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Differential transcendence and walks on self-similar graphs

Yakob Kahane and Marni Mishna

Abstract. Symmetrically self-similar graphs are an important type of fractal graph. Their Green's functions satisfy order one iterative functional equations. We show that when the branching number of a generating cell is two, either the graph is a star consisting of finitely many one-sided lines meeting at an origin vertex, in which case the Green's function is algebraic, or the Green's function is differentially transcendental over $\mathbb{C}(z)$. The proof strategy relies on a result in a recent preprint of Di Vizio, Fernandes, and Mishna. The result adds evidence to a conjecture of Krön and Teufl about the spectra of the difference Laplacian of this family of graphs.

1 Introduction

The discretization of fractals as graphs opens the possibility of using combinatorial methods to study self-similarity. In particular, some types of fractal graphs possess a regularity which can be exploited in enumerative analysis. The focus here is walks on graphs in the family of *symmetrically self-similar* fractal graphs. We shall say more about this family in a moment.

A uniform random walk on a graph $X = (V(X), E(X))$ starts at some vertex, and proceeds along the edges of the graph. We assume that the degree of every vertex is bounded and that every edge incident to a given vertex is equally likely to be chosen. We define the following transition matrix P associated with a graph: For vertices x, y from $V(X)$, the $[x, y]$ entry of P is the transition probability

$$p(x,y) := \begin{cases} \frac{1}{\deg(x)} & \{x,y\} \in E(X) \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

A random walk on X is a Markov chain $(V_i)_{i=0}^{n}$, $V_i \in V(X)$. The probability of the path $V_0 = x_0, V_1 = x_1, \ldots, V_n = x_n$, given that $V_0 = x_0$, is

$$\mathbb{P}_{x_0}[V_0 = x, V_1 = x_1, \ldots, V_n = x_n] = p(x_0, x_1)p(x_1, x_2)\ldots p(x_{n-1}, x_n).$$

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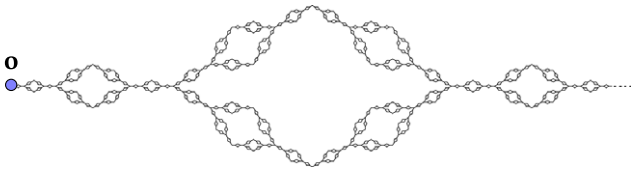


Figure 1: Close up of a symmetrically self-similar graph with branching number 2 near its origin **o**. The generating cell is pictured in Figure 2(a).

We denote the probability of a walk that starts at x and ends at y after n steps ($\mathbb{P}_x[V_n = y]$) by $p^{(n)}(x, y)$. A *Green's function* of (X, P), denoted $G(x, y|z)$, is a generating function of probabilities:

$$G(x, y|z) := \sum_{n\geq 0} p^{(n)}(x, y)\, z^n.$$

When we restrict to walks that start and end at a specified origin vertex labeled **o**, we refer to them as *the* Green's function for a graph, and use the shorthand $G(z)$. Precisely,

$$G(z) := G(\mathbf{o}, \mathbf{o}|z) = \sum_{n\geq 0} p^{(n)}(\mathbf{o}, \mathbf{o})\, z^n.$$

This Green's function, and the functional equations it satisfies, can offer significant information about X, and possibly even objects the graph might encode (like groups in the case of a Cayley graph). We focus here on different notions of transcendency. Recall that a series in the ring of formal power series $\mathbb{C}[[z]]$ is said to be *algebraic* if it satisfies a nontrivial polynomial equation, and is *D-finite* over $\mathbb{C}(z)$ if its derivatives span a finite-dimensional vector space over $\mathbb{C}(z)$. A function is *differentiably algebraic* if it satisfies a nontrivial polynomial differential equation, and is *differentially transcendental*[1] if it is not differentiably algebraic. These categories have demonstrated themselves to be useful in a variety of combinatorial contexts: In the case of Cayley graphs, algebraicity and D-finiteness of a Green's function are each correlated with structural properties of the group [3].

In this work, we focus on a particular family of fractal graphs, introduced by Krön called *symmetrically self-similar graphs* [9]. These graphs are well studied [1, 13, 15] as they are structured, and have many interesting properties. We give the precise definition in the next section, but simple examples can provide some intuition. These graphs are generated by an iterative process involving a finite *cell graph* built from complete graphs and have particular symmetry properties. The graph pictured in Figure 1 is generated from the cell in Figure 2(a). The well-known Sierpiński graph is a second example, generated from the cell in Figure 2(b).

Symmetrically self-similar graphs viewed as a discretization of fractals appear in a variety of different domains, including the study of Brownian motion on fractals [11].

The Green's functions of these graphs are well studied [8, 10], including the asymptotic behavior of the excursion probabilities. Krön and Teufl [10] show that

[1] Also called hypertranscendental in some literature.

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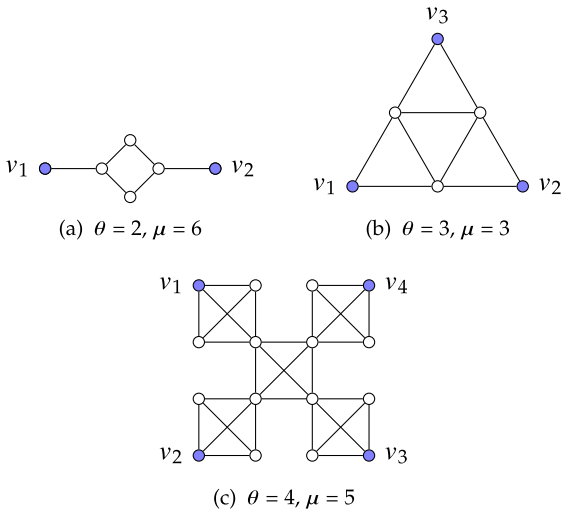


Figure 2: Three cell graphs, with their extremal vertices in blue.

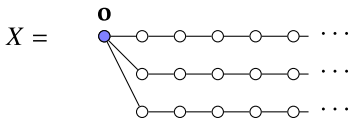


Figure 3: A close up of a star graph near the origin.

asymptotically, as n tends toward infinity, $p^{(n)}(\mathbf{o}, \mathbf{o})$ tends to $n^{\beta} w(n)$, where β is a number related to the geometry of the graph, and $w(n)$ is some oscillating function. In the case of the Sierpiński graph, $\beta = \log 5 / \log 3$. Because β is not rational, we can deduce, using the known structure of coefficients asymptotics of solutions to differential equations (see [16, Theorem 19.1, p. 111]), that the Green's function is not D-finite. Indeed, in almost all examples that have been studied, non-D-finiteness (and hence transcendence) of the Green's function can be deduced immediately from the singular expansion. The trivial fractal graphs consisting of a finite number of lines radiating from a single origin vertex (*star graphs*; see Figure 3) are the only known examples of symmetrically self-similar graphs with algebraic Green's functions. (Although there are other families of fractal-like graphs with algebraic Green's functions.)

