AN ANALYTICAL SOLUTION TO THE THREE-DIMENSIONAL PROBLEM ON ELASTIC EQUILIBRIUM OF AN EXPONENTIALLY-INHOMOGENEOUS LAYER

Y. Tokovyy *

Pidstryhach Institute for Applied Problems of Mechanics and Mathematics
National Academy of Sciences of Ukraine
Lviv, Ukraine

C.-C. Ma

Mechanical Engineering Department
National Taiwan University
Taipei, Taiwan

ABSTRACT

In this paper, we present an analytical solution to the general three-dimensional elasticity problem in a layer, whose Young's modulus varies exponentially within the thickness coordinate and the Poisson's ratio is constant. By making use of the direct integration method, the complete set of the governing equations in terms of stresses has been formed. The latter equations were reduced to separate equations for each stress-tensor component and then solved by means of the Fourier double-integral transformation with respect to the planar coordinates.

Keywords: Three-dimensional problem, Exponentially-inhomogeneous layer, Exact solution, Direct integration method.

1. INTRODUCTION

One of the dominant methods for analysis of inhomogeneous solids rests upon the representation of variable elastic modulii by elementary functions of certain kind with respect to one of the spatial coordinates. Among the variety of such functional representations, the case when the Young's or shearing modulii are given by the exponential function is very attractive due to the fact that it allows for the reduction of original equations with variable coefficients to either the equations with constant coefficients (in the case of the Cartesian coordinate system) or variable coefficients of typical form (in the case of curvilinear coordinate systems, *e.g.*, the cylindrical-polar or spherical ones) with well-known fundamental solutions. This fact makes it quite possible, in many cases, to construct the closed-form analytical solutions for the relevant problems of the elasticity theory. These solutions can be regarded as a perfect tool for the verification of either analytical or numerical methods, which are being developed for the analysis of inhomogeneous solids. Moreover, these solutions are convenient for practical implementation in the engineering and material-processing (functionally graded materials, for instance) design. Due to these reasons, the development of new analytical solutions to the boundary-value problems involving exponentially-inhomogeneous materials is a topical subject in the theory of elasticity.

The exponentially-inhomogeneous materials remain in the focus of the researches for a hundred years. The earlier papers (see, *e.g.*, [1,2]) were concerned with the seismology problems on the waves traveling in a semi-infinite medium with material properties represented by the exponential function of the depth-coordinate. A generalization of the Boussinesq's and Cerutti's solutions for the case of a semispace whose modulus of elasticity is the exponential function of depth has been presented by Holl [3]. The static and quasi-static problems of the elasticity theory, with concern to the geotechnical application, for the analogous solids subjected to local force-loading or punch-indentation, were considered in [4-11] in the two-dimensional (plane or axisymmetric) formulation. It is worth noting that some unfeasible results can be obtained when representing the material properties of infinite or semi-infinite solids by continuously increasing/decreasing functions (in particular, by the exponential one) of a coordinate, which tends to infinity, due to the fact that the material properties go beyond the modeling restrictions of the elasticity theory. With this concern, the material properties, which vary within a coordinate of infinite range, should be given either by bounded functions [12] or a combined representation by two

* Corresponding author (tokovyy@gmail.com)

DOI : 10.1017/jmech.2015.17
Copyright © 2015 The Society of Theoretical and Applied Mechanics, R.O.C.
https://doi.org/10.1017/jmech.2015.17 Published online by Cambridge University Press

constituents: The constant one and the one continuously decreasing when its argument travels to infinity [13]. For the case of bounded solids or when the inhomogeneity is considered within the bounded area, the mentioned problem does not arise.

A number of analytical solutions have been obtained for two-dimensional problems in inhomogeneous solids with exponentially-varying material properties. By making use of the stress-function method, a general solution to the plane thermal-stress problem in a functionally graded plate, whose material properties vary exponentially in the direction which is oblique to the boundary, has been obtained in [14,15]. In [16], the Green's function approach has been used for the analysis of an inhomogeneous medium, whose shearing modulus is presented by the exponential function of the planar coordinates. The constructed Green's function is presented by two constituents involving the modified Bessel functions and one-dimensional Fourier-type integrals. The similar approach has been used in [17] with regard to the heat-conduction problem in materials with exponentially-varying heat-conductivity. The solution to the plane magnetoelectroelasticity problem for a bimaterial plane consisting of two exponentially-graded halfplanes has been given in [18] by means of the Fourier-transform technique. In [19], the heat-conduction problem for the exponentially-graded bimaterial has been attempted in the same manner. Solution of the antiplane problem for an analogous domain has been suggested in [20]. More references concerning two-dimensional problems are given by Wang *et al*. [21, Table 1].

There are a great number of the papers dealing with two-dimensional problems for the bodies of revolution with exponentially-varying elastic modulii (the Young's or shearing ones) with respect to the thickness (radial) coordinate. A general solution to the Lamé equations for such problems has been suggested in [22]. The Green function approach has been used in [23] to study the torsion of an elastic cylinder with the shearing modulus varying exponentially with the radius. A problem on quasi-static thermal loading of a functionally graded thin-walled cylinder with a circumferential crack has been addressed in [24]. The cylinder wall is modeled as a layer resting upon an elastic foundation whose properties are exponential functions of the thickness coordinate. In [25], a solution to the one-dimensional problem for a thin-walled cylinder has been obtained in terms of the power-series expansion. The stress-function approach has been used in [26] to reduce the plane elasticity problem for an inhomogeneous hollow cylinder to a governing equation in terms of the Airy's function. As one of the specifications for the material properties, the case of exponentially-varying Young's modulus has been considered. The analysis of free and forced vibrations in functionally-graded hollow cylinders and spheres with the Young's modulus varying in accordance to the exponential law and the density varying in accordance to the power law, has been performed in [27] by making use of the Laplace-transform method. The transient thermal-stress analysis in an analogous formulation has been performed in [28]. The problem on twisting a sphere by a surface couple is addressed in [29] for the case, when the shearing modulus varies within the radial coordinate exponentially.

Development in the material-processing technologies with regard to the manufacturing and employment of multiphase composites and functionally-graded materials called for the analysis of their thermal and deformation resistance, fracture damage, *etc*. As one of the basic models for this analysis, the exponentially-graded materials were used [30,31]. In [32-35], the effect of the material inhomogeneity on asymptotic stress field in the vicinity of the interfacial (between the inhomogeneous coating with the shearing modulus varying exponentially within the thickness, and the homogeneous substrate) crack tips is discussed for the cases of plane and anti-plane deformation. The two-dimensional problem for bonded halfplanes with an interfacial crack has been addressed in [36]. The axisymmetric problem of a penny-shaped crack in a homogeneous substrate bonded through an interfacial region with exponentially-graded coating was attempted in [37]. In [38-41], the force- and thermal-stress intensity factors in the exponentially-graded materials with cracks have been investigated. The antiplane shear problem for an inhomogeneous medium, consisting of two semiplanes with the shearing modulii being exponentially variable with dissimilar exponential quantity, has been addressed in [42] with the spotlight fixed on the effect of non-smoothness of the specified modulii on the stress field around a crack located in the one of the semiplanes. The plane thermo-mechanical problem for a strip with thermally-insulated crack has been analyzed in [43] by means of the Fourier-transform technique. The Young's modulus along with the thermal-expansion and heat-conductivity coefficients were accepted in the exponential form, meanwhile the Poisson's ratio was presented by a product of the linear and exponential functions of the width-coordinate. Similar material properties were considered earlier in [44,45] for the case of mechanical loading only.

