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STRUCTURED SINGULAR VALUES OF SOME GENERALISED STOCHASTIC MATRICES

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Abstract

We investigate a class of generalised stochastic complex matrices constructed from the class of all doubly stochastic matrices and a special class of circulant matrices. We determine the exact values of the structured singular values of all matrices in the class in terms of the constant row (column) sum.

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1. Introduction

For an unstructured perturbation, the largest singular values of certain matrices provide a clean and numerically tractable method for evaluating robust stability. There is a measure of stability similar to these largest singular values for a system that fails to be robustly stable using *structured singular values* [1]. The structured singular value is a useful tool for analysing the robustness of stability and the performance of dynamical systems [3].

The structured singular value was introduced by Doyle in [2]. It is a matrix map $\mu_{\mathbb{B}} : M_n(\mathbb{C}) \longrightarrow [0, \infty)$ defined by

$$\mu_{\mathbb{B}}(M) := \begin{cases} 0 & \text{if } \det(I + M\Delta) \neq 0 \text{ for all } \Delta \in \mathbb{B}, \\ 1/\min_{\Delta \in \mathbb{B}}\{\|\Delta\|_2 \mid \det(I + M\Delta) = 0\} & \text{otherwise.} \end{cases}$$

Here, $M_n(\mathbb{C})$ denotes the set of all $n \times n$ complex matrices, $\mathbb{B}$ denotes the set of all diagonal blocks

$$\mathbb{B} := \{\operatorname{diag}(\delta_1 I_{k_1}, \ldots, \delta_r I_{k_r}, \Delta_1, \ldots, \Delta_s) \mid \delta_i \in \mathbb{C}, \Delta_j \in M_{n_i}(\mathbb{C})\} \subseteq M_n(\mathbb{C})$$

with $\sum_{i=1}^{r} k_i + \sum_{j=1}^{s} n_j = n$ and $\|A\|_2$ denotes the matrix 2-norm of A (which is also equal to the largest singular value $\sigma_{\max}(A)$ of A for $A \in M_n(\mathbb{C})$). Note that the complex perturbation block structure $\mathbb{B}$ depends on the values of r (number of repeated scalar blocks), s (number of full blocks) and the k_i and n_i (dimensions of each block).

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Some basic properties of the structured singular value can be deduced from the definition (see [2]). For any block structure $\mathbb{B} \subseteq M_n(\mathbb{C})$:

(1) $\mu_{\mathbb{B}}(\alpha M) = |\alpha|\mu_{\mathbb{B}}(M)$ for any $M \in M_n(\mathbb{C})$ and $\alpha \in \mathbb{C}$;
(2) $\mu_{\mathbb{B}}(M) = \sigma_{\max}(M)$ for any $M \in \mathbb{B}$;
(3) $\rho(M) \leq \mu_{\mathbb{B}}(M) \leq \sigma_{\max}(M)$ for any $M \in M_n(\mathbb{C})$, where $\rho(M)$ denotes the spectral radius (the magnitude of the largest eigenvalue) of M;
(4) if we denote $\mathcal{D}_{\mathbb{B}} := \{\mathrm{diag}(d_1 I_{k_1}, \ldots, d_r I_{k_r}, d_{r+1} I_{n_1}, \ldots, d_{r+s} I_{n_s}) \mid d_i \in (0, \infty)\}$ and $\mathcal{U}_{\mathbb{B}} := \{U \in \mathbb{B} \mid U \text{ unitary}\}$, then, for any $M \in M_n(\mathbb{C})$,

$$\max_{U \in \mathcal{U}_{\mathbb{B}}} \rho(MU) \leq \mu_{\mathbb{B}}(M) \leq \inf_{D \in \mathcal{D}_{\mathbb{B}}} \sigma_{\max}(DMD^{-1}). \tag{1.1}$$

In fact, the first inequality in (1.1) is shown to be an equality in [2]:

$$\mu_{\mathbb{B}}(M) = \max_{U \in \mathcal{U}_{\mathbb{B}}} \rho(MU). \tag{1.2}$$

Unfortunately, the optimisation problem in (1.2) is not convex. The lower function $l(U) := \rho(MU)$ can have multiple local maxima which are not global and so the direct calculation of $\max_{U \in \mathcal{U}_{\mathbb{B}}} \rho(MU)$ by gradient search might not find the actual maximum [5]. In contrast, the function $u(D) := \sigma_{\max}(DMD^{-1})$ does not have any local minima which are not global, so computing $\inf_{D \in \mathcal{D}_{\mathbb{B}}} \sigma_{\max}(DMD^{-1})$ is a reasonable task [5]. As shown in [5], the equality $\mu_{\mathbb{B}}(M) = \inf_{D \in \mathcal{D}_{\mathbb{B}}} \sigma_{\max}(DMD^{-1})$ always holds when $2r + s \leq 3$. Conditions under which the structured singular value reaches the upper bound are given in [5, Theorem 4.5] and a condition under which the structured singular value is equal to the spectral norm can be found in [5, Theorem 4.4].

