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THE CLASSICAL CONTINUUM WITHOUT POINTS

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Abstract. We develop a point-free construction of the classical one-dimensional continuum, with an interval structure based on mereology and either a weak set theory or a logic of plural quantification. In some respects, this realizes ideas going back to Aristotle, although, unlike Aristotle, we make free use of contemporary "actual infinity". Also, in contrast to intuitionistic analysis, smooth infinitesimal analysis, and Eret Bishop's (1967) constructivism, we follow classical analysis in allowing partitioning of our "gunky line" into mutually exclusive and exhaustive disjoint parts, thereby demonstrating the independence of "indecomposability" from a nonpunctiform conception. It is surprising that such simple axioms as ours already imply the Archimedean property and the interval analogue of Dedekind completeness (least-upper-bound principle), and that they determine an isomorphism with the Dedekind–Cantor structure of $\mathbb{R}$ as a complete, separable, ordered field. We also present some simple topological models of our system, establishing consistency relative to classical analysis. Finally, after describing how to nominalize our theory, we close with comparisons with earlier efforts related to our own.

§1. Introduction. There is a longstanding view, traced to Aristotle, that a continuous substance is somehow unified. It is *viscous* or *seamless*; its parts show a kind of *cohesiveness* to each other. This is not, by itself, a formal notion, though it has been used to motivate at least two formal properties of continua. Neither of these formal features is enjoyed by the contemporary mainstream Dedekind–Cantor "point-set" account of continuity (see, for example, Dedekind, 1963).

First, Aristotle insisted that continua are *not* composed of points. White (1992, p. 29) attributes the following "principle of nonsupervenience of continuity" to Aristotle:

> N-SC. Each partition of a continuous magnitude into proper parts yields parts each of which is pairwise continuous with at least one other part.

For Aristotle, as for Euclid (and, of course, for the Dedekind–Cantor account), between any two points, there is a line segment. As Aristotle (1941) puts it, " intermediate between points there is always a line and between nows a period of time" (*Physics*, Book 6, 231b9). So no two points or instants can be continuous with each other. If we further assume, as Aristotle seemed to, that any part of continuum can constitute a section of a partition of it, then the principle N-SC entails that a continuum does not have points, or instants, as parts.[1]

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[1] As we will soon see, some theorists deny the antecedent of this conditional, claiming, in effect, that there is no nontrivial partition of a continuum.

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There is a role for points in an Aristotelian continuum. Sticking to one-dimension (as we do below), points are, first and foremost, the *limits*, or endpoints, of line segments. The points that are interior to a line segment only exist potentially. Each such point represents a place where the line *could* be broken. But unless the line *is* broken there, the point does not actually exist.[2]

In Book 5 of the *Physics*, Aristotle defines two things to be *contiguous*, or *in contact*, if they are next to each other in such a way that nothing can go between them: "Things are said to be in contact when their extremities are together" (226b21). Consider, for example, a pair of adjacent books on a tightly packed shelf. Aristotle goes on to define another notion, that of two things being *continuous* with each other, tying the notion to what we call above " viscosity":

> The continuous is a subdivision of the contiguous: things are continuous when the touching limits of each become one and the same and are, as the word implies, contained in each other: continuity is impossible if these extremities are two. This definition makes it plain that continuity belongs to things that naturally in virtue of their mutual contact form a unity. (227a10-15)

So the books on the shelf are not continuous, since no matter how close to each other they get, each retains its own boundaries; they remain *two* books. In contrast, when line segments, for example, are brought into contact, the common boundary between them is absorbed, and they become a single line, a " unity." For a more physical example, think of what happens when two bodies of water or two hunks of wet dough come together.

Aristotle thus defines continuity as a *relation* between things—they are continuous *with each other*, and thus become unified when they come in contact. However, he also speaks of continuous *things*, taking continuity to be a monadic *property*. The idea seems to be that something is continuous if whenever it is divided into two, the parts are continuous with each other, as per White's principle (N-SC). Perhaps, continuous things only have parts potentially. Consider, for example, a limb of a tree. The limb is, intuitively, continuous. To break it up into parts would be to destroy it or at least to destroy its unity.[3]

In Book 6 of *Physics*, Aristotle summarizes the definitions and connects them with the above claims that continuous things are viscous and are not composed of points:

> Now if the terms "continuous" [and] "in contact". . . are understood as defined above—things being "continuous" if their extremities are one [and] "in contact" if their extremities are together". . .—nothing that is continuous can be composed of indivisibles: e.g., a line cannot be composed of points, the line being continuous and the point indivisible. (231a18-25)
>
> . . . since indivisibles have no parts, they must be in contact with one another as whole with whole. And if they are in contact with one another as whole with whole, they will not be continuous: for that which is continuous has distinct parts. . .as we saw, no continuous thing is divisible into things without parts. . .. Moreover, it is plain

[2] This aspect of Aristotle's view may also have been motivated by Zeno's paradoxes.

[3] Thanks to Marco Panza and Keren Wilson.

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> that everything continuous is divisible into divisibles that are infinitely divisible. (231b2-18)

To summarize, then, one formal articulation, or consequence, of viscosity is that continua are not punctiform. The account developed below satisfies this. We do not countenance points, as least not as parts of the line, in any sense of "part". Our line is what metaphysicians call "gunky".[4]

A second theme that is said to flow from viscosity is that continua are *indecomposable*: they cannot be broken, cleanly, into parts. Aristotle argues for a rather weak form of indecomposability, turning on his views on boundaries. If, for example, the aforementioned tree limb is broken into two parts, then something *new* is created, namely the surfaces of the resulting parts. In the simple mathematical case, if a line segment is broken into two parts, then something new is created, or made actual, namely the endpoints of the resulting two segments. In *Metaphysics* 3.5 (1002a28-b4), he writes:

> For when [continuous] bodies come into contact or are separated, their boundaries instantaneously become one at one time—when they touch, and two at another time—when they are separated; so that when they have been put together one boundary does not exist but has perished, and when they have been separated the boundaries exist which before did not exist. For it cannot be said that the point (which is indivisible) was divided into two. Hence, when the bodies have been put together, one boundary does not exist, but *has* ceased to exist, and when they have been divided, the boundaries exist which they did not exist before (for the point, being indivisible, was not divided into two).

The main contemporary theories of continuity that are based on intuitionistic logic, such as intuitionistic analysis and smooth infinitesimal analysis, have a much more robust indecomposability (see Bell, 1998). In a sense, it is opposite to the Aristotelian notion. An attempt to divide a continuum will *leave something out*. Sticking to one dimension, let A be a line or a line segment in one of these intuitionistic theories. Then it is not the case that there are two nonempty, disjoint regions B, C, such that A is the union of B and C. The usual metaphor is that if one cuts a line into pieces, something will stick to the knife.[5]

The contemporary, Dedekind–Cantor conception of continuity has *some* indecomposability. Let A be a line or a line segment. Then it is not the case that there are two disjoint nonempty *open* sets, nor two disjoint nonempty *closed* sets, whose union is A. But the intuitionistic theories are indecomposable in a more thorough way. There are *no* nonempty disjoint sets whose union is A.

In one clear sense, the intuitionistic theories count as punctiform. The intended domain of intuitionistic analysis consists of nothing but points, which can be thought of as

[4] The term seems to have originated in Lewis (1991). Below, we show how to define "points" as items of a superstructure imposed on the gunky line (section 2). But points, so defined, are not, in any sense, in, or parts of, regions. Nor do we distinguish open and closed intervals at the level of our continuum of regions (although the usual open, half-open, and closed intervals are recovered as items of the superstructure of point-sets).

[5] See Bell (2001, 2009) for a detailed study of the indecomposability of intuitionistic analysis and smooth infinitesimal analysis. Bell (2009) calls indecomposability "cohesiveness." We grant that, *prima facie*, indecomposability is one plausible articulation or consequence of the intuitive notion that we call viscosity above, but those are different ideas and can be separated.

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(classes of) *constructive* Cauchy sequences of rational numbers. And, of course, the law of excluded middle fails. In particular, we do not have that for any points a, b, c, either $a < b$, $b < a$, or $a = b$. It seems to us that the restriction to constructive Cauchy sequences does not affect their "discrete" character, which intuitionistic continua share with Dedekind–Cantor continua.

Against this, some might argue that points are *determinate* locations on a continuum. So, for example, if x and y are points on a line, then it is determinate whether they are the same or distinct; and if they are distinct, it is determinate whether x comes before or after y. This builds the validity of (the relevant instances of) excluded middle into the very notion of "point". Of course, if that is what points are, then the intuitionistic continuum is not punctiform. We would not quibble over this piece of terminology, but it remains that the intuitionistic continuum consists of nothing but (classes of) Cauchy sequences.

Smooth infinitesimal analysis has points galore, but it entertains *nilsquare* infinitesimals that form a "micro-neighborhood" Δ around 0 that is translatable anywhere along the smooth line, but cannot be "bent." A crucial axiom is that any (smooth) function on Δ obeys an equation for a straight line—its slope at any locus on the smooth line gives the derivative of the function there. Bell (1998) explicitly motivates smooth infinitesimal analysis by expressions of dissatisfaction with the main contemporary point-based analyses of continua, such as that of Dedekind–Cantor. Bell insists that some of these infinitesimals are not to be thought of as further points, but it is not completely clear why. Perhaps, the idea is that nilsquares are too indefinite or amorphous to be thought of as points. Collectively, the nilsquares are thought to be a kind of glue holding the more definite points of the smooth line together. This connects nicely with above notion of viscosity.

The strong indecomposability achieved by intuitionistic analysis and smooth infinitesimal analysis can be traced to the fact that the background logic is intuitionistic. For example, both theories fail to recognize discontinuous functions. In particular, there is no (total) function on the line that takes a constant value on one nonempty proper segment and a different constant value on the rest. To be sure, we are not saying that the restriction to intuitionistic logic is not well motivated in either case, but merely that strong indecomposability already follows from the restriction in logic. An analysis of the continuum and its constituents is not needed to derive this feature.[6]

It would surely be wonderful to present Aristotle with these contemporary developments. We will leave it to scholars (and imaginative script writers) to conjecture what he might have said. But the modern-day classicist, and perhaps an impartial observer, might say that in the intuitionistic theories, indecomposability is achieved only in a negative sense, that is, by depriving oneself of the logical means of distinguishing things, in the sense stipulated (e.g., a total two-valued function that distinguishes one part of the line from the rest).