The elasticity and thermoelasticity problems for exponentially-inhomogeneous materials were also attempted by making use of the discrete layer approach, which is based on in the treatment of a continuously inhomogeneous solid as a multilayer composite with constant material properties within each layer. This approach has been employed in [46,47] for the analysis of elastic and thermoelastic performance of hollow cylinders with elastic modulii being the exponential functions of the radial coordinate. The discrete layer approach along with the variational principles have been employed in [48] for the analysis of a functionally graded (particularly, exponentially-graded) plate. Having involved the Ritz method, an approximate analytical-numerical solution has been achieved for different-type boundary conditions. A modification of the discrete layer approach, when the material properties of each layer are given by the exponential functions versus the thickness, has been developed in [49] for analysis of a crack problem in an arbitrarily inhomogeneous material.

The representation of the elasticity modulii by the exponential function has found a number of applications

https://doi.org/10.1017/jmech.2015.17 Published online by Cambridge University Press

in the elastic beams, plates, and shells theories. The problem for an exponentially-graded transversely-loaded beam had been discussed by Sankar [50] on the basis of the Euler-Bernoulli theory. Three-dimensional solutions for exponentially graded solids have been obtained on the basis on different plate theories those allowed for reduction of the dimensionality of the problems (usually, to the two-dimensional case) by assuming the dependence on the thickness coordinate in a specific form [51,52]. The Stroh formalism, for instance, has been employed for analysis of elastic behavior of laminated plates in [53]. The piezoelectric response of a simply-supported rectangular plate has been analyzed in [54] by making use of the state space approach. The piezoelectric behavior of exponentially-graded laminates were addressed in [55] for the case of a simply-supported rectangular plate by involving a formalism similar to the Stroh one.

There are several solutions for three-dimensional cases for specific types of loading. An exact solution to the Kelvin's problem on a point loading of an exponentially-graded halfspace has been reported in [56] with concern to the developed Green-function technique. Analogous problem has been attempted in [21] by representing the solution for the halfspace as a superposition of two solutions for full spaces.

Despite a great number of publications devoted to one- and two-dimensional solutions for exponentially-graded materials, there is lack, to the best of the authors' knowledge, of analytical solutions for the general three-dimensional case with no limiting assumptions on the dimensionality or the loading type. This can be explained by the significant complexity of three-dimensional problems even in the case of homogeneous material [57-59]. Moreover, the formulation of three-dimensional elasticity problems in terms of stresses faces serious theoretical drawback concerned with the number of independent compatibility equations [60].

In this paper we employ the direct integration method [60] for analysis of the three-dimensional problem of elasticity for a layer, whose Young's modulus varies within the thickness by the exponential law and the Poisson's ratio is constant. By following the strategy suggested in [61] for the case of homogeneous material, we reduce the solution of the original three-dimensional problem to a set of governing equations for the stress-tensor components. On the basis of direct integration of the equilibrium equations, the relationship between the stress-tensor components is established, which allowed us to uncouple the governing equations. The obtained equations are solved by making use of the Fourier double-integral transformation to provide closed-form analytical expressions for the stress-tensor components.

2. FORMULATION OF THE PROBLEM

Consider a problem on elastic equilibrium of an inhomogeneous layer $-\infty < x < \infty$, $-\infty < y < \infty$, $-1 \le z \le 1$, whose Young's modulus and Poisson's ratio have the form $E(z) = E_0 \exp(\lambda z)$ and $v = v_0 = \text{const}$, respectively. Here x, y, and z are the dimensionless Cartesian coordinates, $\lambda = \text{const}$, and E_0 is a constant in the dimension of stresses. For the considered material, the shearing modulus can be given as $G(z) = G_0 \exp(\lambda z)$, where $G_0 = E_0 /(2 + 2v_0)$. This three-dimensional elasticity problem is governed by the equilibrium equations [62]

$$\nabla \hat{\sigma} = 0 \tag{1}$$

and the Saint-Venant's compatibility equations

$$\frac{\partial^2 e_\alpha}{\partial \beta^2} + \frac{\partial^2 e_\beta}{\partial \alpha^2} = \frac{\partial^2 e_{\alpha\beta}}{\partial \alpha \partial \beta}, \quad 2\frac{\partial^2 e_\alpha}{\partial \beta \partial \gamma} = \frac{\partial}{\partial \alpha}\left(\frac{\partial e_{\alpha\beta}}{\partial \gamma} + \frac{\partial e_{\alpha\gamma}}{\partial \beta} - \frac{\partial e_{\beta\gamma}}{\partial \alpha}\right). \tag{2}$$

Here ∇ is the Del-operator, $\hat{\sigma}$ stands for the symmetric stress tensor for which σ_α and $\sigma_{\alpha\beta}$ are the normal and shearing components, respectively, e_α and $e_{\alpha\beta}$ denote the normal and shearing strains, $\{\alpha, \beta, \gamma\} = \{x, y, z\}$ and $\alpha \ne \beta \ne \gamma$. For the foregoing material properties, the physical relations have the following form:

$$\begin{aligned} E_0 \exp(\lambda z)\, e_\alpha &= \sigma_\alpha - \nu_0(\sigma_\beta + \sigma_\gamma), \\ E_0 \exp(\lambda z)\, e_{\alpha\beta} &= 2(1+\nu_0)\, \sigma_{\alpha\beta}, \end{aligned} \tag{3}$$

where $\{\alpha, \beta, \gamma\} = \{x, y, z\}$ and $\alpha \ne \beta \ne \gamma$.

Assume the limiting planes $z = \pm 1$ to be the exposed to external force loadings

$$\begin{aligned} &\sigma_z(x, y, 1) = -p_1(x, y), \ \sigma_z(x, y, -1) = -p_2(x, y), \\ &\sigma_{\alpha z}(x, y, 1) = q_{\alpha 1}(x, y), \ \sigma_{\alpha z}(x, y, -1) = q_{\alpha 2}(x, y), \end{aligned} \tag{4}$$

where p_j and $q_{\alpha j}$ are given functions vanishing at the points of infinity $(x, y \to \pm\infty)$, $j = 1,2$ and $\alpha = \{x, y\}$.

We intend to solve the problem (1) ~ (3) to determine the stress-tensor components due to the force loading (4). For this goal achievement, we employ the direct integration method [60,61], which consists in reduction of the foregoing boundary-value problem to a set of governing equations with both local and integral conditions.

3. SOLUTION CONSTRUCTION

By following the strategy of [60,61], we can express the shearing stresses in terms of the normal ones as follow:

$$\frac{\partial^2 \sigma_{\alpha\beta}}{\partial \alpha \partial \beta} = \frac{\partial^2 \sigma_\gamma}{\partial \gamma^2} - \frac{\partial^2 \sigma_\alpha}{\partial \alpha^2} - \frac{\partial^2 \sigma_\beta}{\partial \beta^2} \tag{5}$$

on the basis of equilibrium equations (1). Here $\alpha \ne \beta \ne \gamma$ and $\{\alpha, \beta, \gamma\} = \{x, y, z\}$. Note that equations of (5) do not contain the material parameters and, thus, they can be in use with no regard to the material kind. With these equations in mind, the compatibility equations (2) can be given in terms of stresses

https://doi.org/10.1017/jmech.2015.17 Published online by Cambridge University Press

$$\Delta\sigma_z = \frac{1}{1+\nu_0}\left(\frac{\partial^2\sigma}{\partial x^2}+\frac{\partial^2\sigma}{\partial y^2}\right),$$
$$\Delta\Big(\big(\nu_0\sigma-(1+\nu_0)\,\sigma_\alpha\big)\exp(-\lambda z)\Big) = \frac{\partial}{\partial\alpha}\left(\frac{\partial\sigma}{\partial\alpha}+2\lambda(1+\nu_0)\,\sigma_{\alpha z}\right)\exp(-\lambda z) \tag{6}$$

by making use of relations (3). Here $\alpha = \{x, y\}$, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ and

$$\sigma = \sigma_x + \sigma_y + \sigma_z\,. \tag{7}$$

From Eqs. (6) and (7), the equation

$$\Delta\big(\exp(-\lambda z)\,\sigma\big) = \frac{1+\nu_0}{1-\nu_0}\lambda^2\exp(-\lambda z)\,\sigma_z \tag{8}$$

follows.