By using these results, Yamamoto and Kimura showed in [6] that $\mu_{\mathbb{B}}(M) = \sigma_{\max}(M)$ when the matrix M is a square complex matrix satisfying $SM = M^T S$ for some signature matrix S and the perturbation block structure $\mathbb{B}$ has $r = 0$ or $r = 1, s = 1$ (with some conditions). It is well known that if M is a normal matrix, then $\rho(M) = \sigma_{\max}(M)$ and thus by property (3) above, $\rho(M) = \mu_{\mathbb{B}}(M) = \sigma_{\max}(M)$ for any block structure $\mathbb{B} \subseteq M_n(\mathbb{C})$. Also, if M is a doubly stochastic matrix, then $\rho(M) = \sigma_{\max}(A) = 1$ (this fact does not necessarily hold for generalised doubly stochastic matrices) and again $\mu_{\mathbb{B}}(M) = 1$ for any block structure $\mathbb{B} \subseteq M_n(\mathbb{C})$. However, besides these three classes of matrices, to the best of our knowledge, there are no explicit structured singular values for other classes of matrices. Most of the literature related to computation is concerned with the approximation of the upper bound (see, for example, [4] and the references therein).

We investigate a class of generalised row (column) stochastic matrices (square complex matrices having constant row (column) sum) appearing in (2.5). Matrices in this class need not be normal nor radial (a matrix having the spectral norm equal to the spectral radius). The explicit formula for the structured singular values of matrices in this class is given in terms of the constant row (or column) sum.

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2. The results

Throughout, the block structure $\mathbb{B}$ is a subset of $M_n(\mathbb{C})$ having the form

$$\mathbb{B} := \{\mathrm{diag}(\delta_1 I_{k_1}, \ldots, \delta_r I_{k_r}, \Delta_1, \ldots, \Delta_s) \mid \delta_i \in \mathbb{C}, \Delta_j \in M_{n_i}(\mathbb{C})\}$$

for some $r, s \in \mathbb{N}_0$ (the set of all nonnegative integers). Let A be an $n \times n$ generalised row stochastic matrix, meaning that there exists $c \in \mathbb{C}$ such that

$$\sum_{j=1}^{n} A_{ij} = c \quad \text{for } i = 1, \ldots, n, \tag{2.1}$$

and denote the largest singular values of A by σ, so that $\sigma_{\max}(A) = \sigma = \|A\|_2$.

PROPOSITION 2.1. *Let $A \in M_n(\mathbb{C})$ be a generalised row stochastic matrix with the constant row sum $c \in \mathbb{C}$. Suppose that $\|A\|_2 = \sigma$.*

(1) *Then $|c| \le \sigma$.*
(2) *If $|c| = \sigma$, then A must have the constant column sum c, that is, A is a c-generalised doubly stochastic matrix.*
(3) *If $|c| = \sigma$, then $\mu_{\mathbb{B}}(A^m) = \sigma^m$ for all $m \in \mathbb{N}$ and any block structure $\mathbb{B} \subseteq M_n(\mathbb{C})$.*

PROOF. Let $\vec{1} \in \mathbb{C}^n$ be the vector all of whose components are one. By (2.1), $A\vec{1} = c\vec{1}$. By using the standard 2-norm $\|\cdot\|$ on $\mathbb{C}^n$ and the Cauchy–Schwarz inequality,

$$|c|^2\|\vec{1}\|^2 = \langle c\vec{1}, c\vec{1}\rangle = \langle c\vec{1}, A\vec{1}\rangle = \langle cA^*\vec{1}, \vec{1}\rangle = |\langle cA^*\vec{1}, \vec{1}\rangle| \le \|cA^*\vec{1}\|\,\|\vec{1}\|, \tag{2.2}$$

where A^* is the conjugate transpose of A. For any $M \in M_n(\mathbb{C})$ and any nonzero vector $y \in \mathbb{C}^n$,

$$\|M\|_2 := \sup_{x \in (\mathbb{C}^*)^n} \frac{\|Mx\|}{\|x\|} \ge \frac{\|My\|}{\|y\|}.$$