In a recent preprint, Di Vizio et al. [5] showed that not only is the Green's function of the Sierpiński graph not D-finite, but it actually *differentially transcendental*! In the absence of analogous structure theorems for differentially algebraic functions, the proof uses the key result of Grabner and Woess [8] that the Green's function of the Sierpiński graph satisfies an iterative equation of the form

$$G(z) = f(z)G(d(z)), \tag{2}$$

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where f and d are explicit rational functions and are themselves Green's functions of a finite graph related to the generating cell. By Di Vizio et al., series solutions to such equations are either algebraic or are differentially transcendental. In the present work, we generalize this approach to consider the full class of symmetrically self-similar graphs. We have concrete results for a sub-class and give a potential pathway to prove the result in general. This contributes to a growing effort to understand the combinatorial embodiment of differential algebraicity and differential transcendence. Walks in graphs have been a rich domain [6], and Galoisian approaches on other examples have been successful [4]. Such questions have been around at least since Rubel's invitation [14] to the topic.

We conjecture the following, which directly connects differential transcendence to combinatorial structure.

Conjecture 1 *The Green's function of a symmetrically self-similar graph with bounded geometry is algebraic if and only if the graph is a star consisting of finitely many one-sided lines with exactly one origin vertex in common. Otherwise, it is differentially transcendental.*

We prove this conjecture for a subcase in the main theorem of this article, of which Figure 1 is an example. The hypotheses depend on the branching number θ of X, which is defined below. Here, we state our main theorem, which appears in Section 4 as Theorem 12. An example of a star graph is in Figure 3.

Theorem (Main result) *Let X be a symmetrically self-similar graph with bounded geometry, origin $\mathbf{o}$, and branching number $\theta = 2$. Either the graph is a star consisting of finitely many one-sided lines that coincide at $\mathbf{o}$, or the Green's function $G(\mathbf{o}, \mathbf{o}|z)$ of X is differentially transcendental over $\mathbb{C}(z)$.*

The argument at the heart of the proof of our result has the potential to extend to all branching numbers with a little more information on the singularities of G, as we shall discuss near the end. There are two features of the case $\theta = 2$ that make it slightly easier. We have tighter existing bounds on the location of the singularities of the Green function that we can use, and we have a simpler combinatorial situation to prove a bound connecting key parameters (Theorem 16).

What is the potential impact of this result? Malozemov and Teplyaev [13] showed that the spectrum of the difference Laplacian on self-similar graphs consists of the Julia set of a rational function and a (possibly empty) set of isolated eigenvalues that accumulate to the Julia set. Krön and Teufl conjectured that this Julia set is an interval if and only if the graph is a star consisting of finitely many one-sided lines with exactly one origin vertex in common; otherwise, it is a Cantor set. Our work connects the spectrum and the differential transcendency of the Green's function. While we do not resolve the conjecture of Krön and Teufl, the generating function classification provides a new perspective.

Finally, as we alluded to earlier, some infinite Cayley graphs are also fractal, and their Green's functions satisfy functional equations of the type in Equation (2).

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However, they are easily proved to be algebraic [2]. Can we characterize a wider class of fractal graphs with algebraic Green's functions?

2 Symmetrically self-similar graphs

There are two ways to view this family of graphs. Krön describes a blow-up construction applied to a cell in [9], and this is further developed by Krön and Teufl [10]. On the other hand, such a graph can be defined as a fixed point, as was done by Malozemov and Teplyaev [12]. We will consider both definitions below, as they appeal to complementary intuitions.

We start by recalling some basic graph concepts. Let $X = (V(X), E(X))$ be an undirected, simple graph (no loops nor multiple edges). Given a sub-graph C of X, we define its boundary in X, denoted θC, to be the set of all vertices in $V(X)\backslash V(C)$ which are adjacent to some vertex in C. Define the graph $\hat{C}$ as the graph induced by the vertices of C and θC inside X.

2.1 A fixed-point construction

Let B be a sub-graph of a simple graph X. We define the set $\mathrm{Con}_X(B)$ to be the set of connected components in $X\backslash B$. Associated with B, we define the graph X_B with vertex set $V(B)$ and an edge $\{x, y\}$ if there is a component in $\mathrm{Con}_X(B)$ with both x and y in its boundary in X. Roughly, they are both incident to a common component created by deleting B from X. We give a simple example below.

Definition 2 (Self-similar with respect to a sub-graph, and a map) A connected infinite graph X is said to be *self-similar with respect to sub-graph* B and a map

$$\psi : V(X) \to V(X_B)$$

if the following hold:

(1) no two vertices in B are adjacent in X;
(2) the intersection of the boundaries of two different components in $\mathrm{Con}_X(B)$ contains at most one vertex;
(3) $\psi(X) = X_B$ is isomorphic to X.

The components of $\mathrm{Con}_X(B)$ are called *cells*. Consider a map ψ that replaces a cell $\hat{C}$ in X with a complete graph on the vertices of θC. If $\psi(X)$ is isomorphic to X, then we have self-similarity.

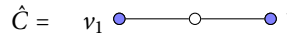
Example 1 Consider the infinite finite path graph X with every second vertex colored blue.

$X =$ 

Let $B \subset X$ be the set of blue vertices, then $\mathrm{Con}_X(B)$ is the set of (isolated) white vertices. These are the cells; remark that the cells are all isomorphic. The closure of

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a cell includes the two adjacent blue vertices. Each looks like:

$\hat{C} =$  .

Let ψ be the map on X that replaces every instance of $\hat{C}$ in X with a single edge between the blue vertices. The image of ψ is again an infinite finite path graph, and is hence in bijection with X. Thus, we can say that X is self-similar with respect to the set of blue vertices and ψ.

Definition 3 (Symmetrically self-similar graph) A self-similar graph X is furthermore said to be *symmetrically self similar* if it is locally finite,[2] and it additionally satisfies the following axioms:

(1) All cells are finite, and for any pair of cells C and D in $C_X(F)$, there exists an isomorphism on vertices from of $\hat{C}$ to $\hat{D}$ such that the image of θC is θD.
(2) The automorphism group $\mathrm{Aut}(\hat{C})$ of $\hat{C}$ acts doubly transitively on θC, which means that it acts transitively on the set of ordered pairs $\{(x, y) \mid x, y \in \theta C, x \neq y\}$, where $g((x, y))$ is defined as $(g(x), g(y))$ for any $g \in \mathrm{Aut}(\hat{C})$.

We see that the finite path graph in the example is symmetrically self-similar as the additional conditions are met.

Under these conditions, an *origin cell* is a cell C such that $\psi(\theta C) \subset C$. A vertex fixed point of ψ is called an *origin vertex*. If X is self-similar with respect to B and ψ, then by Krön, [9, Theorem 1] X has precisely one origin cell and no origin vertex or exactly one origin vertex, which we shall denote **o** when it is the case.