By making use of equilibrium equations (1), boundary conditions (4) for the shearing stresses can be transformed into the boundary conditions

$$\left.\frac{\partial\sigma_z}{\partial z}\right|_{z=1} = -\frac{\partial q_{x1}}{\partial x}-\frac{\partial q_{y1}}{\partial y}, \quad \left.\frac{\partial\sigma_z}{\partial z}\right|_{z=-1} = -\frac{\partial q_{x2}}{\partial x}-\frac{\partial q_{y2}}{\partial y} \tag{9}$$

for the normal derivatives of the stress σ_z.

By correct integration of the Cauchy's equations in the same manner as it has been done in [61], we can derive the following integral compatibility conditions for the strain-tensor components:

$$\int_{-\infty}^{x}\left(e_{xz}(\xi, y, \pm1)-\int_{-\infty}^{\xi}\frac{\partial e_x(\xi_1, y, \pm1)}{\partial z}d\xi_1\right)d\xi = \int_{-\infty}^{y}\left(e_{yz}(x, \eta, \pm1)-\int_{-\infty}^{\eta}\frac{\partial e_y(x, \eta_1, \pm1)}{\partial z}d\eta_1\right)d\eta\,. \tag{10}$$

Note that conditions (10) are equivalent to each other due to the strains compatibility [61]:

$$\int_{-\infty}^{\infty}e_x\,dx = \int_{-\infty}^{\infty}e_y\,dy = 0\,, \quad \int_{-1}^{1}e_z\,dz = u_z(1)-u_z(-1)\,,$$

where u_z is the elastic displacement in z-direction. The superposition of conditions (10) yields the compatibility condition (2.6) in [61]. On the other hand, conditions (10) appeared to be more convenient for the practical implementation. Finally, conditions (10) are formulated in terms of strains and they are irrespective of the material properties in this form. However, the representation of these conditions in terms of stresses involves the material parameters due to the compulsory use of relations (3).

To separate the variables in Eqs. (6) ~ (8) and conditions (4), (9), and (10), we employ the classic Fourier double-integral transformation [63]

$$\overline{f}(z) \equiv \overline{f}(s_x, s_y; z) = \int_{-\infty}^{\infty}f(x, y, z)\exp\big(-i(s_x x+s_y y)\big)\,dxdy\,, \tag{11}$$

where s_x and s_y are the parameters of transformations with respect to x and y, respectively, $i^2 = -1$, and f is an arbitrary function for which the transformation (11) exists [64]. As a result, we obtain the following set of equations

$$\overline{\Delta}\overline{\sigma}_z = -\frac{k^2}{1+\nu_0}\overline{\sigma}\,, \quad \overline{\Delta}\big(\exp(-\lambda z)\overline{\sigma}\big) = \frac{1+\nu_0}{1-\nu_0}\lambda^2\exp(-\lambda z)\overline{\sigma}_z\,,$$
$$\overline{\sigma} = \overline{\sigma}_x + \overline{\sigma}_y + \overline{\sigma}_z\,,$$
$$\overline{\Delta}\Big(\big((1+\nu_0)\overline{\sigma}_\alpha-\nu_0\overline{\sigma}\big)\exp(-\lambda z)\Big) = s_\alpha\big(s_\alpha\overline{\sigma}-2i\lambda(1+\nu_0)\overline{\sigma}_{\alpha z}\big)\exp(-\lambda z)\,, \quad \alpha = \{x, y\} \tag{12}$$

along with the conditions

$$\overline{\sigma}_z(1) = -\overline{p}_1\,, \quad \overline{\sigma}_z(-1) = -\overline{p}_2\,,$$
$$\overline{\sigma}_{\alpha z}(1) = \overline{q}_{\alpha1}\,, \quad \overline{\sigma}_{\alpha z}(-1) = \overline{q}_{\alpha2}\,, \quad \alpha = \{x, y\}\,,$$
$$\frac{d\overline{\sigma}_z(1)}{dz} = -is_x\overline{q}_{x1}-is_y\overline{q}_{y1}\,, \quad \frac{d\overline{\sigma}_z(-1)}{dz} = -is_x\overline{q}_{x2}-is_y\overline{q}_{y2} \tag{13}$$

and

$$\frac{d}{dz}\big(s_y^2\overline{e}_x-s_x^2\overline{e}_y\big) = is_xs_y\big(s_y\overline{e}_{xz}-s_x\overline{e}_{yz}\big) \text{ at } z=-1\,. \tag{14}$$

Here $\overline{\Delta} = d^2/dz^2-k^2$, $k^2 = s_x^2+s_y^2$.

By elimination of the total stress from the first and second equations of (12), we arrive at the equation

$$\overline{\Delta}_1(\overline{\Delta}\overline{\sigma}_z)+\frac{k^2\lambda^2}{1-\nu_0}\overline{\sigma}_z = 0\,,$$

whose solution can be given in the form

$$\overline{\sigma}_z = \sum_{l=1}^{4}a_l\exp(\vartheta_l\,z)\,. \tag{15}$$

Here $\overline{\Delta}_1 = d^2/dz^2-2\lambda(d/dz)+\lambda^2-k^2$ and parameters ϑ_l are found in the form

$$\vartheta_{1,3} = \frac{1}{2}\left(\lambda+\sqrt{\lambda^2+4k^2\pm4i\lambda k\sqrt{\nu_0/(1-\nu_0)}}\right),$$
$$\vartheta_{2,4} = \frac{1}{2}\left(\lambda-\sqrt{\lambda^2+4k^2\pm4i\lambda k\sqrt{\nu_0/(1-\nu_0)}}\right)$$

as the roots of a characteristic equation

$$\vartheta^4-2\lambda\vartheta^3+(\lambda^2-2k^2)\,\vartheta^2+2\lambda k^2\vartheta+k^2\left(k^2+\frac{\lambda^2\nu_0}{1-\nu_0}\right) = 0\,.$$

The constants of integration a_j can be determined by putting (15) into the boundary conditions (13) for the stress $\overline{\sigma}_z$. The expressions for these constants are given in Appendix A.

Having determined the axial stress in the form (15), we can use the first and third equations of (12) to determine the total stress as

https://doi.org/10.1017/jmech.2015.17 Published online by Cambridge University Press

$$\bar{\sigma} = \frac{1+\nu_0}{k^2}\sum_{l=1}^{4} a_l (k^2 - \vartheta_l^2)\exp(\vartheta_l z)\,. \tag{16}$$

From the second and fourth equations of (12) at $\alpha = y$, we can derive the following equation

$$\bar{\Delta}_1(\bar{\sigma}_y) = \frac{\nu_0\lambda^2}{1-\nu_0}\bar{\sigma}_z + \frac{s_y^2}{1+\nu_0}\bar{\sigma} - 2\lambda i s_y \bar{\sigma}_{yz}\,. \tag{17}$$

On the basis of the Eqs. (5), (11), and the third equation of (12), we can obtain

$$2is_y\frac{d\bar{\sigma}_{yz}}{dz} = k^2\bar{\sigma}_y - s_x^2(\bar{\sigma} - \bar{\sigma}_z) - \frac{d^2\bar{\sigma}_z}{dz^2}\,.$$

Putting expressions (15) and (16) into the latter equation yields

$$\bar{\sigma}_y = \frac{2is_y}{k^2}\frac{d\bar{\sigma}_{yz}}{dz} + \frac{1}{k^2}\sum_{l=1}^{4} a_l\left(\nu_0 s_x^2 + \frac{\vartheta_l^2}{k^2}\left(s_y^2 - \nu_0 s_x^2\right)\right)\exp(\vartheta_l z)\,. \tag{18}$$

