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NEWTON'S METHOD FOR STOCHASTIC FUNCTIONAL EVOLUTION EQUATIONS IN HILBERT SPACES

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Abstract. We apply Newton's method to stochastic functional evolution equations in Hilbert spaces using semigroup methods. The first-order convergence is based on our generalization of the Gronwall-type inequality. We also establish a second-order convergence in a probabilistic sense.

§1. *Introduction.* Stochastic evolution equations (SEE) in Hilbert spaces are generalizations of Itô stochastic equations [21] to the infinite dimensional case. The theory has been greatly motivated by various branches of science (e.g. physics, chemistry, biology). For example, stochastic partial differential equations of the evolutionary type are used to describe wave propagation (in random media [14] and turbulence [9]) or the dynamics of populations [13]. They also appear in the study of nonlinear filtering of diffusion processes [15], stochastic flows [25] and control theory [26].

Numerous papers present different approaches to the theory of SEE including the variational approach [30, 34], the martingale (measure) approach [32], the white noise approach [12, 20], the method based on the monotonicity of the drift [29] and finally the semigroup approach [10, 11], which we use in the paper.

The evolution models are considered in different spaces, a consequence of the choice of approach. Semigroup methods are usually applied in Hilbert spaces since real-valued stochastic integration extends directly to processes with values therein. These results by Da Prato and Zabczyk [10] have been generalized to the case of a certain class of Banach spaces. Van Neerven and Veraar reformulate the integration theory for Hilbert spaces and develop the tools and techniques to study SEE in UMD spaces [28] (which for example cover L^p-spaces for $p \in (1, \infty)$) or in martingale type 2 spaces [6] that include L^p-spaces for $p \in [2, \infty)$.

The existence and uniqueness of mild solutions to stochastic functional evolution equations are considered in [31] by using an analytic semigroup approach and fractional power operator arguments. In [22] Jahanipur discusses the problem under assumptions of demicontinuous drift satisfying a semimonotone condition instead of Lipschitz one. In [16, 17] Govindan studies the existence of mild solutions by the comparison theorem. See references therein for more details.

Numerical approximation of solutions to SEE has been discussed by several authors. In [18] Gyöngy applies explicit and implicit numerical schemes

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to SEE with monotone operators in Banach spaces. In [19] Hausenblas considers the Galerkin approximations for space discretization, the Euler and the Crank–Nicholson schemes for time discretization. The work of Bessaih *et al* [4] addresses the numerical approximation of shell models of turbulence. Weak convergence of finite element approximations can be found in [24]. Finite-element-based space-time discretizations of the incompressible Navier–Stokes equations are studied in [7]. The splitting-up method is investigated in [3, 5, 8, 27].

Newton's methods for stochastic differential equations are studied by Kawabata and Yamada in [23] and Amano in [1]. In [2] Amano develops techniques to prove a probabilistic second-order convergence of Newton's methods. In [33] we derive further non-trivial generalizations to the case of stochastic functional differential equations with Hale functionals. The aim of the present paper is to apply Newton's method to stochastic functional evolution equations in a semigroup setting. Consider the stochastic semilinear delay heat equation

$$\frac{\partial}{\partial t}u(t,x) - \frac{\partial^2}{\partial x^2}u(t,x) = \int_{-r_1}^{0}\int_0^1 f(x,y,u(t+s,y))\,dy\,ds + \int_{-r_2}^{0}\int_0^1 b(x,y,u(t+s,y))\,dy\,ds\,\dot{W}(t) \quad (1.1)$$

with the boundary conditions $u(t,0) = u(t,1) = 0$, $t \in [0,T]$, and the initial condition $u(t,x) = \varphi(t,x)$ for $t \in [-r,0]$, $x \in [0,1]$, $r_1, r_2 \geqslant 0$, where

$$\varphi(t,\cdot) \in L^2([0,1]) \quad \text{for } t \in [-r,0], r \geqslant r_1, r_2;$$
$$\varphi(\cdot,x) \in C([-r,0],\mathbb{R}) \quad \text{for } x \in [0,1]$$

and $W(t)$, $t \in [0,T]$ is the real standard Brownian motion. Let $H = L^2([0,1])$ and $A = \partial^2/\partial x^2$ be the linear operator of second-order differentation with respect to the spatial variable with the domain

$$\mathcal{D}(A) = \{u \in L^2([0,1]) : u', u'' \in L^2([0,1]) \text{ and } u(0) = u(1) = 0\}.$$

Then $A : \mathcal{D}(A) \subseteq H \to H$ generates a strongly continuous semigroup $S(t)$. If we define $F, B : C([-r,0],H) \to H$ by

$$F(\psi)(x) = \int_{-r_1}^{0}\int_0^1 f(x,y,\psi(s)(y))\,dy\,ds,$$
$$B(\psi)(x) = \int_{-r_2}^{0}\int_0^1 b(x,y,\psi(s)(y))\,dy\,ds,$$

then the initial-boundary value problem (1.1) can be written as an abstract evolution equation

$$\begin{cases} du(t) = (Au(t) + F(u_t))\,dt + B(u_t)\,dW(t), \\ u_0 = \varphi. \end{cases}$$

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The paper is organized as follows. In §2 we recall some definitions and introduce basic notations. We establish a first-order convergence (§4) and a probabilistic second-order convergence (§5) of Newton's method. The result in §4 is based on the Gronwall-type inequality presented in §3.