The account developed below does not respect indecomposability in any of the above senses. The underlying logic is classical, and so we do not have the strong sense of indecomposability enjoyed by the intuitionistic theories. We show, for example, that for each interval i, there are two congruent, adjacent intervals i_1 and i_2 such that i is the sum of i_1 and i_2 (Theorem 6). Moreover, since, as noted, our system is thoroughly nonpunctiform—what we call gunky—we do not countenance the "creation" of endpoints

[6] For challenges to arguments for intuitionistic logic in smooth infinitesimal analysis based on demands of continuity, see Hellman (2006).

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for the parts. So we do not have Aristotle's weak form of indecomposability. Our approach may reasonably be called "semi-Aristotelian".[7]

§2. Atomless mereological continuum. The system developed in this paper is designed to characterize a one-dimensional continuum consisting of "regions" as parts, including "intervals"—although, as will be explained, the notions of "open," "closed," and "half-open" are not available in our system. This continuum does not have any points as parts, although we will be able to define "point" in terms of intervals. Once we have proved that our continuum is Archimedean, we will demonstrate that it is isomorphic to the classical Dedekind–Cantor continuum, as a complete, separable, linearly ordered field.

Our formalism consists of classical first-order logic with equality supplemented with a standard axiom system for second-order logic (or logic of plural quantification, with an unrestricted comprehension axiom for plurals[8]) and with an adaptation of the standard (Tarski) axioms of mereology together with (something implying) an "atomless" axiom.

2.1. Axioms of mereology.

1a. Axioms on $x \leq y$ ("x is part of y"): reflexive, anti-symmetric, transitive.

Certain of our axioms and theorems are conveniently stated in terms of a binary relation called "overlaps": "*x overlaps y*": $x \circ y \Leftrightarrow^{df} \exists z(z \leq x \,\&\, z \leq y)$.

1b. Axiom on $\leq$ **and** $\circ$: $x \leq y \leftrightarrow \forall z[z \circ x \rightarrow z \circ y]$.

THEOREM 2.1. *Axioms 1a and 1b imply the Extensionality Principle:*

$$x = y \leftrightarrow \forall z[z \circ x \leftrightarrow z \circ y].$$

Proof. From left to right is trivial. (Take y as x, then substitute y for the second x.) From right to left: assume the right of extensionality and rewrite it as the conjunction of two conditionals: $\forall z[z \circ x \rightarrow x \circ y]$ and $\forall z[z \circ y \rightarrow z \circ x]$. By Axiom 1b, the first of these yields $x \leq y$ and the second yields $y \leq x$. By antisymmetry of $\leq$, the conjunction of these is equivalent to $x = y$. □

[7] As noted above, Aristotle and others tend to speak in metaphorical terms derived from the physical realm, where "breaking," "putting together," "creating," and the like make literal sense. But of course, in the domain of contemporary (classical) pure mathematics, such "actions" concern mental operations or conceptions, not continua themselves. Thus, we prove that every interval has a bisection, meaning it is equal to the sum or fusion of two nonoverlapping congruent parts. Any "separating" or "breaking into two" is purely conceptual or, perhaps better, metaphorical. This raises a challenge in interpreting the Aristotelian notion of the endpoints of an interval passing from merely potential to actual on breaking as well as the notion of a (merely) potential infinity. Our objective language and theory entirely sidestep these and related problems. In future work, we hope to develop an account that is even closer to Aristotle's.

[8] This looks very much like second-order logical comprehension for monadic predicates, except that it is conditional upon there being something satisfying the predicate. It may be written:

$$\exists v(\Psi(v)) \rightarrow \exists xx \forall y[y \;\eta\; xx \leftrightarrow \Psi(y)],$$

where "$y \;\eta\; xx$" is read "y is one of (or is among) the xx", and where Ψ is any formula of the language lacking free "xx."

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2. Fusion or whole comprehension: $\exists u\Phi(u) \rightarrow [\exists x\forall y\{y \circ x \leftrightarrow \exists z(\Phi(z)\ \&\ z \circ y)\}]$, where Φ is a predicate of the second-order language (or language of plurals) lacking free x.[9]

At this point, we could add an atomless axiom: $\forall x\exists y(y < x)$, where $y < x \leftrightarrow^{df} y \leq x\ \&\ y \neq x$ (read "*y is a proper part of x*"). But this will follow from a stronger condition imposed below on the interval structure of our "pointless" or gunky line (Axiom 5.).

We write $x + y$ for the mereological sum or fusion of x and y, such that $\forall z[z \circ x + y \leftrightarrow (z \circ x \vee z \circ y)]$, and we use $\sum_{n=0}^{\infty} x_n$ to designate fusions of infinitely many things. Also, if $\exists z(z \leq x\ \&\ z \leq y)$, then we write $x \wedge y$ for the *meet* of x and y, which satisfies $\forall z[z \circ x \wedge y \leftrightarrow z \circ x\ \&\ z \circ y]$. (If x and y have no common part, $x \wedge y$ is undefined.) And we write $x|y$ for $\neg\exists z[z \leq x\ \&\ z \leq y]$, pronounced "$x$ is discrete from y (and vice versa)." Furthermore, if $\exists z(z \circ x\ \&\ \neg(z \circ y))$, then $x - y$ is that part of x which does not overlap y, viz. $\forall z[z \circ x - y \leftrightarrow (z \circ x\ \&\ \neg(z \circ y))]$.(If there is no such z, then $x - y$ is undefined.) By Axiom 2, fusions always exist, and meets and differences also exist wherever defined.

The pointless line we wish to characterize we will label G, for gunky.[10] Below, we will prove that (quite remarkably) our very elementary axioms suffice to characterize G precisely as a certain minimal closure; and then we will prove that, with its interval structure, G is isomorphic to the classical real-numbers structure, $\mathbb{R}$. The point, of course, of having the atomless condition is to insure that, literally, G contains no points at all. Thus, except where we explicitly refer to the natural number structure, the range of our first-order and plural variables can be thought of as all the parts of G, which we also call regions.

It is convenient to introduce a geometric primitive, $L(x, y)$, to mean "x is (entirely) to the left of y". The axioms for L specify that it is irreflexive, asymmetric, and transitive. And we define "$R(x, y)$," "x is (entirely) to the right of y," as $L(y, x)$.

Now we can introduce an important geometric relation, *betweenness:* $Bet\,w(x, y, z)$ for "y is (entirely) between x and z":

$$Bet\ w(x, y, z) \Leftrightarrow^{df} [L(x, y)\ \&\ R(z, y)]\ \vee\ [R(x, y)\ \&\ L(z, y)]$$

It follows that $Bet\,w(x, y, z) \leftrightarrow Bet\,w(z, y, x)$.

$L(x, y)$ obeys the following axioms:

3a. $L(x, y) \vee R(x, y) \rightarrow x|y$**.** (Of course, $x|y$ implies $x \neq y$.)

3b. $L(x, y) \leftrightarrow \forall z, u[\ z \leq x\ \&\ u \leq y \rightarrow L(z, u)]$.

The following can now be inferred:

$$Bet\ w(x, y, z) \rightarrow x|y\ \&\ y|z\&x|z, \quad \text{and}$$
$$Bet\ w(x, y, z)\ \&\ Betw(u, x, z) \rightarrow Betw(u, y, z),$$

where the transitivity of L is used for the latter.

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[9] The formulation in language of plurals takes this form:

$$\forall uu\{\exists w(w\ \eta\ uu) \rightarrow \exists x\forall y[y \circ x \leftrightarrow \exists z(z\ \eta\ uu\ \&\ z \circ y)]\},$$

where "*uu*" is a plural variable, "$w\ \eta\ uu$" is read "w is one of the *uu*" (If plural variables are assumed to have instances, then the antecedent and the main conditional can be omitted.)

[10] Note that, by taking Φ in Axiom 2 as '$x = x$', a universal individual exists.

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Now we can define an essential notion, that of a "*connected* part of G." Intuitively, such a part has no gaps. The definition is straightforward:

$$Conn(x) \Leftrightarrow^{df} \forall y, z, u[z, u \leq x \ \& \ Betw(z, y, u) \rightarrow y \leq x]. \qquad (\text{Df } Conn)$$

("Anything lying between any two parts of x is also a part of x.")

Furthermore, we can define what it means for a connected part of G to be *bounded:* Let $Conn(p)$: then

$$Bounded(p) \Leftrightarrow^{df} \exists x, y[Conn(x) \ \& \ Conn(y) \ \& \ Betw(x, p, y)]. \qquad (\text{Df } Bounded)$$

(A connected region wholly between two others is bounded.) Once we establish that G is bi-infinite, that is, infinite in both directions, it will follow that, for connected regions, boundedness is a necessary condition for *finite in extent*, as commonly understood. And once we have established that G is Archimedean, it will follow that boundedness is also sufficient for "finite in extent."

We call bounded connected regions *intervals* and write $Int(j)$, etc., when needed. However, note that, lacking points, we cannot describe intervals as either open or closed, or half-open.

