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THE DISTRIBUTION OF WEALTH IN THE BLANCHARD–YAARI MODEL

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We study the dynamics of the distribution of wealth in an economy with infinitely lived agents, intergenerational transmission of wealth, and redistributive fiscal policy. We show that wealth accumulation with idiosyncratic investment risk and uncertain lifetimes can generate a double Pareto wealth distribution.

Keywords: Wealth Distribution, Rate of Return Risk, Fat Tails

1. INTRODUCTION

The wealth distribution in the United States has a fat tail. Wolff (2006), using the 2001 Survey of Consumer Finances, finds that the top 1% of households hold 33.4% of the wealth in the United States. Investigating a sample of the richest individuals in the United States, the Forbes 400 data for 1988–2003, Klass et al. (2006) find that the top end of the wealth distribution obeys a Pareto law with an average exponent of 1.49.

In this paper we study a model of wealth accumulation with idiosyncratic investment risk and uncertain lifetime and show that it can generate a double Pareto wealth distribution displaying a Pareto upper tail.[1]

Our model is a continuous-time OLG heterogeneous-agents model. There is a continuum of agents with measure 1 in the economy. Agents have uncertain lifetimes with constant probability of death at each point. The agents have "joy of giving" bequest motives and allocate their wealth among current consumption, a risky asset, a riskless asset, and the purchase of life insurance. The risky asset is a private investment project whose value follows a geometric Brownian motion. Returns from the private investment projects are subject to idiosyncratic risk. The returns from riskless assets and life insurance are the same for all agents. The government taxes capital income and redistributes the proceeds as means-tested subsidies.

The agent's optimal wealth accumulation process follows a geometric Brownian motion and we can calculate the growth rate of aggregate wealth. The ratio

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of individual wealth to aggregate wealth also follows a geometric Brownian motion. The combination of the geometric Brownian motion for accumulation, a means-tested government subsidy policy, and an exponentially distributed age profile (induced by the constant death rate) leads to a stationary distribution for the ratio of individual wealth to aggregate wealth that is a double Pareto distribution.

Our analysis is related to that of Zhu (2010) and Benhabib et al. (2011), who exploit idiosyncratic investment risk to generate fat-tailed wealth distributions in an OLG model where agents have certain finite lifetimes. We provide closed-form solutions for the stationary distribution of wealth, rather than using simulation methods as, e.g., in Aiyagari (1994) and Castaneda et al. (2003).

The rest of this paper is organized as follows. Section 2 contains a continuous-time OLG heterogeneous-agents model with investment risk and lifetime uncertainty. In Section 3 we characterize the wealth distribution of this economy. Section 4 discusses an alternative government policy for redistribution.

2. AN OVERLAPPING GENERATIONS ECONOMY WITH CAPITAL RISK AND BEQUESTS

The economy we study is an extension of those of Yaari (1965) and Blanchard (1985), in which lives are finite and end probabilistically. Yaari (1965) and Blanchard (1985) study only aggregate variables, not the distribution of wealth. Richard (1975) studies consumption choice and portfolio selection in the same environment with a risky asset, also without characterizing the distribution of wealth.

We study the distribution of wealth in a model with uncertain lifetimes and a constant probability of death, where agents have a portfolio choice of risky and riskless assets. The duration of an agent's life is uncertain. Death is governed by a Poisson distribution with rate p.[2] Consequently, the density function of death at any time $t \in [0, +\infty)$ is $\pi(t) = pe^{-pt}$. When the agent dies, the agent's child is born. Each agent has one child.

Let $W(s, t)$ be the wealth at time t of an agent born at time $s \leq t$. An agent allocates individual wealth among current consumption, a risky asset, a riskless asset, and the purchase of life insurance.

2.1. Risky Asset

The risky asset is the source of idiosyncratic capital income risk, e.g., household-owned housing risk and private business risk. We assume that every agent invests his or her wealth in his or her own risky asset. Risk sharing on the return of the risky asset is not possible. The stochastic processes for the agents' idiosyncratic risk are independent, but they follow the same process. For an agent born at time s, the value of the idiosyncratic risky asset at time $t \geq s$, $S(s, t)$, follows a geometric Brownian motion,

$$dS(s, t) = \alpha S(s, t)dt + \sigma S(s, t)dB(s, t),$$

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where $B(s,t)$ is a standard Brownian motion, α is the instantaneous conditional expected percentage change in value per unit of time, and σ is the instantaneous conditional standard deviation per unit of time. The value of the risky asset is then lognormally distributed and its rate of return is independent of its value.

2.2. Riskless Asset

The value of the riskless asset, $Q(t)$, grows exponentially:

$$dQ(t) = rQ(t)dt,$$

where r is the rate of return. Consistently with nonarbitrage, we assume $r < \alpha$.

2.3. Life Insurance

For a price μ, an agent buys life insurance, that is, the right to bequeath to his or her child $\frac{P(s,t)}{\mu}$ if he or she dies at time t. Negative life insurance should be interpreted as an annuity. Life insurance companies are assumed to earn zero profits, and hence $\mu = p$. Let $Z(s,t)$ denote the bequest that an agent born at time s would leave at death at time t. Then[3]

$$Z(s,t) = W(s,t) + \frac{P(s,t)}{p}.$$

2.4. Individual Wealth Accumulation

Agents derive utility from consumption, while alive, and also have a bequest motive of the "joy of giving" form: bequests enter the parents' utility function directly. Both the consumption and the bequest utility indices are assumed CRRA. Let θ denote the time discount rate and χ the strength of the bequest motive. Let $C(s,t)$ denote consumption at time t of an agent born at time s and $\omega(s,t)$ the share of wealth the agent invests in the risky asset at the same time.