§2. *Formulation of the problem.* Fix $T > 0$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and let $(\mathcal{F}_t)_{t\in[0,T]}$ be a normal filtration. Let $(H, \langle\cdot,\cdot\rangle)$ and $(U, \langle\cdot,\cdot\rangle_U)$ be separable Hilbert spaces with the norms denoted by $\|\cdot\|$, $\|\cdot\|_U$ respectively and let $L(U, H)$ be the Banach space of all linear, bounded operators from U into H with the norm

$$\|T\|_{L(U,H)} = \sup_{\|u\|_U\leqslant 1} \|Tu\|, \quad u \in U, T \in L(U, H).$$

Define $L(H) := L(H, H)$. Let Q be an U-valued linear, bounded and symmetric non-negative operator such that $\operatorname{Tr} Q < \infty$. For $U_1 \supset U$ an U_1-valued stochastic process $W(t)$, $t \in [0, T]$ is called a Q-Wiener process on $[0, T]$ if:

$W(0) = 0$;

W has continuous trajectories;

W has independent increments, and;

$W(t) - W(s)$ has normal distribution with mean 0 and variance $(t - s)Q$,

for $0 \leqslant s \leqslant t \leqslant T$.

Let $U_0 = Q^{1/2}U$. By $HS(U_0, H)$ we denote the space of all Hilbert–Schmidt operators from U_0 to H. We recall that a linear and bounded operator $T : U_0 \to H$ is said to be Hilbert–Schmidt if

$$\sum_{k=1}^{\infty} \|Te_k\|^2 < \infty,$$

where $\{e_k\} \in U_0$ is a complete orthonormal base. The space $HS(U_0, H)$ equipped with the norm

$$\|T\|_{HS} = \left(\sum_{k=1}^{\infty} \|Te_k\|^2\right)^{1/2}$$

is a separable Hilbert space with the scalar product

$$\langle S, T\rangle_{HS} = \sum_{k=1}^{\infty} \langle Se_k, Te_k\rangle, \quad S, T \in HS(U_0, H).$$

Fix $0 \leqslant r < \infty$. By $\mathcal{C}_r := C([-r, 0], H)$ denote the Banach space of all continuous functions $v : [-r, 0] \to H$ with the norm

$$\|v\|_{\mathcal{C}_r} = \sup_{-r\leqslant s\leqslant 0} \|v(s)\|.$$

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Set $\mathcal{F}_t := \mathcal{F}_0$ for $t \in [-r, 0]$. For $p > 2$, $\mathcal{X}_{p,T}$ is the Banach space of all continuous $\mathcal{F}_t$-adapted processes $X : [-r, T] \times \Omega \to H$ such that

$$\|X\|_{p,t} = \left(\mathbb{E}\left[\sup_{-r \leqslant s \leqslant t} \|X(s)\|^p\right]\right)^{1/p} < \infty.$$

For a process $X \in \mathcal{X}_{p,T}$ and $t \in [0, T]$, Hale-type operator $X_t : [-r, 0] \times \Omega \to H$ is defined by

$$X_t(\tau) = X(t + \tau) \quad \text{for } \tau \in [-r, 0].$$

We consider on H the following stochastic functional evolution equation:

$$\begin{cases} dX(t) = (AX(t) + F(t, X_t))\, dt + B(t, X_t)\, dW(t), & t \in [0, T] \\ X(t) = \varphi(t), & t \in [-r, 0], \end{cases} \tag{2.1}$$

where:

(i) A is theinfinitesimal generator of a strongly continuous semigroup

$$S(t) = \mathrm{e}^{tA}, t \geqslant 0 \text{ in } H \text{ and } N = \sup_{t \in [0,T]} \|S(t)\|_{L(H)} \text{ for some } N > 0; \tag{2.2}$$

(ii) $F : [0, T] \times \Omega \times \mathcal{C}_r \to H$ is a measurable mapping;

(iii) $B : [0, T] \times \Omega \times \mathcal{C}_r \to HS(U_0, H)$ is a measurable mapping;

(iv) $W(t)$ is a Q-Wiener process on $U_1 \supset U$; and

(v) $\varphi \in \mathcal{X}_{p,0}$.

Since $\mathcal{F}_t = \mathcal{F}_0$ for $t \in [-r, 0)$ the process φ is deterministic and thus independent of the Q-Wiener process on $[0, T]$. We assume that the mappings F and B satisfy the Lipschitz condition with respect to the functional variable:

$$\|F(t, x) - F(t, y)\| \leqslant L\|x - y\|_{\mathcal{C}_r}, \tag{2.3}$$

$$\|B(t, x) - B(t, y)\|_{HS} \leqslant L\|x - y\|_{\mathcal{C}_r} \tag{2.4}$$

for some non-negative constant L, $t \in [0, T]$ and $x, y \in \mathcal{C}_r$. A stochastic process $X \in \mathcal{X}_{p,T}$ is said to be a mild solution of (2.1) if

$$\mathbb{P}\left(\int_{-r}^{T} \|X(s)\|^p\, ds < \infty\right) = 1$$

and for arbitrary $t \in [0, T]$, we have

$$\begin{aligned} X(t) = S(t)\varphi(0) &+ \int_0^t S(t - s)F(s, X_s)\, ds \\ &+ \int_0^t S(t - s)B(s, X_s)\, dW(s), \quad \mathbb{P}\text{-a.s.} \end{aligned} \tag{2.5}$$

with $X(t) = \varphi(t)$ for $t \in [-r, 0]$.