Using L, we can impose a condition of dichotomy for discrete intervals:

4. Dichotomy axiom: $\forall i, j[i, j$ are two discrete intervals $\rightarrow (L(i, j) \vee L(j, i))]$.

Now we can prove a linearity condition among intervals:

THEOREM 2.2 (Linearity). *Let x, y, z be any three pairwise discrete intervals; then exactly one of x, y, z is between the other two.*

Proof. Applying Dichotomy to the hypothesis, assume that (say) $L(x, y)$. If also $L(y, z)$, then $R(z, y)$, so that $Betw(x, y, z)$, and this is unique. If instead $L(z, y)$, then either $L(z, x)$, in which case we have $Betw(z, x, y)$, uniquely, or $L(x, z)$, in which case we have $Betw(x, z, y)$, also uniquely. The argument from assuming at first that $R(x, y)$ is similar. □

To guarantee that arbitrarily small intervals exist everywhere along G, we adopt the following axiom:

5. $\forall x \exists j[Int(j) \ \& \ j < x]$.[11]

An important relation of two intervals is "adjacency," which is defined as follows:

$$Adj(j, k) \Leftrightarrow^{df} j|k \ \& \ \nexists m[Betw(j, m, k)]. \qquad (\text{Df } Adjacent)$$

This nicely captures Aristotles notion of contiguity. Now suppose that $j = \sum_{i=1}^{\infty} j_i$, where $Int(j_i)$ and $R(j_{i+1}, j_i)$ and $Adj(j_{i+1}, j_i)$. Our axioms on L imply that $R(k, j)$ just in case $\forall i[R(k, j_i)]$ and analogously for $L(k, j)$. This will be useful in the proof of Theorem 2.4, below.

The following equivalence relations on intervals will also prove useful: *j and k are left-end equivalent* just in case $\exists p[p \leq j \ \& \ p \leq k \ \& \ \nexists q(\{q \leq j \vee q \leq k\} \ \& \ L(q, p))]$. "*Right-end equivalent*" is defined analogously. (NB: Left- (Right-) end equivalence means, intuitively, that the intervals "share their left (right) ends, or end-points, in common," but our system does not recognize "ends" or endpoints as objects entering into mereological relations.)

[11] This of course implies the "atomless axiom," introduced above.

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One further geometric primitive is very useful both in insuring that G is infinite in extent and in recovering, in effect, the rational numbers as a countable, dense subset of the (arithmetic) continuum, viz. *congruence*, as a binary relation among intervals. Intuitively, $Cong(i, j)$ is intended to mean "the lengths of intervals i and j are equal." Thus, we adopt the usual first-order axioms specifying that *Cong* is an equivalence relation. We will sometimes write this as $|i| = |j|$, but with the understanding that we have not yet given any meaning to "$|i|$" standing alone, but only in certain whole contexts. Similarly, for intervals i, j, we can define, contextually, $|i| < |j|$ as meaning: $\exists j'[j'$ an interval & $j' < j$ & $\text{Cong}(i, j')]$; and we may write $|i| > |j|$ as equivalent to $|j| < |i|$. Our next axiom will guarantee that these comparisons make general sense.

We come now to a key axiom, crucial to our characterization of G:

6. Translation axiom: Given any two intervals, i and j, each is congruent both to a unique left-end equivalent and to a unique right-end equivalent of the other.

In effect, this guarantees that a given length can be "transported" (more accurately, instantiated) anywhere along G, and that these instances are unique as congruent and either left- or right-end equivalent to the given length. In particular, we can prove the following lemma.

LEMMA 2.3. *Given any two intervals i and j such that* $\neg Cong(i, j)$*, either there exists an interval* $i' < j$ *with* $Cong(i, i')$*;or there exists* i' *with* $j < i'$ *with* $Cong(i, i')$.

Proof. By $\neg Cong(i, j)$, $i \neq j$. Assume that $\neg(i' < j)$ for any i' such that $Cong(i, i')$. By the translation axiom, there exists i' such that $Cong(i, i')$ and i' is left-end equivalent to j. We want to show that $j \leq i'$, as that will establish that $j < i'$, as desired. Assume the contrary, that is, that $j \not< i'$. Now, if $i' - j$ does not exist, then, by definition, $i' \leq j$. But, since $Cong(i, i')$, we have $\neg Cong(i', j)$, whence $i' \neq j$, and then we would have $i' < j$, contrary to hypothesis. So assume some $n \leq i' - j$. By the hypothesis for *reductio*, there is also $k \leq j$ and $k \not< i'$, and indeed $\neg(k \circ i')$. Without loss of generality, we may assume that k is an interval (see Axiom 5). Since i' is left-end equivalent to j, it follows that k is *not* left-end equivalent to j. But there is $m \leq j - k$ which is left-end equivalent to both j and i' so satisfies $m \circ i'$. Let m' be a common part of m and i'. Clearly, $L(m', k)$. But $\neg(L(n, k))$, since if it were, it would overlap j, contrary to assumption. (n cannot be left of j, since it is a part of i' and $i'$and j are left-end equivalent.) Therefore, by the Dichotomy axiom on L, we have $L(k, n)$, whence $Betw(m', k, n)$, with both $m', n \leq i'$ but $k \not\leq i'$, contradicting that i' is an interval. □

THEOREM 2.4 (Trichotomy). *For any two intervals, i, j, either* $|i| = |j|$ *or* $|i| < |j|$ *or* $|i| > |j|$.

Proof. Immediate from Lemma 2.3 and the definitions of the disjuncts. □

One further axiom on congruence is useful and intuitively intended, viz. that congruence respects nominalistic summation of adjacent intervals:

7. Additivity: Given intervals i, j, i', j' such that $Adj(i, j)$, $Adj(i', j')$, $Cong(i, i')$, $Cong(j, j')$, then $Cong(k, k')$, where $k = i + j$ and $k' = i' + j'$.

We now turn to the matter of the bi-infinitude of G. In fact, our axioms already guarantee this, as we now prove.

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THEOREM 2.5 (Bi-infinity of G). *Let any interval i be given, then there exist exactly two intervals, j, k, such that $Cong(i, j)$ and $Cong(i, k)$ and $Adj(i, j)$ and $Adj(i, k)$, and one of j, k is left of i and the other is right of i.*

Proof. Given an interval i, by definition it is bounded, so there exists regions that are left of i and regions that are right of i. Assume a region m to (say) the right of i. (The case to the left is handled exactly analogously.) If $Adj(m, i)$, then, by the translation axiom, there is a unique interval j such that $Cong(j, i)$ and j is left-end equivalent with m. If not, $Adj(m, i)$, then let f be the fusion of all intervals p such that $Betw(i, p, m)$. f is an interval. Then, by translation, there is a unique interval j with $Cong(i, j)$ and j left-end equivalent to f. Combining this with the analogous argument for the case to the left of i completes the proof. □

Since bi-extension obviously iterates, this already insures that G is "bi-infinite" in the sense of containing as part the fusion of the minimal closure of any interval i under the operation of *bi-extension* defined in the theorem. (This closure is proved to exist in Lemma 2.7, below.) But we can do better and also insure that G is exhausted by iterating the process of flanking a given interval by two congruent ones as in bi-infinity. This is just the Archimedean property, derived below. Toward this end, call an interval l an (*immediate*) *bi-extension* of interval i—$BiExt(l, i)$, or $biext(i) = l$—just in case $l = j + i + k$, where j, i, k behave as in the bi-infinity theorem.

LEMMA 2.6. *Let i and j be intervals such that $i < j$, then $\neg Cong(i, j)$.*

Proof. For a contradiction, assume $Cong(i, j)$. There are three possible cases: (1) i is left-end equivalent to j, (2) i is right-end equivalent to j, and (3) i is neither. Cases (1) and (2) are argued in exactly the same way. For definiteness, assume case (1). By bi-infinity, there exists i' extending i to the left with $Cong(i', i)$ and $Adj(i', i)$, hence $Adj(i', j)$. But then, by the hypothesis for reductio and transitivity of $Cong$, it follows that both i and j qualify as the unique right extension of i', as required by bi-infinity, and since, by hypothesis of the lemma, $i \neq j$, this is a contradiction. Case (2) is argued exactly analogously, considering i' as extending i to the right.

In case (3), let k_L be the fusion of all parts x of j such that $L(x, i)$ and let k_R be the fusion of all parts x of j such that $R(x, i)$. k_L and k_R are intervals (easy exercise). Clearly, $k_L + i + k_R = j$ and this sum is discrete (all three pairs discrete). Now, let j' be (say) the right extension of j, that is, $Cong(j', j)$ and $Adj(j', j)$. By translation, let i' satisfy $Cong(i', i)$ with i' left-end equivalent to j'. By translation again, let k'_L satisfy $Cong(k_L, k'_L)$ and $Adj(i', k'_L)$ with $L(i', k'_L)$, and let k'_R satisfy $Cong(k_R, k'_R)$ and $Adj(k'_L, k'_R)$ with $L(k'_L, k'_R)$. Then by additivity, $Cong(j, i' + k'_L + k'_R)$, so, by the uniqueness of (right) extension of j as required by bi-infinity, we have $j' = i' + k'_L + k'_R$, whence $i' < j'$, whence $i' \neq j'$, but then both j' and i' qualify as the unique right extension of j, a contradiction. □

Now we can characterize G. Toward that, let X be any class of intervals such that an arbitrary but fixed interval $i \leq G$ is one of the X and such that if $k = biext(j)$ for j any of the intervals of X, then k is also in X. Call such X a "closure of i under $biext$."

LEMMA 2.7. *By Axiom 2, there is an individual which is the common part of the fusions of each class X which is a closure of i under $biext$, which we call their meet or the minimal closure i^* of i under $biext$. (Since i is stipulated to belong to any such X, the meet is nonnull, as required in mereology.)*

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Proof. Immediate from Axiom 2. □

Given a fixed "unit" interval, i, we define the "right-half" or "positive half" i^+ of i^* as the fusion of i and all intervals $j \leq i^*$ such that $R(j, i)$. Then we define the "left-half" or "negative half" i^- of i as the fusion of i and all intervals $j \leq i^*$ such that $L(j, i)$.

