^*-algebra $\mathcal{O}_\mathfrak{L}$ is shown to be simple and purely infinite [**Ma3, Ma7**]. The C^*-algebras $\mathcal{O}_\mathfrak{L}$ are generalizations of the Cuntz–Krieger algebras [**CK**].

An entropic quantity for a λ-graph system has been introduced by Krieger and the present author in [**KM3**], that is called λ-entropy written as $h_\lambda(\mathfrak{L})$. It measures a growth rate of the cardinalities of the vertex sets $\{V_l\}_{l\in\mathbb{Z}_+}$. In this paper, we first introduce a notion of volume entropy for λ-graph system $\mathfrak{L}$, that will be written as $h_{\mathrm{vol}}(\mathfrak{L})$. It measures a growth rate of the cardinalities of the sets of labeled paths. We denote by $P_l(\mathfrak{L})$ the set of all labeled paths starting at a vertex in V_0 and terminating at a vertex in V_l, and by $^\sharp P_l(\mathfrak{L})$ its cardinality. The volume entropy is defined by the formula

$$h_{\mathrm{vol}}(\mathfrak{L}) = \limsup_{l\to\infty} \frac{1}{l} \log {}^\sharp P_l(\mathfrak{L}).$$

We first show that $h_{\mathrm{vol}}(\mathfrak{L})$ is invariant under shift equivalence of λ-graph systems. This implies that the volume entropy yields a topological conjugacy invariant of subshifts (Corollary 1.2). In §2, we show that $h_{\mathrm{vol}}(\mathfrak{L})$ is given by the topological entropy of a topological dynamical system $(\sigma_\mathfrak{L}, X_\mathfrak{L})$ associated with the λ-graph system $\mathfrak{L}$ (Theorem 2.1).

Voiculescu has introduced in [**Vo**] a notion of (non-commutative) topological entropy $ht(\alpha)$ for an automorphism (or an endomorphism) α of a unital nuclear C^*-algebra. The notion extends the classical topological entropy of topological dynamical systems. Brown has extended Voiculescu's non-commutative topological entropy for unital nuclear C^*-algebras to exact C^*-algebras [**Br**]. As an example of computation of non-commutative topological entropy, Choda has computed in [**Ch**, Ch. 2] the topological entropy $ht(\phi_N)$ of the canonical endomorphism ϕ_N on the Cuntz algebra $\mathcal{O}_N$, proving $ht(\phi_N) = \log N$, the topological entropy of the full N-shift. Boca and Goldstein have extended in [**BG**] her result to Cuntz–Krieger algebras. They proved that the topological entropy $ht(\phi_A)$ for the canonical completely positive map ϕ_A on the Cuntz–Krieger algebra $\mathcal{O}_A$ for an

https://doi.org/10.1017/S0143385705000180 Published online by Cambridge University Press

irreducible $0-1$ matrix A is equal to $\log r(A)$, the topological entropy of the associated topological Markov shift, where $r(A)$ denotes the Perron–Frobenius eigenvalue of A. Pinzari *et al* in [**PWY**] have studied non-commutative topological entropy in a more general setting (cf. [**JP**]). They have given some estimates in such entropic quantities for the canonical completely positive maps on the C^*-algebras of Hilbert C^*-bimodules with finite basis. The C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with λ-graph system $\mathfrak{L}$ is regarded as a C^*-algebra of Hilbert C^*-bimodule with finite basis. By applying their result [**PWY**, Theorems 7.4 and Theorem 7.10] to our C^*-algebras $\mathcal{O}_{\mathfrak{L}}$, the inequalities

$$h_{\mathrm{top}}(\Lambda_{\mathfrak{L}}) \leq ht(\gamma_{\mathfrak{L}}) \leq h_{\mathrm{top}}(\Lambda_{\mathfrak{L}}) + 2h_{\lambda}(\mathfrak{L})$$

hold, where $\gamma_{\mathfrak{L}}$ is the canonical completely positive map of $\mathcal{O}_{\mathfrak{L}}$, $\Lambda_{\mathfrak{L}}$ is the subshift presented by $\mathfrak{L}$ and $h_{\lambda}(\mathfrak{L})$ is the λ-entropy defined by

$$h_{\lambda}(\mathfrak{L}) = \limsup_{l\to\infty} \frac{1}{l} \log {}^{\sharp} V_l.$$

Our main purpose in this paper is to compute the entropy $ht(\gamma_{\mathfrak{L}})$ in terms of the λ-graph system. We will prove that the equality

$$ht(\gamma_{\mathfrak{L}}) = h_{\mathrm{vol}}(\mathfrak{L})$$

holds (Theorem 3.2 and Corollary 3.6). As a corollary, the topological entropy $ht(\gamma_{\mathfrak{L}})$ on the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is invariant under shift equivalence of λ-graph systems. Therefore, the topological entropy $ht(\gamma_{\mathfrak{L}^{\Lambda}})$ for the canonical λ-graph system for subshift Λ yields a topological conjugacy invariant of subshifts. Since $ht(\gamma_{\mathfrak{L}^{\Lambda}})$ is equal to $h_{\mathrm{vol}}(\mathfrak{L}^{\Lambda})$ for a subshift Λ, the value $ht(\gamma_{\mathfrak{L}^{\Lambda}})$ is greater than or equal to the topological entropy $h_{\mathrm{top}}(\Lambda)$ of the subshift. If the subshift is sofic, $ht(\gamma_{\mathfrak{L}^{\Lambda}})$ coincides with the topological entropy of the sofic shift. There are, however, examples of non-sofic subshifts Λ such that $ht(\gamma_{\mathfrak{L}^{\Lambda}})$ is strictly greater than its topological entropy $h_{\mathrm{top}}(\Lambda)$. Examples are the Dyck shifts and the Motzkin shifts that are calculated in §4.

1. *The volume entropy for a λ-graph system*

For a λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ over Σ, the vertex set V_l and the edge set $E_{l,l+1}$ at each $l \in \mathbb{Z}_+$ are finite sets. An edge e in $E_{l,l+1}$ has its source vertex $s(e)$ in V_l and its terminal vertex $t(e)$ in V_{l+1}. Every vertex in V has outgoing edges and every vertex in V, except in V_0, has incoming edges. It is then required that there exists an edge in $E_{l,l+1}$ with label α and terminal $v \in V_{l+1}$ if and only if there exists an edge in $E_{l-1,l}$ with label α and terminal $\iota(v) \in V_l$. For $u \in V_{l-1}$ and $v \in V_{l+1}$, we put

$$E^{\iota}(u, v) = \{e \in E_{l,l+1} : t(e) = v, \iota(s(e)) = u\},$$
$$E_{\iota}(u, v) = \{e \in E_{l-1,l} : s(e) = u, t(e) = \iota(v)\}.$$

Then there exists a bijective correspondence between $E^{\iota}(u, v)$ and $E_{\iota}(u, v)$ that preserves labels. This property is called the local property of λ-graph system. We put $\Sigma_i = \Sigma$ and define

$$\Lambda_{\mathfrak{L}}^{+} = \left\{ (\lambda(e_1), \lambda(e_2), \dots) \in \prod_{i\in\mathbb{N}} \Sigma_i : e_i \in E_{i-1,i}, t(e_i) = s(e_{i+1}), i \in \mathbb{N} \right\}$$

https://doi.org/10.1017/S0143385705000180 Published online by Cambridge University Press

and

$$\Lambda_{\mathfrak{L}} = \left\{ (\alpha_i)_{i\in\mathbb{Z}} \in \prod_{i\in\mathbb{Z}} \Sigma_i : (\alpha_i, \alpha_{i+1}, \dots) \in \Lambda_{\mathfrak{L}}^+, i \in \mathbb{Z} \right\}.$$

Then $\Lambda_{\mathfrak{L}}$ is a subshift over Σ called the subshift presented by $\mathfrak{L}$. A finite sequence $\mu = (\mu_1, \dots, \mu_k)$ of $\mu_j \in \Sigma$ that appears in $\Lambda_{\mathfrak{L}}$ is called an admissible word of $\mathfrak{L}$ of length $|\mu| = k$. Denote by $\Lambda_{\mathfrak{L}}^k$ the set of all admissible words of length k of $\mathfrak{L}$ and put $\Lambda_{\mathfrak{L}}^* = \bigcup_{k=0}^{\infty} \Lambda_{\mathfrak{L}}^k$ where $\Lambda_{\mathfrak{L}}^0$ denotes the empty word $\emptyset$. Throughout this paper, $\mathfrak{L}$ is assumed to be left-resolving, which means that $e, e' \in E$ with $\lambda(e) = \lambda(e')$ and $t(e) = t(e')$ implies $e = e'$.