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§3. *Gronwall-type inequality.* By $L(\mathcal{C}_r, H)$ and $L(\mathcal{C}_r, HS)$ we denote the space of all linear and bounded operators from $\mathcal{C}_r$ to H and $\mathcal{C}_r$ to $HS(U_0, H)$ respectively with the norms

$$\|T_1\|_{L(\mathcal{C}_r,H)} := \sup_{\|v\|_{\mathcal{C}_r}\leqslant 1} \|T_1 v\|,$$
$$\|T_2\|_{L(\mathcal{C}_r,HS)} := \sup_{\|v\|_{\mathcal{C}_r}\leqslant 1} \|T_2 v\|_{HS}$$

for $v \in \mathcal{C}_r$, $T_1 \in L(\mathcal{C}_r, H)$ and $T_2 \in L(\mathcal{C}_r, HS)$. The following lemma plays the key role in the proof of the convergence theorem in §4.

LEMMA 3.1. *Suppose that $\alpha_1(t) : [0, T] \to H$, $\alpha_2(t) : [0, T] \to HS(U_0, H)$ are continuous, $T_1 \in L(\mathcal{C}_r, H)$, $T_2 \in L(\mathcal{C}_r, HS)$ and there exists a non-negative constant L such that*

$$\|T_1(t)\|_{L(\mathcal{C}_r,H)} \leqslant L, \qquad \|T_2(t)\|_{L(\mathcal{C}_r,HS)} \leqslant L \tag{3.1}$$

for $t \in [0, T]$. Then for a process $X \in \mathcal{X}_{p,T}$ satisfying the stochastic functional evolution equation

$$\begin{cases} dX(t) = (AX(t) + \alpha_1(t) + T_1(t)X_t)\,dt + (\alpha_2(t) + T_2(t)X_t)\,dW(t), \\ \qquad\qquad t \in [0, T] \\ X(t) = 0, \quad t \in [-r, 0] \end{cases} \tag{3.2}$$

we have

$$\|X\|^p_{p,t} \leqslant 4^{p-1}C_3 e^{4^{p-1}C_3L^p(N^p+1)t} \int_0^t (N^p\|\alpha_1\|^p_{p,s} + \mathbb{E}[\|\alpha_2(s)\|^p_{HS}])\,ds$$

for $t \in [0, T]$.

Proof. By the fact that

$$(|x| + |y|)^p \leqslant 2^{p-1}(|x|^p + |y|^p) \quad \text{for } p > 1 \tag{3.3}$$

we have

$$\begin{aligned} \|X\|^p_{p,t} &= \mathbb{E}\left[\sup_{\tilde{t}\leqslant t} \|X_{\tilde{t}}\|^p\right] \\ &\leqslant 2^{p-1}\mathbb{E}\left[\sup_{\tilde{t}\leqslant t}\left\|\int_0^{\tilde{t}} S(\tilde{t}-s)[\alpha_1(s) + T_1(s)X_s]\,ds\right\|^p\right] \\ &\quad + 2^{p-1}\mathbb{E}\left[\sup_{\tilde{t}\leqslant t}\left\|\int_0^{\tilde{t}} S(\tilde{t}-s)[\alpha_2(s) + T_2(s)X_s]\,dW_s\right\|^p\right] \\ &= 2^{p-1}(I_1 + I_2). \end{aligned}$$

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By (2.2) and the Hölder inequality for some $C_1 > 0$ dependent on T:

$$I_1 \leqslant N^p C_1 \mathbb{E}\left[\sup_{\tilde{t} \leqslant t} \int_0^{\tilde{t}} \|\alpha_1(s) + T_1(s)X_s\|^p \, ds\right]$$
$$\leqslant N^p C_1 \mathbb{E}\left[\int_0^t \|\alpha_1(s) + T_1(s)X_s\|^p \, ds\right].$$

By (3.3) and (3.1) we have

$$I_1 \leqslant 2^{p-1} N^p C_1 \mathbb{E}\left[\int_0^t \|\alpha_1(s)\|^p + \|T_1(s)X_s\|^p \, ds\right]$$
$$\leqslant 2^{p-1} N^p C_1 \int_0^t (\|\alpha_1\|_{p,s}^p + L^p \|X\|_{p,s}^p) \, ds.$$

By [**10**, Proposition 7.3] there exists $C_2 > 0$ dependent on T such that

$$I_2 \leqslant C_2 \mathbb{E}\left[\int_0^t \|\alpha_2(s) + T_2(s)X_s\|_{HS}^p \, ds\right].$$

By (3.3) and (3.1) we have

$$I_2 \leqslant 2^{p-1} C_2 \int_0^t (\mathbb{E}[\|\alpha_2(s)\|_{HS}^p] + L^p \|X\|_{p,s}^p) \, ds.$$

Hence for $C_3 = \max\{C_1, C_2\}$:

$$\|X\|_{p,t}^p \leqslant 4^{p-1} C_3 \int_0^t (N^p \|\alpha_1\|_{p,s}^p + \mathbb{E}[\|\alpha_2(s)\|_{HS}^p]) \, ds$$
$$+ 4^{p-1} C_3 L^p (N^p + 1) \int_0^t \|X\|_{p,s}^p \, ds.$$

For a fixed t_0 such that $0 \leqslant t_0 \leqslant T$ we have

$$\|X\|_{p,t}^p \leqslant 4^{p-1} C_3 \int_0^{t_0} (N^p \|\alpha_1\|_{p,s}^p + \mathbb{E}[\|\alpha_2(s)\|_{HS}^p]) \, ds$$
$$+ 4^{p-1} C_3 L^p (N^p + 1) \int_0^t \|X\|_{p,s}^p \, ds$$

for $0 \leqslant t \leqslant t_0$. Applying the Gronwall inequality we obtain

$$\|X\|_{p,t}^p \leqslant 4^{p-1} C_3 e^{4^{p-1} C_3 L^p (N^p+1)t} \int_0^{t_0} (N^p \|\alpha_1\|_{p,s}^p + \mathbb{E}[\|\alpha_2(s)\|_{HS}^p]) \, ds.$$