By the criterion for identity of mereological objects, the meet i^* of Lemma 2.7 is unique. We now can prove a theorem characterizing G as this meet:

THEOREM 2.8 (Characterization of G). *Let G be the fusion of the objects in the range of the quantifiers of our axioms, then $G = i^*$ the fusion of the minimal closure of i under $biext$.*

Proof. Suppose, for a contradiction, that $G \neq i^*$. Since, by stipulation, $i \leq G$ and G is closed under $biext$, we have that $i^* < G$. Then some part p, indeed (by Axiom 5) an interval $k \leq G$ satisfies $\forall j[Int(j\&j) \leq i^* \rightarrow L(k, j)] \vee \forall j[Int(j\&j) \leq i^* \rightarrow R(k, j)]$; therefore, by definition, $L(k, i^*) \vee R(k, i^*)$. Let us suppose it is $R(k, i^*)$. (The other case is argued exactly analogously.) Recall that i^+ is the positive or right-half of i^*, as defined above. Clearly, i^+ is connected, and by our betweenness criterion, it is also "bounded," so an interval. Therefore, by the translation axiom, there is a unique interval $m \leq i^+$ with the properties (1) m is right-end equivalent to i^+ and (2) $Cong(m, i)$. But this leads to contradiction, as follows: By definition of i^+ as the positive "half" of i^*, the minimal closure of i under biextension, we have that i^+ is the fusion of a sequence $< i_n >$, where $i_0 = i$, and where $i_{n+1} =$ the right extension of i_n under $biext$, that is, $R(i_{n+1}, i_n)$, $Adj(i_{n+1}, i_n)$, and $Cong(i_{n+1}, i)$. Now, since $m \leq i^+$, $m \circ i_k$,some k. Therefore, there is a least such k, call it k_0, such that $m \circ i_{k_0}$. But then, for any $k > k_0$, $i_k < m$. But both i_k and m are congruent to i, so this contradicts Lemma 2.6. This shows that the assumption of such m, hence of a $k \leq G - i^*$, must be wrong. Thus, taking account of the exactly parallel argument for the left-half (negative) of i^* , it follows that $G = i^*$.[12] □

Thus, our axioms on the interval structure of G already imply the Archimedean property directly without a detour through the point-based continuum, $\mathbb{R}$. Indeed, the same can be said of Dedekind-completeness as is shown in the Appendix.

Finally, we need a guarantee that any interval has a unique bisection. But that can now be proved by the following theorem:

THEOREM 2.9 (Existence and uniqueness of bi-sections). *Given any interval i, there exist intervals j, k such that $j < i$ and $k < i$ and $j|k$ & $j + k = i$ and $Cong(j, k)$, and j, k are unique with these properties.*

Proof. Let i be any interval. For any interval j, let j^{+r} be the fusion of j and the right bi-extension of j. So we need to find an interval j that is left-end equivalent with i such that $i = j^{+r}$. Let k be any interval such that $k < i$. Without loss of generality, assume that k is left-end equivalent with i. We have that $i - k$ exists and is an interval. If k is congruent with $i - k$, we are done. So suppose not. Either $|k| < |i - k|$ or $|i - k| < |k|$. Let l be an

[12] Note that this result, expressing that G is Archimedean, is quite surprising as no axiom explicitly contains an "extremal clause" to the effect that the intervals of G are *only* those that are part of the fusion of those obtained by repeated applications of $biext$ starting with a given interval. Nor do we have an induction axiom for properties of intervals, although, of course, in light of Theorem 2.7, such an induction principle could be derived from mathematical induction based on the natural number structure. Alternatively, one could derive that from properties of minimal closures, à la Frege.

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interval that is congruent to the smaller of those two and is left-end equivalent with i. So $l^{+r} \leq i$ (in fact $l^{+r} < i$). Now let j be the fusion of all intervals m such that m is left-end equivalent with i and $m^{+r} \leq i$. Clearly, $j^{+r} \leq i$. If $j^{+r} = i$, we are done. So suppose that $j^{+r} < i$. Let $n = i - j^{+r}$. Then n is an interval. Let p be an interval such that $p < n$ and, without loss of generality, suppose that p is left-end equivalent to n. Let q be $n - p$. Without loss of generality, assume that either $|p| = |q|$ or $|p| < |q|$. So $p^{+r} \leq n$. Let j' be the fusion of j and an interval congruent to p immediately on its right. (First let p_1 be congruent with p and right-end equivalent with j. Then let p_2 be the right bi-extension of p_1. Then $j' = j + p_2$.) An application of additivity shows that $j'^{+r} = j^{+r} + p^{+r}$, and we have $j^{+r} + p^{+r} \leq i$. This contradicts the definition of j as the fusion of all intervals m such that m is left-end equivalent with i and $m^{+r} \leq i$.

For uniqueness, given interval i, suppose both $i = j_L + j_R$ with $Cong(j_L, j_R)$ and $Adj(j_L, j_R)$, and also $i = k_L + k_R$ with $Cong(k_L, k_R)$ and $Adj(k_L, k_R)$, with neither j_L nor $j_R = k_L$ or k_R. Suppose without loss that $k_L < j_L$(and so $j_R < k_R$). Let $m = j_L - k_L$. Then $j_R + m = k_R$, whence $Cong(j_R + m, k_L)$. Now let m' be congruent to m and adjacent to m to its right. Then $Cong(j_R - m', k_L)$. But we have $j_R - m' < j_R < j_R + m = k_R$ (where j_R is discrete from m). Since $Cong(j_R - m', k_R)$, this contradicts our lemma that if $i < j$, then $\neg Cong(i, j)$. □

By repeated application of bi-sections, we can, in effect, approximate any locus along G to within any desired accuracy with sufficiently many nested intervals, whose least after k subdivisions is of norm 2^{-k} assuming the initial "unit interval" i is of norm 1. (Here we are speaking in our metalanguage, not yet having reconstructed the norm function in our object language.) One natural strategy that now suggests itself is to *define* an exact locus or "point" as a "Cauchy sequence" of such decreasing intervals. As a warm-up example, let us construct an endpoint—say the left—for a given arbitrary interval i. That will simply be the set of all subintervals j of i obtained by successive subdivisions into equal parts such that for all j, there is no $p < i$ such that $L(p, j)$. In point-based one-dimensional geometry, if we arbitrarily set the left-endpoint of $i = 0$, this corresponds to the Cauchy sequence: $< \frac{1}{2^k} >$, $k = 1, 2, \ldots n \ldots$, converging to 0. Indeed, we can introduce "0" (as a *real*) in exactly this way: let i be an arbitrary but fixed interval, oriented as just described. Then $0 =^{df} \cap[S$: S contains i and contains the left-half of any subinterval j of i such that $j \in S]$. Similarly, we could define 1, replacing "left" (L) with "right" (R). (Below, however, we give a definition in terms of Cauchy sequences of intervals increasing to the right, in conformity to the rest of the positive half of G.) Notice that these "objects," whether thought of as numbers or as points, are not claimed to be parts of G. On the contrary, they are part of a *superstructure* that we construct over the mereological-interval structure of G.

In general, we define a sequence $< j_i >$ of intervals increasing to the right (or left, for negative reals) to be $Cauchy$ just in case, for any interval, ε, there exists N such that for any $m > k > N$, $j_m - j_k$ is an interval, $R(j_m - j_k, j_k)$, and $|j_m - j_k| < |\varepsilon|$.(This last expression was defined contextually above. By assumption that $< j_i >$ is increasing to the right, $j_m - j_k$ exists and $R(j_m - j_k, j_k)$,similarly for sequences increasing to the left. Note the role of Axiom 5, guaranteeing that arbitrarily small intervals are values of ε.)

By repeated application of bi-infinity and translation, we can always avail ourselves of Cauchy interval sequences $s = < s_j >$ increasing to the right, beginning with our fixed unit interval, i, for positive reals (to the left, beginning with $-i$ for negative reals), that is, such that $R(s_{k+1} - s_k, s_k)$ $(L(s_{k+1} - s_k, s_k))$; the fusion of all the intervals s_j forms an

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interval.[13] This proves convenient in giving a second representation of real numbers as intervals in G itself, which in turn—as we shall see below—augments the reach of reconstructions that do not rely on set theory. Thus, we will have available two relative interpretations of the classical continuum, $\mathbb{R}$, based on G: (1) equivalence classes of Cauchy sequences of intervals of G or canonical ones from each class and (2) fusions of canonical Cauchy interval sequences, as just indicated.

We will return to (2) below. First, let us pursue (1) in some more detail. We want to construct the reals over the fixed interval i already associated with [0,1]. The first step is to identify the binary rationals as the appropriate subintervals left-endpoint equivalent to i, obtained by iterated subdivisions licensed by the bisection axiom. Thus, each rational of the form $\frac{n}{2^k}$, where $n = 1, 2, \ldots, 2^k - 1$, corresponds 1–1 with the left-endpoint equivalent subintervals of i determined by the kth stage of bisections. (The reader will have noted that the full binary tree of Baire space is in effect generated by these subdivisions.) The next step is to identify arbitrary reals in (0, 1] with increasing Cauchy sequences of these subintervals.[14]

The natural ordering of the binary subintervals of i implicit above is this: $j \prec k \leftrightarrow^{df} k - j$ exists (is nonnull) and $R(k - j, j)$.

The next step is to extend this ordering to the increasing Cauchy sequences of intervals. We set $< r_i > \prec < s_i >$ just in case $\exists\varepsilon\exists N\forall k > N[\varepsilon$ an interval & $|s_k - r_k| > |\varepsilon|$ & $R(s_k - r_k, r_k)$ (Recall that the norm-notation was defined contextually using $Cong$, above.)

§3. Recovering $\mathbb{R}$. The pieces are now in place to prove a first recovery theorem:

THEOREM 3.1. *The ordered structure of binary intervals within i together with the Cauchy sequences of them is order-isomorphic to the classical real numbers of* (0, 1] *(in their natural ordering, <).*

Proof. There is a 1–1 invertible map φ from the binary intervals of i to the binary rationals of (0, 1]. Define φ as follows: set φ of the left interval, call it $\frac{i}{2}$, of i resulting from the first subdivision = $\frac{1}{2}$; after the kth subdivision, set φ of the left-most = $\frac{1}{2^k}$, of the next left-most = $\frac{2}{2^k}$,..., of the next to right-most = $\frac{2^k-1}{2^k}$. (φ of the right-most is of course always = 1.)[15] Clearly, φ is order-preserving. To extend this to the increasing Cauchy interval sequences, map each such of the form $< s_1, s_2, \ldots, s_n \ldots >$ to the increasing rational Cauchy sequence determined by φ, viz. $< \varphi(s_1), \varphi(s_2), \ldots, \varphi(s_n), \ldots >$. Call this extension of φ φ'. That φ' is 1–1 and onto the increasing binary rational Cauchy sequences is immediate from the properties of φ. That φ' is order-preserving is also clear: if $< r_k >$

[13] Since we do not recognize a null interval, 0 is conventionally defined either as above or as a rightward proceeding (nested) sequence starting with an interval, call it $-\frac{i}{2}$, congruent to the left-half of i, choosing at each stage, k, the right-half subinterval of $-\frac{i}{2^{k-1}}$.