The agent's utility maximization problem is

$$\max_{C,\,\omega,\,P} E_t \int_t^{+\infty} e^{-(\theta+p)(v-t)} \left\{ \frac{[C(s,v)]^{1-\gamma}}{1-\gamma} + p\chi \frac{[Z(s,v)]^{1-\gamma}}{1-\gamma} \right\} dv, \qquad \textbf{(1)}$$

subject to

$$dW(s,v) = [rW(s,v) + (\alpha - r)\omega(s,v)W(s,v) - C(s,v) - P(s,v)]\,dv$$
$$+ \sigma\omega(s,v)W(s,v)dB(s,v) \qquad \textbf{(2)}$$

and the transversality condition.[4,5]

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PROPOSITION 1. *The agent's optimal policies are characterized by*

$$C(s,t) = A^{-\frac{1}{\gamma}} W(s,t),$$
$$\omega(s,t) = \frac{\alpha - r}{\gamma \sigma^2},$$
$$Z(s,t) = \rho W(s,t),$$

with $A = \{\frac{\theta+p-(1-\gamma)[r+p+(\alpha-r)^2/2\gamma\sigma^2]}{\gamma(1+p\chi^{1/\gamma})}\}^{-\gamma}$ *and* $\rho = (\frac{\chi}{A})^{1/\gamma}$. *Furthermore,*

$$dW(s,t) = gW(s,t)dt + \kappa W(s,t)dB(s,t), \tag{3}$$

with $g = \frac{r-\theta}{\gamma} + \frac{1+\gamma}{2\gamma}\frac{(\alpha-r)^2}{\gamma\sigma^2}$ *and* $\kappa = \frac{\alpha-r}{\gamma\sigma}$.

Several properties of the solution, using the CRRA form for the utility function and the complete insurance markets, deserve notice. First of all, the mean growth rate of the agent's wealth, $g = \frac{r-\theta}{\gamma} + \frac{1+\gamma}{2\gamma}\frac{(\alpha-r)^2}{\gamma\sigma^2}$, is independent of the bequest parameter χ. Also, the share of the risky asset, $\omega(s,t) = \frac{\alpha-r}{\gamma\sigma^2}$, is only influenced by the risk premium of the risky asset, the degree of risk aversion, and the volatility of the return of the risky asset. The volatility of the growth rate of the agent's wealth, $\kappa = \frac{\alpha-r}{\gamma\sigma}$, does not depend on the bequest motive parameter, χ, but is negatively related to the standard deviation of the price of the risky asset, σ.

Equation (3) means that individual wealth follows a geometric Brownian motion.[6] Note that $dB(s,t)$ represents a positive shock to the return of the risky asset. It is these shocks, as well as mortality, that induce wealth inequality in our economy.

2.5. Redistribution Policies

At time t, the size of the cohort born at time s is $pe^{p(s-t)}$. The mean wealth of the cohort born at time s is denoted by $E_s W(s,t)$, where the expectation is calculated with respect to the cross-sectional wealth distribution of agents born at time s who are still alive at time t. $E_s W(s,t)$ grows at a rate of g,[7]

$$E_s W(s,t) = E_s W(s,s) e^{g(t-s)}. \tag{4}$$

Aggregate wealth $W(t)$ can be calculated as[8]

$$W(t) = \int_{-\infty}^{t} E_s W(s,t) p e^{p(s-t)} ds. \tag{5}$$

We assume that government subsidies are distributed so as to guarantee all newborns a threshold level of initial wealth proportional to aggregate wealth. Any newborn receiving an inheritance $Z(s,t) = \rho W(s,t)$ at time t higher than the threshold level does not obtain any subsidy. Let $x^* W(t)$ be such a threshold level.

Government subsidies are financed by a capital income tax. The interest rate on the riskless asset, r, is net of the tax rate τ: $r = \tilde{r} - \tau$, where $\tilde{r}$ is the before-tax

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interest rate on the riskless asset. The mean return on the risky asset, α, is also net of the tax rate τ: $\alpha = \tilde{\alpha} - \tau$, where $\tilde{\alpha}$ is the before-tax mean return on the risky asset. The government collects capital income taxes and pays subsidies to newborns. A balanced budget is maintained at all times. Thus,

$$\tau W(t) = p \int_0^{\frac{x^*}{\rho} W(t)} [x^* W(t) - \rho W] h(W, t) dW,$$

where $h(W, t)$ is the wealth distribution at time t.

2.6. Aggregate Wealth

The aggregate wealth growth equation is derived from equations (5) and (4):

$$\dot{W}(t) = g W(t) - p W(t) + p E_t W(t, t).$$

The first term of $gW(t)$ is due to individual wealth growth. The second term of $-pW(t)$ is due to death. And the third term of $pE_t W(t, t)$ is the reinjection of wealth through the starting wealth of newborns.