A symbolic matrix system is a matrix presentation of a λ-graph system. Strong shift equivalence and shift equivalence between two symbolic matrix systems have been algebraically formulated in [**Ma2**] as a generalization of strong shift equivalence and shift equivalence for non-negative matrices [**Wi**] and for symbolic matrices [**BK, Na**]. Topological conjugacy classes of subshifts are completely characterized as strong shift equivalence classes of their canonical symbolic matrix systems. Strong shift equivalence implies shift equivalence. λ-graph systems are said to be shift equivalent if the corresponding symbolic matrix systems are shift equivalent. Several shift equivalence invariants of λ-graph systems have been studied in, e.g., [**Ma2**]. In this paper we define a new invariant $h_{\mathrm{vol}}(\mathfrak{L})$ of λ-graph system, called the volume entropy, that will be caught in operator algebraic entropy in §3. For a λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$, we denote by $P_l(\mathfrak{L})$ the set of all labeled paths starting at V_0 and terminating at V_l, and by $^\sharp P_l(\mathfrak{L})$ its cardinality.

PROPOSITION 1.1. *The volume entropy $h_{\mathrm{vol}}(\mathfrak{L})$ defined by setting*

$$h_{\mathrm{vol}}(\mathfrak{L}) = \limsup_{l\to\infty} \frac{1}{l} \log {}^\sharp P_l(\mathfrak{L})$$

is invariant under shift equivalence of λ-graph systems.

Proof. Let $\mathfrak{L}$ and $\mathfrak{L}'$ be λ-graph systems over Σ and Σ' with vertex sets V and V', respectively. Suppose that $\mathfrak{L}$ and $\mathfrak{L}'$ are shift equivalent with lag N for some $N \in \mathbb{N}$, that is, the corresponding symbolic matrix systems $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ are shift equivalent with lag N. We denote by $m(l)$ and $m'(l)$ the cardinal numbers $^\sharp V_l$ and $^\sharp V_l'$, respectively. They are the row sizes of $\mathcal{M}_{l,l+1}$ and $\mathcal{M}'_{l,l+1}$, respectively. Then $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ are said to be shift equivalent (with lag N) if there exist alphabets C, D and specifications

$$\varphi_1 : \Sigma \cdot C \to C \cdot \Sigma', \quad \varphi_2 : \Sigma' \cdot D \to D \cdot \Sigma$$

and

$$\psi_1 : (\Sigma)^N \to C \cdot D, \quad \psi_2 : (\Sigma')^N \to D \cdot C$$

such that for each $l \in \mathbb{Z}_+$, there exist an $m(l) \times m'(l+N)$ matrix $\mathcal{H}_l$ over C and an $m'(l) \times m(l+N)$ matrix $\mathcal{K}_l$ over D satisfying the following equations:

$$\mathcal{M}_{l,l+1}\mathcal{H}_{l+1} \overset{\varphi_1}{\simeq} \mathcal{H}_l\mathcal{M}'_{l+N,l+N+1}, \quad \mathcal{M}'_{l,l+1}\mathcal{K}_{l+1} \overset{\varphi_2}{\simeq} \mathcal{K}_l\mathcal{M}_{l+N,l+N+1},$$
$$I_{l,l+N}\mathcal{M}_{l+N,l+2N} \overset{\psi_1}{\simeq} \mathcal{H}_l\mathcal{K}_{l+N}, \quad I'_{l,l+N}\mathcal{M}'_{l+N,l+2N} \overset{\psi_2}{\simeq} \mathcal{K}_l\mathcal{H}_{l+N}$$

https://doi.org/10.1017/S0143385705000180 Published online by Cambridge University Press

and

$$I_{l,l+1}\mathcal{H}_{l+1} = \mathcal{H}_l I'_{l+N,l+N+1}, \quad I'_{l,l+1}\mathcal{K}_{l+1} = \mathcal{K}_l I_{l+N,l+N+1}.$$

Here a specification $\varphi : \Sigma_1 \to \Sigma_2$ between finite alphabets Σ_1, Σ_2 means a bijection from a subset of Σ_1 onto a subset of Σ_2, and the notation $\mathcal{M}_1 \overset{\varphi}{\simeq} \mathcal{M}_2$ for symbolic matrices $\mathcal{M}_1$ over Σ_1 and $\mathcal{M}_2$ over Σ_2 and a specification $\varphi : \Sigma_1 \to \Sigma_2$ means the equality $\varphi(\mathcal{M}_1(i, j)) = \mathcal{M}_2(i, j)$ for each component of matrices. Hence, each vertex of V'_{l+N} has an incoming edge corresponding to a component of $\mathcal{H}_l$, which is labeled by a symbol of C. Similarly each vertex of V_{l+2N} has an incoming edge corresponding to a component of $\mathcal{K}_{l+N}$, that is labeled by a symbol of D. Hence, the inequalities

$$^\sharp P_l(\mathfrak{L}) \le {}^\sharp P_{l+N}(\mathfrak{L}') \le {}^\sharp P_{l+2N}(\mathfrak{L})$$

hold for all $l \in \mathbb{Z}_+$. Therefore, we have the assertion. □

For a subshift Λ, we denote by $\mathfrak{L}^\Lambda$ the canonical λ-graph system for Λ. We then define the volume entropy $h_{\mathrm{vol}}(\Lambda)$ of Λ as the volume entropy $h_{\mathrm{vol}}(\mathfrak{L}^\Lambda)$ of the λ-graph system $\mathfrak{L}^\Lambda$. By [**Ma2**] (cf. [**Ma4**]), two subshifts Λ and Λ' are topologically conjugate if and only if their canonical λ-graph systems $\mathfrak{L}^\Lambda$ and $\mathfrak{L}^{\Lambda'}$ are strong shift equivalent. Strong shift equivalence implies shift equivalence. Hence, we have the following.

COROLLARY 1.2. *The volume entropy $h_{\mathrm{vol}}(\Lambda)$ is invariant under topological conjugacy of subshifts.*

PROPOSITION 1.3. *Let $\Lambda_{\mathfrak{L}}$ be the subshift presented by a λ-graph system $\mathfrak{L}$. Then the inequalities*

$$h_{top}(\Lambda_{\mathfrak{L}}) \le h_{\mathrm{vol}}(\mathfrak{L}) \le h_{top}(\Lambda_{\mathfrak{L}}) + h_\lambda(\mathfrak{L}) \le 2h_{\mathrm{vol}}(\mathfrak{L})$$

hold, where $h_{top}(\Lambda_{\mathfrak{L}})$ is the topological entropy of the subshift $\Lambda_{\mathfrak{L}}$.

Proof. Let ${}^\sharp\Lambda^l_{\mathfrak{L}}$ be the cardinal number of the set of admissible words of $\Lambda_{\mathfrak{L}}$ with length l. Since ${}^\sharp\Lambda^l_{\mathfrak{L}} \le {}^\sharp P_l(\mathfrak{L})$, the inequality $h_{\mathrm{top}}(\Lambda_{\mathfrak{L}}) \le h_{\mathrm{vol}}(\mathfrak{L})$ is obvious. As $\mathfrak{L}$ is left-resolving, a path in $P_l(\mathfrak{L})$ is determined by its label sequence and its terminal vertex. Hence, we have

$$^\sharp P_l(\mathfrak{L}) \le {}^\sharp\Lambda^l_{\mathfrak{L}} \times {}^\sharp V_l$$

so that we obtain the inequality

$$h_{\mathrm{vol}}(\mathfrak{L}) \le h_{\mathrm{top}}(\Lambda_{\mathfrak{L}}) + h_\lambda(\mathfrak{L}).$$

As $h_{\mathrm{top}}(\Lambda_{\mathfrak{L}}), h_\lambda(\mathfrak{L}) \le h_{\mathrm{vol}}(\mathfrak{L})$, we get the desired inequalities. □

Wolfgang Krieger has kindly suggested to the author that the volume entropy $h_{\mathrm{vol}}(\mathfrak{L})$ is realized as the λ-entropy $h_\lambda(\widetilde{\mathfrak{L}})$ of the path λ-graph system $\widetilde{\mathfrak{L}}$ of $\mathfrak{L}$, where the vertex set $\widetilde{V}_l$ of $\widetilde{\mathfrak{L}}$ consists of the path set $P_l(\mathfrak{L})$. The author would like to thank Wolfgang Krieger for his suggestion.

https://doi.org/10.1017/S0143385705000180 Published online by Cambridge University Press