[14] Of course, sequences may omit both left and right subintervals at a given stage, corresponding to a 0 in the binary numeral representation of the real in question.

[15] Since, for example, $\frac{2}{2^k} = \frac{1}{2^{k-1}}$, etc., it appears that φ is many one; but really it is not as the the result of proceeding stepwise to the right simply adds a congruent interval adjacent to the preceding, so that the result is an interval, and in the case of an even number of steps, it is always equal to an interval obtained at an earlier subdivision. For example, in one rightward step at the kth subdivision, we get left-most-$\frac{i}{2^k}$ + next-left-most-$\frac{i}{2^k}$ = left-most-$\frac{i}{2^{k-1}}$, etc. (where the "+" here is mereological summation).

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$\prec < s_k >$, then beyond some N (given in the definition of this ordering, above), $\exists n$ such that the corresponding rational differences, $\varphi(s_m) - \varphi(r_m) > 2^{-n}$, for any $m > N$, which defines order on these rational sequences. □

Now we can extend this to the whole-positive half-line, $(0, \infty)$ by applying the same procedure to right-extensions of i by any number of intervals each congruent with i. For example, we map the interval $i + j$, where $Cong(i, j)$ and $Adj(i, j)$ and $R(j, i)$ to $(0, 2]$ (appealing to the bi-infinity theorem), iterating this procedure to cover all intervals of the form $(0, n]$. Thus, we have:

COROLLARY 3.2. *The theorem statement holds for all intervals of the form* $(0, n]$, *hence for the whole-positive part* G^+ *of* G *as order-isomorphic to the positive reals,* $(0, \infty)$.

Proof. The only thing to check, in addition to what has already been established, is that the map just introduced—call it φ''—from G^+ to $(0, \infty)$ is indeed defined on all G^+ in the sense that no part $p \leq G^+$ is discrete from all the intervals on which φ'' is defined. Suppose, to the contrary, there is some such part, p. Then p must be discrete from each binary rational interval left-end equivalent to i, and in particular discrete from any of the intervals obtained from i by any finite number of applications of the $biext$ operation introduced above. But then p would be discrete from the fusion of all such intervals, which fusion = exactly the meet (= minimal closure of i under $biext$) that we proved = G in Theorem 2.8, above. This contradicts that $p \leq G^+$. □

COROLLARY 3.3. *The theorem statement holds for all intervals of the form* $[-n, 0]$, *hence for the whole negative part* G^- *of* G *as order-isomorphic to the nonpositive reals,* $(-\infty, 0]$.

Proof. Applying Translation and Bi-infinity, the constructions for the positive part of G can be shifted accordingly. (For the sake of the field operations, introduced below, it is simplest to reflect the positive intervals " about 0", that is, proceeding leftward starting with $-i$ or subdivisions thereof.) The proof that the "mirror image" of the map φ'' is defined on all of G^- is exactly analogous to the proof of the first Corollary that φ'' is defined on all of G^+. □

To extend the isomorphisms of the corollaries to cover the whole of G simply stipulate that, for every interval j of the negative part of G and every interval k of the positive part, $j \prec k$.Thus, we have established the following theorem:

THEOREM 3.4. *G (i.e., $(G, \prec)$) is order isomorphic to R (ordered by $<$).*

Call the isomorphism of the latter theorem Φ.

With respect to the field-algebraic structure of $\mathbb{R}$, we can proceed in either of two ways. (1) We could regard it as additional structure of $\mathbb{R}$, built up in the usual way from the Cauchy sequences defining the reals, not bothering about any additional structure of G or (2) we can introduce operations of "addition" and "multiplication" of intervals of G and prove that the order-isomorphism Φ is also an algebraic isomorphism. (2) is more interesting so let us pursue it. To define an interval sum operation, call it $i \oplus j$, we can first define this for binary intervals,which we can already express as $\Phi^{-1}(q)$, for q a binary rational of $\mathbb{R}$, where such an interval is either left-end equivalent to i, if $q > 0$, or right-end equivalent to $-i$, if $q < 0$, where $-i$ is the interval satisfying $Cong(i, -i)$ and $Adj(i, -i)$ and $R(i, -i)$.Then $\oplus$ will simply be vector-addition along G. $\oplus$ is then extended to all $\mathbb{R}$ by applying it "pointwise" to inverse images under Φ of the (binary) rational Cauchy sequences defining the reals in question. To obtain "multiplication" of intervals, $i \otimes j$,

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either of two methods may be used. Remaining entirely within G, we first define this product for inverse images of binary rationals as iterated interval-addition, $\oplus$, treating binary fractions of intervals in accordance with the distributive law of multiplication over addition.[16] Then $\otimes$ is extended to all $\mathbb{R}$ by applying it pointwise to inverse images under Φ of the rational Cauchy sequences defining the reals in question. Alternatively, we can adapt the established Euclidean geometric method of introducing product of two lengths by working in the Euclidean plane. Here, of course, we work in $G \times G$, diagrammatically representing one copy of G as our abcissa, the other as ordinate, such that the left-end of our unit, i, is the origin, where the two axes cross.[17] Then the product $j \otimes k$ is obtained by taking j as left-end equivalent to i along the abcissa, k as extending from the origin along the ordinate (either "up" if we are operating on $+k$ or "down" if we are operating on $-k$). Next we construct the "hypotenuse" segment σ connecting the right-end of i with the "top"-end of k (if we are consider $j \otimes +k$, "bottom"-end of k if it is $j \otimes -k$). The value of the product is then represented as the interval m along the ordinate from the origin to where the segment, call in σ', meets the ordinate, where σ' extends from the right-end of j and lies parallel to σ. (σ' forms a second "hypotenuse," so we have two similar triangles. Then the definition of product derives directly from the fact that $\frac{|m|}{|k|} = \frac{|j|}{1}$.)[18] Now the definition of the operation $\otimes$ is extended to all $\mathbb{R}$ via rational Cauchy sequences as in the first method.

Recalling that $\Phi^{-1}(0)$ was introduced via a rational-interval Cauchy sequence, one checks that the algebraic laws of the field operations are preserved under Φ. Thus, we have

THEOREM 3.5. *G $(G, \prec, \oplus, \otimes)$ is ordered-field isomorphic to R $(R, <, +, \cdot)$.*

Although these isomorphism theorems rely on the characterization theorem (5) of G, which already expresses that G is Archimedean, it is worth pointing out that Theorem 3.5 implies that G satisfies a statement of the Archimedean property closer to the one commonly given for $\mathbb{R}$. Here is one way to formulate this. As will be suggested below, assume as part of our mereological universe a natural numbers structure, N, constituted entirely of atoms, which we designate $0_N, 1_N, 2_N, \ldots$.[19] Then we can speak of functions from N $('$, $+_N$, $\bullet_N)$ to intervals of G by quantifying plurally over (unordered) pairs, $(n, j) =^{df} n + j$ (where this is nominalistic summing). Now inductively define a map φ: $N \rightarrow$ Intervals of G via

(i) $\varphi(0_N) = i$

(ii) $\varphi(n_N +_N 1_N) = biext(\varphi(n_N))$.

---

[16] For example, $\Phi^{-1}(2.5) \otimes \Phi^{-1}(3.25)$ is computed by vectorially adding twice the (unit) interval i and half-i, call this $2.5i$, and then vectorially tripling this and adding a quarter of $2.5i$ to obtain the answer.

[17] Justifying this method on the present pointless basis would require adopting some further primitives and axioms to extend our methods to the Euclidean plane. We would need further equivalence relations of "end equivalence" to replace reference to "the point where two noncollinear intervals meet"; and we would need a relation of angle-congruence for purposes of constructing paralells. All this will be carried out in further work on two-dimensional continua.

[18] Note that we are now in a position to define the norm, $|j|$, for any interval j, based on the isomorphism Φ.

[19] This can itself be carried out along lines of Hellman (1996) but ignoring modal operators for present purposes.

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Now define "i divides k (mod 3)" to mean: $k = \varphi(m_N)$, some m_N. Then we say G *is Archimedean* just in case

$$\forall j \exists h[j \text{ an interval } \rightarrow h \text{ an interval } \& \ i \text{ divides } h \text{ (mod 3) } \& \ j \leq h].$$

Thus, we have

COROLLARY 3.6. *G is Archimedean.*

Proof. This follows from the characterization theorem on G as the minimal closure of i under $biext$. For a contradiction, suppose that some interval $j \leq G$ is not covered by any interval h obtained by iterating $biext$, starting with i, any finite number of times. Then, by construction of G, for some $j' \leq j$, $j'|G$, so $j \not< G$, contradiction (cf. the proofs of Corollaries 3.2 and 3.3, via the definedness of the maps described there). □

Finally, we would like to be assured that the above development of $\mathbb{R}$ over G is independent of the starting interval, i, that is, that starting with any other, j, leads to essentially the same recovery theorems, and even that the minimal closure of j under $biextension$ is indeed G itself. That can be arrived at as follows.