With Eq. (18) in view, Eq. (17) can be given as

$$\bar{\Delta}_1\left(\frac{d\bar{\sigma}_{yz}}{dz}\right) + \lambda k^2\bar{\sigma}_{yz} = \sum_{l=1}^{4} a_l\,\alpha_l\exp(\vartheta_l z)\,, \tag{19}$$

where

$$\alpha_l = \frac{1}{2is_y}\left(\frac{\nu_0\lambda^2k^2}{1-\nu_0} + (k^2 - \vartheta_l^2)s_y^2 - \left(\nu_0 s_x^2 + \frac{\vartheta_l^2}{k^2}(s_y^2 - \nu_0 s_x^2)\right)\left(\vartheta_l^2 - 2\lambda\vartheta_l + \lambda^2 - k^2\right)\right).$$

The solution of Eq. (19) can be found as

$$\bar{\sigma}_{yz} = \sum_{m=1}^{3} b_m\exp(\mu_m z) + \sum_{l=1}^{4} a_l\beta_l\exp(\vartheta_l z)\,, \tag{20}$$

where $\beta_l = \alpha_l / \left(\vartheta_l^3 - 2\lambda\vartheta_l^2 + (\lambda^2 - k^2)\,\vartheta_l + \lambda k^2\right)$, μ_m are the roots of the characteristic equation $\mu^3 - 2\lambda\mu^2 + (\lambda^2 - k^2)\,\mu + \lambda k^2 = 0$ having the form $\mu_1 = \lambda$, $\mu_{2,3} = (\lambda \pm \sqrt{\lambda^2 + 4k^2})/2$, and b_m are arbitrary constants of integration. To determine the latter constants, we can use two boundary conditions (13) for the shearing stress $\bar{\sigma}_{yz}$ and condition (14) along with physical relations (3) and expressions (15) and (16). The expressions for b_m are given in Appendix B. Note that in contrast to ϑ_l, the characteristic parameters μ_m are irrespective of the Poisson's ratio ν_0.

Having found the shearing stress $\bar{\sigma}_{yz}$ in the form (20), we can use Eq. (18) to find

$$\bar{\sigma}_y = \frac{2is_y}{k^2}\sum_{m=1}^{3}\mu_m b_m\exp(\mu_m z) + \frac{1}{k^2}\sum_{l=1}^{4} a_l\left(\nu_0 s_x^2 + (s_y^2 - \nu_0 s_x^2)\,\frac{\vartheta_l^2}{k^2} + 2is_y\vartheta_l\beta_l\right)\exp(\vartheta_l z)\,. \tag{21}$$

With formulae (15), (16), (21) and the third Eq. (12) in view, we can determine the remaining normal stress as follows:

$$\bar{\sigma}_x = -\frac{2is_y}{k^2}\sum_{m=1}^{3}\mu_m b_m\exp(\mu_m z) + \frac{1}{k^2}\sum_{l=1}^{4} a_l\left(\nu_0 s_y^2 - \vartheta_l^2 - (1+\nu_0)\,s_y^2\frac{\vartheta_l^2}{k^2} - 2is_y\vartheta_l\beta_l\right)\exp(\vartheta_l z)\,. \tag{22}$$

Now the stresses $\bar{\sigma}_{xz}$ and $\bar{\sigma}_{xy}$ can be found from equilibrium equations (1) in the following form:

$$\bar{\sigma}_{xz} = -\frac{s_y}{s_x}\sum_{m=1}^{3} b_m\exp(\mu_m z) + \frac{1}{s_x}\sum_{l=1}^{4} a_l(i\vartheta_l - s_y\beta_l)\exp(\vartheta_l z)\,, \tag{23}$$

$$\bar{\sigma}_{xy} = i\frac{s_x^2 - s_y^2}{s_x k^2}\sum_{m=1}^{3} b_m\,\mu_m\exp(\mu_m z) + \frac{i}{s_x k^2}\sum_{l=1}^{4} a_l\left((s_x^2 - s_y^2)\vartheta_l\beta_l + is_y\left(\nu_0 s_x^2 + (s_y^2 - \nu_0 s_x^2)\right)\frac{\vartheta_l^2}{k^2}\right)\exp(\vartheta_l z)\,. \tag{24}$$

As the stress-tensor components are determined by Eqs. (15), (20) ~ (24) in the Fourier-image space, we can employ the inverse transformation

$$f(x, y, z) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\bar{f}(z)\exp\left(i(s_x x + s_y y)\right)ds_x ds_y \tag{25}$$

to determine the original stresses satisfying Eqs. (1) ~ (4).

4. NUMERICAL EXAMPLE AND DISCUSSION

Consider a numerical implementation of the constructed solution for computation of the stresses in an exponentially-inhomogeneous layer with $\lambda = 0$, 1, 2, see Fig. 1, and $\nu_0 = 0.3$ due to the symmetrical normal loading (4), where

$$p_1 = p_2 = p\exp(-c_x x^2 - c_y y^2)\,,$$
$$q_{\alpha j} = 0\,,\quad \alpha = \{x, y\}\,,\quad j = 1, 2\,, \tag{26}$$

and p is an arbitrary constant in the dimension of stresses. For the below-presented results, we imposed $c_x = 3$ and $c_y = 5$. The tractions p_1 and p_2 here are given by continuous functions, which vanish with x, $y \to \pm\infty$ (Fig. 2).

https://doi.org/10.1017/jmech.2015.17 Published online by Cambridge University Press

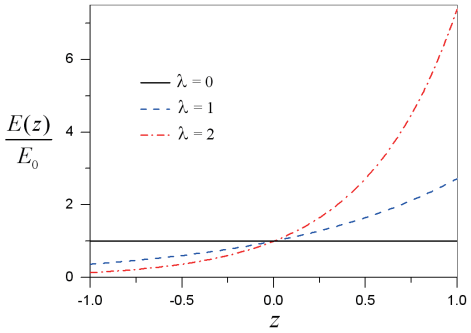


Fig. 1 Distributions of the dimensionless Young's modulus versus the thickness for $\lambda = 0, 1, 2$.

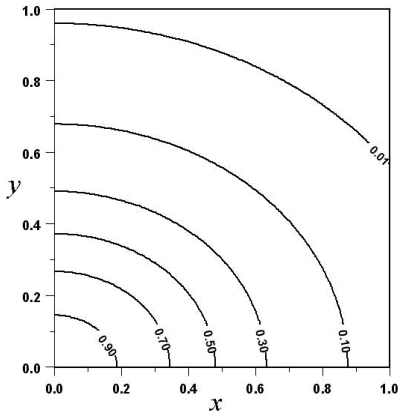


Fig. 2 Visualization of the loading factors p_j / p, $j = 1, 2$, given by (26).

Note that if $\lambda = 0$, then the material is regarded as the homogeneous one (solid line in Fig. 1) with the solution given in [61]. In this case ($\lambda = 0$) for the loading (26), the normal stress-tensor components are even functions of all coordinates, the stress σ_{yz} is even function by x and odd by y and z, the stress σ_{xz} is even function by y and odd by x and z, and the stress σ_{xy} is even function by z and odd by x and y. Thus, the stress $\sigma_{\alpha\beta}$ is zero at the cross-sections $\alpha = 0$ and $\beta = 0$, where $\alpha, \beta = \{x, y, z\}$. Distributions of these stress-tensor components are shown in Fig. 3 at some cross-sections of the layer. As we can observe, the normal stresses are maximum in the vicinity of the loading zones of the boundary planes $z = \pm 1$; they decrease when moving away from the boundaries in the z-direction and vanish when $x, y \to \pm\infty$. The stress σ_z (Fig. 3(a)) is compressive within the affected area, meanwhile the stresses σ_y (Fig. 3(b)) and σ_x (Fig. 3(c)) turn form compressive to tensile within the inner area, which is equidistant from the poles of tractions p_1 and p_2. The maximum value of the tension for the latter stresses is lesser, however, than the maximum value of compression by order of ten. The shearing stresses σ_{yz} (Fig. 3(d)) and σ_{xz} (Fig. 3(e)) have similar distributions versus z in the sections $x = C$ and $y = C$, respectively (C = const). The marginal quantitative difference is caused by the fact that $c_x \neq c_y$ in (26). The stress σ_{xy} (Fig. 3(f)) is even function of the thickness and is equal to zero at $x = 0$ and $y = 0$. Its maximum values are much lower than ones for the foregoing shearing stress-tensor components.