Aggregate starting wealth of the newborns at time t, $pE_t W(t, t)$, is the sum of private bequests and a public subsidy. By Proposition 1, private bequests are, in the aggregate, $p\rho W(t)$. And from Section 2.5, aggregate subsidies are equal to total tax revenue $\tau W(t)$. Thus

$$p E_t W(t, t) = (p\rho + \tau) W(t).$$

Therefore the aggregate wealth has a growth rate of $\tilde{g}$:

$$dW(t) = \tilde{g} W(t) dt = (g + p\rho + \tau - p) W(t) dt. \quad \textbf{(6)}$$

3. THE DISTRIBUTION OF WEALTH

We now investigate the cross-sectional distribution of wealth in our economy. It is in fact convenient to study the ratio of individual to aggregate wealth,

$$X(s, t) = \frac{W(s, t)}{W(t)}, \quad \textbf{(7)}$$

which displays a stationary distribution.[9] It is straightforward to see that $X(s, t)$ also follows a geometric Brownian motion,[10]

$$dX(s, t) = (g - \tilde{g}) X(s, t) dt + \kappa X(s, t) dB(s, t).$$

To investigate the cross-sectional distribution of $X(s, t)$, we need to know not only the evolution function of $X(s, t)$ during an agent's lifetime, but also the

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change of $X(s,t)$ between two consecutive generations, which reflects the role of inheritance and subsidies. Let the cross-sectional distribution of $X(\cdot,t)$ at time t, be denoted by $f(x,t)$. In Section 5 of the Appendix we derive the forward Kolmogorov equation for $f(x,t)$:

$$\frac{\partial f(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}[\kappa^2 x^2 f(x,t)] - \frac{\partial}{\partial x}[(g-\tilde{g})xf(x,t)] - pf(x,t) + pf\left(\frac{x}{\rho},t\right)\frac{1}{\rho}, \quad x > x^*, \tag{8}$$

$$\frac{\partial f(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}[\kappa^2 x^2 f(x,t)] - \frac{\partial}{\partial x}[(g-\tilde{g})xf(x,t)] - pf(x,t), \quad x < x^*. \tag{9}$$

Partial differential equations (8) and (9) do not hold at $x = x^*$.[11] But $f(x^*,t)$ is determined by the boundary conditions

$$\int_0^{+\infty} f(x,t)dx = 1 \quad \text{and} \quad \int_0^{+\infty} xf(x,t)dx = 1, \ \forall t \geq 0.$$

In turn, x^* is determined by government budget balance, given the capital income tax rate τ:

$$\tau = p\int_0^{\frac{x^*}{\rho}} (x^* - \rho x) f(x,t)dx.$$

It is difficult to solve the partial differential equations with an arbitrary initial distribution. Instead, we investigate the behavior of the equations in the long run, the stationary wealth distribution. In a stationary distribution, we have $\frac{\partial f(x,t)}{\partial t} = 0$. A stationary distribution $f(x)$ then satisfies the following ordinary differential equations:

$$\frac{1}{2}\kappa^2 x^2 f''(x) + [2\kappa^2 - (g-\tilde{g})]xf'(x) + [\kappa^2 - (g-\tilde{g}) - p]f(x) + pf\left(\frac{x}{\rho}\right)\frac{1}{\rho} = 0, \quad x > x^*, \tag{10}$$

$$\frac{1}{2}\kappa^2 x^2 f''(x) + [2\kappa^2 - (g-\tilde{g})]xf'(x) + [\kappa^2 - (g-\tilde{g}) - p]f(x) = 0, \quad x < x^*, \tag{11}$$

as well as the boundary conditions

$$\int_0^{+\infty} f(x)dx = 1 \quad \text{and} \quad \int_0^{+\infty} xf(x)dx = 1.$$

We are now ready to state the main result of this paper.

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PROPOSITION 2. *The stationary distribution* $f(x)$ *has the following form:*

$$f(x) = \begin{cases} C_1 x^{-\beta_1} & \text{when } x \le x^* \\ C_2 x^{-\beta_2} & \text{when } x \ge x^*, \end{cases} \tag{12}$$

with

$$C_1 = \left(1 - \frac{1-\beta_2}{2-\beta_2} x^*\right) (x^*)^{\beta_1 - 2} \frac{(2-\beta_1)(2-\beta_2)(1-\beta_1)}{\beta_2 - \beta_1}$$

and

$$C_2 = \left(1 - \frac{1-\beta_1}{2-\beta_1} x^*\right) (x^*)^{\beta_2 - 2} \frac{(2-\beta_1)(2-\beta_2)(1-\beta_2)}{\beta_2 - \beta_1}.$$

Furthermore, $\beta_1 < 1$ *is the smaller root of the characteristic equation*

$$\frac{\kappa^2}{2}\beta^2 - \left[\frac{3}{2}\kappa^2 - (g - \tilde{g})\right]\beta + \kappa^2 - p - (g - \tilde{g}) = 0, \tag{13}$$

$\beta_2 > 2$ *is the larger root of the characteristic equation*

$$\frac{\kappa^2}{2}\beta^2 - \left[\frac{3}{2}\kappa^2 - (g - \tilde{g})\right]\beta + \kappa^2 - p - (g - \tilde{g}) + p\rho^{\beta - 1} = 0, \tag{14}$$

and

$$x^* = \frac{\frac{\tau}{p} + \rho - \frac{2-\beta_1}{\beta_2 - \beta_1}\rho^{\beta_2 - 1}}{1 - \frac{1-\beta_1}{\beta_2 - \beta_1}\rho^{\beta_2 - 1}}. \tag{15}$$

The distribution $f(x)$ is a double Pareto distribution. The parameter β_2 controls the tail of this distribution: the smaller β_2 is, the fatter is the tail. The integrability of $f(x)$ and $xf(x)$ on $(0, +\infty)$ is implied by $\beta_1 < 1$ and $\beta_2 > 2$. This ensures that $f(x)$ is a distribution function with a finite mean, but its variance does not necessarily exist.