Thus, $\|cA^*\vec{1}\| \le \|cA^*\|_2\|\vec{1}\| = |c|\,\|A^*\|_2\|\vec{1}\|$. Since $\|A^*\|_2 = \|A\|_2$ for any $A \in M_n(\mathbb{C})$, (2.2) becomes

$$|c|^2\|\vec{1}\|^2 \le \langle cA^*\vec{1}, \vec{1}\rangle \le |c|\,\|A\|_2\|\vec{1}\|^2 = \sigma|c|\,\|\vec{1}\|^2. \tag{2.3}$$

By (2.3), if $c \neq 0$, then $\sigma \ge |c|$. Since σ is always a nonnegative real number, if $c = 0$, then $\sigma \ge 0 = |c|$. Hence, the statement (1) holds.

If $0 = |c| = \sigma := \|A\|_2$, then $A = O$ (the zero matrix) and thus the statement (2) is obviously true. Suppose that $0 \neq |c| = \sigma$. Then the inequality (2.3) and hence the Cauchy–Schwarz inequality become equalities. This happens when $cA^*\vec{1} = d'\vec{1}$ for some $d' \in \mathbb{C}$, that is, $A^T\vec{1} = d\vec{1}$, where $d := \bar{d}'/\bar{c} \in \mathbb{C}$, giving the constant column sums

$$\sum_{i=1}^{n} A_{ij} = d \quad \text{for } j = 1, \ldots, n.$$

By (2.3) with equality,

$$d'\|\vec{1}\|^2 = \langle d'\vec{1}, 1\rangle = \langle cA^*\vec{1}, \vec{1}\rangle = |c|\sigma\|\vec{1}\|^2,$$

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which means that $d' = |c|\sigma = |c|^2 = c\bar{c}$ and $d = \bar{d}'/\bar{c} = c$. Hence, (2) holds.

To prove (3), for the case $|c| = \sigma = \|A\|_2$, we write $c = \sigma e^{i\theta}$ for some $\theta \in \mathbb{R}$. Then

$$A\vec{1} = c\vec{1} = \sigma e^{i\theta}\vec{1},$$

so that $\sigma e^{i\theta}$ is an eigenvalue of A. Now,

$$\sigma = |\sigma e^{i\theta}| \le \rho(A) \le \|A\|_2 = \sigma$$

and hence $\rho(A) = \sigma$. For each $m \in \mathbb{N}$, $\rho(A^m) = (\rho(A))^m$ and $\|A^m\|_2 \le \|A\|_2^m$ because $\|\cdot\|_2$ is a sub-multiplicative norm. By using the fact that $\rho(M) \le \mu_{\mathbb{B}}(M) \le \|M\|_2$ for any block structure $\mathbb{B} \subseteq M_n(\mathbb{C})$ and matrix $M \in M_n(\mathbb{C})$,

$$\sigma^m = (\rho(A))^m = \rho(A^m) \le \mu_{\mathbb{B}}(A^m) \le \|A^m\|_2 \le \|A\|_2^m = \sigma^m$$

for each natural number m, which proves (3). □

The next result is a consequence of Proposition 2.1(3).

COROLLARY 2.2. *Let $D_1, D_2, \ldots, D_k$ be doubly stochastic matrices of size $n \times n$ and $\delta_1, \delta_2, \ldots, \delta_k$ be positive real numbers. Let*

$$S_k(D) := \delta_1 D_1 + \delta_2 D_2 + \cdots + \delta_k D_k.$$

Then, for each $m \in \mathbb{N}$ and any block structure $\mathbb{B} \subseteq M_n(\mathbb{C})$,

$$\mu_{\mathbb{B}}(S_k(D)^m) = (\delta_1 + \delta_2 + \cdots + \delta_k)^m.$$

PROOF. Note that $S_k(D)$ is a δ-generalised doubly stochastic matrix with constant row sum $\delta := \delta_1 + \delta_2 + \cdots + \delta_k$. Since $S_k(D)\vec{1} = \delta\vec{1}$, we have $\rho(S_k(D)) \ge \delta$. Also, $\|D_i\|_2 = 1$ for each $i = 1, \ldots, k$. By the triangle inequality for the spectral norm $\|\cdot\|_2$,

$$\delta \le \rho(S_k(D)) \le \|S_k(D)\|_2 = \|\delta_1 D_1 + \delta_2 D_2 + \cdots + \delta_k D_k\|_2 \le \sum_{i=1}^{k} \delta_i \|D_i\|_2 = \delta.$$

Hence, $\|S_k(D)\|_2 = \delta$ and thus the result follows from Proposition 2.1(3). □

It is still possible to calculate the structured singular value if the constant condition on the row sum in the proposition is slightly relaxed.