Distribution of the stress σ_z within the thickness coordinate on the central line $x = y = 0$, is depicted in Fig. 4 for different values of the parameter of inhomogeneity λ. Due to dependence of the Young's modulus on the thickness coordinate, the distributions of this stress for $\lambda \neq 0$ are not symmetric with respect to the midplane $z = 0$ of the layer. As the Young's modulus increases from the side $z = -1$ to $z = 1$ (see the dashed and dashed-doted lines in Fig. 1), the minimum (by the magnitude) values of compression move in the same direction and slightly decrease. In the case of $\lambda = 1$, the minimum of this stress on the line $x = y = 0$ is located at $z = 0.09$ with the relative difference to the homogeneous case ($\lambda = 0$) within 2%; for $\lambda = 2$, the minimum is located at $z = 0.17$ with the difference of 7.5%.

Figure 5 demonstrates the effect of inhomogeneity in the stresses σ_y and σ_x. These stresses are shown versus the thickness coordinate at $x = y = 0$ for different values of λ. As we can observe, the variation of the Young's modulus with z causes the significant difference between the values of these stresses at the opposite points of the limiting planes $z = \pm 1$: The increment of this modulus effects in the greater values of the stresses σ_y and σ_x at $z = 1$ and lesser values at $z = -1$. At $z = 1$ and $x = y = 0$, the value of σ_y, for instance, for $\lambda = 1$ is about 29% greater by its magnitude than one for the case $\lambda = 0$. For $\lambda = 2$, the analogous difference is 77%. At $z = -1$, the difference between the values of these stresses for different values of the parameter of inhomogeneity is not that significant: It falls (for the stress σ_y, again) within 16% and 25% for $\lambda = 1$ and $\lambda = 2$, respectively. The values of σ_x on the boundary $z = 1$ are slightly lower that ones for σ_y, meanwhile the qualitative behavior across the layer is the same.

The shearing stress σ_{yz} is shown in Fig. 6 at $x = 0$ and $y = \pm 0.5$ (note that this stress is zero at $y = 0$). Again, the dependence of the Young's modulus on the thickness coordinate effects in the symmetry breakdown with respect to the midplane $z = 0$. It is worth noting that the values of maximum magnitude of this stress, which are located near to the side $z = 1$, travel to the boundary $z = 1$ with the increment of λ (the direction of the Young's modulus increment), meanwhile the maximums located near to the side $z = -1$ move to the side $z = -1$ (the direction, which is opposite to the Young's modulus increment). For greater values of λ, the maximum values of this stress increase (see Fig. 7).

It is worth noting that the expressions (15), (20) ~ (24) for the stress-tensor components vanish by $s_x, s_y \to \pm\infty$ so that the numerical implementation of the inversion formula (25) was performed by the sufficiently finite intervals for both mentioned variables ($|s_\alpha| < K_\alpha$ = const, $\alpha = \{x, y\}$). As a criterion for determination of K_α, we use the accuracy in satisfaction of the original boundary conditions (4) and (26) by expressions (15) and (20) after applying to them formula (25) with the infinity symbols substituted by K_α. For instance, if $K_x = K_y = 15$, the error in satisfaction of the boundary conditions falls within 0.01%. This fact makes the obtained solution to be quite convenient for the numerical implementations.

https://doi.org/10.1017/jmech.2015.17 Published online by Cambridge University Press

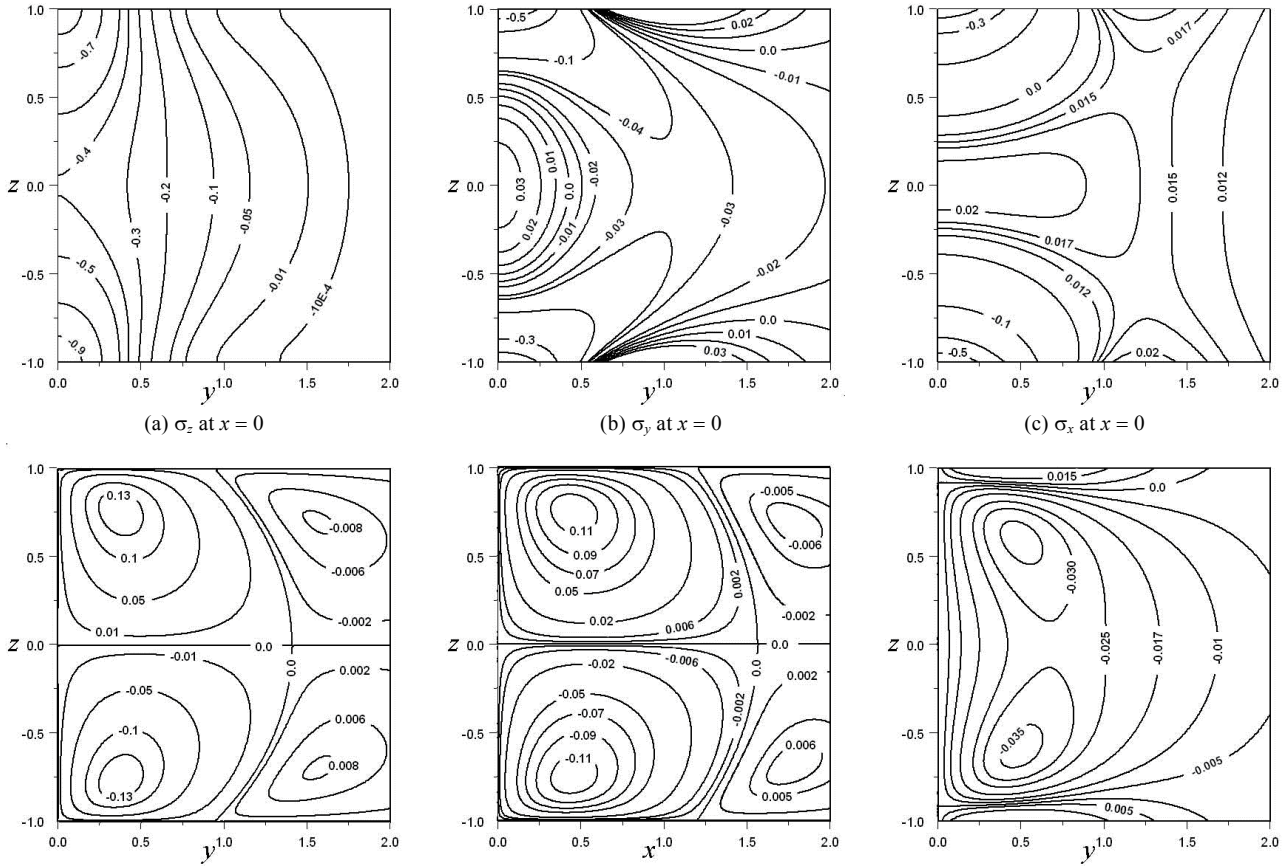


Fig. 3 Distributions of the stresses (normalized by p) in a homogeneous layer ($\lambda = 0$) due to the normal force loading (26).

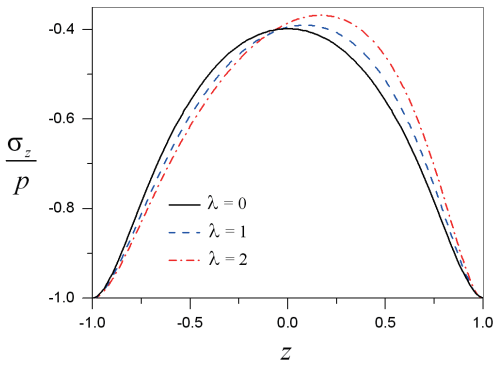


Fig. 4 Distribution of the stress σ_z / p versus the thickness coordinate at $x = y = 0$ in the layer for different cases of inhomogeneity.