Finally, we show that $f(x)$ in equation (12) is the unique stationary distribution and starting from any initial distribution, $f(x, 0)$, the stochastic process $X(\cdot, t)$ converges to the stationary distribution.

PROPOSITION 3. *The stochastic process,* $X(\cdot, t)$*, is ergodic.*

Even though β_1 has a closed-form solution, β_2 does not, generally.[12] We can, however, solve equation (14) numerically. A calibration exercise for this economy is performed in Benhabib and Zhu (2008) with the following parameters: $\theta = 0.03$, $\chi = 15$, $\sigma = 0.26$, $\gamma = 3$, $r = 1.8\%$, $\alpha = 8.8\%$, $p = 0.016$, $q = 0.012$, and $\tau = 0.004$. In that parameterization two more parameters are introduced, q and ζ, which, for simplicity, we omit in the preceding model. These two additional features represent the fraction of agents having a bequest motive, $\frac{q}{p} = 0.75$, and a calibrated estate tax rate $\zeta = 0.19$. The estate taxes then add to government revenues and are incorporated into the balanced budget each period. The stationary wealth distribution $f(x)$ for this calibration is shown in Figure 1.[13] We can

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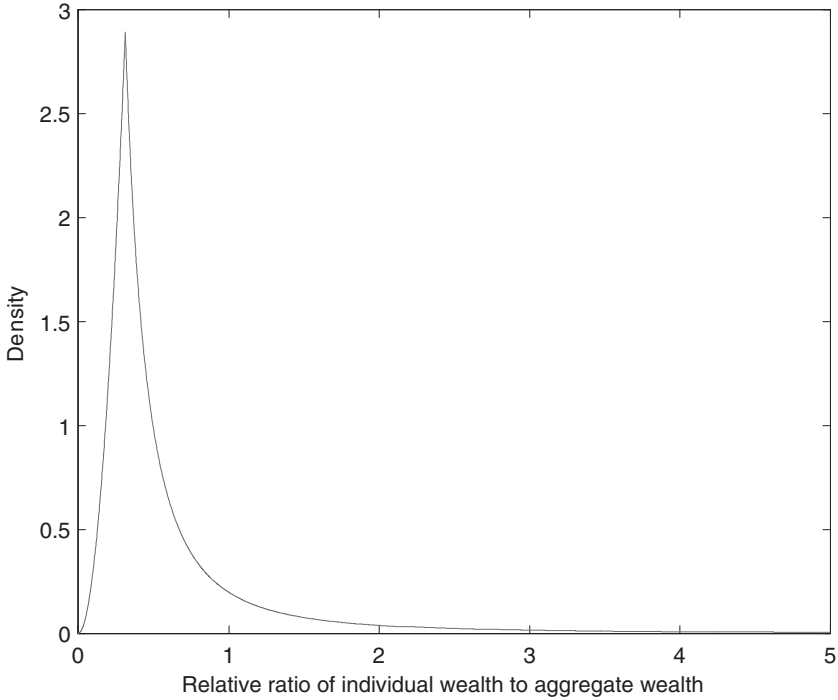


FIGURE 1. Ratio of individual wealth to aggregate wealth.

compare this with the wealth distribution for the United States, using data from the 2004 Survey of Consumer Finances, as shown in Figure 2.[14] Benhabib and Zhu (2008) also investigate the impact of fiscal policy on wealth inequality. Their calibration exercise shows numerically that a higher capital income tax rate implies a lower Gini coefficient of the wealth distribution and a higher estate tax rate implies a lower Gini coefficient: redistribution policies tend to reduce wealth inequality.

3.1. No Investment Risk

The special case of our economy in which agents do not face any investment risk is worth considering explicitly, because it can be solved more completely. In this economy, an agent allocates individual wealth among current consumption, a riskless asset, and the purchase of life insurance. The only risk that agents face is mortality. For simplicity, we also assume in the following analysis that utility indices are logarithmic, that is, $\gamma = 1$. Each agent's optimal policies are easily characterized and induce the following equation for the dynamics of individual

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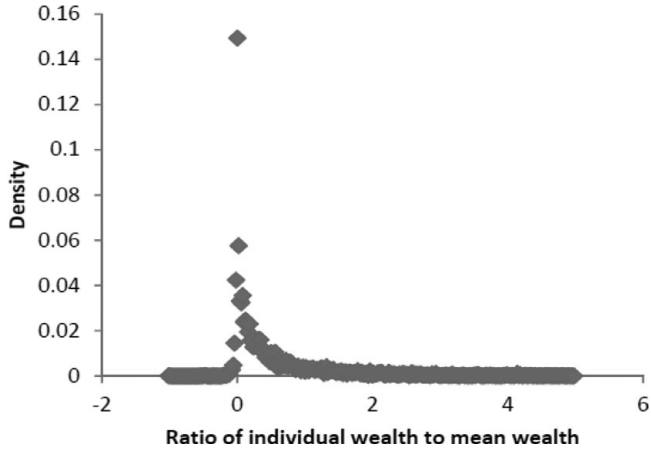