Let j be any interval of G other than i. Now minimally close j under $biextension$, calling the result j^*. Then we proceed in two steps: (1) we can carry out the whole of the above construction to produce an ordered-field isomorphism Φ' from the binary intervals and Cauchy sequences thereof between G' and $\mathbb{R}$,where G' and Φ' are introduced just as G and Φ were but substituting reference to j for that to i throughout. (Think of a transformation of the interval structure of G based on i combining a suitable translation and either a shrinking, in case $|j| < |i|$, or a stretching, in case $|i| < |j|$, or neither, if $Cong(i, j)$.) Step (2) consists of demonstrating that, in light of (1), the point-free continuum $G' = j^*$ based on j is indeed $= G$. That makes essential use of the Archimedean property of G', afforded by carrying out the proof of the last corollary with G' in place of G, *mutatis mutandis.*

THEOREM 3.7. *The point-free continuum $G' = j^*$ based on interval j ($\leq G$ and $\leq G'$) $= G$.*

Proof. By definition of G as minimal closure of i under $biextension$, $G' \leq G$. For since, by hypothesis, $j \leq G$, we have that for some finite n, $j \leq biext^n(i)$, that is, j is part of the result of the nth iterate of $biext$ applied to i. Then it is straightforward to show that any k covered by finitely many applications of $biext$ starting with j is also covered by finitely many applications of $biext$ starting with i. We need to show the converse that $G \leq G'$. First we claim that $i \leq G'$. This follows from the Archimedean condition displayed above, applied to G', interchanging the roles of i and j to produce an h ($\leq G'$) such that $i \leq h$. Next we argue by induction that any interval k obtained from i by repeated application of the $biextension$ operation will be accessible from j in the same sense as i is, that is, by appealing again to the Archimedean condition above for G', substituting k for j and j for i. Then, from the definition of G, since every interval m of G is covered by the result of some finite number of iterations of $biextension$ based on i, it follows from the induction that $m \leq G'$, as well, whence $G \leq G'$, and therefore $G' = G$. □

§4. Topological models. We now present two topological models for our axiomatization. These illustrate some of the Aristotelian notions of contiguity and

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continuity.[20] The exercise will also serve to remove any lingering doubts concerning the consistency of our axiomatization, if there are any.

For both models, the background meta-theory is the ordinary Dedekind–Cantor account of the real numbers, with their usual topological properties. An open set S of real numbers is said to be *regular* if S is identical to the interior of its closure.[21]

Define a real number r to be an *interior boundary* of a set S, if $r \notin S$, but there are numbers s, t such that $s < r < t$ and the open set $(s, t) - \{r\}$ is a subset of S. So, for example, the number 1 is an interior boundary of the union of $(0, 1)$ and $(1, 2)$. Regular open sets are those open sets that have no interior boundaries.

The domain of our first model consists of all nonempty, regular open sets of real numbers. The parthood relation is just the subset relation, as might be expected. Let Π be a nonempty set (or plurality) of nonempty, regular sets. Define $SUM(\Pi)$ to be the interior of the closure of the union of Π. This is the fusion relation. So to get the fusion of a set of regions in our model, first take the union of the sets, then the closure of the result, and then the interior of that. The result is, again, a regular, open set. And it is straightforward to verify that our Axiom 2, of fusion (or whole comprehension) is satisfied:

$$(\forall ww)(\exists x)(\forall y)[y \circ x \leftrightarrow (\exists z)(z \; \eta \; ww \; \& \; z \circ y)]$$

This model nicely recapitulates some of Aristotle's account of continuous objects. Consider, again, the intervals $(0, 1)$ and $(1, 2)$. Those are " contiguous", since there is nothing of the same kind in between them. Indeed, there are no members of our domain in between. The only thing " between" them is the real number 1, and $\{1\}$ is, of course, not a regular open set. These two intervals are also " continuous", in Aristotle's sense, since when we put them together—when we take their sum—we obtain the interval $(0, 2)$. That is, the "boundary" disappears and they become a single interval, a unity.

The proper definitions of the other primitives in our axiomatization are as straightforward as can be. Recall that $L(x, y)$ intuitively means that the region x is entirely to the left of y. Let X and Y be nonempty, regular open sets of real numbers. Then define X to be $LEFT$ of Y just in case every member of X is smaller than every member of Y. It is trivial to verify that the relevant axioms are satisfied.

Recall that an "interval" is defined to be a connected, bounded region. In the model, the intervals are just the open intervals, (a, b), with $a < b$. And, of course, the Gunkiness Axiom 5 is also trivial:

$$(\forall x)(\exists j)(Int(j) \; \& \; j < x).$$

The notion of "left-end equivalence" is also straightforward. Two intervals are left-end equivalent just in case they have the same left endpoint. And, of course, "congruence" of regions is defined to be congruence of regular, open sets. Verifying the remaining axioms is also trivial.

Our second topological model is a sort of dual to the first, as its domain consists of certain *closed* sets of real numbers. It is, in one sense, a little more Aristotelian, since it does allow that intervals have endpoints—although these endpoints are not regions in the domain.

[20] For a discussion and comparison of these concepts, see Hellman & Shapiro (2012).

[21] Cartwright (1975) argues that in 3-space, all and only regular open sets are "receptacles," regions of space that physical objects can occupy. Cartwright's theory is at least partly Aristotelian.

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Say that a set S of real numbers is *regular closed* if S is identical to the closure of its interior. A real number r is an isolated point of a set B if $r \in B$, and there are numbers s, t such that $s < r < t$ and the open set $(s, t) - \{r\}$ is disjoint from B. So, for example, 1 is an isolated point of $\{1\}$, and also of the union of $\{1\}$ with $[2, 3]$. Regular closed sets have no isolated points.

The domain of our second model is the set of nonempty, regular closed sets of real numbers. As with our first model, the parthood relation is the subset relation. If Π is a nonempty set of nonempty regular closed sets, define the SUM of Π to be the closure of union of Π. It is straightforward to verify that this SUM is regular closed, and that Axiom 2 of fusion is satisfied.

Perhaps, this model better captures the Aristotelian notion of continuous objects. The sum of $[0, 1]$ and $[1, 2]$ is, of course, $[0, 2]$. Here, the boundary point(s) of the contiguous intervals is "absorbed" into the sum.

With this model, we must be a little more careful when characterizing some of the relations in our theory, even the defined ones. Consider, for example, the two closed intervals $[0, 1]$ and $[1, 2]$, both of which are in our model. As sets, those are not disjoint, as they have a member, 1, in common. Recall, however, that in our definition of overlap:

$$x \circ y \leftrightarrow (\exists z)(z \leq x \ \& \ z \leq y),$$

the quantifier ranges over the *regions* in the model. So $[0, 1]$ does not overlap $[1, 2]$ since there is no member of the domain—no regular closed set—that is a part of (i.e., a subset of) both. The regions are indeed discrete. In our second model, intervals—bounded and connected regions—are closed intervals $[a, b]$.

Let A be the union of $[0, 1]$, $[2, 3]$, $[4, 5]$,..., and let B be the union of $[1, 2]$, $[3, 4]$, $[5, 6]$,.... Then A does not overlap B—they are discrete—even though their intersection, as sets, is infinite.

Our definition of "Left" is similarly nuanced. If A and B are regular closed sets, then define A to be $LEFT$ of B just in case for every r in A and every s in B, either $r < s$ or both $r = s$ and r is a boundary of both. So $[0, 1]$ is to the left of $[1, 2]$. It is trivial to verify that the axioms of our theory are all satisfied in this second model.

This second model does have some rather strange or, at least intuitively un-Aristotelian regions. Consider the union of the closed intervals $[.1, .9]$, $[.01, .09]$, $[.001, .009]$,... and $\{0\}$. This set is regular closed and so is a region in our model. However, it has an actual infinity of discrete parts and a sort of loose point $\{0\}$ at its left end.

§5. Interlude: a brief dialogue. *Objection: If this is really coherent, it would indeed reënforce that "nonpunctiform" and indecomposable are very different attributes as applied to continua.*[22] *But the connection between them may be greater than the above seems to imply. Indeed, one may question whether it really makes sense to say of G that it has any proper parts at all! Let's take your interval i: You suppose that it makes sense to speak of i and its " negate", Neg(i) = G − i. But consider the question of the two places where i "meets" Neg(i): Do they touch there? Surely there must be a point at each of those places, either as part of i itself or of Neg(i). But of course, as your system allows, i can*

[22] As discussed in Section 1 above, the intuitionistic continuum is indecomposable but, in some sense of "point," is entirely composed of points. Although not all pairs of reals are orderable (so that "exact location" cannot be attributed), still they are specified with infinite precision relative to the everywhere dense rationals.

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be shifted anywhere along G so that any place can serve as a boundary of an interval, in which case G is composed of points after all.

Reply: It is indeed part of the standard punctiform conception that boundaries of (bounded) intervals must exist, and then they must be part of one or the other of an interval and its complement. But the above theory of G makes sense on the basis of a *different* conception. According to it, let us consider your question, whether i and Neg(i) "touch." Well, we do not make sense of that, except to say that, if you mean, "Is there anything in between i and Neg(i)?," the answer is clearly no; and this is perfectly compatible with the two parts not overlapping. After all, a very similar thing happens in the case of your Dedekind cuts in the rationals used to define irrational numbers: the lower and upper sections are disjoint (the set-theoretic analogue of our discrete), yet there is no " boundary" in the sense of an lub of the lower section or a glb of the upper, until *by fiat* one is introduced by defining it to be the cut (or a section thereof) itself! We all know and love this move by Dedekind, and we know how well it works. But that move does not establish that a corresponding point really exists ("was there all along") on any actual one-dimensional continuum (if there are any), much less on any that can be coherently conceived. And it is simply not a move that we are forced to make, either on conceptual or on practical grounds. Instead, we can, if we like, carry out a Dedekind style construction as a *superstructure* over G, preserving all the advantages of classical analysis while resisting the attribution of a point-ontology.

Objection: The analogy with Dedekind cuts (for irrationals) is flawed: Your G is supposed to model a continuum, so the analogue of a division of G into two discrete yet adjacent parts would be a Dedekind cut in the reals, not in the rationals. But the Dedekind continuum is complete, so there are no cuts in the reals without the corresponding real belonging to one segment or the other. This shows, does it not, that for example, your Bisection axiom really makes no sense unless a point is added at the place where the two parts meet.