Since t_0 is fixed arbitrarily we have

$$\|X\|_{p,t}^p \leqslant 4^{p-1} C_3 e^{4^{p-1} C_3 L^p (N^p+1)t} \int_0^t (N^p \|\alpha_1\|_{p,s}^p + \mathbb{E}[\|\alpha_2(s)\|_{HS}^p]) \, ds.$$

This completes the proof. □

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§4. *Convergence of Newton's method.* We formulate the Newton scheme for problem (2.1). Let $X^{(0)} \in \mathcal{X}_{p,T}$, $X^{(0)}(t) = \varphi(t)$ for $t \in [-r, 0]$ and

$$\begin{aligned} dX^{(k+1)}(t) &= [AX^{(k+1)}(t) + F(t, X_t^{(k)}) + F_v(t, X_t^{(k)})(X_t^{(k+1)} - X_t^{(k)})]\,dt \\ &\quad + [B(t, X_t^{(k)}) + B_v(t, X_t^{(k)})(X_t^{(k+1)} - X_t^{(k)})]\,dW(t), \\ X^{(k+1)}(0) &= \varphi(t), \quad t \in [-r, 0], \end{aligned} \tag{4.1}$$

where $F_v \in L(\mathcal{C}_r, H)$ and $B_v \in L(\mathcal{C}_r, HS)$ are Fréchet derivatives of F and B with respect to the functional variable $v \in \mathcal{C}_r$. Let $\Delta X^{(k)}(t) = X^{(k+1)}(t) - X^{(k)}(t)$,

$$\begin{aligned} \Delta F^{(k)}(t, X_t) &:= F(t, X_t^{(k+1)}) - F(t, X_t^{(k)}), \\ \Delta B^{(k)}(t, X_t) &:= B(t, X_t^{(k+1)}) - B(t, X_t^{(k)}). \end{aligned}$$

We have the following differential equations for increments $\Delta X^{(k+1)}$:

$$\begin{aligned} &d\Delta X^{(k+1)}(t) \\ &\quad = [A\Delta X^{(k+1)}(t) + \Delta F^{(k)}(t, X_t) - F_v(t, X_t^{(k)})\Delta X_t^{(k)} \\ &\qquad + F_v(t, X_t^{(k+1)})\Delta X_t^{(k+1)}]\,dt \\ &\qquad + [\Delta B^{(k)}(t, X_t) - B_v(t, X_t^{(k)})\Delta X_t^{(k)} + B_v(t, X_t^{(k+1)})\Delta X_t^{(k+1)}]\,dW(t). \end{aligned} \tag{4.2}$$

THEOREM 4.1. *Suppose that there exists a non-negative constant L such that*

$$\|F_v(t, v)\|_{L(\mathcal{C}_r, H)} \leqslant L, \tag{4.3}$$

$$\|B_v(t, v)\|_{L(\mathcal{C}_r, HS)} \leqslant L \tag{4.4}$$

for $t \in [0, T]$, $v \in \mathcal{C}_r$ which implies the Lipschitz condition

$$\|F(t, x) - F(t, y)\| \leqslant L\|x - y\|_{\mathcal{C}_r}, \qquad \|B(t, x) - B(t, y)\|_{HS} \leqslant L\|x - y\|_{\mathcal{C}_r}$$

for $t \in [0, T]$, $x, y \in \mathcal{C}_r$. Then the Newton sequence $(X^{(k)})_{k\in\mathbb{N}}$ defined by (4.1) converges to the unique solution X of equation (2.1) in the following sense:

$$\lim_{k\to\infty} \|X^{(k)} - X\|_{p,T} = 0.$$

Proof. We show that $(X^{(k)})_{k\in\mathbb{N}}$ satisfies the Cauchy condition with respect to the norm $\|\cdot\|_{p,T}$. We apply Lemma 3.1 with

$$\begin{aligned} \alpha_1(s) &= \Delta F^{(k)}(s, X_s) - F_v(s, X_s^{(k)})\Delta X_s^{(k)}, & T_1(s) &= F_v(s, X_s^{(k+1)}), \\ \alpha_2(s) &= \Delta B^{(k)}(s, X_s) - B_v(s, X_s^{(k)})\Delta X_s^{(k)}, & T_2(s) &= B_v(s, X_s^{(k+1)}) \end{aligned}$$

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and obtain

$$\begin{aligned}
&\|\Delta X^{(k+1)}\|_{p,t}^p \\
&\quad \leqslant 4^{p-1}C_3 e^{4^{p-1}C_3L^p(N^p+1)t}N^p \int_0^t \|\Delta F^{(k)}(\cdot, X_\cdot) - F_v(\cdot, X_\cdot^{(k)})\Delta X_\cdot^{(k)}\|_{p,s}^p \, ds \\
&\qquad + 4^{p-1}C_3 e^{4^{p-1}C_3L^p(N^p+1)t} \int_0^t \mathbb{E}\,[\|\Delta B^{(k)}(s, X_s) \\
&\qquad - B_v(s, X_s^{(k)})\Delta X_s^{(k)}\|_{HS}^p]\, ds.
\end{aligned}$$

By (3.3), (2.3), and (2.4) we have

$$\|\Delta F^{(k)}(s, X_s) - F_v(s, X_s^{(k)})\Delta X_s^{(k)}\|^p \leqslant 2^p L^p \|\Delta X_s^{(k)}\|_{\mathcal{C}_r}^p$$

and

$$\|\Delta B^{(k)}(s, X_s) - B_v(s, X_s^{(k)})\Delta X_s^{(k)}\|_{HS}^p \leqslant 2^p L^p \|\Delta X_s^{(k)}\|_{\mathcal{C}_r}^p.$$

