A rich calculus problem about a frustum

The problem is this:

> *What is the maximum vertical height of a frustum of a right-circular cone with fixed base and given curved surface area?*

Intuitively, one might anticipate that one of the extreme cases (a cone or a cylinder) might be the maximising shape, but the reality is considerably more complicated. For the frustum shown in Figure 1, its curved surface area S is given by the difference between the curved surface area of a large cone (base radius R) and that of a smaller cone (base radius r) both of which are similar to the cone shown in Figure 2 with respective area scale factors of $\left(\frac{R}{R-r}\right)^2$ and $\left(\frac{r}{R-r}\right)^2$. We thus have

$$S = \pi(R-r)d\left(\frac{R^2}{(R-r)^2} - \frac{r^2}{(R-r)^2}\right) = \pi(R+r)d = \pi(R+r)\sqrt{(R-r)^2+h^2}$$

from which $h = \sqrt{\frac{S^2}{\pi^2(R+r)^2} - (R-r)^2}$.

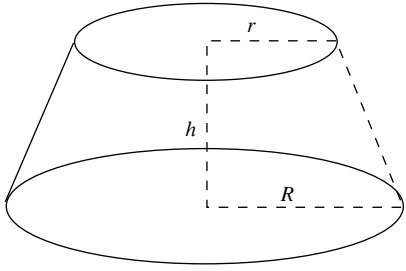


FIGURE 1

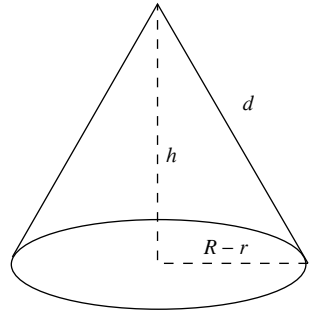


FIGURE 2

It is more convenient to work with the dimensionless variables $x = \frac{r}{R}$, $z = \frac{h}{R}$, and constant $K = \frac{S^2}{\pi^2 R^4}$ for which

$$z = \sqrt{\frac{K}{(1+x)^2} - (1-x)^2}. \tag{1}$$

The relevant domain for which z in (1) is well-defined is $0 \leqslant x \leqslant 1$ for $K \geqslant 1$ but

$$\sqrt{1-\sqrt{K}} \leqslant x \leqslant 1 \text{ for } 0 < K < 1. \tag{2}$$

[Notice that the latter restriction corresponds to $S \geqslant \pi(R^2 - r^2)$ in Figure 1; the curved surface area has to be greater than its annular projection onto the base.]

https://doi.org/10.1017/mag.2015.55 Published online by Cambridge University Press

Differentiating (1) gives

$$(1 + x)^3 z\frac{dz}{dx} = (1 - x)(1 + x)^3 - K, \tag{3}$$

so the key to analysing this problem lies in the graph of $y = (1 - x)(1 + x)^3$ shown in Figure 3.

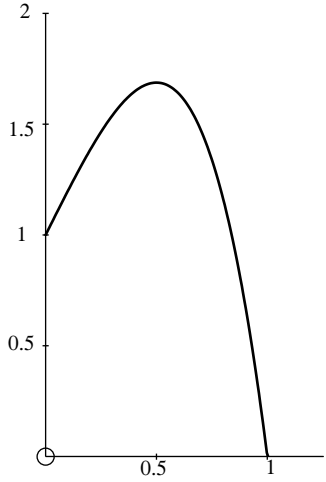


FIGURE 3: The graph of $y = (1 - x)(1 + x)^3$ with maximum point $\left(\frac{1}{2}, \frac{27}{16}\right)$

- *Case* 1: $0 < K \leqslant 1$

Here, (3) shows that $\frac{dz}{dx} < 0$ when $x = 1$. At $x = \sqrt{1 - \sqrt{K}}$, the other end of the domain in (2), for $0 < K < 1$ we have

$$\begin{aligned}(1 - x)(1 + x)^3 - K &= \left(1 - x^2\right)\left(1 + 2x + x^2\right) - K \\ &> 1 - x^4 - K \\ &= 1 - \left(1 - \sqrt{K}\right)^2 - K \\ &= 2\sqrt{K}\left(1 - \sqrt{K}\right) > 0.\end{aligned}$$

Thus, from Figure 3, there is always a stationary point in the relevant domain which is a global maximum, together with a vertical tangent at $x = \sqrt{1 - \sqrt{K}}$ where $z = 0$; see Figure 4(a).