FIGURE 2. The empirical distribution of wealth. The horizontal axis represents the ratio of individual wealth to the mean wealth of the economy; the mean of this ratio equals 1.

wealth:

$$dW(s,t) = gW(s,t)dt,$$

with $g = r - \theta$. The aggregate growth rate of wealth is $\tilde{g} = g + p\rho + \tau - p$. The stochastic process $X(s,t)$ then follows

$$dX(s,t) = (g - \tilde{g})X(s,t)dt.$$

Assuming $g - \tilde{g} > 0$, we proceed to obtain the forward Kolmogorov equation for the distribution $f(x,t)$ and show that a stationary distribution $f(x)$ satisfies

$$-(p + g - \tilde{g})\, f(x) + pf\left(\frac{x}{\rho}\right)\frac{1}{\rho} - (g - \tilde{g})\, x\frac{df(x)}{dx} = 0. \qquad \textbf{(16)}$$

We proceed then by guessing a Pareto distribution for $f(x)$:

$$f(x) = (\lambda - 1)(x^*)^{\lambda-1}x^{-\lambda}, \qquad \textbf{(17)}$$

where $\lambda > 1$. Plugging equation (17) into equation (16), we find that λ must solve

$$(g - \tilde{g})\,\lambda - (p + g - \tilde{g}) + p\,(\rho)^{\lambda-1} = 0. \qquad \textbf{(18)}$$

Benhabib and Bisin (2007) show that this economy has a unique stationary distribution $f(x)$ that is Pareto with λ corresponding to the (unique) root of (18) that is greater than 1, and with $x^* = \frac{\tau/p}{1-[(\lambda-1)/(\lambda-2)]\rho+[1/(\lambda-2)](\rho)^{\lambda-1}}$.[15] Furthermore, Benhabib and Bisin (2007) show that the stochastic process $X(\cdot,t)$ is ergodic.

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Consider now the case in which agents have no preferences for bequests: $\chi = 0$ and hence $\rho = 0$. In this case, equation (18) has a closed-form solution, $\lambda = \frac{p}{p-\tau} + 1$, and the stationary distribution of wealth is

$$f(x) = \frac{p}{p-\tau}\left(\frac{\tau}{p}\right)^{\frac{p}{p-\tau}} x^{-\left(\frac{p}{p-\tau}+1\right)}.$$

Furthermore, in this case, Benhabib and Bisin (2007) show that the Kolmogorov equation implies that, for any initial distribution of x, $h(x) = f(x, 0)$, the distribution $f(x, t)$ is a truncated Pareto distribution in the range $[x^*, x^* e^{(p-\tau)t}]$:

$$f(x,t) = \begin{cases} \dfrac{p}{p-\tau}\, \underline{w}^{\frac{p}{p-\tau}}\, z^{-\left(\frac{p}{p-\tau}+1\right)} & \text{for } x \in \left[x^*, x^* e^{(p-\tau)t}\right] \\ e^{-(p+p-\tau)t} h\left[x e^{-(p-\tau)}\right] & \text{for } x \geq x^* e^{(p-\tau)t}. \end{cases}$$

In the economy with no investment risk, the stationary wealth distribution is a Pareto distribution, not a double Pareto distribution as in our general economy. It is the negative shocks to investment returns that send wealth below the threshold for a fraction of the agents in the economy, generating the left Pareto tail. With no investment risk, individual wealth increases for all agents while they are alive, so that the ratio of individual wealth to aggregate wealth is greater than x^* for any agent.

4. A LUMP-SUM REDISTRIBUTION POLICY

In this section we discuss the effects on the wealth distribution of an alternative redistribution policy, lump-sum redistribution, under which all newborns receive the same subsidy:[16] the government collects $\tau W(t)$ of capital income tax and each newborn at t receives the same subsidy, $b(t) = \frac{\tau}{p} W(t)$. The individual wealth accumulation equation and the growth of aggregate wealth are unchanged under lump-sum redistribution, but the stationary distribution $f(x)$ must satisfy the following Kolmogorov equation:

$$\frac{1}{2}\kappa^2 x^2 f''(x) + [2\kappa^2 - (g - \tilde{g})] x f'(x) + [\kappa^2 - (g - \tilde{g}) - p] f(x) + pf\left(\frac{x - x^*}{\rho}\right)\frac{1}{\rho} = 0 \quad \text{when } x > \frac{x^*}{1-\rho}. \tag{19}$$

Equation (19) differs from equation (10) of Section 3 only in that the last term is $pf(\frac{x-x^*}{\rho})\frac{1}{\rho}$ rather than $pf(\frac{x}{\rho})\frac{1}{\rho}$. However, for large x, the influence of the shift term, x^*, can be ignored and the stationary wealth distribution under lump-sum redistribution has an asymptotic Pareto tail, which admits the same Pareto exponent as in the distribution associated with the means-tested redistribution used in in Proposition 2. We summarize this result as follows.[17]

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PROPOSITION 4. *The stationary distribution is* $f(x) \sim x^{-\beta_2}$ *as* $x \to +\infty$, *where* β_2 *is the larger solution of the characteristic equation*

$$\frac{\kappa^2}{2}\beta^2 - \left[\frac{3}{2}\kappa^2 - (g - \tilde{g})\right]\beta + \kappa^2 - p - (g - \tilde{g}) + p\rho^{\beta-1} = 0.$$

5. CONCLUSION

We set up a continuous-time OLG heterogeneous-agents model to show that investment risk and uncertain lifetimes, plus a specific government subsidy policy, can generate a double Pareto wealth distribution.[18] We also show that uncertain lifetimes and the specific government subsidy policies can technically generate a Pareto wealth distribution. Finally, we show that our model is robust to the government subsidy policy. An alternative government policy, that is, a lump-sum subsidy policy, can produce an asymptotic Pareto tail that admits the same Pareto exponent as in our benchmark model.