Hence we have the estimate

$$\|\Delta X^{(k+1)}\|_{p,t}^p \leqslant 2^{3p-1}C_3 L^p e^{4^{p-1}C_3L^p(N^p+1)t}(N^p+1)\int_0^t \|\Delta X^{(k)}\|_{p,s}^p \, ds.$$

Since $t \leqslant T$ we have

$$\|\Delta X^{(k+1)}\|_{p,t}^p \leqslant C\int_0^t \|\Delta X^{(k)}\|_{p,s}^p \, ds, \tag{4.5}$$

where

$$C = 2^{3p-1}C_3 L^p e^{4^{p-1}C_3L^p(N^p+1)T}(N^p+1).$$

The recursive use of (4.5) leads to

$$\|\Delta X^{(k+1)}\|_{p,t}^p \leqslant \frac{C^{k+1}t^{k+1}}{(k+1)!}\|\Delta X^{(0)}\|_{p,t}^p.$$

Thus

$$\|X^{(k+p)} - X^{(k)}\|_{p,t}^p \leqslant \left(\frac{C^{k+p-1}t^{k+p-1}}{(k+p-1)!} + \cdots + \frac{C^k t^k}{k!}\right)\|\Delta X^{(0)}\|_{p,t}^p.$$

We conclude that $(X^{(k)})_{k\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $\mathcal{X}_{p,T}$. Therefore it is convergent to X, which is a solution to equation (2.1). This completes the proof. □

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§5. *Probabilistic second order convergence of Newton's method.*

THEOREM 5.1. *Suppose that (4.3) and (4.4) hold and assume additionally that there exists a non-negative constant M such that*

$$\|F_v(t,v)-F_v(t,\bar v)\|_{L(\mathcal{C}_r,H)}\leqslant M\|v-\bar v\|_{\mathcal{C}_r}, \tag{5.1}$$
$$\|B_v(t,v)-B_v(t,\bar v)\|_{L(\mathcal{C}_r,HS)}\leqslant M\|v-\bar v\|_{\mathcal{C}_r} \tag{5.2}$$

for $v,\bar v\in\mathcal{C}_r$. Then for any $T>0$, $p>2$ there exists $\tilde C>0$ such that

$$P\left(\sup_{0\leqslant t\leqslant T}\|\Delta X^{(k)}(t)\|\leqslant\rho\quad\Rightarrow\quad\sup_{0\leqslant t\leqslant T}\|\Delta X^{(k+1)}(t)\|\leqslant R\rho^2\right)\geqslant 1-\tilde C T R^{-p}$$

for all $R>0$, $0<\rho\leqslant 1$, $k=0,1,2,\ldots$.

Proof. Define the sets

$$A^{(k)}_{\rho,t}=\left\{\omega:\sup_{0\leqslant s\leqslant t}\|\Delta X^{(k)}(s)\|\leqslant\rho\right\}\quad\text{for } 0<\rho\leqslant 1,\ 0\leqslant t\leqslant T,\ k=0,1,2,\ldots.$$

We consider the sequence $(\Delta X^{(k)})_{k\in\mathbb{N}}$ restricted to the sets $A^{(k)}_{\rho,t}$. For this reason we multiply equation (2.5) by $\mathbf{1}_{A^{(k)}_{\rho,t}}$, the characteristic function of the set $A^{(k)}_{\rho,t}$:

$$\begin{aligned}\mathbf{1}_{A^{(k)}_{\rho,t}}\Delta X^{(k+1)}(t)&=\mathbf{1}_{A^{(k)}_{\rho,t}}\int_0^t S(t-s)\tilde F^{(k,k+1)}(s,X_s)\,ds\\&\quad+\mathbf{1}_{A^{(k)}_{\rho,t}}\int_0^t S(t-s)\tilde B^{(k,k+1)}(s,X_s)\,dW(s),\end{aligned}$$

where

$$\begin{aligned}\tilde F^{(k,k+1)}(t,X_t)&:=\Delta F^{(k)}(t,X_t)-F_v(t,X_t^{(k)})\Delta X_t^{(k)}+F_v(t,X_t^{(k+1)})\Delta X_t^{(k+1)},\\ \tilde B^{(k,k+1)}(t,X_t)&:=\Delta B^{(k)}(t,X_t)-B_v(t,X_t^{(k)})\Delta X_t^{(k)}+B_v(t,X_t^{(k+1)})\Delta X_t^{(k+1)}.\end{aligned}$$

Applying (3.3) and by the fact that for $s\leqslant t$,

$$A^{(k)}_{\rho,t}\subset A^{(k)}_{\rho,s}\quad\Rightarrow\quad\mathbf{1}_{A^{(k)}_{\rho,t}}=\mathbf{1}_{A^{(k)}_{\rho,t}}\mathbf{1}_{A^{(k)}_{\rho,s}}$$

we have

$$\begin{aligned}&\|\mathbf{1}_{A^{(k)}_{\rho,t}}\Delta X^{(k+1)}\|^p_{p,t}\\&\leqslant 2^{p-1}\mathbb{E}\left[\mathbf{1}_{A^{(k)}_{\rho,t}}\sup_{\tilde t\in[0,t]}\left\|\int_0^{\tilde t}S(\tilde t-s)\tilde F^{(k,k+1)}(s,X_s)\,ds\right\|^p\right]\\&\quad+2^{p-1}\mathbb{E}\left[\mathbf{1}_{A^{(k)}_{\rho,t}}\sup_{\tilde t\in[0,t]}\left\|\int_0^{\tilde t}S(\tilde t-s)\tilde B^{(k,k+1)}(s,X_s)\,dW(s)\right\|^p\right]\end{aligned}$$