COROLLARY 2.3. *Let $A \in M_n(\mathbb{C})$ be a matrix such that $\|A\|_2 = \sigma$. Let $r_i := \sum_{k=1}^{n} A_{ik}$ be the ith row sum for each $i = 1, \ldots, n$ and $c_j := \sum_{k=1}^{n} A_{kj}$ be the jth column sum for each $j = 1, \ldots, n$. If $|r_1| = \cdots = |r_n| = \sigma$ or $|c_1| = \cdots = |c_n| = \sigma$, then*

$$\mu_{\mathbb{B}}(A^m) = \sigma^m$$

for each $m \in \mathbb{N}$ and any block structure $\mathbb{B} \subseteq M_n(\mathbb{C})$ containing, respectively,

$$\operatorname{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}) \quad \text{or} \quad \operatorname{diag}(e^{i\gamma_1}, e^{i\gamma_2}, \ldots, e^{i\gamma_n}),$$

where $r_j := \sigma e^{i\theta_j}$ and $c_j := \sigma e^{i\gamma_j}$ for each $j = 1, \ldots, n$.

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PROOF. Note that $\rho(M^*) = \rho(M) = \rho(M^T)$ and $\rho(MN) = \rho(NM)$ for $M, N \in M_n(\mathbb{C})$. Also, by (1.2), $\mu_{\mathbb{B}}(M) = \max_{U \in \mathcal{U}_{\mathbb{B}}} \rho(UM)$. So, for $U \in \mathcal{U}_{\mathbb{B}}$,

$$\mu_{\mathbb{B}}(M^T) = \mu_{\mathbb{B}}(M) = \mu_{\mathbb{B}}(UM) = \mu_{\mathbb{B}}(MU). \tag{2.4}$$

If $|r_1| = \cdots = |r_n| = \sigma$ and $r_j = \sigma e^{i\theta_j}$, where $\theta_j \in \mathbb{R}$ for each $j = 1, \ldots, n$, then $A = UA_0$, where $U = \operatorname{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n})$ and A_0 is a generalised row stochastic matrix with the constant row sum $\sigma \in \mathbb{C}$. By the condition on the perturbation block structure $\mathbb{B}$, we conclude that $A_0 = U^*A$ with $U^* \in \mathcal{U}_{\mathbb{B}}$. Hence, by (2.4), $\mu_{\mathbb{B}}(A_0) = \mu_{\mathbb{B}}(A)$ and the result follows from Proposition 2.1(3).

If $|c_1| = \cdots = |c_n| = \sigma$ and $c_j = \sigma e^{i\gamma_j}$, where $\gamma_j \in \mathbb{R}$ for each $j = 1, \ldots, n$, then $A^T = WB_0$, where $W = \operatorname{diag}(e^{i\gamma_1}, e^{i\gamma_2}, \ldots, e^{i\gamma_n})$ and B_0 is a generalised row stochastic matrix with the constant row sum $\sigma \in \mathbb{C}$. Then $B_0 = W^*A^T$ and $W^* \in \mathcal{U}_{\mathbb{B}}$, by the assumption of $\mathbb{B}$. Hence, by (2.4), $\mu_{\mathbb{B}}(B_0) = \mu_{\mathbb{B}}(A)$ and again the result follows from Proposition 2.1(3). □

Note that when $r = 0$ or $k_1 = k_2 = \cdots = k_r = 1$, the assumption on the perturbation block structure $\mathbb{B}$ in Corollary 2.3 will always be fulfilled. Furthermore, any matrix satisfying the assumptions of Corollary 2.3 must have the following property.