Reply: Agreed, the analogy is imperfect; still, imperfect analogies can be of heuristic value. But it does not follow that points need to be added "where [the] two parts meet," for, as we said above, "where they meet" is language belonging to the Cantor–Dedekind theories of a pointful continuum, but it is foreign language that cannot really be translated into the theory of G. Point-like "places" simply are not recognized; *that* is the point! Yes, they can be introduced as superstructure as shown above, but that does not require revising the description of G or thinking that "really" the points are "there" on G. Just as we can say that two people believe in the same god(s) without thereby implying that there exist god(s) in which they both believe, so we can speak (as we have spoken) of two intervals' being, say, "left-end equivalent" without thereby implying that there is a special further entity called "the left-end (as a point)" shared by both segments. Even in cases where abstraction based on an equivalence relation may be reasonable and useful, for example, as when we speak of "income levels" based on persons' or families' "sharing similar enough incomes," that does not mean—at least, without much further argument—that we have to recognize "income levels" as entities or that it is somehow incoherent not to.

§6. Doing without set theory. There were two main places in the above reconstruction where set theory seemed to play an indispensable role: (1) in the use of Cauchy sequences of binary intervals and (2) in the use of isomorphisms between G and $\mathbb{R}$. In the latter case, appeal to set theory is, in a sense, not really problematic with regard to the autonomy of the interval structure of G, since, after all, we are reasoning about the relation

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of that structure to that of the classical continuum, which is fundamentally set-theoretic. The first case, however, is problematic, as the autonomy of G is threatened by reliance on enough set theory to build up $\mathbb{R}$, with its ontology of points, etc. Is there an alternative way of introducing interval sequences that improves on this?

Indeed, there is. All that is needed is to expand the theory of G by specifying that, additional to G but entirely discrete from it, there is a denumerable infinity of mereological atoms that form a natural numbers structure ("N-structure"). That is readily expressed with mereology and plural quantification (cf., e.g., Hellman, 1996). Now the role of ordered pairs of the form $< n, j >$, where n is an atom of the N-structure and j is a binary rational interval of G, can be served by the (unordered) mereological sum, $n + j$.[23] Plurally quantifying over such things then enables *plurally quantifying* over binary rational interval sequences. Those that satisfy the Cauchy convergence condition serve as our reals. The upshot is that, now, we bypass set theory entirely in the development of our G-interval structure, *even to the extent of being able to carry out a great deal of classical analysis without even recognizing real numbers as objects!* Instead of speaking of reals as individual objects, we quantify plurally over the increasing binary rational Cauchy interval sequences corresponding to them (as their limit points, on standard classical theory). This fulfills at least the nonpunctiform aspect of the Aristotelian conception of continua in a clearly nonparasitic way. Points are not introduced at all; rather, their mathematical roles are performed by surrogates within the thoroughly nonpunctiform framework of G. We can even claim to capture Aristotle's idea that "points exist only *potentially*" since, as already seen, we *can* introduce real numbers or points as objects serving as the limits of our converging interval sequences, but that is not forced on us.

But, in order to make good on this last claim, it needs to be checked that at least a good portion of classical analysis can indeed be constructed within the G-framework (with plural quantification but no set theory, as just described). In particular, how are we to reconstruct quantification over functions from reals to reals if we can only plurally quantify over Cauchy interval sequences. This is not a problem so long as our functions are continuous (or at least continuous on a cocountable domain of reals). The key here is that continuous functions are determined by their behavior at rational arguments. Now binary rationals are available directly as intervals. Suppose we have a sequence of all these representing rationals in the domain of definition of f, that is, we refer plurally to countably many individuals of the form $< n, j >$ as defined above. A continuous function f assigns each of these a real value, represented over G as a Cauchy interval sequence, $f(< n, j >) = f_1^{n+j}, f_2^{n+j}, \ldots$. Now we would like to code all these (countably many) Cauchy interval sequences as a single interval sequence (which itself, of course, need not be convergent—it merely serves to encode the behavior of f at rational arguments, enumerated in an arbitrary fashion). One convenient way is to work with just the atoms n of the $< n, j >$ serving in the enumeration of rational intervals, coding the (set-theoretic) ordered-pair $< n, f(< n, j >) >$ as the sequence $n + f_1^{n+j}, n + f_2^{n+j}, \ldots$. Then the restriction $f \restriction Q$ of f to rationals can be coded up in a single sequence by a dove-tailing construction

[23] Although the intervals of G already allow encoding of natural numbers and the arithmetic operations, representation of sequences of intervals and other functions within a nominalist framework becomes cumbersome at best since a stock of intervals would be serving two roles at once, that of natural numbers and that of items of sequences. Such problems are bypassed by positing an atomic N-structure discrete from G.

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on the countably many sequences of this latter form.[24] In this way, we bring continuous and cocountably continuous functions within the purview of plural quantification of the language of the G-interval structure.

With one more reductive step, we can improve further on this to bring also the isomorphism recovery theorems within the purview of the theory of G. Instead of recovering real numbers as Cauchy interval sequences, we can avail ourselves of the second relative interpretation of $\mathbb{R}$ over G referred to above, toward the end of section 2, viz representing reals as certain intervals of G, viz. fusions of the right-(or left-)increasing Cauchy interval sequences introduced above. Here is how this helps in representing the isomorphisms of the recovery theorems without set-theoretic machinery.

As in the reconstruction just sketched, we reconstruct interval sequences via an atomic N-structure. But now instead of treating quantification over reals as plural quantification over the wholes, $n + j$, coding the (canonical) Cauchy interval sequences defining them, instead we quantify singularly over the fusions of the intervals making up such a sequence, which, as noted, are themselves intervals of G.

Before we can introduce functions from such "real intervals" of G (as we may now call them) to reals of the classical continuum, we need (or at least prefer) to identify a suitable target structure that is itself not essentially set-theoretic. Here, we can appeal to arithmetic methods applied to our given N-structure. Coding signed integers $+m$ $(-m)$ conventionally as certain pairs of naturals, for example, as $2^1 \cdot 3^m$ $(2^2 \cdot 3^m)$, and then rationals as (reduced) pairs of signed integers, we then have canonical Cauchy sequences $< q_j >$ of rationals coded as number-theoretic ordered pairs of the form $< j, \overline{q}_j >$, where $\overline{q}_j$ is the number coding the rational q_j.[25] Any two distinct such sequences differ at some place; thus, the fusion of any such sequence is just a uniquely determined whole of atoms of our original N-structure, with different sequences yielding different fusions. These fusions then can serve as our classical reals (which we call "reals over $\mathbb{N}$"). It is tedious, but routine, to define linear ordering and the field operations on these.

We still need a way of representing or coding mappings between real intervals of G and fusions over the N-structure serving as classical reals . First, how can we code an ordered pair of the form $< k, \sigma >$, where k is a real interval of G and σ is a (nominalistic) sum of "naturals" (atoms of the N-structure) coding a canonical Cauchy sequence of rationals representing a real? Well, if we stipulate that the mapping goes in the direction $k \rightarrowtail \sigma$, we can make do with an *un*ordered pair. But that can simply be taken as the fusion of k and σ, as these are always nonoverlapping, as they are from different structures, one atomless, the other atomic. Finally, by quantifying plurally over fusions of this form $(k + \sigma)$, we achieve the effect of quantifying over mappings or functions from our G-structure to the reals over $\mathbb{N}$. In this manner, we can reconstruct the isomorphisms of the recovery theorems and prove their required properties within our theory of G together with an atomic N-structure, without ever using set-membership or quantifying over sets.

Finally, there is a readily available way of further reinforcing the idea that real numbers as points "exist only potentially": the above extension of the universe of G by an N-structure and the whole subsequent development of an $\mathbb{R}$-structure over it can be carried

[24] This is modeled on the way countably many real numbers can be coded as a single real number.

[25] Here, of course, j is a numerical index referring to an atom of our N-structure, not an interval of G.

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out under the supposition that such an extension is merely *logically possible*.[26] Indeed, even the theory of G itself can be carried out relative to the assumption that such a mereological-interval structure is merely a logical possibility. Even so, extensions of G by further structures, for example, an atomic N-structure as above, are taken as *further* possibilities, relative to any given hypothetical G-structure, so that the "merely possible" status of real numbers as points is still recognized even within a thorough-going modal-structural treatment.

§7. Comparisons with some other constructions. Without attempting anything like complete coverage of alternatives already in the literature, we present some comparisons that we hope will be instructive.

As our title states, we are concentrating on point-free constructions of the classical continuum, $\mathbb{R}$, and in further work, we will develop a recovery of $\mathbb{R} \times \mathbb{R}$, which extends to higher dimensions, without primitives for any objects of lower dimension. In addition to forming the basis of classical functional analysis, these systems can be adapted to study geometric spaces of various sorts, including hyperbolic and spherical geometries (as we will show in future work). So we are clearly working in the area (category) of *metric spaces*, not purely topological spaces. There have, indeed, been a number of efforts to develop topological spaces based on axioms governing *regions*, rather than sets of points, going back at least to work of Karl Menger.[27] Our own work only touches indirectly on these precedents, insofar as well-known topological spaces can be constructed from $\mathbb{R}$, $\mathbb{R}^2$, and higher dimensions. But those are derivative from the metrical structure of these spaces and so are not purely topological.

Focusing, then, on point-free geometries, there is the noteworthy reconstruction of three-dimensional Euclidean geometry by Tarski (1956). This explicitly uses (atomless) mereology to which is added the single primitive *sphere*. (In addition, the construction uses some set theory, e.g., in the key definition of *point* as the set of all spheres *concentric with* a given sphere and also in recovering definitions of various kinds of sets of points, for example, *regular open*, etc.) A few clever definitions introduce the notions *sphere A is externally tangent to sphere B*, *sphere A is internally tangent to sphere B*, *spheres A and B are externally diametrical to sphere C*, *spheres A and B are internally diametrical to sphere C*, leading finally to *sphere A is concentric with sphere B*: where $A \neq B$, and where, say A, is proper part of B, given two spheres, X, Y, externally diametrical to A and internally tangent to B, X, and Y are also internally diametrical to B. Then "*point*" is introduced as already explained, and then the notion of two points being *equidistant from* a third is defined.