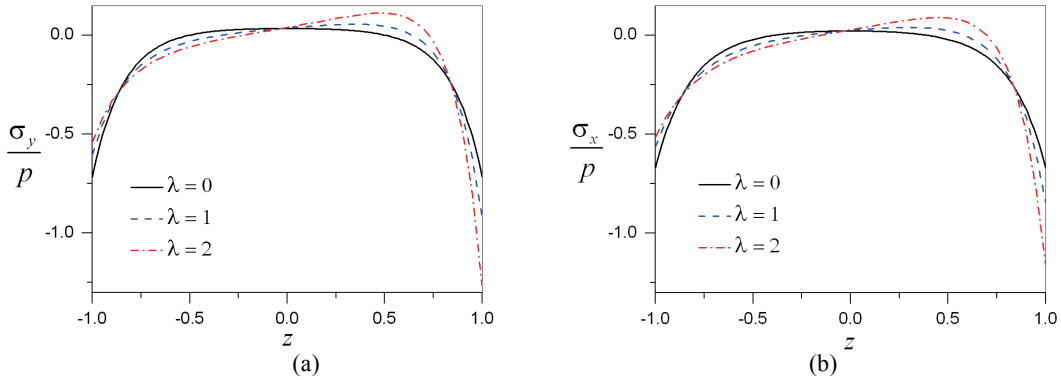


Fig. 5 Distribution of the stresses (a) σ_y / p and (b) σ_x / p versus the thickness coordinate at $x = y = 0$ in the layer for different cases of inhomogeneity.

https://doi.org/10.1017/jmech.2015.17 Published online by Cambridge University Press

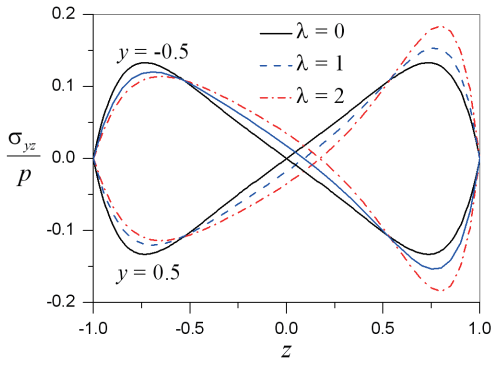


Fig. 6 Distribution of the shearing stress σ_{yz}/p versus the thickness coordinate at $x = 0$, $y = \pm 0.5$ in the layer for $\lambda = 0, 1, 2$.

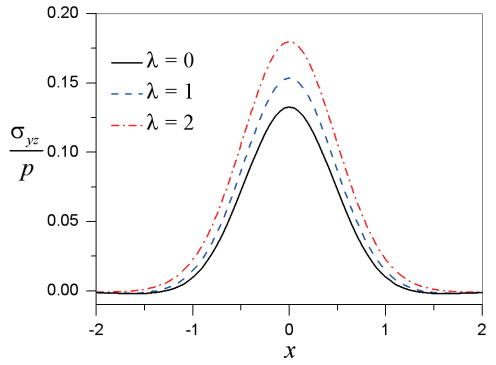


Fig. 7 Distribution of the shearing stress σ_{yz}/p versus x at $y = 0.5$, $z = 0.75$ in the layer for $\lambda = 0$, 1, 2.

5. CONCLUSIONS

By making use of the direct integration method, we constructed an analytical solution to the general three-dimensional problem of the elasticity theory for a layer, whose Young's modulus varies within the thickness coordinate in accordance to the exponential law and the Poisson's ratio is constant. The layer is exposed to the external force loading on its limiting planes. The problem is reduced to solution of the governing equations, which were obtained on the basis of the compatibility equations in terms of strains. By making use of the Fourier double-integral transformation, the governing equations were solved to provide explicit analytical expressions for the stress-tensor components. The latter feature of our solution makes it to be potentially attractive for verification of the newly developed analytical or numerical methods for analysis of three-dimensional problems of elasticity for inhomogeneous solids.

APPENDIX A

The constants of integration a_l $(l = \overline{1,4})$ appearing in Eq. (15) have the following form

$$a_1 = \tilde{a}_1(\vartheta_2, \vartheta_3, \vartheta_4)/\tilde{a},$$
$$a_2 = \tilde{a}_2(\vartheta_1, \vartheta_3, \vartheta_4)/\tilde{a},$$
$$a_3 = \tilde{a}_3(\vartheta_1, \vartheta_2, \vartheta_4)/\tilde{a},$$
$$a_4 = \tilde{a}_4(\vartheta_1, \vartheta_2, \vartheta_3)/\tilde{a},$$

where

$$\tilde{a} = 2(\vartheta_1 - \vartheta_3)(\vartheta_2 - \vartheta_4)\cosh(\vartheta_1 - \vartheta_2 + \vartheta_3 - \vartheta_4) + 2(\vartheta_2 - \vartheta_1)(\vartheta_3 - \vartheta_4)\cosh(\vartheta_1 + \vartheta_2 - \vartheta_3 - \vartheta_4) + 2(\vartheta_1 - \vartheta_4)(\vartheta_3 - \vartheta_2)\cosh(\vartheta_1 - \vartheta_2 - \vartheta_3 + \vartheta_4),$$

$$\begin{aligned}\tilde{a}_l(t_1,t_2,t_3) = (-1)^l &\Big(\big[t_2(t_3-t_1)\exp(t_2-t_1-t_3)+t_1(t_2-t_3)\exp(t_1-t_2-t_3)\\ &+t_3(t_1-t_2)\exp(t_3-t_1-t_2)\big]\bar{p}_1\\ &+\big[t_2(t_3-t_1)\exp(t_1+t_3-t_2)+t_1(t_2-t_3)\exp(t_2+t_3-t_1)\\ &+t_3(t_1-t_2)\exp(t_1+t_2-t_3)\big]\bar{p}_2\\ &+\big[(t_3-t_2)\exp(t_1-t_2-t_3)+(t_2-t_1)\exp(t_3-t_1-t_2)\\ &+(t_1-t_3)\exp(t_2-t_1-t_3)\big](s_x\bar{q}_{x1}+s_y\bar{q}_{y1})\,i\\ &+\big[(t_3-t_2)\exp(t_2+t_3-t_1)+(t_2-t_1)\exp(t_1+t_2-t_3)\\ &+(t_1-t_3)\exp(t_1+t_3-t_2)\big](s_x\bar{q}_{x2}+s_y\bar{q}_{y2})\,i\Big).\end{aligned}$$