As the cells are isomorphic, we can talk about the cell of a symmetrically self-similar graph. Denote this cell by C and its boundary in X by θC. The boundary vertices θC are called the *extremal vertices*. If there are θ of them, then θ is the *branching number* of X. If we write $\hat{C}$ as *the* cell associated with the graph, we are referring to the closure in X.

It can be shown that the edges of $\hat{C}$ can be partitioned into μ complete graphs on θ vertices, and that ψ maps instances of $\hat{C}$ to complete graphs on the θ extremal vertices of $\hat{C}$ (i.e., $\psi(\hat{C})$ is isomorphic to K_θ). By simple accounting, we have $\mu = \frac{2|E|}{\theta(\theta-1)}$ and so μ corresponds to the usual mass scaling factor of self-similar sets. Figure 2 has three examples of cell graphs (indicating the extremal vertices in blue), and showing the values of θ and μ. If a self-similar graph is bipartite, its cell graph is bipartite and $\theta = 2$.

2.2 A blowup construction

Krön and Teufl [10, Theorem 1] describe the reverse of this construction as an iterative process whose limit is a symmetrically self-similar graph. The initial graph is $\hat{C}$, and at each step, the next graph is generated using an inverse action to ψ: the μ copies of K_θ in $\hat{C}$ are "blown up" and are replaced by a copy of the $\hat{C}$. The sequence of substitutions

[2] All vertices have finite degree.

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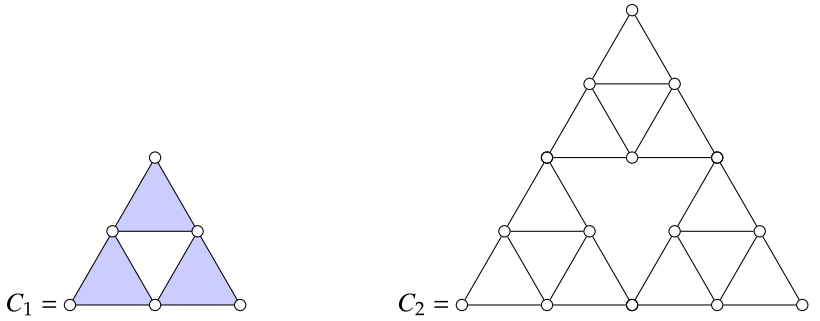


Figure 4: $C_1 = \hat{C}$ and C_2. From C_i to C_{i+1}, the shaded triangles are each replaced by a copy of $\hat{C}$. The limit graph in this case is the Sierpiński graph.

converges to a unique graph X, which is fixed under ψ. The first few iteration that builds the Sierpiński graph is pictured in Figure 4.

Remark that the graph with a finite number of copies of this graph centered at $\mathbf{o}$ is also a fixed point of ψ. That is, given $\hat{C}$, we can generate a countable family of symmetrically self-similar graphs. However, remark that as copies are added, the outward probability of the origin is reduced by the same factor, and hence the Green's functions will all be the same.

3 Green's functions

Our work relies heavily on the existing literature on Green's functions.

3.1 Finite graphs

The study of Green's functions of finite graphs is a classic topic, going back to at least the 1950s. We will recall some results as presented in [7], as they are very helpful for our purpose. Very roughly, in the case of a finite graph, we can express Green's functions as a quotient of two determinants of the probability matrix. Thus, in this case, the Green's function will be a rational function.

Proposition 4 (Application of Proposition V.6 of Flajolet and Sedgewick) *Let W be a digraph on the vertex set $\{v_1, \ldots, v_k\}$ and let T be the $k \times k$ matrix of edge weights $T[i,j] = \text{weight}(v_i, v_j)$. The weight of a path is the product of the weights of the edges in the path. The series $F_{i,j}(z)$ defined so the coefficient of z^n is the sum of the weights of all paths from v_i to v_j of length n in W, is the entry $[i,j]$ of the matrix $(I - zT)^{-1}$. Otherwise stated,*

$$F_{i,j}(z) = \left((1 - zT)^{-1}\right) = (-1)^{i+j} \frac{\Delta^{j,i}(z)}{\Delta(z)}. \tag{3}$$

Here, $\Delta(z) = \det(I - zT)$ and $\Delta^{j,i}(z)$ is the determinant of the minor of index j, i of $I - zT$.

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Thus, in the finite case, the Green's function is a rational function. We can deduce further information about a singular expansion near its dominant singularity. Recall that matrices constructed this way are said to be irreducible if the graph system is strongly connected, that is, there is a path between any two vertices.

Lemma 5 (Lemma V.1 of [7] (Iteration of irreducible matrices)) *Let the nonnegative matrix T be irreducible and aperiodic, with λ_1 its dominant eigenvalue. Then, the residue matrix such that*

$$(I - zT)^{-1} = \frac{\Phi}{1 - z\lambda_1} + O(1) \quad (z \to \lambda^{-1})$$

has entries given by $\phi_{i,j} = \frac{r_i \ell_j}{\langle r, \ell \rangle}$, where r and ℓ are, respectively, right and left eigenvectors of T corresponding to the eigenvalue λ_1.

Theorem 6 (Theorem V.7 of [7] (Asymptotics of paths in finite graphs)) *Consider the matrix*

$$F(z) = (I - zT)^{-1},$$

where T is a scalar nonnegative matrix, in particular, the adjacency matrix of a graph equipped with positive weights. Assume that T is irreducible. Then, all entries $F_{i,j}(z)$ of $F(z)$ have the same radius of convergence ρ, which can be defined in two equivalent ways:

(1) *as $\rho = \lambda_1^{-1}$, where λ_1 is the largest positive eigenvalue of T;*
(2) *as the smallest positive root of the determinantal equation:* $\det(I - zT) = 0$.

Furthermore, the point $\rho = \lambda_1^{-1}$ is a simple pole of each $F_{i,j}(z)$. If T is irreducible and aperiodic, then $\rho = \lambda_1^{-1}$ is the unique dominant singularity of each $F_{i,j}(z)$, and

$$[z^n]F_{i,j}(z) = \phi_{i,j}\lambda_1^n + O(\Lambda^n), 0 \le \Lambda < \lambda_1,$$

for computable constants $\phi_{i,j} > 0$.

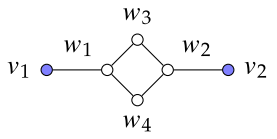
Given a finite graph with vertex set $\{v_1, \dots v_k\}$ if we set T to be P, the matrix of transition probabilities as defined in Equation (1), then $(T^n)_{ij} = p^{(n)}(v_i, v_j)$, and $F_{ij}(z) = G(v_i, v_j|z)$. From this result, we can understand why all Green's functions of a fixed finite graph have the same radius of convergence: they are all the multiplicative inverse of the dominant eigenvalue of T.