PROPOSITION 2.4. *Let $A \in M_n(\mathbb{C})$ be a matrix such that $\|A\|_2 = \sigma$. Let $r_i := \sum_{k=1}^n A_{ik}$ be the ith row sum for each $i = 1, \ldots, n$ and $c_j := \sum_{k=1}^n A_{kj}$ be the jth column sum for each $j = 1, \ldots, n$.*

(1) *The row sums satisfy $|r_1| = \cdots = |r_n| = \sigma$ if and only if $A = \sigma W_\theta D$, where $W_\theta := \operatorname{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n})$, $r_k := \sigma e^{i\theta_k}$, for each $k = 1, \ldots, n$, and D is a 1-generalised doubly stochastic matrix with $\|D\|_2 = 1$.*
(2) *The column sums satisfy $|c_1| = \cdots = |c_n| = \sigma$ if and only if $A = \sigma D W_\gamma$, where $W_\gamma := \operatorname{diag}(e^{i\gamma_1}, e^{i\gamma_2}, \ldots, e^{i\gamma_n})$, $c_k := \sigma e^{i\gamma_k}$, for each $k = 1, \ldots, n$, and D is a 1-generalised doubly stochastic matrix with $\|D\|_2 = 1$.*

PROOF. Suppose that $|r_1| = \cdots = |r_n| = \sigma$ and write $r_k = \sigma e^{i\theta_k}$ for each $k = 1, \ldots, n$. Then $A = \sigma W_\theta D$ such that D is a generalised row stochastic matrix with constant row sum 1. Since $\sigma = \|A\|_2 = \|\sigma W_\theta D\|_2 = \sigma \|D\|_2$, it follows that $\|D\|_2 = 1$. By Proposition 2.1(2), D must be a 1-generalised doubly stochastic matrix. The converse is obvious. Statement (2) is proved simply by using the fact that $\|A^T\|_2 = \|A\|_2 = \sigma$ and by applying statement (1) to A^T. □

By Proposition 2.4, any matrix of the form

$$\sigma \operatorname{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}) D,$$

where D is a doubly stochastic matrix and $\theta_1, \theta_2, \ldots, \theta_n \in \mathbb{R}$, $\delta \in \mathbb{R}^+$, will satisfy the assumption of Corollary 2.3. For example, the matrix

$$A := \begin{bmatrix} i/20 & 0 & i/20 \\ -i/20 & -i/20 & 0 \\ 0 & (1 + i\sqrt{3})/40 & (1 + i\sqrt{3})/40 \end{bmatrix}$$

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can be written in the form

$$A = \frac{1}{10}\operatorname{diag}(e^{i(\pi/2)}, e^{i(-\pi/2)}, e^{i(\pi/3)})\begin{bmatrix}1/2 & 0 & 1/2\\ 1/2 & 1/2 & 0\\ 0 & 1/2 & 1/2\end{bmatrix},$$

where the last matrix is a doubly stochastic matrix. By Corollary 2.3, $\mu_{\mathbb{B}}(A^m) = (0.1)^m$ for any $m \in \mathbb{N}$ and any block structure $\mathbb{B}$ containing $\operatorname{diag}(e^{i(\pi/2)}, e^{i(-\pi/2)}, e^{i(\pi/3)})$. We also observe that the largest (in absolute value) eigenvalue of A is $0.0488352 + 0.0447302i$, which means that $\rho(A) < \|A\|_2 = 0.1$. Thus, A is not a radial matrix and hence not normal.

Note further that if D is a 1-generalised doubly stochastic matrix with $\|D\|_2 = 1$, then $|D_{ij}| \le 1$ for each $i, j \in \{1, 2, \dots, n\}$. To see this, note that if $\|D\|_2 = 1$, then $\|Dx\| \le \|x\|$ for any nonzero vector $x \in \mathbb{C}^n$. So, if $|D_{ij}| > 1$, by choosing $x = e_i := [0, \dots, 1, \dots, 0]^T \in \mathbb{C}^n$ (where 1 appears only in the ith position),

$$\|Dx\|^2 = |D_{1j}|^2 + \cdots + |D_{ij}|^2 + \cdots + |D_{nj}|^2 \ge |D_{ij}|^2 > 1 = \|x\|^2,$$

which is a contradiction. If D is a 1-generalised doubly stochastic matrix, then $\|D\|_2$ need not be 1. For example, the checkerboard matrix

$$D := \begin{bmatrix}1 & -1 & 1 & -1 & 1\\ -1 & 1 & -1 & 1 & -1\\ 1 & -1 & 1 & -1 & 1\\ -1 & 1 & -1 & 1 & -1\\ 1 & -1 & 1 & -1 & 1\end{bmatrix}$$

is a 1-generalised doubly stochastic matrix with $\|D\|_2 = 5$. To see this, note first that $Dv = 5v$ with $v = [1, -1, 1, -1, 1]^T$, so that $\rho(D) \ge 5$. Now,

$$5 \le \rho(D) \le \|D\|_2 \le \|D\|_F := \sqrt{\textstyle\sum_{i=1}^{5}\sum_{j=1}^{5}|D_{ij}|^2} = 5.$$

However, there are still many 1-generalised doubly stochastic matrices having spectral norm 1 which are not doubly stochastic, as we can see from the following theorem.