APPENDIX B

The constants of integration b_m $(m = 1, 2, 3)$ appearing in Eq. (20) have the form

$$b_1 = \tilde{b}_1(\mu_2, \mu_3)/\tilde{b},$$
$$b_2 = \tilde{b}_2(\mu_1, \mu_3)/\tilde{b},$$
$$b_3 = \tilde{b}_3(\mu_1, \mu_2)/\tilde{b},$$

where

$$\tilde{b} = \big(\mu_3(\mu_3-\lambda)-\mu_2(\mu_2-\lambda)\big)\exp(\mu_1-\mu_2-\mu_3) + \big(\mu_2(\mu_2-\lambda)-\mu_1(\mu_1-\lambda)\big)\exp(-\mu_1-\mu_2+\mu_3) + \big(\mu_1(\mu_1-\lambda)-\mu_3(\mu_3-\lambda)\big)\exp(-\mu_1+\mu_2-\mu_3),$$

$$\begin{aligned}\tilde{b}_m(t_1,t_2) &= (-1)^{m+1}\Big(\chi\sinh(t_1-t_2)+\big(t_2(t_2-\lambda)-t_1(t_1-\lambda)\big)\\ &\cdot\exp(-t_1-t_2)\Big(\bar{q}_{y1}-\sum_{l=1}^{4}a_l\beta_l\exp(\vartheta_1)\Big)\Big)+\big(t_1(t_1-\lambda)\exp(t_2-t_1)\\ &-t_2(t_2-\lambda)\exp(t_1-t_2)\big)\Big(\bar{q}_{y2}-\sum_{l=1}^{4}a_l\beta_l\exp(-\vartheta_1)\Big)\Big),\end{aligned}$$

$$\chi = \lambda i s_y\bar{p}_2 + s_x(s_x\bar{q}_{y2}-s_y\bar{q}_{x2}) + k^2\bar{q}_{y2} - \sum_{l=1}^{4}a_l(\vartheta_l-\lambda)\left(is_y\left(1-2\frac{\vartheta_l^2}{k^2}\right)+2\vartheta_l\beta_l\right)\exp(-\vartheta_l).$$

https://doi.org/10.1017/jmech.2015.17 Published online by Cambridge University Press

ACKNOWLEDGMENTS

The authors gratefully acknowledge the financial support of this research by the National Science Council (Republic of China) under Grant NSC 102-2811-E-002-018.

REFFERENCES

1. Aichi, K., "On the Transversal Seismic Waves Travelling Upon the Surface of Heterogeneous Material," *Proceedings of the Physico-Mathematical Society of Japan*, 3rd Series, **4**, pp. 137–142 (1922).
2. Lock, M. H., "Axially Symmetric Elastic Waves in an Unbounded Inhomogeneous Medium with Exponentially Varying Properties," *Bulletin of the Seismological Society of America*, **53**, pp. 527–538 (1963).
3. Holl, D. L., "Stress Transmission in Earths," *Proceedings of the Twentieth Annual Meeting*, Highway Research Board, Washington D.C., **20**, pp. 709–721 (1940).
4. Gibson, R. E., "Some Results Concerning Displacements and Stresses in a Non-Homogeneous Elastic Half-Space," *Géotechnique*, **17**, pp. 58–67 (1967).
5. Popov, G. I., "On a Method of Solution of the Axisymmetric Contact Problem of the Theory of Elasticity," *Journal of Applied Mathematics and Mechanics*, **25**, pp. 105–118 (1961).
6. Teodorescu, P. P. and Predeleanu, M., "Quelques Considérations Sure le Problème des Corps Élastiques Hétérogènes," Olszak, W. Ed., *Non- Homogeneity in Elasticity and Plasticity*, Olszak, W. Ed., Pergamon Press, pp. 31–38 (1959).
7. Rostovtsev, N. A., "On the Theory of Elasticity of a Nonhomoqeneous Medium," *Journal of Applied Mathematics and Mechanics*, **28**, pp. 745–757 (1964).
8. Mossakovskii, V. I., "Pressure of a Circular Die [Punch] on an Elastic Half-Space, Whose Modulus of Elasticity is an Exponential Function of Depth," *Journal of Applied Mathematics and Mechanics*, **22**, pp. 168–171 (1958).
9. Giannakopoulos, A. E. and Suresh, S., "Indentation of Solids with Gradients in Elastic Properties. I. Point Force," *International Journal of Solids and Structures*, **34**, pp. 2357–2392 (1997).
10. Selvadurai, A. P. S. and Katebi, A., "Mindlin's Problem for an Incompressible Elastic Half-Space with an Exponential Variation in the Linear Elastic Shear Modulus," *International Journal of Engineering Science*, **65**, pp. 9–21 (2013).
11. Katebi, A. and Selvadurai, A. P. S., "Undrained Behaviour of a Non-Homogeneous Elastic Medium: The Influence of Variations in the Elastic Shear Modulus with Depth," *Géotechnique*, **63**, pp. 1159–1169 (2013).
12. Plevako, V. P., "Equilibrium of a Nonhomogeneous Half-Plane under the Action of Forces Applied to the Boundary," *Journal of Applied Mathematics and Mechanics*, **37**, pp. 858–866 (1973).
13. Tokovyy, Y. and Ma, C.-C., "Analytical Solutions to the 2D Elasticity and Thermoelasticity Problems for Inhomogeneous Planes and Half-Planes," *Archive of Applied Mechanics*, **79**, pp. 441–456 (2009).
14. Ohmichi, M. and Noda, N., "Plane Thermoelastic Problem in a Functionally Graded Plate with an Oblique Boundary to the Functional Graded Direction," *Journal of Thermal Stresses*, **30**, pp. 779–799 (2007).
15. Teodorescu, P. P. and Predeleanu, M., "Über das Ebene Problem Nichthomogener Elastischer Körper," *Acta Technica Academiae Scientiarum Hungaricae*, **27**, pp. 349–369 (1959).
16. Chan, Y.-S., Gray, L. J., Kaplan, T. and Paulino, G. H., "Green's Function for a Two-Dimensional Exponentially Graded Elastic Medium," *Proceedings of the Royal Society of London A*, **460**, pp. 1689–1706 (2004).
17. Gray, L. J., Kaplan, T., Richardson, J. D. and Paulino, G. H., "Green's Functions and Boundary Integral Analysis for Exponentially Graded Materials: Heat Conduction," *Journal of Applied Mechanics, Transactions of ASME*, **70**, pp. 543–549 (2003).
18. Ma, C. C. and Lee, J. M., "Theoretical Analysis of In-Plane Problem in Functionally Graded Nonhomogeneous Magnetoelectroelastic Bimaterials," *International Journal of Solids and Structures*, **46**, pp. 4208–4220 (2009).
19. Ma, C. C. and Chen, Y. T., "Theoretical Analysis of Heat Conduction Problems of Nonhomogeneous Functionally Graded Materials for a Layer Sandwiched Between Two Half-Planes," *Acta Mechanica*, **221**, pp. 223–237 (2011).
20. Lee, J. M. and Ma, C. C., "Analytical Solutions for an Antiplane Problem of Two Dissimilar Functionally Graded Magnetoelectroelastic Half-Planes," *Acta Mechanica*, **212**, pp. 21–38 (2010).
21. Wang, C. D., Tzeng, C. S., Pan, E. and Liao, J. J., "Displacements and Stresses due to a Vertical Point Load in an Inhomogeneous Transversely Isotropic Half-Space," *International Journal of Rock Mechanics & Mining Sciences*, **40**, pp. 667–685 (2003).
22. Ter-Mkrtichian, L. N., "Some Problems in the Theory of Elasticity of Nonhomogeneous Elastic Media," *Journal of Applied Mathematics and Mechanics*, **25**, pp. 1667–1675 (1961).
23. George, O. D., "Torsion of an Elastic Solid Cylinder With a Radial Variation in the Shear Modulus," *Journal of Elasticity*, **6**, pp. 229–244 (1976).
24. Dağ, S., Dadıoğlu, S. and Selçuk Yakşi, O., "Circumferential Crack Problem for an FGM Cylinder under Thermal Stresses," *Journal of Thermal Stresses*, **22**, pp. 659–687 (1999).
25. Tutuncu, N., "Stresses in Thick-Walled FGM Cylin-

https://doi.org/10.1017/jmech.2015.17 Published online by Cambridge University Press