Since our cells are connected undirected graphs, T is irreducible, and it is only periodic when the cell is bipartite. We have a good understanding of this case, and principally, we can show (by considering T^2, which will not be periodic) that in the value we are interested in:

$$[z^n]F_{1,1}(z) = \begin{cases} \phi_{1,1}\lambda_1^n + O(\Lambda^n), 0 \le \Lambda < \lambda_1, & n \text{ even} \\ 0, & \text{otherwise.} \end{cases}$$

Recall the following two facts from classic matrix theory. If T is a stochastic matrix, then 1 is a dominant eigenvalue of T. If T is a sub-stochastic matrix, that is, there is at least one rowsum strictly less than 1, then the dominant eigenvalue is strictly less

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$\hat{C} =$ 

$$P = \begin{array}{c|cccccc} & v_1 & v_2 & w_1 & w_2 & w_3 & w_4 \\ \hline v_1 & 0 & 0 & 1 & 0 & 0 & 0 \\ v_2 & 0 & 0 & 0 & 1 & 0 & 0 \\ w_1 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ w_2 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ w_3 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ w_4 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{array}$$

Figure 5: A graph and its transition matrix.

than 1, and hence the dominant singularity of any $F_{i,j}(z)$ will be strictly greater than 1 since it is the multiplicative inverse of the dominant singularity.

Example 2 (Diamond) Consider the labeled graph and its transition matrix given in Figure 5.

This matrix has two dominant eigenvalues 1 and −1, and we can see that

$$G_{\hat{C}}(v_1, v_1|z) = F_{1,1}(z) = \frac{z^4 - 9z^2 + 9}{3\left(z^2 - 3\right)\left(z^2 - 1\right)}.$$

The series starts $G(v_1, v_1|z) = 1 + \frac{1}{3}z^2 + \frac{2}{9}z^4 + \frac{5}{27}z^6 + \frac{14}{81}z^8 + \mathrm{O}\left(z^{10}\right)$. For example, we see that $p^{(8)}(v_1, v_1) = \frac{14}{81}$. Remark that the periodicity is evident as $G_{\hat{C}}(v_1, v_1|z)$ is a function of z^2. This is a direct consequence of the bipartiteness of the graph.

In the next section, we apply these theorems to the cells of a symmetrically self-similar graph, setting some edge weights to 0.

3.2 Symmetrically self-similar graphs

Krön and Teufl translate the iterative fractal nature of X into a functional equation for the Green's function in one of the main results of [10], exploiting the iterative nature of the blow up construction. Define d to be the *transition function*: the generating function of the probabilities that the simple random walk on $\hat{C}$ starting at $\mathbf{o}$ in θC hits a vertex in $\theta C \backslash \{\mathbf{o}\}$ for the first time after exactly n steps. It may return to the origin before hitting this other cell.

Next, define f to be the *return function*: the generating function of f_n, the probability that the random walk on $\hat{C}$ starting at $\mathbf{o}$ returns to $\mathbf{o}$ after n stages without hitting a vertex in $\theta C \backslash \{\mathbf{o}\}$. Since the start is considered the first visit, $f_0 = 1$, and thus $f(0) = 1$. We also use $r(z)$, the probability function of first return which satisfies $f(z) = \frac{1}{1-r(z)}$.

Because $\hat{C}$ is finite, f and d are expressed in terms of Green's functions of a finite graph. They are well understood: they are rational functions that can be computed as the determinant of a specific matrix. We first give the functional equation, and then consider singular expansions of G, d, and f.

Given the closed cell $\hat{C}$, and let $P_{\hat{C}}$ be the probability transition matrix associated with it. Set P_f to be the matrix $P_{\hat{C}}$ with the modification $P_f[i, j] := 0$ for $1 < j \leq \theta$,

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$i > \theta$. Then, $f(z) = \left[(I - zP_f)\right]^{-1}[1,1]$. Set P_d to be the matrix $P_{\hat{C}}$ with the modification $P_d[i,j] := 0$ for $1 < i \leq \theta$, $j > \theta$. Then, $d(z) = \sum_{j=2}^{\theta} \left[(1 - zP_d)\right]^{-1}[1,j]$. Thus, both f and d can be determined by straightforward matrix inverse computations (possibly followed by some sums).

Lemma 7 (Krön and Teufl [10, Lemma 3]) *Let $G(z)$ be the Green's function for a symmetrically self-similar graph with origin* **o**. *Then, the following equation holds for all $z \in U(0,1)$:*

$$G(z) = f(z)G(d(z)). \tag{4}$$

This functional equation was first established and exploited in the case of the Sierpiński graph by Grabner and Woess [8] before being generalized for all self-similar graphs by Krön [9].

Example 3 (Star graphs) Consider a cell that is a path on three vertices, as in Example 1. It is straightforward to show that $f(z) = \frac{2}{2-z^2}$ and $d(z) = \frac{z^2}{2-z^2}$, and that any star graph thus has $G(z) = \frac{1}{\sqrt{1-z^2}}$, as it satisfies $G(z) = f(z)G(d(z))$. Remark that this is an algebraic function, a solution y to $y^2(1-z^2) - 1 = 0$. We further deduce $p^{(n)}(\mathbf{o},\mathbf{o}) = \binom{2n}{n}2^{-n}$, for even n, 0 otherwise.

Example 4 (Diamond example continued) We can define f and d for the diamond example using the following matrices of graph weights:

$$P_f = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix} \quad P_d = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix}. \tag{5}$$

It follows that

$$f(z) = (1 - zP_f)^{-1}[1,1] = -\frac{3\left(2z^2 - 3\right)}{z^4 - 9z^2 + 9} = 1 + \frac{1}{3}z^2 + \frac{2}{9}z^4 + \mathrm{O}\left(z^{10}\right)$$

$$\text{and} \quad d(z) = (1 - zP_d)^{-1}[1,2] = \frac{z^4}{z^4 - 9z^2 + 9} = \frac{1}{9}z^4 + \frac{1}{9}z^6 + \frac{8}{81}z^8 + \mathrm{O}\left(z^{10}\right).$$

The dominant eigenvalues of both P_f and P_d are $\frac{1+\sqrt{5}}{2\sqrt{3}}$ and hence $\rho_f = \rho_d = \frac{2\sqrt{3}}{1+\sqrt{5}} \approx 1.070$. Using these, we can determine the initial series expansion of the Green's function,

$$G(\mathbf{o},\mathbf{o}|z) \equiv G(z) = f(z)\, f(d(z))\, f(d(d(z)))\ldots \tag{6}$$

$$= 1 + \frac{1}{3}z^2 + \frac{2}{9}z^4 + \frac{5}{27}z^6 + O(z^7). \tag{7}$$

We inspect the components of this equation more closely. The Green's function $G(u,v|z)$ of a finite graph with transition probability matrix P is the corresponding

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$[u, v]$ entry of $(I - zP)^{-1}$. Analytically, we know a fair amount about G in this case. Remark that the result does not apply directly to bipartite graphs, as P fails the condition of aperiodicity, but there are workarounds using P^2, for example. A key result is the singular expansion of f (and d) is of the form $\kappa(1 - z/\rho)^{-1} + O(1)$, as $z \to \rho$ for a real, positive κ. Furthermore, as $\hat{C}$ is finite, d and f are rational.