2. *Topological entropy of the associated dynamical system*

For a λ-graph system $\mathfrak{L}$, let $\Omega_{\mathfrak{L}}$ be the projective limit

$$\Omega_{\mathfrak{L}} = \left\{ (u^l)_{l\in\mathbb{Z}_+} \in \prod_{l\in\mathbb{Z}_+} V_l : \iota_{l,l+1}(u^{l+1}) = u^l, l \in \mathbb{Z}_+ \right\}$$

of the system $\iota_{l,l+1} : V_{l+1} \to V_l, l \in \mathbb{Z}_+$. We endow $\Omega_{\mathfrak{L}}$ with the projective limit topology from the discrete topologies on $V_l, l \in \mathbb{Z}_+$ so that it is a compact Hausdorff space. Let $E_{\mathfrak{L}}$ be the set of all triplets $(u, \alpha, w) \in \Omega_{\mathfrak{L}} \times \Sigma \times \Omega_{\mathfrak{L}}$ such that there exists $e_{l,l+1} \in E_{l,l+1}$ satisfying $u^l = s(e_{l,l+1})$, $w^{l+1} = t(e_{l,l+1})$ and $\alpha = \lambda(e_{l,l+1})$ for each $l \in \mathbb{Z}_+$ where $u = (u^l)_{l\in\mathbb{Z}_+}, w = (w^l)_{l\in\mathbb{Z}_+} \in \Omega_{\mathfrak{L}}$. The set $E_{\mathfrak{L}} \subset \Omega_{\mathfrak{L}} \times \Sigma \times \Omega_{\mathfrak{L}}$ is a continuous graph in the sense of Deaconu [**De, De2**] (see also [**Ma3**, Proposition 2.1]). It has been also studied in [**KM**] as a Shannon graph. Following Deaconu [**De2**] and Krieger [**Kr2**], we consider the set $X_{\mathfrak{L}}$ of all one-sided paths of $E_{\mathfrak{L}}$:

$$X_{\mathfrak{L}} = \Bigg\{ (\alpha_i, u_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} (\Sigma \times \Omega_{\mathfrak{L}}) : (u_i, \alpha_{i+1}, u_{i+1}) \in E_{\mathfrak{L}} \text{ for all } i \in \mathbb{N}$$
$$\text{and } (u_0, \alpha_1, u_1) \in E_{\mathfrak{L}} \text{ for some } u_0 \in \Omega_{\mathfrak{L}} \Bigg\}.$$

The set $X_{\mathfrak{L}}$ becomes a zero-dimensional compact Hausdorff space under the relative topology from the infinite product topology of $\Sigma \times \Omega_{\mathfrak{L}}$. The shift map $\sigma_{\mathfrak{L}} : (\alpha_i, u_i)_{i\in\mathbb{N}} \in X_{\mathfrak{L}} \to (\alpha_{i+1}, u_{i+1})_{i\in\mathbb{N}} \in X_{\mathfrak{L}}$ is a local homeomorphism by [**Ma3**, Lemma 2.2]. We have a topological dynamical system $(X_{\mathfrak{L}}, \sigma_{\mathfrak{L}})$ of a compact Hausdorff space $X_{\mathfrak{L}}$ with a continuous map $\sigma_{\mathfrak{L}}$.

THEOREM 2.1. $h_{\mathrm{vol}}(\mathfrak{L}) = h_{\mathrm{top}}(\sigma_{\mathfrak{L}})$: *the topological entropy of* $(X_{\mathfrak{L}}, \sigma_{\mathfrak{L}})$.

Proof. For $\mu = (\mu_1, \ldots, \mu_k) \in \Lambda^k, v_i^l \in V_l$, put

$$U(\mu, v_i^l) = \{(\alpha_n, u_n)_{n\in\mathbb{N}} \in X_{\mathfrak{L}} : \alpha_1 = \mu_1, \ldots, \alpha_k = \mu_k, u_k^l = v_i^l\}$$

where $u_k = (u_k^l)_{l\in\mathbb{Z}_+}$. Note that the following identities hold:

$$U(\mu, v_i^l) = \bigcup_{\iota(v_j^{l+1})=v_i^l} U(\mu, v_j^{l+1}),$$

$$U(\mu, v_i^l) = \bigcup_{\alpha\in\Sigma} \bigcup_{j=1}^{m(l+1)} \{U(\mu\alpha, v_j^{l+1}) : \text{if } s(e) = v_i^l, \lambda(e) = \alpha, t(e) = v_j^{l+1}$$
$$\text{for some } e \in E_{l,l+1}\},$$

$$\sigma_{\mathfrak{L}}^{-1}(U(\mu, v_i^l)) = \bigcup_{\beta\in\Sigma} U(\beta\mu, v_i^l).$$

Put for $1 \le k \le l$

$$\mathfrak{A}_{k,l} = \{U(\mu, v_i^l) : \mu \in \Lambda^k, v_i^l \in V_l, i = 1, \ldots, m(l)\}.$$

https://doi.org/10.1017/S0143385705000180 Published online by Cambridge University Press

Then $\mathfrak{A} = \bigcup_{\substack{k,l \\ k \le l}} \mathfrak{A}_{k,l}$ is a family of generating open sets of $X_{\mathfrak{L}}$. For $v_i^l \in V_l$, $v_j^{l+1} \in V_{l+1}$, we denote by $A_{l,l+1}(i, j)$ the number of edges starting at v_i^l and ending at v_{j+1}^{l+1}. The matrix $A_{l,l+1}$ is obtained from $\mathcal{M}_{l,l+1}$ by substituting 1 for all the symbols in $\mathcal{M}_{l,l+1}$, where $(\mathcal{M}_{l,l+1}, I_{l,l+1})$ is the symbolic matrix system of $\mathfrak{L}$. Put $A_{0,n} = A_{0,1}A_{1,2}\cdots A_{n-1,n}$. For an open covering $\mathfrak{B}$ of $X_{\mathfrak{L}}$, denote by $N(\mathfrak{B})$ the minimum number of a subcovering of $\mathfrak{B}$. Since $\mathfrak{A}_{1,1}$ is the minimum subcovering of $\mathfrak{A}$ of $X_{\mathfrak{L}}$, one has

$$N(\mathfrak{A}) = \sum_{i=0}^{m(0)} \sum_{j=1}^{m(1)} A_{0,1}(i, j).$$

For open coverings $\mathfrak{B}, \mathfrak{B}'$ we denote by $\mathfrak{B} \vee \mathfrak{B}'$ the open covering

$$\mathfrak{B} \vee \mathfrak{B}' = \{U \cap V : U \in \mathfrak{B}, V \in \mathfrak{B}'\}.$$

As $\mathfrak{A} \vee \sigma_{\mathfrak{L}}^{-1}(\mathfrak{A}) \not\supset \mathfrak{A}_{1,1}$ and $\mathfrak{A} \vee \sigma_{\mathfrak{L}}^{-1}(\mathfrak{A}) \supset \mathfrak{A}_{2,2}$, the minimum subcovering of $\mathfrak{A} \vee \sigma_{\mathfrak{L}}^{-1}(\mathfrak{A})$ is $\mathfrak{A}_{2,2}$. Hence, one has

$$N(\mathfrak{A} \vee \sigma_{\mathfrak{L}}^{-1}(\mathfrak{A})) = \sum_{i=0}^{m(0)} \sum_{j=1}^{m(2)} A_{0,2}(i, j).$$

Similarly one knows that the minimum subcovering of $\mathfrak{A} \vee \sigma_{\mathfrak{L}}^{-1}(\mathfrak{A}) \vee \cdots \vee \sigma_{\mathfrak{L}}^{-n}(\mathfrak{A})$ is $\mathfrak{A}_{n,n}$. Hence, one has

$$N(\mathfrak{A} \vee \sigma_{\mathfrak{L}}^{-1}(\mathfrak{A}) \vee \cdots \vee \sigma_{\mathfrak{L}}^{-n}(\mathfrak{A})) = \sum_{i=0}^{m(0)} \sum_{j=1}^{m(n)} A_{0,n}(i, j).$$

Since $\mathfrak{A}$ is a generator of the topology of $X_{\mathfrak{L}}$, one obtains

$$h_{\mathrm{top}}(\sigma_{\mathfrak{L}}) = \limsup_{n\to\infty} \frac{1}{n} \log N(\mathfrak{A} \vee \sigma_{\mathfrak{L}}^{-1}(\mathfrak{A}) \vee \cdots \vee \sigma_{\mathfrak{L}}^{-n}(\mathfrak{A}))$$

(cf. [**Wa**]). As ${}^{\sharp}P_n(\mathfrak{L}) = \sum_{i=0}^{m(0)} \sum_{j=1}^{m(n)} A_{0,n}(i, j)$, one concludes that $h_{\mathrm{top}}(\sigma_{\mathfrak{L}}) = h_{\mathrm{vol}}(\mathfrak{L})$. □

Hiroki Matui has pointed out that the dynamical system $(X_{\mathfrak{L}^\Lambda}, \sigma_{\mathfrak{L}^\Lambda})$ for a subshift Λ is obtained by a completion of the right one-sided subshift (Λ^+, σ) of (Λ, σ) with respect to a certain metric on Λ^+.