ders with Exponentially-Varying Properties," *Engineering Structures*, **29**, pp. 2032–2035 (2007).
26. Nie, G. J. and Batra, R. C., "Material Tailoring and Analysis of Functionally Graded Isotropic and Incompressible Linear Elastic Hollow Cylinders," *Composite Structures*, **92**, pp. 265–274 (2010).
27. Keles, I. and Tutuncu, N., "Exact Analysis of Axisymmetric Dynamic Response of Functionally Graded Cylinders (or Disks) and Spheres," *Journal of Applied Mechanics, Transactions of ASME*, **78**, pp. 061014-1-7 (2011).
28. Shao, Z. S., Wang, T. J. and Ang, K. K., "Transient Thermo-Mechanical Analysis of Functionally Graded Hollow Circular Cylinders," *Journal of Thermal Stresses*, **30**, pp. 81–104 (2007).
29. Chakravorty, J. G., "Twisting of a Non-Homogeneous Sphere," *Pure and Applied Geophysics*, **98**, pp. 26–28 (1972).
30. Suresh, S., "Graded Materials for Resistance to Contact Deformation and Damage," *Science*, **292**, pp. 2447–2451 (2001).
31. Jin, Z.-H. and Batra, R. C., "Some Basic Fracture Mechanics Concepts in Functionally Graded Materials," *Journal of Mechanics and Physics of Solids*, **44**, pp. 1221–1235 (1996).
32. Erdogan, F., "Fracture Mechanics of Functionally Graded Materials," *Composites Engineering*, **5**, pp. 753–770 (1995).
33. Erdogan, F., "The Crack Problem for Bonded Nonhomogeneous Materials under Antiplane Shear Loading," *Journal of Applied Mechanics, Transactions of ASME*, **52**, pp. 823–828 (1985).
34. Chen, Y. F. and Erdogan, F., "The Interface Crack Problem for a Nonhomogeneous Coating Bonded to a Homogeneous Substrate," *Journal Mechanical Physics and Solids*, **44**, pp. 771–87 (1996).
35. Jin, Z.-H. and Batra, R. C., "Interface Cracking Between Functionally Graded Coatings and a Substrate under Antiplane Shear," *International Journal of Engineering Science*, **34**, pp. 1705–1716 (1996).
36. Delale, F. and Erdogan, F., "On the Mechanical Modeling of the Interfacial Region in Bonded Half Planes," *Journal of Applied Mechanics, Transactions of ASME*, **55**, pp. 317–324 (1988).
37. Ozturk, M. and Erdogan, F., "Axisymmetric Crack Problem in Bonded Materials with a Graded Interfacial Region," *International Journal of Solids and Structures*, **33**, pp. 193–219 (1996).
38. Jin, Z.-H. and Noda, N., "Transient Thermal Stress Intensity Factors for a Crack in a Semi-Infinite Plate of a Functionally Gradient Material," *International Journal of Solids and Structures*, **31**, pp. 203–218 (1994).
39. Jin, Z. H. and Noda, N., "Edge Crack in a Nonhomogeneous Half Plane under Thermal Loading," *Journal of Thermal Stresses*, **17**, pp. 591–599 (1994).
40. Noda, N. and Jin, Z. H., "Thermal Stress Intensity Factors for a Crack in a Strip of a Functionally Gradient Material," *International Journal of Solids and Structures*, **30**, pp. 1039–1056 (1993).
41. Gu, P. and Asaro, J., "Crack in Functionally Graded Materials," *International Journal of Solids and Structures*, **34**, pp. 1–17 (1997).
42. Erdogan, F., "The Crack Problem for Bonded Nonhomogeneous Materials Under Antiplane Shear Loading," *Journal of Applied Mechanics, Transactions of ASME*, **52**, pp. 823–828 (1985).
43. Noda, N. and Jin, Z. H., "Thermal Stress Intensity Factors for a Crack in a Strip of a Functionally Gradient Material," *International Journal of Solids and Structures*, **30**, pp. 1039–1056 (1993).
44. Delale, F. and Erdogan, F., "On the Mechanical Modeling of the Interfacial Region in Bonded Half-Planes," *Journal of Applied Mechanics, Transactions of ASME*, **55**, pp. 317–324 (1988).
45. Gu, P. and Asaro, R. J., "Cracks in Functionally Graded Materials," *International Journal of Solids and Structures*, **34**, pp. 1–17 (1997).
46. Zhang, X. and Hasebe, N., "Elasticity Solution for a Radially Nonhomogeneous Hollow Circular cylinder," *Journal of Applied Mechanics, Transactions of ASME*, **66**, pp. 598–606 (1999).
47. Liew, K. M., Kitipornchai, S., Zhang, X. Z. and Lim, C. W., "Analysis of the Thermal Stress Behaviour of Functionally Graded Hollow Circular Cylinders," *International Journal of Solids and Structures*, **40**, pp. 2355–2380 (2003).
48. Ramirez, R., Heyliger, P. R. and Pan, E., "Static Analysis of Functionally Graded Elastic Anisotropic Plates Using a Discrete Layer Approach," *Composites Part B: Engineering*, **37**, pp. 10–20 (2006).
49. Guo, L.-C. and Noda, N., "Modeling Method for a Crack Problem of Functionally Graded Materials with Arbitrary Properties — Piecewise-Exponential Model," *International Journal of Solids and Structures*, **44**, pp. 6768–6790 (2007).
50. Sankar, B. V., "An Elasticity Solution for Functionally Graded Beams," *Composites Science and Technology*, **61**, pp. 689–696 (2001).
51. Abade, S., "Functionally Graded Plates Behave Like Homogeneous Plates," *Composites Part B: Engineering*, **39**, pp. 151–158 (2008).
52. Kashtalyan, M., "Three-Dimensional Elasticity Solution for Bending of Functionally Graded Rectangular Plates," *European Journal of Mechanics A/Solids*, **23**, pp. 853–864 (2004).
53. Pan, E., "Exact Solution for Functionally Graded Anisotropic Elastic Composite Laminates," *Journal of Composite Materials*, **37**, pp. 1903–1920 (2003).
54. Zhong, Z. and Shang, E. T., "Three-Dimensional Exact Analysis of a Simply Supported Functionally Gradient Piezoelectric Plate," *International Journal of Solids and Structures*, **40**, pp. 5335–5352 (2003).

https://doi.org/10.1017/jmech.2015.17 Published online by Cambridge University Press

55. Lu, P., Lee, H. P. and Lu, C., "Exact Solutions for Simply Supported Functionally Graded Piezoelectric Laminates by Stroh-like Formalism," *Composite Structures*, **72**, pp. 352–363 (2006).
56. Martin, P. A., Richardson, J. D., Gray, L. J. and Berger, J. R., "On Green's Function for a Three-Dimensional Exponentially Graded Elastic Solid," *Proceedings of the Royal Society of London A*, **458**, pp. 1931–1947 (2002).
57. Lur'e, A. I., *Three-Dimensional Problems of the Theory of Elasticity*, Interscience Publisher, New York, p. 493 (1964).
58. Borodachev, N. M., "Three-Dimensional Elasticity-Theory Problem in Terms of the Stress," *International Applied Mechanics*, **31**, pp. 991–996 (1995).
59. Teodorescu, P. P., *Treatise on Classical Elasticity. Theory and Related Problems*, Springer, Dordrecht, p. 802 (2013).
60. Tokovyy, Y. V., "Direct Integration Method," in R. B. Hetnarski, *Encyclopedia of Thermal Stresses*, Springer, New Jersey, **2**, pp. 951–960 (2014).
61. Vigak, V. M. and Rychagivskii, A. V., "Solution of a Three-Dimensional Elastic Problem for a Layer," *International Applied Mechanics*, **38**, pp. 1094–1102 (2002).
62. Sadd, M. H., *Elasticity: Theory, Applications, and Numerics*, Elsevier Academic Press, Oxford, p. 533 (2009).
63. Brigham, E. O., *The Fast Fourier Transform and Its Applications*, Prentice-Hall Inc., New Jersey, p. 448 (1988).
64. Brychkov, Y. A. and Prudnikov, A. P., *Integral Transforms of Generalized Functions*, Gordon and Breach Science Publishers, New York, p. 343 (1989).

(Manuscript received September 23, 2014, accepted for publication October 20, 2014.)

https://doi.org/10.1017/jmech.2015.17 Published online by Cambridge University Press