NOTES

1. A double Pareto distribution is a distribution exhibiting power-law behavior in both tails. The name is due to William Reed; see Reed (2001).
2. An agent alive at t dies with probability $p\Delta t$ in the time interval $(t, t + \Delta t)$.
3. In the presence of perfect life insurance markets there are no accidental bequests.
4. The transversality condition is

$$\lim_{t \to +\infty} e^{-(\theta+p)(t-s)} E\{J[W(s,t)]\} = 0,$$

where $J[W(s,t)]$ is the agent's optimal value and $E[\cdot]$ is the expectation operator; see Merton (1992).

5. For a generalization to bequest functions for altruistic agents facing redistributive policies, see Appendix A of Benhabib and Bisin (2007) or Footnote 15 in Benhabib and Zhu (2008). Also, Section 5 in Benhabib and Bisin (2007) shows that the Blanchard model can be mapped into a dynastic model with Poisson shocks to individual wealth.
6. The growth rate is independent of wealth, and so individual wealth follows Gibrat's law.
7. This follows easily from the fact that the stochastic return shock is idiosyncratic and the stochastic growth rate is independent of the individual wealth level. We thank Zheng Yang for this point.
8. The aggregate wealth equals the mean wealth in our model because the measure of agents is 1.
9. Note that, under our normalization that total population is 1, $X(s,t)$ represents also the ratio of individual to mean wealth.
10. Then $X(s,t)/X(s,s)$ is lognormally distributed, and

$$X(s,t) = X(s,s)\exp\left\{\left(g - \tilde{g} - \frac{1}{2}\kappa^2\right)(t-s) + \kappa[B(s,t) - B(s,s)]\right\}.$$

11. For this point we greatly benefited from discussions with Matthias Kredler and Professor Henry P. McKean.
12. A closed-form solution for β_2 can be obtained, however, when there is no bequest motive, i.e., $\chi = 0$ (note that $\chi = 0$ implies $\rho = 0$). The mathematical result of the double Pareto distribution in this special case is also in Reed (2001), though in a different economic environment.
13. The horizontal axis represents the ratio of individual wealth to aggregate wealth, or mean wealth in our normalization.

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14. Note that the densities on the vertical axis are not directly comparable across figures, as the calibrated distribution is obtained from the continuous model, whereas the distribution in the data is naturally obtained from discrete bins. Furthermore, note that in the U.S. data a small fraction of agents have negative physical wealth, whereas in our simulated data wealth is positive for all agents. In Benhabib and Zhu (2008), the wealth of agents is defined to include discounted future labor earnings (human capital) as well as physical wealth, which can be a factor that makes aggregate wealth non-negative.

15. A different method of obtaining the Pareto distribution, via a "change of variable," can also be applied to this economy and dates back to Cantelli (1921).

16. See Huggett (1996) for a calibrated model where accidental bequests are distributed equally to everyone, not just newborns. De Nardi (2004) also has some specifications of calibrated models where accidental bequests are equally distributed to the population.

17. A formal proof using the theory of Kesten processes is available from the authors on request. Note that in this economy, the government subsidy plays the role of reflecting barrier which pushes the wealth accumulation process away from zero. In Zhu (2010) and Benhabib et al. (2011), it is instead labor income that operates as a reflecting barrier.

18. Government subsidies essentially play the role of a reflecting barrier for the inheritance process. Without government subsidies, we are not able to obtain a nontrivial stationary distribution of wealth. Thus death also plays an important role because government subsidies only occur when an agent dies.

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APPENDIX

We collect most of the technical proofs here.

A.1. PROOF OF PROPOSITION 1

Let $J[W(s,t)]$ be the optimal value function of the agent with wealth $W(s,t)$. Following and Kamien and Schwartz (1991) and Merton (1992), we set up the Hamilton–Jacobi–Bellman equation of the maximization problem,

$$(\theta + p)J[W(s,t)] = \max_{C,\omega,P} \left\{ \begin{array}{c} \frac{[C(s,v)]^{1-\gamma}}{1-\gamma} + p\chi\frac{[Z(s,v)]^{1-\gamma}}{1-\gamma} \\ +J_W[W(s,t)]\,[rW(s,t) + (\alpha - r)\omega(s,t)W(s,t) - C(s,t) - P(s,t)] \\ +\frac{1}{2}J_{WW}[W(s,t)]\sigma^2\omega^2(s,t)W^2(s,t) \end{array} \right\}.$$