3. *Non-commutative topological entropy in $\mathcal{O}_{\mathfrak{L}}$*

We first recall basic definitions of non-commutative topological entropy defined in [**Vo**] and [**Br**] (see also [**RS**]). Let A be an exact C^*-algebra. We denote by $\mathcal{P}f(A)$ the set of all finite subsets of A. For a faithful $*$-representation of A on a Hilbert space H, we denote by $CPA(\pi, A)$ the set of triples (ϕ, ψ, B), where B is a finite-dimensional C^*-algebra and $\phi : A \to B$, $\psi : B \to B(H)$ are contractive completely positive maps. For $\omega \in \mathcal{P}f(A)$ and $\delta > 0$, the completely positive δ-rank

$$rcp(\pi, \omega; \delta) = \inf\{\mathrm{rank}\, B : (\phi, \psi, B) \in CPA(\pi, A), \|\psi \circ \phi(a) - \pi(a)\| < \delta \text{ for } a \in \omega\}$$

https://doi.org/10.1017/S0143385705000180 Published online by Cambridge University Press

where rank B denotes the dimension of a maximal abelian $*$-subalgebra of B. As proved in [**Br**], $rcp(\pi, \omega; \delta)$ is independent of the choice of π. We may assume that A is faithfully represented on H and denote $rcp(\pi, \omega; \delta)$ by $rcp(\omega; \delta)$. Let $\gamma : A \to A$ be a completely positive map. Define

$$ht(\gamma, \omega; \delta) = \limsup_{n\to\infty} \frac{1}{n} \log rcp(\omega \cup \gamma(\omega) \cup \cdots \cup \gamma^{n-1}(\omega); \delta),$$
$$ht(\gamma, \omega) = \sup_{\delta>0} ht(\gamma, \omega; \delta).$$

The (non-commutative) topological entropy $ht(\gamma)$ of $\gamma : A \to A$ is defined by setting

$$ht(\gamma) = \sup_{\omega\in\mathcal{P}f(A)} ht(\gamma, \omega).$$

Let us now briefly review the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with λ-graph system $\mathfrak{L}$, that has been originally constructed in [**Ma3**] to be a groupoid C^*-algebra of a groupoid of a continuous graph obtained by $\mathfrak{L}$ (cf. [**De, De2, Re**]). Let $\mathfrak{L} = (V, E, \lambda, \iota)$ be a left-resolving λ-graph system over Σ. The vertex set V_l is denoted by $\{v_1^l, \ldots, v_{m(l)}^l\}$. Define the transition matrices $A_{l,l+1}, I_{l,l+1}$ of $\mathfrak{L}$ by setting for $i = 1, 2, \ldots, m(l),\ j = 1, 2, \ldots, m(l+1),\ \alpha \in \Sigma$,

$$A_{l,l+1}(i, \alpha, j) = \begin{cases} 1 & \text{if } s(e) = v_i^l, \lambda(e) = \alpha, t(e) = v_j^{l+1} \text{ for some } e \in E_{l,l+1}, \\ 0 & \text{otherwise,} \end{cases}$$
$$I_{l,l+1}(i, j) = \begin{cases} 1 & \text{if } \iota_{l,l+1}(v_j^{l+1}) = v_i^l, \\ 0 & \text{otherwise.} \end{cases}$$

The C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is realized as the universal unital C^*-algebra generated by partial isometries $S_\alpha, \alpha \in \Sigma$ and projections $E_i^l, i = 1, 2, \ldots, m(l), l \in \mathbb{Z}_+$ subject to the following operator relations called ($\mathfrak{L}$):

$$\sum_{\alpha\in\Sigma} S_\alpha S_\alpha^* = 1, \tag{3.1}$$
$$\sum_{i=1}^{m(l)} E_i^l = 1, \quad E_i^l = \sum_{j=1}^{m(l+1)} I_{l,l+1}(i, j) E_j^{l+1}, \tag{3.2}$$
$$S_\beta S_\beta^* E_i^l = E_i^l S_\beta S_\beta^*, \tag{3.3}$$
$$S_\beta^* E_i^l S_\beta = \sum_{j=1}^{m(l+1)} A_{l,l+1}(i, \beta, j) E_j^{l+1}, \tag{3.4}$$

for $\beta \in \Sigma, i = 1, 2, \ldots, m(l), l \in \mathbb{Z}_+$. It is nuclear [**Ma3**, Proposition 5.6] and purely infinite, simple if $\mathfrak{L}$ is irreducible with condition (I) (cf. [**Ma, Ma3**]). Suppose that $\mathfrak{L}$ satisfies condition (I). The C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is the unique C^*-algebra subject to the above relations ($\mathfrak{L}$). Hence, the correspondence $S_\alpha \to zS_\alpha$, $E_i^l \to E_i^l$ for $z \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ yields an action $\alpha_{\mathfrak{L}}$ of $\mathbb{T}$ called the gauge action. Let $\mathcal{A}_l$ be the finite-dimensional commutative C^*-subalgebra of $\mathcal{O}_{\mathfrak{L}}$ generated by $E_i^l, i = 1, 2, \ldots, m(l)$. By the relations (3.2), the algebra $\mathcal{A}_l$ is naturally regarded as a subalgebra of $\mathcal{A}_{l+1}$.

https://doi.org/10.1017/S0143385705000180 Published online by Cambridge University Press

For $k \leq l$, let $\mathcal{F}_k^l$ be the finite-dimensional C^*-subalgebra of $\mathcal{O}_{\mathfrak{L}}$ generated by $S_\mu E_i^l S_\nu^*$, $\mu, \nu \in \Lambda_{\mathfrak{L}}^k, i = 1, 2, \ldots, m(l)$. By the relations (3.1) and (3.4), $\mathcal{F}_k^l$ is naturally regarded as a subalgebra of $\mathcal{F}_{k+1}^{l+1}$. Let $\mathcal{F}_{\mathfrak{L}}$ be the C^*-subalgebra of $\mathcal{O}_{\mathfrak{L}}$ generated by the algebras $\mathcal{F}_k^l$. It is an AF-algebra realized as the fixed point algebra $\mathcal{O}_{\mathfrak{L}}^{\alpha_{\mathfrak{L}}}$ of $\mathcal{O}_{\mathfrak{L}}$ under $\alpha_{\mathfrak{L}}$. In what follows, we assume that $\mathfrak{L}$ is irreducible with condition (I).

Let $\gamma_{\mathfrak{L}} : \mathcal{O}_{\mathfrak{L}} \to \mathcal{O}_{\mathfrak{L}}$ be the canonical completely positive map defined by setting

$$\gamma_{\mathfrak{L}}(X) = \sum_{\alpha \in \Sigma} S_\alpha X S_\alpha^*, \quad X \in \mathcal{O}_{\mathfrak{L}}.$$

We will compute the topological entropy $ht(\gamma_{\mathfrak{L}})$ of the completely positive map $\gamma_{\mathfrak{L}}$.

Let φ be a KMS-state for the gauge action on $\mathcal{O}_{\mathfrak{L}}$. A KMS-state for gauge action always exists at maximal inverse temperature by a similar argument to [**MWY**] or [**PWY**]. Since $\mathfrak{L}$ is irreducible, a KMS-state is automatically faithful (cf. [**MWY**, Lemma 4.3]). We first note the following lemma.

LEMMA 3.1. *Let φ be a KMS-state for the gauge action on $\mathcal{O}_{\mathfrak{L}}$ at inverse temperature $\log \beta$ for some $\beta > 1$. For $s \leq t$, $s, t \in \mathbb{Z}_+$, there exists a conditional expectation $\mathcal{E}_s^t : \mathcal{F}_{\mathfrak{L}} \to \mathcal{F}_s^t$ with respect to the state φ.*

Proof. Define $\mathcal{E}_s^t : \mathcal{F}_{\mathfrak{L}} \to \mathcal{F}_s^t$ by setting

$$\mathcal{E}_s^t(S_{\mu_1 \cdots \mu_k} E_i^l S_{\nu_1 \cdots \nu_k}^*)$$

$$= \begin{cases} S_{\mu_1 \cdots \mu_k} E_i^l S_{\nu_1 \cdots \nu_k}^* & \text{if } k \leq s \text{ and } l \leq t, \\ \dfrac{\varphi(E_i^l)}{\varphi(P_{i(\mu,\nu,s)}^t)} S_{\mu_1 \cdots \mu_k} P_{i(\mu,\nu,s)}^t S_{\nu_1 \cdots \nu_k}^* & \text{if } k \leq s \text{ and } l > t, \\ \dfrac{\varphi(S_{\mu_{s+1} \cdots \mu_k} E_i^l S_{\nu_{s+1} \cdots \nu_k}^*)}{\varphi(P_{i(\mu,\nu,s)}^t)} S_{\mu_1 \cdots \mu_s} P_{i(\mu,\nu,s)}^t S_{\nu_1 \cdots \nu_s}^* & \\ & \text{if } k > s \text{ and } \mu_{s+1} \cdots \mu_k = \nu_{s+1} \cdots \nu_k, \\ 0 & \text{if } k > s \text{ and } \mu_{s+1} \cdots \mu_k \neq \nu_{s+1} \cdots \nu_k, \end{cases}$$

where the projection $P_{i(\mu,\nu,s)}^t \in \mathcal{A}_t$ is defined as in the following way. Assume that $S_{\mu_1 \cdots \mu_k} E_i^l S_{\nu_1 \cdots \nu_k}^* \neq 0$. There exist two labeled paths from vertices in V_{l-k} to the vertex v_i^l, whose label sequences are $\mu_1 \cdots \mu_k$ and $\nu_1 \cdots \nu_k$.