What about axioms? As his *Postulate I*, Tarski stipulates that "The notion of *point* and that of *equidistant from* satisfy all the postulates of ordinary Euclidean geometry of three dimensions." Then follow three "auxiliary postulates" governing *solids*, connecting *solid*

[26] Thus, the above development of an extension of G by an atomic N-structure, etc. can be set in the modal-structural framework of Hellman (1989), as improved in Hellman (1996) and subsequent presentations. We do not claim that this captures what Aristotle meant by "potentially infinite."

[27] See, for example, Menger (1940). Modern point-free topology has taken an algebraic turn, investigating Heyting lattices of open sets of a topological space, with important results in the spirit of Stone's representation theorem for Boolean lattices. For an overview, see Johnstone (1983), and for key results, see Gierz et al. (1980), especially chapter 5, 5.16, p. 271. We return briefly to these developments below. For related approaches, see also Roeper (1997) and Gruszczyński & Pietruszczak (2009).

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and *part* with *regular open set* and *inclusion*. He then writes: "The postulate system given above is far from being simple and elegant; it seems very likely that this postulate system can be essentially simplified by using intrinsic properties of the geometry of solids." For the one-dimensional case, that is precisely what our system above accomplishes. We may call an approach such as our own in which no axioms governing even *defined* points are listed (or, in higher-dimensional cases, governing defined concepts for any lower-dimensional objects), one of "honest toil." When it comes to two-dimensions, we will present a recovery of the key Archimedean property meriting the honorific "honest toil." Another departs from this but only fairly modestly; we can describe this as an instance of "petty theft" (first offense). In comparison, Tarski's method seems an instance of "grand larceny." As his self-critical remark suggests, it is one thing to show that key primitive notions can be adequately defined by terms designating objects and relations of a given level (dimension); it is quite another to achieve a full-scale *reduction* of a point-based theory by *deriving* translations of its axioms induced by the definitions as *theorems* from axioms governing concepts pertaining entirely to the given level. In fairness, it should be mentioned that, as Tarski states, his postulates can be proved categorical, and they can be proved mutually relatively interpretable with standard point-based Euclidean geometry. Nevertheless, the achievement of a full-scale reduction is clearly more desirable, not only for the unity it achieves but also for establishing the autonomy and sufficiency of the conceptual machinery operating at a given level or dimension.

Now it is true that the classical Dedekind–Cantor continuum can be arrived at quite elegantly by means of the modern lattice-theoretic approach to point-free topology, cited above (note 27). The method involves defining points via a 1–1 correspondence between points of the underlying topological space and the maximal open sets of the Heyting algebra of open sets resulting from "subtracting singletons," identifying a "left-half" and a "right-half," and then defining one such point to be "left" of another if its left-half is a subset (or part) of the other's. However, the axioms governing these regions are simply identified as those specifying a complete linear ordering. In effect, Dedekind completeness and, as a consequence, the Archimedean principle are just stipulated. In contrast, as we have already shown in detail, we are able to derive suitable expressions of these as theorems in our regions-based system, without a detour through isomorphism with $\mathbb{R}$. We thus preserve, in our approach, one of the few lasting achievements of "logicism," viz. *deriving* the continuity principle in our logical framework, as did Dedekind in his, rather than simply postulating it as an extra axiom. Presumably, a derivation like ours could also be carried out in the algebraic framework by translating the steps we have taken. But from an axiomatic standpoint, the algebraic approach does not provide an independent, equally advantageous system. (This, of course, is not to deny that it affords other insights into the sources of continuity and related mathematical phenomena.)

Closer in content and method to the reconstructions of continua presented here is the recent work of Roeper (2006), on what he calls "the Aristotelian continuum." Like our system presented above, his system characterizes a linear continuum on a point-free basis in terms of a structure of regions and intervals, and it is shown categorical and isomorphic to a classical, point-based continuum. The following are the main points of comparison:

1. Roeper's axioms describe a continuum as a certain kind of region-based topological space, connected, locally connected, second-countable, also linearly ordered, complete, and separable. Our axioms not only do this but also describe the metrical structure of an ordered field. Thus, we have *congruence* of intervals as one of our primitives, whereas Roeper's system omits this.

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2. As part of his logic, in place of mereology and logic of plurals, Roeper uses a (first- and second-order) logic of mass terms. Thus, where we would express, e.g., "region r is entirely to the left of region s" as "any part of r is left of every part of s," Roeper's language expresses "r is everywhere left of s (everywhere)." Probably, the two languages, as used in these respective reconstructions of continua, that is, in the presence of other primitives and axioms governing them, are intertranslatable.
3. Roeper draws heavily on region-based topology, a theory with two nonlogical primitives, *limited* (i.e., bounded or "finite in extent"), and α *is connected with* β, written $\alpha \infty \beta$, intuitively meaning either α and β overlap or they "abut one another," along with 13 axioms governing these, including *Coherence* (if $\gamma = \alpha \cup \beta$, then $\alpha \infty \beta$), *Continuity* (a topological version of a least-upperbound or greatest-lowerbound principle), and *Countable Convex Cover* (akin to separability). It should be emphasized that these topological axioms are not axioms of the Aristotelian continuum but are instead (required and shown to be) derivable therefrom. However, the two primitives just listed are taken over in Axioms 6, 7, and 8 of the latter system. Axiom 6 is essentially our definition of *bounded*; Axiom 7 is a version of *continuity*; and Axiom 8 is a version of *separability.* For purposes of describing the classical linear continuum, we define the two primitives, *limited* and *connected with*: for α *is limited*, see the definition of *bounded* above, section 2; for *connected*, we defined *Adj(r,s)*, "r, s are adjacent," and of course we have $r \circ s$, from mereology. Moreover, the crucial properties of *coherence*, *continuity*, and *separability* are not taken as axioms of our system, but rather proved as theorems, as we have shown (above and in the appendix), where for the first two properties, we use the plurals comprehension scheme of mereology (describing minimal closures and certain fusions), and for the third, we use the representation of rational-length intervals via the theorem of bisectability. Regarding the important Archimedean property, Roeper's system can derive this from its version of continuity or completeness (as standardly done in point-based frameworks, e.g., à la Dedekind), whereas we prove this directly from our comprehension axioms and translation.

Indeed, we can claim more: all Roeper's axioms are in fact derivable as theorems in our system, under the translation just given of his primitives, *limited* and *connected*. Thus, we have a nice unification of systems, our geometry-*cum*-classical analysis and Roeper's regions-based topology-*cum*-ordered continuum. In sum, "honest toil" has paid off.

§8. Appendix: proof that the interval structure of *G* is Dedekind complete (satisfies LUB principle). We use j, k, m as variables for intervals (recall that intervals are totally bounded, connected regions), X, Y, etc. for sets or pluralities of intervals, and 'R' ('L') for right (left).

Some definitions:

DEFINITION 8.1 *Let X be a set of intervals. Say X contains a left-most interval iff there is an interval k of X such that, for any interval m of X that is not left-end equivalent with k, a part k′ of k is left of m.*

DEFINITION 8.2 *Let X be a set of intervals, and let $Y \subseteq X$; then we say that Y extends to the right exactly as far as X iff no part of any of the Y is right of all the X and no part of any of the X is to right of all the Y.*

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DEFINITION 8.3 *Let X be a set of intervals. We say that X is R-bounded iff* $\exists j \forall k[k \in X \rightarrow \exists j'(j' \leq j \ \& \ R(j', k))]$.

For the next definition, we need the following construction: given a set or plurality X of intervals, let Y be any subset of X such that Y contains a left-most interval but extends to the right exactly as far as X. Now let Y' be the result of adding to Y any interval between any two of the Y intervals (calling such Y' "the gap-filled extension of Y"); and let F be the fusion of Y'. By an axiom of mereology on forming fusions, F is the meet of all individuals H which overlap all individuals overlapping some interval of Y'. We say that F is *the least* of such H. F then overlaps exactly all individuals that overlap some interval of Y', by definition *the fusion of* Y'.

DEFINITION 8.4 *Let X be a R-bounded set of intervals. Then we say that j R-bounds X iff j witnesses that X is R-bounded (Definition 8.3), or for some* $Y \subseteq X$ *with a left-most interval and extending to the right exactly as far as X, j is R-end equivalent to F, the fusion of* Y', *the gap-filled extension of Y (as above).*

Note that, by construction, F is an interval: as the fusion of Y', F is connected. Furthermore, Y has a left-most interval so F is left-bounded; and by the hypothesis of the definition, X and hence Y and F are right-bounded. Thus, the condition "j is R-end equivalent to F" makes sense. (End-equivalences are relations of *intervals.*)

DEFINITION 8.5 *j is a least R-bound of X iff j R-bounds X &*$\forall k[k$ *R-bounds X &* k *not R-end equivalent to* $j \rightarrow \exists k'(k' \leq k \ \& \ R(k', j))]$.

THEOREM 8.6. *(LUB) Let X be a R-bounded set of intervals, then there exists a least R-bound of X, unique up to R-end-equivalence.*

Proof. (Sketch): Step 1: Carry out the construction of F as described above after Definition 8.3. As noted after Definition 8.4, by construction F is an interval.

Step 2: **Claim**: F is a least R-bound of X.

Proof of Claim: (1) F R-bounds X, for, by construction of F, F extends at least as far right as any k of X. (2) F is a *least* R-bound of X: let F' be any R-bound of X not R-end equivalent to F. Then by the construction of F as the least region satisfying the "gap-filling" condition, we must have $\exists m[m \leq F' \ \& \ R(m, F)]$, for otherwise, since F' is not R-end equivalent of F, there would be some $n \leq F$ with R(n, F'). But by construction of F, some part of an X is also right of F', contradicting the hypothesis that F' is a R-bound of X.

That the uniqueness clause is satisfied follows from axioms of mereology. □

The argument that there is a greatest lower bound of any L-bounded set of intervals is exactly analogous.

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