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$$\begin{aligned}
&\leqslant 2^{p-1}\mathbb{E}\left[\sup_{\tilde{t}\in[0,t]}\left\|\int_0^{\tilde{t}}\mathbf{1}_{A_{\rho,s}^{(k)}}S(\tilde{t}-s)\tilde{F}^{(k,k+1)}(s,X_s)\,ds\right\|^p\right]\\
&\quad+2^{p-1}\mathbb{E}\left[\sup_{\tilde{t}\in[0,t]}\left\|\int_0^{\tilde{t}}\mathbf{1}_{A_{\rho,s}^{(k)}}S(\tilde{t}-s)\tilde{B}^{(k,k+1)}(s,X_s)\,dW(s)\right\|^p\right]\\
&=2^{p-1}(J_1+J_2).
\end{aligned}$$

By the Hölder inequality for some $C_4>0$ dependent on T, (2.2) and (3.3) we get

$$\begin{aligned}
J_1&\leqslant N^pC_4\mathbb{E}\left[\sup_{\tilde{t}\in[0,t]}\int_0^{\tilde{t}}\mathbf{1}_{A_{\rho,s}^{(k)}}\|\tilde{F}^{(k,k+1)}(s,X_s)\|^p\,ds\right]\\
&\leqslant N^pC_4\mathbb{E}\left[\int_0^{t}\mathbf{1}_{A_{\rho,s}^{(k)}}\|\tilde{F}^{(k,k+1)}(s,X_s)\|^p\,ds\right]\\
&\leqslant 2^{p-1}N^pC_4\mathbb{E}\left[\int_0^{t}\mathbf{1}_{A_{\rho,s}^{(k)}}\|\Delta F^{(k)}(s,X_s)-F_v(s,X_s^{(k)})\Delta X_s^{(k)}\|^p\,ds\right]\\
&\quad+2^{p-1}N^pC_4\mathbb{E}\left[\int_0^{t}\mathbf{1}_{A_{\rho,s}^{(k)}}\|F_v(s,X_s^{(k+1)})\Delta X_s^{(k+1)}\|^p\,ds\right].
\end{aligned}$$

From the fundamental theorem of calculus and (5.1) it follows that

$$\begin{aligned}
&\|\Delta F^{(k)}(s,X_s)-F_v(s,X_s^{(k)})\Delta X_s^{(k)}\|\\
&\leqslant\sup_{\tilde{s}\in[0,s]}\|\Delta X^{(k)}(\tilde{s})\|\int_0^1\|F_v(s,X_s^{(k)}+\theta\Delta X_s^{(k)})-F_v(s,X_s^{(k)})\|_{L(\mathcal{C}_r,H)}\,d\theta\\
&\leqslant\frac{1}{2}M\sup_{\tilde{s}\in[0,s]}\|\Delta X^{(k)}(\tilde{s})\|^2.
\end{aligned}$$

Hence by (4.3),

$$\begin{aligned}
J_1&\leqslant\frac{1}{2}N^pM^pC_4\mathbb{E}\left[\int_0^t\mathbf{1}_{A_{\rho,s}^{(k)}}\|\Delta X^{(k)}(s)\|^{2p}\,ds\right]\\
&\quad+2^{p-1}N^pL^pC_4\mathbb{E}\left[\int_0^t\mathbf{1}_{A_{\rho,s}^{(k)}}\|\Delta X^{(k+1)}(s)\|^p\,ds\right].
\end{aligned}$$

Recall that $\|\Delta X^{(k)}(\tilde{s})\|\leqslant\rho$ on $A_{\rho,s}^{(k)}$ for $0\leqslant\tilde{s}\leqslant s$. Thus we have

$$J_1\leqslant\frac{1}{2}N^pM^pC_4\rho^{2p}t+2^{p-1}N^pL^pC_4\int_0^t\|\mathbf{1}_{A_{\rho,s}^{(k)}}\Delta X^{(k+1)}\|_{p,s}^p\,ds.$$

We estimate J_2. Since

$$\mathbf{1}_{A_{\rho,s}^{(k)}}\tilde{B}^{(k,k+1)}(s,X_s)$$

is an $HS(U_0,H)$-valued predictable process, we apply [**10**, Proposition 7.3] with some constant $C_5>0$ dependent on T and (3.3):

$$J_2=\mathbb{E}\left[\sup_{\tilde{t}\in[0,t]}\left\|\int_0^{\tilde{t}}\mathbf{1}_{A_{\rho,s}^{(k)}}S(\tilde{t}-s)\tilde{B}^{(k,k+1)}(s,X_s)\,dW_s\right\|^p\right]$$

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$$\leqslant C_5\mathbb{E}\bigg[\int_0^t \|\mathbf{1}_{A^{(k)}_{\rho,s}}\tilde{B}^{(k,k+1)}(s,X_s)\|^p_{HS}\,ds\bigg]$$
$$\leqslant 2^{p-1}C_5\mathbb{E}\bigg[\int_0^t \mathbf{1}_{A^{(k)}_{\rho,s}}\|\Delta B^{(k)}(s,X_s)-B_v(s,X^{(k)}_s)\Delta X^{(k)}_s\|^p_{HS}\,ds\bigg]$$
$$+2^{p-1}C_5\mathbb{E}\bigg[\int_0^t \mathbf{1}_{A^{(k)}_{\rho,s}}\|B_v(s,X^{(k+1)}_s)\Delta X^{(k+1)}_s\|^p_{HS}\,ds\bigg].$$