A graph with bounded geometry here refers to a graph where the number of neighbors each vertex has is finite and has a uniform upper bound. In the case of symmetrically self-similar graphs with bounded geometries, Krön and Teufl exploit their functional equation to determine the singular expansion of $G(\mathbf{o}, \mathbf{o}|z)$ near $z = 1$. Here, $\tau = d'(1)$ and $\alpha = f(1)$. From their combinatorial definition, note that both quantities are greater than 1.

Theorem 8 (Krön and Teufl [10, Theorem 5]) *Let X be a symmetrically self-similar graph with bounded geometry and origin vertex $\mathbf{o}$. Then, there exists a 1-periodic, holomorphic function ω on some horizontal strip around the real axis such that the Green's function G has the local singular expansion*

$$\forall |z| < 1, \quad G(z) = (z-1)^{\eta} \left(\omega\left(\frac{1-z}{\tau}\right) + o(z-1) \right) \tag{8}$$

with $\eta = \frac{\log \mu}{\log \tau} - 1$.

Recall that μ is the number of cliques that form $\hat{C}$. When $\theta = 2$, this is the number of edges in $\hat{C}$. Moreover, even though it is not directly stated, it follows from a quick analysis of the asymptotics of G at 1 that $\eta = -\frac{\log \alpha}{\log \tau}$. Because α and τ are both greater than 1, η is a negative number.

Example 5 (Diamond example continued) Since we have exact expressions for f and d, we compute η:

$$\eta = \frac{\log \mu}{\log \tau} - 1 = \frac{\log 6}{\log 18} - 1 = -\frac{\log 3}{\log 18} = \frac{f(1)}{d'(1)} \approx -0.3800.$$

The domain of analyticity of G is governed by the dynamic behavior of f and d, according to Krön [9] and Krön and Teufl [10]. If ω is not constant, a singularity analysis shows that G is not algebraic. Krön and Teufl conjecture that: if ω is constant, then $\hat{C}$ is a path graph.

Using a result of Di Vizio, Fernandes, and Mishna, which appeared recently in a preprint [5], we can say more.

Theorem 9 (Vizio, Fernandes, and Mishna [5, Special case of Theorem C]) *Suppose $y(z) \in \mathbb{C}[[z]]$ is a series that satisfies the following iterative equation with some $a, b \in \mathbb{C}(z)$:*

$$y(z) = a(z)y(b(z)).$$

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If $b(z) \in \mathbb{C}(z)$ satisfies the following conditions: $b(0) = 0$; $b'(0) \in \{0, 1, \text{roots of unity}\}$; and no iteration of $b(z)$ is equal to the identity, then either there is some $N \in \mathbb{N}^*$, such that $y(z)^N$ is rational, or y is differentially transcendental over $\mathbb{C}(z)$.

Putting these results together, we have our workhorse result.

Corollary 10 *Let $G(z)$ be the Green's function of a symmetrically self-similar graph with origin* **o**. *Then, $G(z)$ is either algebraic or differentially transcendental over $\mathbb{C}(z)$. If G is algebraic, then there exists a minimal N such that $G^N = P/Q$ with $P, Q \in \mathbb{C}[z]$, P and Q co-prime and Q monic. In this case, the equality*

$$f(z)^N \frac{P(d(z))}{Q(d(z))} = \frac{P(z)}{Q(z)} \tag{9}$$

is defined and true for all complex z except possibly a finite set of points.

Proof First, we recall that there is a unique analytic continuation of any Green's function $G(u, v|z)$ to some domain beyond the radius of convergence. If G is algebraic, using the polynomial equation it satisfies, we can verify that Equation (2) holds in that domain.

Next, under these circumstances, we verify that the hypotheses of Theorem 9 hold. We set b to d, f to a. These are both rational since the cells are finite. We then set y to G. By definition, two vertices on the boundary of the cell have distance at least two away, so $0 = d_0 = d(0)$ and $0 = d_1 = d'(0)$. No iteration of $d(z)$ is the identity, since each iteration increases the number of initial terms equal to 0. The rest follows from Theorem 9 since the hypotheses hold.

As to the validity of Equation (9), this follows from the fact that it can be reduced to a polynomial equation that is valid in the entire open unit disc. The exceptions are contained in the set of poles of f and d, the zeros of Q and any z such that $d(z)$ is a zero of Q. ∎

3.3 A few technical results on *f* and *d*

The derivative $d'(1)$ is called τ and is the expected number of steps for a walk on $\hat{C}$ that starts at **o** to hit another external vertex. By definition, external vertices are not adjacent, so $d'(1) > 2$. Since d_n, (f_n and r_n) are all probabilities, and hence nonnegative numbers, by Pringsheim's theorem, the radius of convergence ρ_d (respectively ρ_f and ρ_r) of d, (f and r) are singularities. Now, as d is a probability generating function, it follows that $d(1) = 1$ and thus $\rho_d > 1$ since d converges at 1. As these are Green's functions, we can deduce key facts about these elements that we use in subsequent proofs. The proofs are a mix of basic facts on Green's functions and series analysis. The results are demonstrated in Example 4.

Lemma 11 *For ρ_d, ρ_f, and ρ_r as defined above, the following are true:*

(1) $\rho_d = \rho_f$;
(2) ρ_d *and* ρ_f *are each poles of order 1;*
(3) $\rho_f < \rho_r$;

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(4) $\forall z \in (1, \rho_d)$, $d'(z) > 1$ *and therefore* $d(z) > z$;
(5) *for* $z \in [0, \rho_r]$ *if* $f(z) = \infty$, *then* $z = \rho_f$.

Proof of Lemma 11

(1) We apply Equation (3) to P_f and P_d to show that the radii of convergence of f and d are the same: These two matrices differ in the following ways – The columns of P_f associated with v_i, $i > 1$ are zero since there are no edges to these vertices. The rows of P_d associated with these v_i are zero. The determinants of $I - zP_f$ and $I - zP_d$ can be seen to be the same by iteratively taking cofactor expansions recursively along the rows associated with v_i and the columns associated with v_i, respectively. In both cases, the only nonzero element is the 1 on the diagonal, and up to a sign, the value of the determinant is equal to the determinant of the matrix with that row and column removed. Iterating this process, we can see that as P_d and P_f with all columns and rows associated with the v_i are removed give the same matrix, both determinant computations will eventually reduce to the same value. Thus, P_f and P_d have the same dominant eigenvalue, say λ. The dominant singularity in both cases is thus $\rho_f = \frac{1}{\lambda} = \rho_d$.
(2) This is a direct application of [7, Theorems V.7.*iii* and V.7.*i*].
(3) Since they are sums of nonnegative terms, f and r are continuous increasing functions along the interval from 0 to their radius of convergence. The inequality is a consequence of the relation $f(z) = \frac{1}{1-r(z)}$: $f(z)$ is singular at the minimum of ρ_r and a value where $r(z) = 1$. Since $r(0) = 0$, and $r(z)$ is increasing, the value of z where it is 1 is strictly less than ρ_r.
(4) We note that for z in the indicated interval $z > 0$ and so $d''(z) = \sum_{n \leq 2} n(n-1)d_n z^{n-2} > 0$, therefore, d' is increasing. The value $\tau = d'(1)$ is the expected time to travel between distinct elements in θC. By construction, the distance between two points in θC is at least two, so $\tau \geq 2$.
(5) The singularities of $f(z) = \frac{1}{1-r(z)}$ are contained in the set of solutions to $1 - r(z) = 0$ and the singularities of $r(z)$. Recall that ρ_f is the dominant singularity: the smallest positive real singularity of f. By definition, there are no positive real singularities of r smaller than its dominant singularity, ρ_r, so either if $f(z) = \infty$, then either $z = \rho_r$ or $1 - r(z) = 0$. Let us consider the latter case first. As $r(z)$ is increasing on this interval, there is a single value for z for which $1 - r(z) = 0$. Since this will give a pole for $f(z)$, this value is the dominant singularity of $f(z)$, ρ_f. There are no other singularities of f in the interval $[0, \rho_r)$. If $\rho_f = \rho_r$, then since $r(z)$ is positive and increasing, it must either evaluate to a positive number (in the case of a branch cut) or $r(\rho_r) = \infty$. If it evaluates to 1 then the previous argument holds. If it does not, then the conclusion is still logically valid (but it is empty). ■