Case 1 ($k \leq s$ and $l > t$). Put $v_{i(\mu,\nu,s)}^t = \iota^{l-t}(v_i^l) \in V_t$. By the local property of λ-graph system, there exist paths labeled by $\mu_1 \cdots \mu_k$ and $\nu_1 \cdots \nu_k$ that end at $v_{i(\mu,\nu,s)}^t$. We put $P_{i(\mu,\nu,s)}^t = E_{i(\mu,\nu,s)}^t$.

Case 2 ($k > s$ and $\mu_{s+1} \cdots \mu_k = \nu_{s+1} \cdots \nu_k$). There exists a vertex $v_{i(\mu,\nu,s)}^{l-(k-s)}$ in $V_{l-(k-s)}$ such that the labeled path $\mu_{s+1} \cdots \mu_k (= \nu_{s+1} \cdots \nu_k)$ starts at $v_{i(\mu,\nu,s)}^{l-(k-s)}$ and ends at v_i^l. We have two subcases.

Case 2.1 ($l - (k - s) \leq t$). Put $P_{i(\mu,\nu,s)}^t = E_{i(\mu,\nu,s)}^{l-(k-s)}$. The projection belongs to the algebra $\mathcal{A}_{l-(k-s)}$ that is a subalgebra of $\mathcal{A}_t$. Hence, $S_{\mu_1 \cdots \mu_s} P_{i(\mu,\nu,s)}^t S_{\nu_1 \cdots \nu_s}^*$ belongs to $\mathcal{F}_s^t$.

https://doi.org/10.1017/S0143385705000180 Published online by Cambridge University Press

Case 2.2 ($l-(k-s)>t$). Put $v^t_{i'(\mu,\nu,s)}=\iota^{l-(k-s)-t}(v^{l-(k-s)}_{i(\mu,\nu,s)})\in V_t$. By the local property of λ-graph system, there exist paths labeled by $\mu_1\cdots\mu_s$ and $\nu_1\cdots\nu_s$ that end at $v^t_{i'(\mu,\nu,s)}$. This means $S_{\nu_1\cdots\nu_s}E^t_{i'(\mu,\nu,s)}S^*_{\nu_1\cdots\nu_s}\neq 0$. We then put $P^t_{i(\mu,\nu,s)}=E^t_{i'(\mu,\nu,s)}$.

As the following equalities are straightforward,

$$\mathcal{E}^t_s(E^l_i)=\sum_{j=1}^{m(l+1)}I_{l,l+1}(i,j)\mathcal{E}^t_s(E^{l+1}_j),$$

$$\mathcal{E}^t_s(S_{\mu_1\cdots\mu_k}E^l_iS^*_{\nu_1\cdots\nu_k})=\sum_{\alpha\in\Sigma}\sum_{j=1}^{m(l+1)}A_{l,l+1}(i,\alpha,j)\mathcal{E}^t_s(S_{\mu_1\cdots\mu_k\alpha}E^{l+1}_jS^*_{\nu_1\cdots\nu_k\alpha}),$$

the map $\mathcal{E}^t_s:\mathcal{F}_{\mathfrak{L}}\to\mathcal{F}^t_s$ is well-defined. Since φ is a KMS-state at inverse temperature $\log\beta$, the equality

$$\varphi(S_\mu XS^*_\mu)=\delta_{\mu,\nu}\frac{1}{\beta^{|\mu|}}\varphi(XS^*_\mu S_\mu),\quad X\in\mathcal{F}_{\mathfrak{L}},\mu\in\Lambda^*_{\mathfrak{L}}$$

holds (cf. [**PWY**, Proposition 9.1] and [**MWY**, Proposition 3.3]). Hence, the equality $\varphi\circ\mathcal{E}^t_s=\varphi$ is direct. □

Our purpose is to show the following theorem.

THEOREM 3.2. $ht(\gamma_{\mathfrak{L}})\le h_{\mathrm{vol}}(\mathfrak{L})$.

We will prove Theorem 3.2, combining appropriate analogues of Boca and Goldstein [**BG**] for Cuntz–Krieger algebras and Pinzari *et al* [**PWY**] for full C^*-dynamical systems. Assume that the algebra $\mathcal{O}_{\mathfrak{L}}$ is faithfully represented on a Hilbert space. Put for $h,k\le l$

$$\Omega^l_{h,k}=\{S_\mu E^l_iS^*_\nu:|\mu|\le h,|\nu|\le k,i=1,2,\dots,m(l)\}.$$

For a finite set $\omega\subset\Omega^l_{h,k}$, put

$$L(\omega)=\max\{|\mu|,|\nu|:S_\mu E^l_iS^*_\nu\in\omega\},$$
$$\Lambda^*_{L(\omega)}=\{\mu\in\Lambda^*:|\mu|\le L(\omega)\},$$
$$\theta(\omega)={}^\sharp\Lambda^*_{L(\omega)}.$$

For $p\in\mathbb{Z}$, we denote by $\mathcal{O}_{\mathfrak{L}}{}^{\alpha_{\mathfrak{L}}}[p]$ the pth spectral subspace for the gauge action $\alpha_{\mathfrak{L}}$:

$$\mathcal{O}_{\mathfrak{L}}{}^{\alpha_{\mathfrak{L}}}[p]=\{X\in\mathcal{O}_{\mathfrak{L}}:(\alpha_{\mathfrak{L}})_z(X)=z^pX,z\in\mathbb{T}\}.$$

Let $\mathcal{E}_p:\mathcal{O}_{\mathfrak{L}}\to\mathcal{O}_{\mathfrak{L}}{}^{\alpha_{\mathfrak{L}}}[p]$ be defined by setting

$$\mathcal{E}_p(X)=\int_{z\in\mathbb{T}}\bar{z}^p(\alpha_{\mathfrak{L}})_z(X)\,dz,\quad X\in\mathcal{O}_{\mathfrak{L}}.$$

In particular $\mathcal{E}_0:\mathcal{O}_{\mathfrak{L}}\to\mathcal{O}_{\mathfrak{L}}{}^{\alpha_{\mathfrak{L}}}=\mathcal{F}_{\mathfrak{L}}$ is a conditional expectation. Motivated by [**PWY**], we define contractive completely positive maps. We put $\phi_{\theta(\omega)}:\mathcal{O}_{\mathfrak{L}}\to M_{\theta(\omega)}(\mathcal{F}_{\mathfrak{L}})$ by setting

$$\phi_{\theta(\omega)}(X)=[S^*_\mu\mathcal{E}_{|\mu|-|\nu|}(X)S_\nu]_{\mu,\nu\in\Lambda^*_{L(\omega)}},\quad X\in\mathcal{O}_{\mathfrak{L}}.$$

https://doi.org/10.1017/S0143385705000180 Published online by Cambridge University Press

Let $\mathcal{E}_s^t : \mathcal{F}_{\mathfrak{L}} \to \mathcal{F}_s^t$ be the conditional expectation with respect to a KMS-state φ for the gauge action on $\mathcal{O}_{\mathfrak{L}}$ defined in Lemma 3.1. For $n \in \mathbb{Z}_+$, define $\phi_\omega^n : \mathcal{O}_{\mathfrak{L}} \to M_{\theta(\omega)} \otimes \mathcal{F}_{L(\omega)+n}^{l+n}$ by setting

$$\phi_\omega^n(X) = (\mathrm{id} \otimes \mathcal{E}_{L(\omega)+n}^{l+n}) \circ \phi_{\theta(\omega)}(X), \quad X \in \mathcal{O}_{\mathfrak{L}}.$$