When $K = 1$, the shape of the graph is similar to this but (from (3)) the vertical tangent is replaced by one at $x = 0$ of gradient $\sqrt{2}$ because

$$\lim_{x\to 0^+}\frac{dz}{dx} = \lim_{x\to 0^+}\frac{(1-x)(1+x)^3 - 1}{(1+x)^3 z} = \lim_{x\to 0^+}\frac{2 - 2x^2 - x^3}{(1+x)^2\sqrt{2-x^2}} = \sqrt{2}.$$

(By contrast, we shall see below that, for $K > 1$, the tangent at $x = 0$ has a negative gradient.)

https://doi.org/10.1017/mag.2015.55 Published online by Cambridge University Press

- *Case* 2: $1 < K \leqslant \frac{27}{16}$

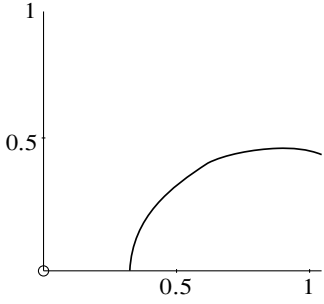
For $1 < K < \frac{27}{16}$, Figure 3 shows that there are two stationary points in the relevant domain, $0 \leqslant x \leqslant 1$. The smaller value of x gives a local minimum, the larger x a local maximum as in Figure 4(b). Just below $K = \frac{27}{16}$, there are one or more endpoint extrema as in Figure 4(c).

When $K = \frac{27}{16}$, the stationary points merge to give a point of inflection at $x = 0.5$, Figure 4(d).

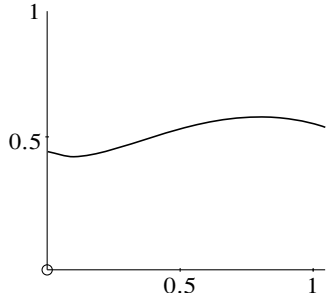
- *Case* 3: $K > \frac{27}{16}$

Here, from (3), $\frac{dz}{dx} < 0$ for $0 \leqslant x \leqslant 1$ so z is strictly decreasing with $x = 0$ giving an endpoint maximum.

It is interesting to note that $x = 1$ (the cylinder) never gives a (global) maximum value of h; $x = 0$ (the cone) gives a (global) maximum value of h only for $K > 6\sqrt{3} - 9 \approx 1.392$. The latter value may either be found numerically using a graph plotter or algebraically. For, by (3), a local maximum occurs where $(1 - x)(1 + x)^3 = K$ and, by (1), this gives the same value for z as that for $x = 0$ if we also have $K - 1 = \dfrac{K}{(1 + x)^2} - (1 - x)^2$. Eliminating K leads to the quadratic $x^2 + 2x - 2 = 0$ from which $x = \sqrt{3} - 1$ and $K = 6\sqrt{3} - 9$.

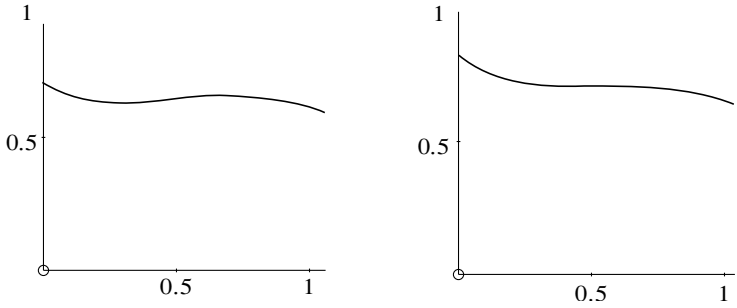


(a) $K = 0.8$



(b) $K = 1.2$

https://doi.org/10.1017/mag.2015.55 Published online by Cambridge University Press



(c) $K = 1.5$ (d) $K = \frac{27}{16} = 1.6875$

FIGURE 4: Graphs of $z = \sqrt{\dfrac{K}{(1+x)^2} - (1-x)^2}$ for the values of K as shown

Finally, a confession. I stumbled across this problem by misremembering the similarly counterintuitive problem in [1]. There, Johnson considers:

> *What is the maximum (or minimum) curved surface area of a frustum with fixed base and fixed vertical height?*

Since $S = \pi(R + r)\sqrt{(R - r)^2 + h^2}$, this corresponds to investigating the curve $w = (1 + x)\sqrt{L + (1 - x)^2}$ with the dimensionless variables $w = \dfrac{S}{\pi R^2}$, $x = \dfrac{r}{R}$ and constant $L = \dfrac{h^2}{R^2}$. Because $\dfrac{dw}{dx} = \dfrac{2x^2 - 2x + L}{\sqrt{L + (1 - x)^2}}$, the sign of $\frac{dw}{dx}$ is the same as that of $x^2 - x + \frac{1}{2}L$. Here the domain is $0 \leqslant x \leqslant 1$ throughout and the split of cases is:

(a) $0 < L < \frac{1}{2}$: As x increases from 0 to 1, w first has a local maximum, then a local minimum (again, with the possibility of the endpoint cone or cylinder yielding global maxima and minima).

(b) $\frac{1}{2} \leqslant L$: Here, w is strictly increasing as x increases from 0 to 1 (with a stationary point of inflection at $x = \frac{1}{2}$ when $K = \frac{1}{2}$).

Acknowledgement

I am very grateful to the referee whose careful reading of the first draft of this article led to several significant corrections, clarifications and improvements.

Reference

1. R. A. Johnson, A problem in maxima and minima, *Amer. Math. Monthly* **35** (1928) pp. 187-188.

doi:10.1017/mag.2015.55

NICK LORD
Tonbridge School, Kent TN9 1JP

https://doi.org/10.1017/mag.2015.55 Published online by Cambridge University Press