4 Proof of the main theorem

Throughout this section, we assume that X is a symmetrically self-similar graph with bounded geometry, origin $\mathbf{o}$ and branching number $\theta = 2$. If ever we additionally

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assume that $G(z) = G(\mathbf{o}, \mathbf{o}|z)$ is algebraic, then we write $G^N = P/Q$ as in Corollary 10. Recall the statement of the Main Theorem.

Theorem 12 *Let X be a symmetrically self-similar graph with bounded geometry, origin $\mathbf{o}$ and branching number $\theta = 2$. Either the graph is a star consisting of finitely many one-sided lines coinciding at $\mathbf{o}$, or the Green's function $G(\mathbf{o}, \mathbf{o}|z)$ is differentially transcendental over $\mathbb{C}(z)$.*

Outline of Proof First, suppose X is bipartite. We show in Theorem 18 below that if G is algebraic, then $\hat{C}$ is a finite path graph. By [9, Lemma 2], X is a star centered at an origin vertex precisely when $\hat{C}$ is a path. Otherwise, G is differentially transcendental.

Otherwise, if X is not bipartite, we show, in Theorem 19, that G is differentially transcendental. Putting all of these components together, the result is established. ■

We can reframe this result as a case of the conjecture of Krön and Teufl.

Corollary 13 *Let X be a symmetrically self-similar graph with bounded geometry, origin $\mathbf{o}$, and branching number $\theta = 2$. Suppose that ω in Theorem 8 is constant, and G is algebraic. Then, the graph is a star consisting of finitely many one-sided lines coinciding at $\mathbf{o}$.*

4.1 The structure of an algebraic $G(z)$

Krön and Teufl [10, p. 13] showed that if X is not bipartite, then $z = 1$ is the only singularity of G, the boundary of the unit disk and if X is bipartite, then the only singularities on the unit disk are 1 and −1. We start by extending this result in the case that G is algebraic.

Lemma 14 *If $G(z)$ is algebraic, then the rational function G^N has only real zeros and poles. The only positive pole is at 1 and the only possible nonnegative 0 is at 0.*

Proof If G is algebraic, then ω in Theorem 8 is constant and by Krön [9, Theorem 7] the singularities of $G(z)$ are real, and contained in the set $(-\infty, -1] \cup [1, \infty)$. Writing $G^N = P/Q$, we have that the singularities of G are contained in the roots of P and Q, as these are the poles and zeros of G^N.

Toward a contradiction, assume there is some real z_0, with $z_0 > 1$ that is either a pole or a zero of G^N. Since G^N has only a finite number of zeros and poles, we can assume that z_0 is the smallest one strictly greater than 1. Since ρ_d is a pole of $d(z)$, $d((1, \rho_d)) = (1, \infty)$, and so there exists a $z' \in (1, \rho_d)$ such that $d(z') = z_0$. Similarly, f is analytic and defined on the interval $[1, \rho_d)$. By Lemma 11, $\rho_d = \rho_f$, so z' is not a pole of f. Since $f(0) = f_0 = 1$ (by convention, the length of a walk whose first contact with the origin for a walk that starts at the origin is 0) and f is increasing on this interval, so z' is not a zero of f either.

From Equation (9), $\frac{G(z)^N}{f(z)^N} = G(d(z))^N$, and so

$$\lim_{z \to z_0} G(z)^N = \lim_{z \to z'} G(d(z))^N = \lim_{z \to z'} \frac{G(z)^N}{f(z)^N}. \tag{10}$$

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Hence, as $\lim_{z\to z'} f(z) = f(z')$ is finite and nonzero, if z_0 is a zero or pole of G^N greater than 1, then z' is also a zero or pole, but is closer to 1, a contradiction.

This contradiction establishes that there is no real-valued zero or pole greater than 1.

From the singular expansion of G at 1 of Krön and Teufl, we have that 1 is the dominant singularity, and hence there is no other singularity of smaller modulus. Consequently, there is no positive real pole of G^N less than 1.

Since G is defined as a series with nonnegative coefficients, any evaluation of $G(z)$ at a positive real number in its domain of convergence is nonzero, and similarly for $G^N(z)$. Again, from the singular expansion, we can exclude the $G(1) = 0$ case since α and τ are both greater than 1. ∎

Lemma 15 *In this context, if G is algebraic, with $G^N = P/Q$, then* $\deg Q \geq N$.

Proof We start with Equation (9) and consider expansions of G, f, and d around $\rho_f = \rho_d$. Now, $G(z)$ is not singular nor 0 at ρ_f, since $\rho_f > 1$ by Lemma 14. Here, κ_f and κ_d are positive real constants by [7, Theorem 7]. Let $u = (1 - z/\rho_f)^{-1}$. We compute the following limits in $\mathbb{R}$:

$$\begin{aligned}
G(\rho_f)^N &= \lim_{z\to\rho_f} G(z)^N \\
&= \lim_{z\to\rho_f} f(z)^N \frac{P(d(z))}{Q(d(z))} \\
&= \lim_{z\to\rho_f} \left(\frac{\kappa_f^N}{(1-z/\rho_f)^N} + O(z-\rho_f)^{-N+1} \right) \frac{P(\kappa_d(1-z/\rho_f)^{-1} + O(1))}{Q(\kappa_d(1-z/\rho_f)^{-1} + O(1))} \\
&= \lim_{u\to\infty} (\kappa_f^N u^N + O(u)^{N-1}) \frac{P(\kappa_d u + O(1))}{Q(\kappa_d u + O(1))} \\
&= \lim_{u\to\infty} \kappa_f^N u^N \frac{P(\kappa_d u)}{Q(\kappa_d u)}.
\end{aligned}$$

Remark that as z approaches ρ_f, u is arbitrarily large. The degree of Q as a polynomial in z must be larger than N since the value of the last limit is nonzero. ∎

4.2 Results when $\theta = 2$

Theorem 16 *If $\theta = 2$, then $\alpha \leq \mu$ with equality if and only if $\hat{C}$ is a finite path graph.*

The proof is a little technical and is delayed to the next section. Recall that μ is the number of edges in $\hat{C}$, and α is the expected number of visits to v_1 before a visit to v_2. The proof but relies on a straightforward intuition that among all finite cell graphs $\hat{C}$ where v_1 and v_2 are a fixed distance apart, the average number of times a random walk starting at v_1 hits v_1 before v_2 (i.e., $f(1)$) is minimized when the graph is a line.