Let $f \in l^2(\mathbb{Z})$ be a function on $\mathbb{Z}$ such that $\|f\|_2 \le 1$ and $|f * \tilde{f}(p) - 1| < \delta$ for $-L(\omega) \le p \le L(\omega)$ where $\tilde{f}(q) = \overline{f(-q)}$. Define a contractive completely positive map $\psi^f : M_{\theta(\omega)} \otimes \mathcal{F}_{L(\omega)+n}^{l+n} \to \mathcal{O}_{\mathfrak{L}}$ by setting

$$\psi^f([X_{\mu,\nu}]_{\mu,\nu\in\Lambda_{L(\omega)}^*}) = \sum_{\mu,\nu\in\Lambda_{L(\omega)}^*} f(|\mu|)\bar{f}(|\nu|)S_\mu X_{\mu,\nu} S_\nu^*.$$

LEMMA 3.3. *For $X \in \omega$, if $X \in \mathcal{O}_{\mathfrak{L}}{}^{\alpha_{\mathfrak{L}}}[p]$, one has*

$$\|\psi^f \circ \phi_\omega^n(X) - X\| < \delta \quad \textit{for } n \in \mathbb{Z}_+.$$

Proof. Since $\mathrm{id} \otimes \mathcal{E}_{L(\omega)+n}^{l+n}(\phi_{\theta(\omega)}(X)) = \phi_{\theta(\omega)}(X)$, it suffices to show that

$$\|\psi^f \circ \phi_{\theta(\omega)}(X) - X\| < \delta.$$

One sees that

$$\begin{aligned}
\psi^f \circ \phi_{\theta(\omega)}(X) &= \sum_{\mu,\nu\in\Lambda_{L(\omega)}^*} f(|\mu|)\bar{f}(|\nu|)S_\mu S_\mu^* \mathcal{E}_{|\mu|-|\nu|}(X) S_\nu S_\nu^* \\
&= \sum_{|\mu|-|\nu|=p} f(|\mu|)\bar{f}(|\nu|)S_\mu S_\mu^* X S_\nu S_\nu^* \\
&= \sum_{s-r=p} f(s)\bar{f}(r)\Big(\sum_{|\mu|=s} S_\mu S_\mu^*\Big) X \Big(\sum_{|\nu|=s} S_\nu S_\nu^*\Big) \\
&= f * \tilde{f}(p) X.
\end{aligned}$$

Hence, we have

$$\|\psi^f \circ \phi_{\theta(\omega)}(X) - X\| = |f * \tilde{f}(p) - 1| < \delta$$

because $\|X\| = 1$. □

LEMMA 3.4. *For $\omega \subset \Omega_{h,k}^l$, $\delta > 0$ and $n \in \mathbb{N}$, there exist contractive completely positive maps*

$$\phi_\omega^n : \mathcal{O}_{\mathfrak{L}} \to M_{\theta(\omega)} \otimes \mathcal{F}_{L(\omega)+n}^{l+n}, \quad \psi^f : M_{\theta(\omega)} \otimes \mathcal{F}_{L(\omega)+n}^{l+n} \to \mathcal{O}_{\mathfrak{L}}$$

such that

$$\|\psi^f \circ \phi_\omega^n(Y) - Y\| < \delta, \quad \textit{for } Y \in \omega \cup \gamma_{\mathfrak{L}}(\omega) \cup \cdots \cup \gamma_{\mathfrak{L}}^{n-1}(\omega).$$

https://doi.org/10.1017/S0143385705000180 Published online by Cambridge University Press

Proof. Take $Y = \gamma_{\mathfrak{L}}^{j}(X)$ for some $X \in \omega$ and $j = 0, 1, \ldots, n-1$. Since $\|\gamma\| = 1$, one sees $\|Y\| = 1$. As $X \in \mathcal{O}_{\mathfrak{L}}{}^{\alpha_{\mathfrak{L}}}[p]$ for some $p \in \mathbb{Z}$, one has $Y = \gamma_{\mathfrak{L}}^{j}(X) \in \mathcal{O}_{\mathfrak{L}}{}^{\alpha_{\mathfrak{L}}}[p]$. By the preceding lemma, one gets the desired inequality. □

PROPOSITION 3.5. *For $\omega \subset \Omega_{h,k}^{l}$, $\delta > 0$ and $n \in \mathbb{N}$, the inequality*

$$rcp(\omega \cup \gamma_{\mathfrak{L}}(\omega) \cup \cdots \cup \gamma_{\mathfrak{L}}^{n-1}(\omega); \delta) \leq \theta(\omega)^{\sharp} P_{l+n}(\mathfrak{L})$$

holds.

Proof. By the preceding lemma, we have

$$rcp(\omega \cup \gamma_{\mathfrak{L}}(\omega) \cup \cdots \cup \gamma_{\mathfrak{L}}^{n-1}(\omega); \delta) \leq \operatorname{rank}(M_{\theta(\omega)} \otimes \mathcal{F}_{L(\omega)+n}^{l+n}).$$

Since one sees that

$$\begin{aligned} \operatorname{rank}(\mathcal{F}_{L(\omega)+n}^{l+n}) &= \sum_{\substack{i=1,\ldots,m(l-L(\omega)),\\ j=1,\ldots,m(l+n)}} A_{l-L(\omega),l+n}(i, j) \\ &\leq \sum_{\substack{i=1,\ldots,m(0),\\ j=1,\ldots,m(l+n)}} A_{0,l+n}(i, j) = {}^{\sharp}P_{l+n}(\mathfrak{L}), \end{aligned}$$

the desired inequality holds. □

Proof of Theorem 3.2. For $\omega \subset \Omega_{h,k}^{l}$, $\delta > 0$, by the preceding proposition, it follows that

$$ht(\gamma_{\mathfrak{L}}, \omega; \delta) \leq \limsup_{n\to\infty} \frac{1}{n} \log \theta(\omega)^{\sharp} P_{l+n}(\mathfrak{L}) = h_{\mathrm{vol}}(\mathfrak{L})$$

and hence $ht(\gamma_{\mathfrak{L}}, \omega) \leq h_{\mathrm{vol}}(\mathfrak{L})$. As the linear span of $\Omega_{h,k}^{l}, h, k \leq l, l \in \mathbb{Z}_{+}$ is dense in $\mathcal{O}_{\mathfrak{L}}$, by the Kolmogorov–Sinai type property, one has

$$ht(\gamma_{\mathfrak{L}}) = \sup_{\omega \subset \Omega_{h,k}^{l}} ht(\gamma_{\mathfrak{L}}; \omega).$$

Therefore, we get the inequality $ht(\gamma_{\mathfrak{L}}) \leq h_{\mathrm{vol}}(\mathfrak{L})$. □

We reach the following main result.

COROLLARY 3.6. $ht(\gamma_{\mathfrak{L}}) = h_{\mathrm{vol}}(\mathfrak{L})$.

Proof. Let $\mathcal{D}_{\mathfrak{L}}$ be the commutative C^*-subalgebra of all diagonal elements of the AF-subalgebra $\mathcal{F}_{\mathfrak{L}}$. The algebra $\mathcal{D}_{\mathfrak{L}}$ is generated by the elements of the form $S_{\mu} E_i^{l} S_{\mu}^{*}$, $\mu \in \Lambda^{*}, i = 1, \ldots, m(l), l \in \mathbb{Z}_{+}$. It is invariant under the completely positive map $\gamma_{\mathfrak{L}}|_{\mathcal{D}_{\mathfrak{L}}}$ such that the dynamical system $(\mathcal{D}_{\mathfrak{L}}, \gamma_{\mathfrak{L}}|_{\mathcal{D}_{\mathfrak{L}}})$ is canonically isomorphic to the dynamical system $(C(X_{\mathfrak{L}}), \sigma_{\mathfrak{L}}^{*})$. By the endomorphism version of [**Vo**, Proposition 4.8], we know that $ht(\sigma_{\mathfrak{L}}^{*}) = h_{\mathrm{top}}(\sigma_{\mathfrak{L}})$ and that is $h_{\mathrm{vol}}(\mathfrak{L})$ by Theorem 2.1. As there exists a projection of norm one from $\mathcal{O}_{\mathfrak{L}}$ to $\mathcal{D}_{\mathfrak{L}}$, one sees that $ht(\gamma_{\mathfrak{L}}) = ht(\gamma_{\mathfrak{L}}|_{\mathcal{D}_{\mathfrak{L}}})$ by [**Vo**, Proposition 4.4]. Therefore one concludes that $h_{\mathrm{vol}}(\mathfrak{L}) \leq ht(\gamma_{\mathfrak{L}})$ so that $h_{\mathrm{vol}}(\mathfrak{L}) = ht(\gamma_{\mathfrak{L}})$. □

COROLLARY 3.7. $ht(\gamma_{\mathfrak{L}}) = ht(\gamma_{\mathfrak{L}}|_{\mathcal{F}_{\mathfrak{L}}}) = ht(\gamma_{\mathfrak{L}}|_{\mathcal{D}_{\mathfrak{L}}}) = h_{\mathrm{top}}(\sigma_{\mathfrak{L}}) = h_{\mathrm{vol}}(\mathfrak{L})$.

https://doi.org/10.1017/S0143385705000180 Published online by Cambridge University Press

Let Λ be a subshift and $\mathcal{O}_\Lambda$ be the C^*-algebra associated with Λ. The algebra $\mathcal{O}_\Lambda$ is simply the C^*-algebra $\mathcal{O}_{\mathfrak{L}^\Lambda}$ associated with the canonical λ-graph system $\mathfrak{L}^\Lambda$ (cf. [**CM, Ma, Ma3**]). We denote by γ_Λ the canonical completely positive map $\gamma_{\mathfrak{L}^\Lambda}$ on the algebra $\mathcal{O}_{\mathfrak{L}^\Lambda}$. We put $h_{\mathrm{vol}}(\Lambda) = h_{\mathrm{vol}}(\mathfrak{L}^\Lambda)$. Hence, we have the following.