From the fundamental theorem of calculus and (5.2) it follows that

$$\|\Delta B^{(k)}(s,X_s)-B_v(s,X^{(k)}_s)\Delta X^{(k)}_s\|_{HS}$$
$$\leqslant \sup_{\tilde{s}\in[0,s]}\|\Delta X^{(k)}(\tilde{s})\|\int_0^1\|B_v(s,X^{(k)}_s+\theta\Delta X^{(k)}_s)-B_v(s,X^{(k)}_s)\|_{L(\mathcal{C}_r,HS)}\,d\theta$$
$$\leqslant \frac{1}{2}M\sup_{\tilde{s}\in[0,s]}\|\Delta X^{(k)}(\tilde{s})\|^2.$$

Hence

$$J_2\leqslant\frac{1}{2}M^pC_4\mathbb{E}\bigg[\int_0^t\mathbf{1}_{A^{(k)}_{\rho,s}}\|\Delta X^{(k)}(s)\|^{2p}\,ds\bigg]$$
$$+2^{p-1}L^pC_5\mathbb{E}\bigg[\int_0^t\mathbf{1}_{A^{(k)}_{\rho,s}}\|\Delta X^{(k+1)}(s)\|^p\,ds\bigg].$$

Recall that $\|\Delta X^{(k)}(\tilde{s})\|\leqslant\rho$ on $A^{(k)}_{\rho,s}$ for $0\leqslant\tilde{s}\leqslant s$. Thus we have

$$J_2\leqslant\frac{1}{2}M^pC_5\rho^{2p}t+2^{p-1}L^pC_5\int_0^t\|\mathbf{1}_{A^{(k)}_{\rho,s}}\Delta X^{(k+1)}\|^p_{p,s}\,ds.$$

Finally, for $C_6=\max\{C_4,C_5\}$ we have the estimate

$$\|\mathbf{1}_{A^{(k)}_{\rho,t}}\Delta X^{(k+1)}\|^p_{p,t}\leqslant 2^{p-1}(J_1+J_2)$$
$$\leqslant 2^{p-2}M^pC_6\rho^{2p}(N^p+1)t+4^{p-1}L^pC_6(N^p+1)$$
$$\times\int_0^t\|\mathbf{1}_{A^{(k)}_{\rho,s}}\Delta X^{(k+1)}\|^p_{p,s}\,ds.$$

Applying the Gronwall inequality we obtain

$$\|\mathbf{1}_{A^{(k)}_{\rho,t}}\Delta X^{(k+1)}\|^p_{p,t}\leqslant 2^{p-2}M^pC_6\rho^{2p}(N^p+1)te^{4^{p-1}L^pC_6(N^p+1)}.$$

The Chebyshev inequality yields

$$P\bigg(\sup_{0\leqslant s\leqslant t}\|\Delta X^{(k)}(s)\|\leqslant\rho\wedge\sup_{0\leqslant s\leqslant t}\|\Delta X^{(k+1)}(s)\|>R\rho^2\bigg)$$
$$=P\bigg[\mathbf{1}_{A^{(k)}_{\rho,t}}\sup_{0\leqslant s\leqslant t}\|\Delta X^{(k+1)}(s)\|>R\rho^2\bigg]\leqslant\frac{1}{R^p\rho^{2p}}\|\mathbf{1}_{A^{(k)}_{\rho,t}}\Delta X^{(k+1)}\|^p_{p,t}$$
$$\leqslant\frac{1}{R^p}2^{p-2}M^pC_6(N^p+1)te^{4^{p-1}L^pC_6(N^p+1)}=\tilde{C}tR^{-p}.$$

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Hence for any $R > 0$ we have

$$P\left(\sup_{0\leqslant s\leqslant t}\|\Delta X^{(k)}(s)\|\leqslant\rho \quad\Rightarrow\quad \sup_{0\leqslant s\leqslant t}\|\Delta X^{(k+1)}(s)\|\leqslant R\rho^2\right)\geqslant 1-\tilde{C}tR^{-p}.$$

This completes the proof. □

§6. *Applications.*

Example 1. Taking $A\equiv 0$, $H=\mathbb{R}$ and $\mathcal{C}_r = C([-r,0],\mathbb{R})$ in problem (2.1) we arrive at the stochastic functional differential equation of the form

$$\begin{cases} dX(t) = F(t,X_t)\,dt + B(t,X_t)\,dW(t), \quad t\in[0,T] \\ X(t)=\varphi(t), \quad t\in[-r,0]. \end{cases} \tag{6.1}$$

The Newton scheme for problem (6.1):

$$\begin{aligned} dX^{(k+1)}(t) &= [F(t,X_t^{(k)}) + F_v(t,X_t^{(k)})(X_t^{(k+1)}-X_t^{(k)})]\,dt \\ &\quad + [B(t,X_t^{(k)}) + B_v(t,X_t^{(k)})(X_t^{(k+1)}-X_t^{(k)})]\,dW(t), \quad t\in[0,T] \\ X^{(k+1)}(0) &= \varphi(t), \quad t\in[-r,0] \end{aligned}$$

is investigated in [33]. The techniques to prove the probabilistic second-order convergence in the non-functional setting are developed in [2] by Amano, whose proof is based on solving an error stochastic differential equation and introducing a certain representation of the solution (see [2, Lemma 2.5]) with components easy to estimate. There is a significant difference between the functional and non-functional case: Amano uses Itô isometry whereas the functional case demands the Doob inequality with some tricks to overcome difficulties arising from the impossibility of constructing explicit solutions to functional SDEs.