Corollary 17 *If $\theta = 2$, then η in the development in Equation (8) satisfies*

$$\eta \geq -\frac{1}{2},$$

with equality if and only if $\hat{C}$ is a finite path graph.

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Proof Starting from the definition of η, we compute the following bound:

$$\begin{aligned}\eta = -\frac{\log \alpha}{\log \tau} &= \frac{\log \mu}{\log \tau} - 1 \\ &\geq \frac{\log \alpha}{\log \tau} - 1 \quad \text{since} \quad \mu \geq \alpha \\ &= -\eta - 1.\end{aligned}$$

Solving for η gives $\eta \geq -\frac{1}{2}$. ∎

4.3 Case 1: *X* is bipartite

First, we consider the case of bipartite graphs. In this case, it must be that $\theta = 2$, since any complete graph on three or more vertices is not bipartite. The main result of this section is the following.

Theorem 18 *Let X be a symmetrically self-similar graph with bounded geometry, origin $\mathbf{o}$, and branching number $\theta = 2$. If, additionally, X is bipartite, and G is algebraic, then $\hat{C}$ is a path.*

Proof First, note that as X is bipartite, then for any n, $p^{2n+1}(\mathbf{o}, \mathbf{o}) = 0$, since bipartite graphs have no odd circuits. Thus, G is a series in z^2.

By Corollary 10, either G is differentially transcendental, or it is algebraic. Assume that G is algebraic. By Lemma 14, the only positive singularity of G is at 1. Since G^N is a series in z^2, its poles appear in signed pairs, and so -1 is also a pole, and no other negative real numbers. Since the poles are all real numbers, the only poles are at 1 and -1.

By Theorem 8, the exponent at the singular expansion is η. The singularities of G are contained in the roots of P and Q. Since they are co-prime P and Q have no common roots, we deduce that both 1 and -1 are roots of order $-nN$ of Q. Up to a constant, $Q(z)$ is $(1-z)^{-\eta N}(1+z)^{-\eta N}$. Now, $\deg Q = -2\eta N$ and $\deg Q \geq N$ from Lemma 15. Putting the two relations together, we have $\eta \leq -1/2$. This, coupled with Corollary 17, suggests that $\eta = -1/2$ precisely. By Corollary 17, this equality holds if and only if $\hat{C}$ is a path. ∎

4.4 Case 2: *X* is not bipartite

Next, we consider the case that X is not bipartite.

Theorem 19 *Let X be a non-bipartite symmetrically self-similar graph with origin $\mathbf{o}$, bounded geometry, and branching number $\theta = 2$. Then, the Green's function $G(z)$ is differentially transcendental over $\mathbb{C}(z)$.*

Proof We assume that G is algebraic and derive a contradiction, and thus G must be differentially transcendental. If X is not bipartite, then $z = 1$ is the only singularity on the boundary of the unit disc by [10, p. 13]. Since G is algebraic, then ω in Theorem 8 is constant, and thus the singularities of G are contained in the set $(-\infty, -1) \cup [1, \infty)$ but

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we have already shown that are no singularities in $(1, \infty)$ by Lemma 14. We next show that there are also no singularities in $(-\infty, -1)$. Since ρ_d is a pole of order 1 of d by Theorem 6, and d is increasing, we can deduce that $\lim_{z \to \rho_d^+} d(z) = -\infty$. Using a limit argument applied to Equation (9), We can show that f has neither zero nor pole in the interval (ρ_f, ρ_r), and d has also no pole in the interval (ρ_f, ρ_r). Therefore, by the intermediate value theorem, for any $z_0 \in (-\infty, -1)$, there is some $z' \in (\rho_f, \rho_r)$ such that $d(z') = z_0$. Since $z' > \rho_f$, z' is a positive real number greater than 1, therefore, z' is neither a pole nor a zero of G.

We substitute this into Equation (9)

$$G^N(z') = \frac{P(z')}{Q(z')} = f(z')^N \frac{P(d(z'))}{Q(d(z'))} = f(z')^N \frac{P(z_0)}{Q(z_0)} = f(z')G(z_0).$$

We can conclude that $G(z_0)$ is finite and nonzero.

Thus, given the singular expansion of G at 1, we have that Q has a single zero and it is at 1, and is of order $-N\eta$. Combining prior results:

$$N \leq \deg Q = -N\eta \implies \eta \leq -1.$$

This directly contradicts that $\eta \geq -\frac{1}{2}$. Thus, G cannot be algebraic. ∎

Remark that for $\theta > 2$, it suffices to find a similarly incompatible lower bound on η. This same proof should apply.

5 Proof of Theorem 16

To finish, we now prove Theorem 16 which states that if $\theta = 2$, then $\mu \geq \alpha$ with equality if and only if C is a finite path graph. In fact, we conjecture that the inequality holds for all cell graphs, and that when $\theta > 2$, the inequality is strict.

5.1 Discrete harmonic functions on graphs

Let $\hat{C}$ be a cell of a symmetrically self-similar graph with $\theta = 2$. For $v \in V(\hat{C})$, let $H(v)$ be the probability that a random walk starting at v will reach v_1 before v_2. Hence, $H(v_1) = 1$ and $H(v_2) = 0$. By decomposing the first step of the random walk, we have

$$H(v) = \frac{1}{\deg u} \sum_{u \in N(v)} H(u).$$

Then, H is a harmonic function[3] of the graph C with boundary conditions given by the values on v_1 and v_2. It is a classic result that there is a unique harmonic function on a graph with fixed boundary conditions. Since $\theta = 2$, v_1 has only one neighbor, call it w. Then, $\alpha - 1$ is the average time that a random walk starting at v_1 makes the step $w \to v_1$. After that step, the only possible step is the step $v_1 \to w$. A random walk on $\hat{C}$ starting at v_1 can have a unique decomposition[4] on the steps from v_1 to w (which

[3] A discrete harmonic function is a function on a graph where the value at any node equals the average of the values at its neighbors. It satisfies a discrete version of the Laplace equation.

[4] See Flajolet and Sedgewick [7] for a detailed treatise on combinatorial grammars that formalize such decompositions.