COROLLARY 3.8. $h_{\mathrm{top}}(\Lambda) \leq ht(\gamma_\Lambda) = h_{\mathrm{vol}}(\Lambda) \leq h_{\mathrm{top}}(\Lambda) + h_\lambda(\Lambda)$ *where* $h_\lambda(\Lambda)$ *is the* λ*-entropy* $h_\lambda(\mathfrak{L}^\Lambda)$ *defined in* [**KM3**].

The last inequality sharpens the inequality of [**PWY**, Theorem 7.10].

4. *Examples*

4.1. *The sofic shifts.* Let $\mathcal{G} = (V, E, \lambda)$ be a left-resolving finite labeled graph with vertex set V, edge set E and labeling map $\lambda : E \to \Sigma$. As in [**Ma3**, Lemma 2.1], the labeled graph naturally defines a λ-graph system $\mathfrak{L}_\mathcal{G}$. Since each level of the vertex set of $\mathfrak{L}_\mathcal{G}$ is V, the volume entropy $h_{\mathrm{vol}}(\mathfrak{L}_\mathcal{G})$ of $\mathfrak{L}_\mathcal{G}$ coincides with the topological entropy of the sofic shift $\Lambda_\mathcal{G}$ presented by the labeled graph $\mathcal{G}$ (cf. [**Kit, LM**]). The C^*-algebra $\mathcal{O}_{\mathfrak{L}_\mathcal{G}}$ associated with the λ-graph system $\mathfrak{L}_\mathcal{G}$ is isomorphic to a Cuntz–Krieger algebra (cf. [**Ma3**, Proposition 7.1, [**Ca, To**]]). We have in this case

$$ht(\gamma_{\mathfrak{L}_\mathcal{G}}) = h_{\mathrm{top}}(\Lambda_\mathcal{G}).$$

On the other hand, a topological Markov shift is a sofic shift and any Cuntz–Krieger algebra is realized as a C^*-algebra $\mathcal{O}_{\mathfrak{L}_\mathcal{G}}$ for some finite labeled graph $\mathcal{G}$. The above equality shows Boca and Goldstein's result [**BG**] for Cuntz–Krieger algebras.

4.2. *The β-shifts.* For a real number $1 < \beta \in \mathbb{R}$, let Λ_β be the β-shift, that is defined by β-adic expansion of real numbers [**Ré, Pa, Bl**]. The symbolic matrix systems of the canonical λ-graph systems $\mathfrak{L}_\beta$ of Λ_β have been seen in [**KMW**]. As in that paper, if Λ_β is sofic, the equalities $ht(\gamma_{\Lambda_\beta}) = h_{\mathrm{vol}}(\Lambda_\beta) = h_{\mathrm{top}}(\Lambda_\beta)$ are clear. Otherwise, the cardinal number $\sharp P_n(\Lambda_\beta)$ of the vertex set at level n is $n + 1$. Hence, the λ-entropy $h_\lambda(\Lambda_\beta) = 0$ so that we also have

$$ht(\gamma_{\Lambda_\beta}) = h_{\mathrm{vol}}(\Lambda_\beta) = h_{\mathrm{top}}(\Lambda_\beta).$$

The equality $ht(\gamma_{\Lambda_\beta}) = h_{\mathrm{top}}(\Lambda_\beta)$ was first shown in [**PWY**, Theorem 7.10], and $h_{\mathrm{top}}(\Lambda_\beta)$ is known to be $\log \beta$ (cf. [**Bl, Wa**]).

4.3. *The Dyck shifts.* The theory of symbolic dynamics has a close relationship with automata theory and language theory. In the theory of language, there is a class of universal languages due to W. Dyck. The corresponding symbolic dynamics generated by the languages are called the Dyck shifts D_N for $2 \leq N \in \mathbb{N}$ (cf. [**Kr, Kr3, Kr4**]). They are non-sofic subshifts. Its alphabet Σ_N consists of the $2N$ brackets: $(_1, \ldots, (_N,)_1, \ldots,)_N$. Let us denote the brackets by $\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_N$, respectively. The Dyck inverse monoid [**Kr3, Kr4**] has the relations

$$\alpha_i\beta_i = \mathbf{1} \quad \text{for } i = 1, \ldots, N, \quad \alpha_i\beta_j = 0 \quad \text{for } i \neq j \tag{4.1}$$

https://doi.org/10.1017/S0143385705000180 Published online by Cambridge University Press

and a word $\gamma_1, \ldots, \gamma_n$ of Σ_N is admissible for D_N precisely if $\prod_{m=1}^{n} \gamma_m \neq 0$. This means that the forbidden words consist of words that do not obey the standard bracket rules.

In [**KM2**], the Cantor horizon λ-graph system $\mathfrak{L}^{Ch(D_N)}$ of the Dyck shift D_N has been studied. The Cantor horizon λ-graph system of D_N is a minimal irreducible component of the canonical λ-graph system of D_N. The C^*-algebra $\mathcal{O}_{\mathfrak{L}^{Ch(D_N)}}$ is purely infinite, simple with infinite-order K-theory groups (cf. [**Ma8**]). Let $V_l^{Ch(D_N)}$ be the vertex set at level l of $\mathfrak{L}^{Ch(D_N)}$. The cardinal number of $V_l^{Ch(D_N)}$ is N^l and every vertex has $N+1$ incoming edges. Hence, every vertex of $V_l^{Ch(D_N)}$ has $(N+1)^l$ predecessors so that

$$\sharp P_l(\mathfrak{L}^{Ch(D_N)}) = (N+1)^l \times N^l.$$

Therefore, we have

$$h_{\mathrm{vol}}(\mathfrak{L}^{Ch(D_N)}) = \log(N+1) + \log N.$$

In [**Ma6**], the canonical λ-graph system $\mathfrak{L}^{D_N}$ of the Dyck shift D_N has been studied. Let $V_l^{D_N}$ be the vertex set at level l of $\mathfrak{L}^{D_N}$. The cardinal number of $V_l^{D_N}$ is $(1/(N-1))$ $(N^{l+1}-1)$ and every vertex of $V_l^{D_N}$ except one vertex has $N+1$ incoming edges, and the one exceptional vertex has $2N$ incoming edges. It is easy to see that the inequalities

$$\sharp P_l(\mathfrak{L}^{Ch(D_N)}) \leq \sharp P_l(\mathfrak{L}^{D_N}) \leq \sharp P_{l+1}(\mathfrak{L}^{Ch(D_N)})$$

hold so that

$$h_{\mathrm{vol}}(\mathfrak{L}^{D_N}) = h_{\mathrm{vol}}(\mathfrak{L}^{Ch(D_N)}) = \log(N+1) + \log N.$$

As the topological entropy of the Dyck shift D_N is $\log(N+1)$ (see [**Kr**]), the value $h_{\mathrm{vol}}(\mathfrak{L}^{Ch(D_N)}) = h_{\mathrm{vol}}(\mathfrak{L}^{D_N}) = \log(N+1) + \log N$ is strictly larger than the topological entropy $h_{\mathrm{top}}(D_N)$.

5. *The Motzkin shifts*

There is another natural grammar called Motzkin system that has first appeared in Krieger's papers [**Kr3, Kr4**]. It is a grammar generated by the Dyck system and its unit. The subshift for the Motzkin system is called the Motzkin shift [**Kr3, Kr4**]. Consider the alphabet $\Sigma_N^{\mathbf{1}} = \{\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_N, \mathbf{1}\}$ such that $\alpha_1, \ldots \alpha_N, \beta_1, \ldots, \beta_N$ satisfy (4.1). The Motzkin shift M_N of order N is a subshift over $\Sigma_N^{\mathbf{1}}$ such that a word $\zeta_1 \cdots \zeta_n$ of $\Sigma_N^{\mathbf{1}}$ is admissible for M_N precisely if $\prod_{m=1}^{n} \zeta_m \neq 0$ (see [**Kr3, Kr4**]).