THEOREM 2.5. *Let k be a positive integer. Let $a, \alpha_1, \alpha_2, \dots, \alpha_k$ be positive real numbers and b be a real number.*

(1) *If $n = 2k$ and $a \ge |b|$, then the circulant matrix $C^e(a, b, \alpha)$ having the row vector*

$$[a + bi,\ a - bi,\ \alpha_2(a + bi),\ \alpha_2(a - bi), \dots,\ \alpha_k(a + bi),\ \alpha_k(a - bi)]$$

as the first row will be a δ_e-generalised doubly stochastic matrix of size $n \times n$ with the spectral norm $\|C^e(a, b, \alpha)\|_2 = \delta_e$, where

$$\delta_e := 2a(1 + \alpha_2 + \cdots + \alpha_k).$$

(2) *If $n = 2k + 1$, then the circulant matrix $C^o(a, b, \alpha)$ having the row vector*

$$[a + bi,\ a - bi,\ \alpha_2(a + bi),\ \alpha_2(a - bi), \dots,\ \alpha_k(a + bi),\ \alpha_k(a - bi),\ \alpha_1]$$

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as the first row will be a δ_o-generalised doubly stochastic matrix of size $n \times n$ with the spectral norm $\|C^o(a,b,\alpha)\|_2 = \delta_o$, where

$$\delta_o := 2a(1+\alpha_2+\cdots+\alpha_k)+\alpha_1.$$

PROOF. We first consider the case $n = 2k$ and $a \geq |b|$. It is well known that all eigenvalues of the circulant matrix $C^e(a,b,\alpha)$ are

$$\lambda_j = z + \bar{z}\omega^j + \alpha_2 z\omega^{2j} + \alpha_2\bar{z}\omega^{3j} + \cdots + \alpha_k z\omega^{2k-2} + \alpha_k\bar{z}\omega^{2k-1}, \quad j = 0,1,\ldots,n-1,$$

where $z = a+bi$ and $\omega = e^{2\pi i/n}$. Then, for each $j \in \{0,1,\ldots,n-1\}$,

$$\begin{aligned}|\lambda_j| &= |(z+\bar{z}\omega^j) + \alpha_2(z+\bar{z}\omega^j)\omega^{2j} + \alpha_3(z+\bar{z}\omega^j)\omega^{4j} + \cdots + \alpha_k(z+\bar{z}\omega^j)\omega^{2(k-1)j}|\\ &= (|z+\bar{z}\omega^j|)(|1+\alpha_2\omega^{2j}+\alpha_3\omega^{4j}+\cdots+\alpha_k\omega^{2(k-1)j}|)\\ &\leq (|z+\bar{z}\omega^j|)(1+\alpha_2+\alpha_3+\cdots+\alpha_k).\end{aligned}$$

The triangle inequality becomes an equality when there are nonnegative real numbers x_t such that

$$\alpha_t\omega^{2tj} = x_t \cdot 1 = x_t \quad \text{for each } t = 1,\ldots,k-1.$$

Since $\alpha_t > 0$, x_t is a positive real number and thus $\omega^{2tj} = 1$ for each $t = 1,\ldots,k-1$. This implies that

$$\omega^{2j} + \omega^{4j} + \cdots + \omega^{2(k-1)j} = k-1$$

and that

$$\omega^{4j} + \omega^{6j} + \cdots + \omega^{2kj} = \omega^{2j}(k-1).$$

Subtracting the first equation from the second equation and using $\omega^{2kj} = \omega^{nj} = 1^j = 1$, we obtain $\omega^{2j} - 1 = (k-1)(1-\omega^{2j})$. If $\omega^{2j} \neq 1$, then $0 \leq k-1 = -1$, which is a contradiction. So, the maximum of the $|\lambda_j|$ can only be reached when $\omega^{2j} = 1$, that is, when $j = 0$ or $j = n/2$. If $j = n/2$, then $\omega^j = -1$ and thus

$$|\lambda_{n/2}| = (|z-\bar{z}|)(1+\alpha_2+\alpha_3+\cdots+\alpha_k) = 2|b|(1+\alpha_2+\alpha_3+\cdots+\alpha_k).$$