We give examples of stochastic partial differential equations to illustrate the main assumptions implying the first-order and probabilistic second-order convergence of Newton's method.

Example 2. Consider a special case of problem (1.1):

$$\begin{aligned} \frac{\partial}{\partial t}u(t,x) - \frac{\partial^2}{\partial x^2}u(t,x) &= \int_{-r_1}^{0}\int_0^1 \sin(\beta(x,y)u(t+s,y))\,dy\,ds \\ &\quad + \int_{-r_2}^{0}\int_0^1 \sin(\beta(x,y)u(t+s,y))\,dy\,ds\dot{W}(t), \end{aligned} \tag{6.2}$$

where $\beta:[0,1]^2\to\mathbb{R}$ is a continuous function and thus bounded by a positive constant K. Define $F, B:\mathcal{C}_r\to L^2([0,1])$ by

$$F(\psi)(x) = \int_{-r_1}^{0}\int_0^1 \sin(\beta(x,y)\psi(s,y))\,dy\,ds,$$

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$$B(\psi)(x) = \int_{-r_2}^{0} \int_{0}^{1} \sin(\beta(x, y)\psi(s, y))\, dy\, ds.$$

The Fréchet derivatives $F_\psi(\psi),\ B_\psi(\psi) : \mathcal{C}_r \to L^2([0, 1])$,

$$(F_\psi(\psi)h)(x) = \int_{-r_1}^{0} \int_{0}^{1} \cos(\beta(x, y)\psi(s, y))\beta(x, y)h(s, y)\, dy\, ds,$$
$$(B_\psi(\psi)h)(x) = \int_{-r_2}^{0} \int_{0}^{1} \cos(\beta(x, y)\psi(s, y))\beta(x, y)h(s, y)\, dy\, ds$$

satisfy the following conditions:

$$\|F_\psi(\psi)\|_{L(\mathcal{C}_r, L^2([0,1]))} \leqslant Kr_1,$$
$$\|B_\psi(\psi)\|_{L(\mathcal{C}_r, L^2([0,1]))} \leqslant Kr_2.$$

Hence the assumptions of Theorem 4.1 are satisfied and the Newton sequence for (6.2) converges to the solution u of equation (6.2). Moreover, the Fréchet derivatives $F_\psi(\psi)$, $B_\psi(\psi)$ satisfy the Lipschitz condition

$$\|F_\psi(t, \psi_1) - F_\psi(t, \psi_2)\|_{L(\mathcal{C}_r, L^2([0,1]))} \leqslant K^2 r_1 \|\psi_1 - \psi_2\|_{\mathcal{C}_r},$$
$$\|B_\psi(t, \psi_1) - B_\psi(t, \psi_2)\|_{L(\mathcal{C}_r, L^2([0,1]))} \leqslant K^2 r_2 \|\psi_1 - \psi_2\|_{\mathcal{C}_r}.$$

Thus the assumptions of Theorem 5.1 are satisfied. Therefore we obtain the probabilistic second-order convergence of Newton's method for (6.2).

Similar results can be obtained in the modification of (6.2):

$$\frac{\partial}{\partial t}u(t, x) - \frac{\partial^2}{\partial x^2}u(t, x) = \int_0^1 \sin(\beta(x, y)u(\gamma(t), y))\, dy + \int_0^1 \sin(\beta(x, y)u(\gamma(t), y))\, dy \dot{W}(t),$$

where $\gamma : [0, T] \to [-r, T]$, $\gamma(t) \leqslant t$.

Example 3. Consider the stochastic semilinear delay heat equation (cf. [**22**, Example 1, pp. 1250–1252])

$$\frac{\partial}{\partial t}u(t, x) - \frac{\partial^2}{\partial x^2}u(t, x) = \int_{-r_1}^{0} f(u(t + s, x))\, ds + \int_{-r_2}^{0} b(u(t + s, x))\, ds \dot{W}(t) \tag{6.3}$$

with the initial and boundary conditions as in problem (1.1). Let $H = L^2([0, 1])$, $A = \partial^2/\partial x^2$ and $f, g : \mathbb{R} \to \mathbb{R}$, $f(u) = g(u) = \sin u$. Define $F, B : \mathcal{C}_r \to L^2([0, 1])$ by

$$F(\psi)(x) = \int_{-r_1}^{0} \sin(\psi(s, x))\, ds, \qquad B(\psi)(x) = \int_{-r_2}^{0} \sin(\psi(s, x))\, ds.$$

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The Fréchet derivatives $F_\psi(\psi), B_\psi(\psi) : \mathcal{C}_r \to L^2([0,1])$ satisfy the following conditions:

$$\|F_\psi(\psi)\|_{L(\mathcal{C}_r, L^2([0,1]))} \leqslant r_1,$$
$$\|B_\psi(\psi)\|_{L(\mathcal{C}_r, L^2([0,1]))} \leqslant r_2.$$

Therefore by Theorem 4.1 the first-order convergence of Newton's method for (6.3) is obtained. However, since the Fréchet derivatives of F and B are not Lipschitz, the assumptions of Theorem 5.1 are not satisfied. Furthermore, second-order convergence problems appear even if we reduce the equation to the following form:

$$du(t) = (Au(t) + F(t, u_t))\,dt + dW(t),$$

where $F(t, \psi)(x) = \sin \psi(\gamma(t) - t, x)$, $\gamma : [0, T] \to [-r, T]$, $\gamma(t) \leqslant t$.

It is surprising that for the corresponding deterministic problem ($B \equiv 0$) one obtains the second-order convergence of Newton's method.

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