Using the relationship

$$Z(s,t) = W(s,t) + \frac{P(s,t)}{p},$$

we find the first-order conditions

$$C(s,t)^{-\gamma} = J_W,$$

$$\chi Z(s,t)^{-\gamma} = J_W,$$

and

$$(\alpha - r)J_W W(s,t) = -J_{WW}\sigma^2\omega(s,t)W^2(s,t).$$

We guess the value function

$$J[W(s,t)] = \frac{A}{1-\gamma}W(s,t)^{1-\gamma},$$

where A is an undetermined constant. Then we find the expressions for $C(s,t)$, $Z(s,t)$, $P(s,t)$, and $\omega(s,t)$ from the first-order conditions

$$C(s,t) = A^{-\frac{1}{\gamma}}W(s,t),$$

$$Z(s,t) = \left(\frac{\chi}{A}\right)^{\frac{1}{\gamma}}W(s,t),$$

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$$P(s,t) = \left[p\left(\frac{\chi}{A}\right)^{\frac{1}{\gamma}} - p \right] W(s,t),$$

$$\omega(s,t) = \frac{\alpha - r}{\gamma \sigma^2}.$$

Plugging these equations into the Hamilton–Jacobi–Bellman equation, we can determine the constant

$$A = \left\{ \frac{\theta + p - (1-\gamma)\left[r + p + \frac{(\alpha-r)^2}{2\gamma\sigma^2}\right]}{\gamma\left(1 + p\chi^{1/\gamma}\right)} \right\}^{-\gamma}.$$

From the budget constraint, we obtain the wealth accumulation equation,

$$dW(s,t) = \left[\frac{r-\theta}{\gamma} + \frac{1+\gamma}{2\gamma} \frac{(\alpha - r)^2}{\gamma\sigma^2} \right] W(s,t)dt + \frac{\alpha - r}{\gamma\sigma} W(s,t) dB(s,t). \quad \blacksquare$$

A.2. DERIVATION OF THE FORWARD KOLMOGOROV EQUATIONS

Following Ross (1983), we heuristically derive the forward Kolmogorov equations (8) and (9).

Let $f(x,t;y)$ be the probability density of $X(t)$, given $X(0) = y$. Note that

$$\Pr\{X(t) = x | X(0) = y, X(t - \Delta t) = a\} = \Pr\{X(\Delta t) = x | X(0) = a\}.$$

Let

$$f_{\mathrm{DB}}(x;a) = \Pr\{X(\Delta t) = x | X(0) = a\}.$$

Then

$$f_{\mathrm{DB}}(x;a) = \frac{1}{x\sqrt{2\pi\kappa^2\Delta t}} \exp\left(-\frac{1}{2\kappa^2\Delta t} \left\{ \log x - \left[\log a + (g - \tilde{g} - \frac{1}{2}\kappa^2)\Delta t \right] \right\}^2 \right),$$

because $X(\Delta t)|X(0)$ is a lognormal distribution. When $x > x^*$, we have

$$\begin{aligned}
&f(x,t;y) \\
&= (1 - p\Delta t) \int_0^{+\infty} f(a, t - \Delta t; y) f_{\mathrm{DB}}\left(\frac{x}{a}\right) da + p\Delta t \cdot f\left(\frac{x}{\rho}, t - \Delta t; y\right) \frac{1}{\rho} \\
&= (1 - p\Delta t) \int_0^{+\infty} \left[\begin{array}{c} f(x,t;y) + (a-x)\frac{\partial}{\partial x} f(x,t;y) - \Delta t \frac{\partial}{\partial t} f(x,t;y) \\ + \frac{(a-x)^2}{2} \frac{\partial^2}{\partial x^2} f(x,t;y) \end{array} \right] f_{\mathrm{DB}}(x;a) da \\
&+ p\Delta t \cdot f\left(\frac{x}{\rho}, t - \Delta t; y\right) \frac{1}{\rho} \\
&= (1 - p\Delta t) \left[\begin{array}{c} \left(1 - (g - \tilde{g})\Delta t + \kappa^2 \Delta t\right) f(x,t;y) \\ + x\left(-(g - \tilde{g})\Delta t + 2\kappa^2 \Delta t\right) \frac{\partial}{\partial x} f(x,t;y) - \Delta t \frac{\partial}{\partial t} f(x,t;y) + x^2 \frac{\kappa^2}{2} \Delta t \frac{\partial^2}{\partial x^2} f(x,t;y) \end{array} \right] \\
&+ p\Delta t \cdot f\left(\frac{x}{\rho}, t - \Delta t; y\right) \frac{1}{\rho} + o(\Delta t),
\end{aligned}$$

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where we use the Taylor expansion in the second and third equalities. Dividing by Δt on both sides and letting $\Delta t \to 0$, we have

$$\frac{\partial}{\partial t} f(x,t;y) = [\kappa^2 - p - (g-\tilde{g})] f(x,t;y) + [2\kappa^2 - (g-\tilde{g})] x \frac{\partial}{\partial x} f(x,t;y)$$
$$+ \frac{\kappa^2}{2} x^2 \frac{\partial^2}{\partial x^2} f(x,t;y) + p f(\frac{x}{\rho}, t; y) \frac{1}{\rho}, \quad x > x^*.$$

Thus

$$\frac{\partial f(x,t)}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial x^2} \left[\kappa^2 x^2 f(x,t)\right] - \frac{\partial}{\partial x}[(g-\tilde{g}) x f(x,t)] - p f(x,t) + p f\left(\frac{x}{\rho}, t\right) \frac{1}{\rho}, \; x > x^*.$$

Similarly, we have

$$\frac{\partial f(x,t)}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial x^2} \left[\kappa^2 x^2 f(x,t)\right] - \frac{\partial}{\partial x}[(g-\tilde{g}) x f(x,t)] - p f(x,t), \; x < x^*. \; \blacksquare$$