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we denote $v_1 \to w$):

$$\mathcal{W}_1 = (v_1 \to w) \times \mathbf{SEQ}(\mathcal{W}_2 \times (v_1 \to w)) \times \mathcal{W}_3.$$

Here,

- $\mathcal{W}_1$ is the set of walks starting at v_1 that stop after reaching v_2, but can return to v_1 multiple times.
- $\mathcal{W}_2$ is the set of walks starting at w that never reach v_2 and stop after reaching v_1.
- $\mathcal{W}_3$ is the set of walks starting at w that never reach v_1 and stop after reaching v_2.

We denote by α the expected number of visits to v_1 in a walk in $\mathcal{W}_1$ (including the start of the walk). From the decomposition, we see that this is one more than the expected number of $\mathcal{W}_2$ sub-walks, as each one ends in a unique return to v_1.

Let $\mathcal{W}_4$ be the set of walks starting at w that stop the first time they reach either v_1 or v_2. We can decompose this set by endpoint (v_1 and v_2, respectively) into disjoint sets:

$$\mathcal{W}_4 = \mathcal{W}_2 + \mathcal{W}_3.$$

The probability that a walk in $\mathcal{W}_4$ is also in $\mathcal{W}_2$ is precisely $H(w)$. Therefore, the number of $\mathcal{W}_2$ in the decomposition of walk in $\mathcal{W}_1$ follows a geometric law of parameter $H(w)$. Hence, $\alpha - 1 = \dfrac{1}{1 - H(w)}$. Finding an upper bound of α is therefore equivalent to finding an upper bound of $H(w)$.

5.2 The finite path graph

We can determine α explicitly for a path L on μ edges, as the harmonic function $H_L(v)$ is straightforward to compute. Remark that $d(v_2, v_1) = \mu$ in this case. Classically,

$$H_L(v) = \frac{d(v_2, v)}{d(v_2, v_1)}. \tag{11}$$

Remark that $H_L(v_1) = 1$, $H_L(v_2) = 0$ and for the vertex v at distance d from v_2, $d \in \{1, \ldots, d(v_1, v_2) - 1\}$,

$$H_L(v) = \frac{1}{2}\left(\frac{d-1}{d(v_1, v_2)} + \frac{d+1}{d(v_1, v_2)}\right) = \frac{d}{d(v_1, v_2)}.$$

Therefore, in the case of a path,

$$\alpha = 1 + \frac{1}{1 - H_L(w)} = 1 + \frac{1}{1 - \frac{d(v_1,v_2)-1}{d(v_1,v_2)}} = d(v_1, v_2) = \mu.$$

5.3 General case

Consider a cell graph $\hat{C}$. We compare values of H, the harmonic function defined on $\hat{C}$ and boundary values $H(v_1) = 1, H(v_2) = 0$ to H_L as defined in Equation (11).

First, consider the following lemma.

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Lemma 20 *Let $\hat{C}$ be a symmetric graph with $\theta = 2$ and F be the harmonic function defined on $\hat{C}$ the value on the boundary: $H(v_1) = 1$ and $H(v_2) = 0$. Let $u \in V(C)$ be a vertex such that there is a connected component $\tilde{C}$ of, $C\backslash\{u\}$ which contains both v_1 and v_2. Let $\tilde{H}$ be the harmonic function defined on $\tilde{C}$ by the same boundaries as for F. Then,*

$$\forall x \in V(\tilde{C}), \quad H(u) \le H(x) \implies H(x) \le \tilde{F}(x).$$

That is, if the probability of a reaching v_1 before v_2 is greater starting at x than at u, this probability is bounded above the analogous probability in the graph where u is removed, provided that v_1, v_2, and x are all in the same component after the removal.

Proof The walks in $\hat{C}$ starting at x absorbed by a boundary can be decomposed into three disjoint sets, according to which point v_1, v_2, u it will reach first. We assign the probabilities, $a, b, 1-a-b$, to be the respective probabilities that a walk starting at x will reach v_1, v_2, or u first. Then, by linearity of expectation,

$$H(x) = aH(v_1) + bH(v_2) + (1-a-b)H(u) \tag{12}$$

$$= a + (1-a-b)H(u) \tag{13}$$

$$\implies H(u) = \frac{H(x)-a}{1-a-b}. \tag{14}$$

Let A be the set of walks that reach v_1 before u, and let B be the set of walks that never touch u. Then, $a = P(A) = P(A|B)P(B) = \tilde{H}(x)(a+b)$ since the set of walks in $\hat{C}$ that start at x and never touch u are in bijection with the set of walks that start at x in $\tilde{C}$. Therefore, $\tilde{H}(x) = \frac{a}{a+b}$, and if $H(u) \le H(x)$,

$$\frac{H(x)-a}{1-a-b} = H(u) \le H(x) \implies H(x) \le \frac{a}{a+b}$$
$$\implies H(x) \le \tilde{H}(x). \qquad \blacksquare$$

We can now prove the main result of the section.

Proof of Theorem 16 Given a sequence of graphs, we create the sequence of harmonic functions on those graphs with boundary values on v_1 and v_2 of 1 and 0, respectively. Let $(u_1, = v_1, u_2 = w, \ldots, u_{d+1} = v_2)$ be a shortest path from v_1 to v_2 in $\hat{C}$. The graph sequence is built by iteratively moving along the path, and removing all the neighbors of vertices on the path, except $u_1, \ldots, u_{d+1}$. In every subgraph, as w is the only vertex adjacent to v_1, every walk starting from a different vertex must pass through w. Thus, the harmonic function on this graph, H', satisfies $H'(v) \le H'(w)$ for all $v \ne v_1$ in the subgraph. If v is off of the path, then v_1, v_2, and w will be in the same component if v is deleted from the graph. Thus, in the graph with v removed, the value of its harmonic function $\tilde{H}$ satisfies $H(w) \le \tilde{H}(w)$ by the above lemma.

Thus, in the sequence of harmonic functions generated by the sequence of graphs, the value on w (or indeed, any point on the path) is monotonically increasing. The process terminates at a path, the smallest connected graph, since removing any vertex will separate v_1 and v_2.

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Therefore, in the general case, $H(w) \leq H_L(w) = \frac{d(v_1,v_2)-1}{d(v_1,v_2)}$ and so $\alpha \leq d(v_1, v_2) \leq \mu$. The last inequality is an equality if and only if $\hat{C}$ is a finite path graph since μ is the total number of edges in the graph. ■

6 Perspectives

A proof of Conjecture 1 with this argument framework could be accomplished with two additional results. First, we could prove a version of Theorem 16 for $\theta > 2$. Our arguments use the fact that v_1 has only one neighbor. With some additional bookkeeping, perhaps the argument could be extended. The core is doubly vertex transitive, and this might be useful. Next, one must show that when X is not bipartite, and G is algebraic, Q only has a zero at 1. The fact that G^N is rational is a very strong characterization and tells that each singularity is isolated. This, in combination with known information about the Julia set may yield the desired result. Establishing the more general result of Krön and Teufl would then be reduced to showing that the only case of a constant ω occurs with an algebraic Green's function.

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