If $j = 0$, then $\omega^j = 1$ and thus

$$|\lambda_0| = (|z+\bar{z}|)(1+\alpha_2+\alpha_3+\cdots+\alpha_k) = 2a(1+\alpha_2+\alpha_3+\cdots+\alpha_k) := \delta_e.$$

Since any circulant matrix is always normal and $a \geq |b|$, we conclude that

$$\|C^e(a,b,\alpha)\|_2 = \max\{|\lambda_j| \mid j = 0,1,\ldots,n-1\} = |\lambda_0| = \delta_e.$$

For the case $n = 2k+1$, we compute, for each $j = 0,1,\ldots,n-1$,

$$\begin{aligned}|\lambda_j| &= |(z+\bar{z}\omega^j)(1+\alpha_2\omega^{2j}+\alpha_3\omega^{4j}+\cdots+\alpha_k\omega^{2(k-1)j}) + \alpha_1\omega^{2kj}|\\ &\leq |(z+\bar{z}\omega^j)(1+\alpha_2\omega^{2j}+\alpha_3\omega^{4j}+\cdots+\alpha_k\omega^{2(k-1)j})| + |\alpha_1\omega^{2kj}|\\ &\leq (|z+\bar{z}\omega^j|)(1+\alpha_2+\alpha_3+\cdots+\alpha_k)+\alpha_1.\end{aligned}$$

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By using the same process as the even case, the inequalities become equalities when $j = 0$ or $j = n/2$. Since n is odd, it follows that $j = 0$ and

$$\begin{aligned}\|C^o(a, b, \alpha)\|_2 &= \max\{|\lambda_j| \mid j = 0, 1, \ldots, n-1\} \\ &= |\lambda_0| = 2a(1 + \alpha_2 + \alpha_3 + \cdots + \alpha_k) + \alpha_1 := \delta_o.\end{aligned}$$

□

For example, if $n = 3$ and $a = 1/20, \alpha_1 = 9/10$ and $b = -\sqrt{3}/20$, then the matrix

$$C^o := \begin{bmatrix} (1 - i\sqrt{3})/20 & (1 + i\sqrt{3})/20 & 9/10 \\ 9/10 & (1 - i\sqrt{3})/20 & (1 + i\sqrt{3})/20 \\ (1 + i\sqrt{3})/20 & 9/10 & (1 - i\sqrt{3})/20 \end{bmatrix}$$

is 1-generalised doubly stochastic (but not of course doubly stochastic) having $1, 1, 0.7$ as its singular values. By Corollary 2.3, $\mu_{\mathbb{B}}((C^o)^m) = 1$ for any block structure $\mathbb{B} \subseteq M_3(\mathbb{C})$ and

$$\mu_{\mathbb{B}}((\sigma W_\theta C^o)^m) = \sigma^m$$

for any $m \in \mathbb{N}$, $\sigma \in \mathbb{R}^+$, $W_\theta := \operatorname{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \in M_3(\mathbb{C})$ and any block structure $\mathbb{B}$ containing W_θ.

If $n = 4$ and $a = 1, b = -1/2$ and $\alpha_2 = 1/3$ so that $a > |b|$, then

$$C^e := \begin{bmatrix} (2 - i)/2 & (2 + i)/2 & (2 - i)/6 & (2 + i)/6 \\ (2 + i)/6 & (2 - i)/2 & (2 + i)/2 & (2 - i)/6 \\ (2 - i)/6 & (2 + i)/6 & (2 - i)/2 & (2 + i)/2 \\ (2 + i)/2 & (2 - i)/6 & (2 + i)/6 & (2 - i)/2 \end{bmatrix}$$

is an $8/3$-generalised doubly stochastic matrix with $\|C^e\|_2 = 8/3$. The condition in statement (1) of Theorem 2.5 is essential. For example, if we choose $n = 4, a = 1, b = 2$ and $\alpha_2 = 3$, then $a < |b|$ and the matrix

$$E := \begin{bmatrix} 1 + 2i & 1 - 2i & 3 + 6i & 3 - 6i \\ 3 - 6i & 1 + 2i & 1 - 2i & 3 + 6i \\ 3 + 6i & 3 - 6i & 1 + 2i & 1 - 2i \\ 1 - 2i & 3 + 6i & 3 - 6i & 1 + 2i \end{bmatrix}$$

is an 8-generalised doubly stochastic matrix having $\|E\|_2 = 16$.