In [**Ma5**], the Cantor horizon λ-graph system $\mathfrak{L}^{Ch(M_N)}$ of the Motzkin shift M_N has been studied. The C^*-algebra $\mathcal{O}_{\mathfrak{L}^{Ch(M_N)}}$ is purely infinite, simple with infinite order K-theory groups. Let $V_l^{Ch(M_N)}$ be the vertex set at level l of $\mathfrak{L}^{Ch(M_N)}$. The cardinal number of $V_l^{Ch(M_N)}$ is N^l and every vertex has $N+2$ incoming edges. Hence, every vertex of $V_l^{Ch(M_N)}$ has $(N+2)^l$ predecessors so that

$$\sharp P_l(\mathfrak{L}^{Ch(M_N)}) = (N+2)^l \times N^l.$$

Therefore, we have

$$h_{\mathrm{vol}}(\mathfrak{L}^{Ch(M_N)}) = \log(N+2) + \log N.$$

The canonical λ-graph system $\mathfrak{L}^{M_N}$ for the Motzkin shift M_N can be calculated in the same manner as the canonical λ-graph system $\mathfrak{L}^{D_N}$ for the Dyck shift D_N as in [**Ma6**].

https://doi.org/10.1017/S0143385705000180 Published online by Cambridge University Press

We know the volume entropy for the canonical λ-graph system $\mathfrak{L}^{M_N}$ of the Motzkin shift M_N in a similar way to the Dyck shift. We then have

$$h_{\mathrm{vol}}(\mathfrak{L}^{M_N}) = h_{\mathrm{vol}}(\mathfrak{L}^{Ch(M_N)}) = \log(N+2) + \log N.$$

The value $h_{\mathrm{vol}}(\mathfrak{L}^{Ch(M_N)}) = h_{\mathrm{vol}}(\mathfrak{L}^{M_N}) = \log(N+2) + \log N$ is strictly larger than the topological entropy $h_{\mathrm{top}}(M_N) = \log(N+2)$ of the Motzkin shift M_N.

References

[BG] F. P. Boca and P. Goldstein. Topological entropy for the canonical endomorphism of the Cuntz–Krieger algebras. *Bull. London Math. Soc.* **32** (2000), 345–352.

[BK] M. Boyle and W. Krieger. Almost Markov and shift equivalent sofic systems. *Proceedings of Maryland Special Year in Dynamics 1986–87 (Lecture Notes in Mathematics, 1342)*. Springer, Berlin, 1988, 33–93.

[Bl] F. Blanchard. β-expansions and symbolic dynamics. *Theoret. Comput. Sci.* **65** (1989), 131–141.

[Br] N. Brown. Topological entropy in exact C^*-algebras. *Math. Ann.* **314** (1999), 347–367.

[Ca] T. M. Carlsen. On C^*-algebras associated with sofic shifts. *J. Operator Th.* **49** (2003), 203–212.

[Ch] M. Choda. Endomorphisms of shift type (entropy for endomorphisms of Cuntz algebras). *Proceedings of Operator Algebras and Quantum Field Theory (Rome, 1996)*. International Press, 469–475.

[Ch2] M. Choda. Entropy of Cuntz's canonical endomorphism. *Pacific. J. Math.* **190** (1999), 235–245.

[CK] J. Cuntz and W. Krieger. A class of C^*-algebras and topological Markov chains. *Invent. Math.* **56** (1980), 251–268.

[CM] T. M. Carlsen and K. Matsumoto. Some remarks on the C^*-algebras associated with subshifts. *Math. Scand.* **95** (2004), 145–160.

[De] V. Deaconu. Groupoids associated with endomorphisms. *Trans. Amer. Math. Soc.* **347** (1995), 1779–1786.

[De2] V. Deaconu. Generalized solenoids and C^*-algebras. *Pacific. J. Math.* **190** (1999), 247–260.

[JP] J. A. Jeong and G. H. Park. Topological entropy for the canonical completely positive maps of graph C^*-algebras. *Bull. Austral. Math. Soc.* **70** (2004), 101–116.

[Kit] B. P. Kitchens. *Symbolic Dynamics*. Springer, Berlin, 1998.

[KM] W. Krieger and K. Matsumoto. Shannon graphs, subshifts and lambda-graph systems. *J. Math. Soc. Japan* **54** (2002), 877–899.

[KM2] W. Krieger and K. Matsumoto. A lambda-graph system for the Dyck shift and its K-groups. *Doc. Math.* **8** (2003), 79–96.

[KM3] W. Krieger and K. Matsumoto. A class of invariants of the topological conjugacy of subshifts. *Ergod. Th. & Dynam. Sys.* **24** (2004), 1155–1172.

[KMW] Y. Katayama, K. Matsumoto and Y. Watatani. Simple C^*-algebras arising from β-expansion of real numbers. *Ergod. Th. & Dynam. Sys.* **18** (1998), 937–962.

[Kr] W. Krieger. On the uniqueness of the equilibrium state. *Math. Syst. Th.* **8** (1974), 97–104.

[Kr2] W. Krieger. On subshifts and topological Markov chains. *Proceedings of Numbers, Information and Complexity (Bielefeld, 1998)*. Kluwer, Boston, MA, 2000, 453–472.

[Kr3] W. Krieger. On subshift presentations. *Preprint*.

[Kr4] W. Krieger. On a class of algebraic subshifts. *Preprint*.

[LM] D. Lind and B. Marcus. *An Introduction to Symbolic Dynamics and Coding*. Cambridge University Press, Cambridge, 1995.

[Ma] K. Matsumoto. On C^*-algebras associated with subshifts. *Int. J. Math.* **8** (1997), 357–374.

[Ma2] K. Matsumoto. Presentations of subshifts and their topological conjugacy invariants. *Doc. Math.* **4** (1999), 285–340.

[Ma3] K. Matsumoto. C^*-algebras associated with presentations of subshifts. *Doc. Math.* **7** (2002), 1–30.

[Ma4] K. Matsumoto. On strong shift equivalence of symbolic matrix systems. *Ergod. Th. & Dynam. Sys.* **23** (2003), 1551–1774.

https://doi.org/10.1017/S0143385705000180 Published online by Cambridge University Press

[Ma5] K. Matsumoto. A simple purely infinite C^*-algebra associated with a lambda-graph system of the Motzkin shift. *Math. Z.* **248** (2004), 369–394.

[Ma6] K. Matsumoto. K-theoretic invariants and conformal measures on the Dyck shifts. *Int. J. Math.* **16** (2005), 213–248.

[Ma7] K. Matsumoto. Construction and pure infiniteness of C^*-algebras associated with lambda-graph systems. *Math. Scand.* to appear.

[Ma8] K. Matsumoto. On the simple C^*-algebras arising from Dyck systems. *Preprint*.

[MWY] K. Matsumoto, Y. Watatani and M. Yoshida. KMS states for gauge actions on C^*-algebras associated with subshifts. *Math. Z.* **228** (1998), 489–509.

[Na] M. Nasu. Textile systems for endomorphisms and automorphisms of the shift. *Mem. Amer. Math. Soc.* **114**(546) (1995).

[Pa] W. Parry. On the β-expansion of real numbers. *Acta Math. Acad. Sci. Hung.* **11** (1960), 401–416.

[PWY] C. Pinzari, Y. Watatani and K. Yonetani. KMS states, entropy and the variational principle in full C^*-dynamical systems. *Commun. Math. Phys.* **213** (2000), 331–381.

[Re] J. N. Renault. A groupoid approach to C^*-algebras. *(Lecture Notes Math., 793)*. Springer, Berlin, 1980.

[Ré] A. Rényi. Representations for real numbers and their ergodic properties. *Acta Math. Acad. Sci. Hung.* **8** (1957), 477–493.

[RS] M. Rørdom and E. Størmer. Classification of nuclear C^*-algebras. *Entropy in Operator Algebras (Encyclopedia Math. Sci., 126)*. Springer, Berlin, 2002.

[To] M. Tomforde. C^*-algebras of labeled graphs. *Preprint*.

[Vo] D. Voiculescu. Dynamical approximation entropies and topological entropy in operator algebras. *Commun. Math. Phys.* **164** (1994), 181–193.

[Wa] P. Walters. *An Introduction to Ergodic Theory (Graduate Texts Math., 79)*. Springer, Berlin, 1982.

[Wi] R. F. Williams. Classification of subshifts of finite type. *Ann. Math.* **98** (1973), 120–153 (Erratum, *Ann. Math.* **99** (1974), 380–381).

https://doi.org/10.1017/S0143385705000180 Published online by Cambridge University Press