A.3. PROOF OF PROPOSITION 2

Plugging $f(x) = Cx^{-\beta}$ into equation (11), we have

$$\frac{\kappa^2}{2}\beta^2 - \left[\frac{3}{2}\kappa^2 - (g-\tilde{g})\right]\beta + \kappa^2 - p - (g-\tilde{g}) = 0.$$

Therefore,

$$\beta_1 = \frac{\frac{3}{2}\kappa^2 - (g-\tilde{g}) - \sqrt{\left[\frac{1}{2}\kappa^2 - (g-\tilde{g})\right]^2 + 2\kappa^2 p}}{\kappa^2}.$$

To show that $\beta_1 < 1$, note that

$$\beta_1 < 1 \Leftrightarrow \frac{\frac{3}{2}\kappa^2 - (g-\tilde{g}) - \sqrt{\left[\frac{1}{2}\kappa^2 - (g-\tilde{g})\right]^2 + 2\kappa^2 p}}{\kappa^2} < 1$$
$$\Leftrightarrow \frac{1}{2}\kappa^2 - (g-\tilde{g}) < \sqrt{\left[\frac{1}{2}\kappa^2 - (g-\tilde{g})\right]^2 + 2\kappa^2 p}.$$

The last inequality holds because $\kappa > 0$ and $p > 0$. Thus $\beta_1 < 1$. Plugging $f(x) = Cx^{-\beta}$ into equation (10), we have

$$\frac{\kappa^2}{2}\beta^2 - \left[\frac{3}{2}\kappa^2 - (g-\tilde{g})\right]\beta + \kappa^2 - p - (g-\tilde{g}) + p\rho^{\beta-1} = 0.$$

We next show that $\beta_2 > 2$. Let $\Gamma(\beta) = \frac{\kappa^2}{2}\beta^2 - [\frac{3}{2}\kappa^2 - (g-\tilde{g})]\beta + \kappa^2 - p - (g-\tilde{g}) + p\rho^{\beta-1}$. Note that $\Gamma(1) = 0$. Because $\frac{\kappa^2}{2} > 0$, we have

$$\lim_{\beta \to +\infty} \Gamma(\beta) = +\infty.$$

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Also, $\Gamma(2) = g - \tilde{g} - p + p\rho = -\tau$ by equation (6). Thus $\Gamma(2) < 0$. By the continuity of $\Gamma(\beta)$, we know that there exists $\beta > 2$ such that $\Gamma(\beta) = 0$. Because the function $\Gamma(\beta)$ is strictly convex, it can have at most two roots. Then there exists a unique β_2, which is greater than 2. ■

A.4. PROOF OF PROPOSITION 3

We use the embedded Markov chain method to establish the ergodicity of the wealth distribution of newborns, which then implies the ergodicity of the wealth distribution of the whole economy.

As in Karlin and Taylor (1981), we construct the embedded Markov chain from the continuous-time process $X(\cdot, t)$. Let $t_1, t_2, t_3, \cdots$, denote the birth times of generation 1, generation 2, generation 3,$\cdots$. In our notation, their starting wealth is $X(t_1, t_1)$, $X(t_2, t_2)$, $X(t_3, t_3), \cdots$. Let

$$\Phi_0 = X(\cdot, 0), \quad \Phi_n = X(t_n, t_n), \quad n = 1, 2, 3, \cdots .$$

Then Φ_n is the newborns' starting wealth. Note that the state space for Φ_n is $S = [x^*, +\infty)$ by the subsidy policy of the government. The stochastic process Φ_n is a Markov chain. Note that the duration of life follows an exponential distribution with parameter p. When the agent is alive, her wealth follows a geometric Brownian motion, as in equation (3) in the text. Given the government subsidy policy for the newborns, the transition probability of Φ_n is

$$P(\Phi_{n+1} = x^* \mid \Phi_n = x) = \int_0^{x^*} \int_0^{+\infty} pe^{-pt} \times \frac{1}{y\sqrt{2\pi t\kappa^2}} \exp\left(-\frac{1}{2\kappa^2 t}\left\{\log\left(\frac{y}{\rho}\right) - \left[\log x + (g - \tilde{g} - \frac{1}{2}\kappa^2)t\right]\right\}^2\right) dt\, dy,$$

and for $y > x^*$

$$P(\Phi_{n+1} = y \mid \Phi_n = x) = \int_0^{+\infty} pe^{-pt} \times \frac{1}{y\sqrt{2\pi t\kappa^2}} \exp\left(-\frac{1}{2\kappa^2 t}\left\{\log\left(\frac{y}{\rho}\right) - \left[\log x + (g - \tilde{g} - \frac{1}{2}\kappa^2)t\right]\right\}^2\right) dt.$$

By Theorem 13.3.3 of Meyn and Tweedie (1993), $\{\Phi_n\}_{n=0}^{\infty}$ will be ergodic whenever it is positive Harris and aperiodic.

We need to show the following conditions to draw the conclusion of Proposition 3:

(1) Φ_n is ψ-irreducible, (2) Φ_n admits an invariant probability measure. (3) Φ_n is Harris recurrent, and (4) Φ_n is aperiodic.

It is easy to prove (1), because of the special lower bound of x^*. (2) is also true because of Proposition 2 in the text. The existence of the stationary wealth distribution, $f(x)$, implies the existence of the invariant probability measure of Φ_n. (3) is true because the government subsidy policy guarantees that, starting from any place in S, Φ_n visits x^* almost surely. (4) is obviously true. [For these mathematical concepts, see Meyn and Tweedie (1993).] ■

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