Note that any sum or product of circulant matrices is a circulant matrix. However, a sum of a circulant matrix and a doubly stochastic matrix need not be normal (and hence need not be circulant). For example, the matrix

$$S = C^e + \begin{bmatrix} 1/4 & 0 & 3/8 & 3/8 \\ 1/4 & 0 & 3/8 & 3/8 \\ 1/4 & 1/2 & 5/32 & 3/32 \\ 1/4 & 1/2 & 3/32 & 5/32 \end{bmatrix} = \begin{bmatrix} (5 - 2i)/4 & (2 + i)/2 & (17 - 4i)/24 & (17 + 4i)/24 \\ (7 + 2i)/12 & (2 - i)/2 & (11 + 4i)/8 & (17 - 4i)/24 \\ (7 - 2i)/12 & (5 + i)/6 & (37 - 16i)/32 & (35 + 16i)/32 \\ (5 + 2i)/4 & (5 - i)/6 & (41 + 16i)/96 & (37 - 16i)/32 \end{bmatrix}$$

is an $11/3$-generalised doubly stochastic matrix with $\|S\|_2 = 11/3$, but it is not normal, as we can see by a direct calculation that $S^*S \neq SS^*$.

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The conclusion of Corollary 2.3 can be extended to give the following result.

COROLLARY 2.6. *Let $\tilde{D}_1, \tilde{D}_2, \ldots, \tilde{D}_k$ be respectively δ_i-generalised row (or column) stochastic matrices in $M_n(\mathbb{C})$ such that $\|\tilde{D}_i\|_2 = \delta_i > 0$ for each $i = 1, 2, \ldots, k$. Then $S_k(\tilde{D})$ is a δ-generalised doubly stochastic matrix in $M_n(\mathbb{C})$ such that $\|S_k(\tilde{D})\|_2 = \delta$, where $\delta := \delta_1 + \delta_2 + \cdots + \delta_k$ and, for each $m \in \mathbb{N}$ and any block structure $\mathbb{B} \subseteq M_n(\mathbb{C})$,*

$$\mu_{\mathbb{B}}(S_k(\tilde{D})^m) = \delta^m.$$

PROOF. By Theorem 2.1(2), the D_i must be δ_i-generalised doubly stochastic matrices. The result follows by exactly the same arguments as in the proof of Corollary 2.3. □

Let $DC(n)$ be the set of all $n \times n$ doubly circulant matrices in Theorem 2.5 and $D(n)$ be the set of all doubly stochastic matrices. Let $\Omega_D^{Cir}(n)$ be the set of all $n \times n$ complex matrices of the form

$$\Omega_D^{Cir}(n) := \left\{ \sum_{i=1}^{s} d_i D_i + \sum_{j=1}^{t} \alpha_j C_j \;\middle|\; D_i \in D(n), C_j \in DC(n), d_i, \alpha_i \in \mathbb{R}_0^+, s, t \in \mathbb{N}_0 \right\}, \tag{2.5}$$

where $\mathbb{R}_0^+$ denotes the set of all nonnegative real numbers. Then, by Corollary 2.6, any matrix in $\Omega_D^{Cir}(n)$ satisfies the assumptions of Corollary 2.3 and hence the structured singular value, spectral norm and spectral radius are equal to the constant row sum or column sum, which are independent from the perturbation block structure. From the above discussion, matrices in this class are generalised doubly stochastic and have spectral norm equal to the constant row sum. They are radial but need not be normal.

To sum up, the structured singular values can be calculated explicitly for the class of matrices (which need not be radial nor normal) of the form

$$\Omega_n^m(\theta, \gamma, \delta) := \{\delta(W_\theta X W_\gamma)^m \mid X \in \Omega_D^{Cir}(n);, W_\theta, W_\gamma \in \mathrm{Diag}(n), m \in \mathbb{N}, \delta \in \mathbb{R}^+\},$$

where $\mathrm{Diag}(n) := \{\mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}) \mid \theta_1, \ldots, \theta_n \in \mathbb{R}\} \subseteq M_n(\mathbb{C})$; namely, by Corollary 2.3, if $X \in \Omega_D^{Cir}(n)$ has the constant row (column) sum r, then

$$\mu_{\mathbb{B}}(\delta(W_\theta X W_\gamma)^m) = \delta \cdot r^m$$

for any $W_\theta, W_\gamma \in \mathrm{Diag}(n)$, $\delta \in \mathbb{R}^+$, $m \in \mathbb{N}$ and any complex block structure $\mathbb{B} \subseteq M_n(\mathbb{C})$ containing W_θ and W_γ.

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