Combinatorics, Probability and Computing (2026), **35**, pp. 714–733
doi:10.1017/S0963548326100492

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ARTICLE

Computing the probability of intersection

Alexander Barvinok

University of Michigan, USA
Email: barvinok@umich.edu

(Received 13 September 2025 UTC; revised 15 March 2026 UTC; accepted 22 May 2026 UTC;
first published online 19 June 2026 UTC)

Abstract

Let $\Omega_1, \dots, \Omega_m$ be probability spaces, let $\Omega = \Omega_1 \times \cdots \times \Omega_m$ be their product and let $A_1, \dots, A_n \subset \Omega$ be events. Suppose that each event A_i depends on r_i coordinates of a point $x \in \Omega$, $x = (\xi_1, \dots, \xi_m)$, and that for each event A_i there are Δ_i other events A_j that depend on some of the coordinates that A_i depends on. Let $\Delta = \max\{5,\ \Delta_i : i = 1, \dots, n\}$ and let $\mu_i = \min\{r_i,\ \Delta_i + 1\}$ for $i = 1, \dots, n$. We prove that if $\mathbb{P}(A_i) < (3\Delta)^{-3\mu_i}$ for all i, then for any $0 < \epsilon < 1$, the probability $\mathbb{P}\left(\bigcap_{i=1}^n \overline{A}_i\right)$ of the intersection of the complements of all A_i can be computed within relative error ϵ in polynomial time from the probabilities $\mathbb{P}\left(A_{i_1} \cap \ldots \cap A_{i_k}\right)$ of k-wise intersections of the events A_i for $k = e^{O(\Delta)} \ln(n/\epsilon)$.

Keywords: Probability; zeros of polynomials; partially dependent events; Lovász local lemma; algorithm

2020 MSC Codes: Primary: 60C05; Secondary: 68Q87, 68W05, 30C15

1. Introduction and main results

1.1 The setup

Let Ω be a probability space and let $A_1, \dots, A_n \subset \Omega$ be events. We want to compute (or approximate)

$$\mathbb{P}\left(\bigcap_{i=1}^n \overline{A}_i\right) = 1 - \mathbb{P}\left(\bigcup_{i=1}^n A_i\right), \tag{1.1}$$

the probability of the intersection of their complements. This is of course a fairly general problem, with many applications. In the usual interpretation, A_i are some 'bad' events and so we are interested in the probability that none of them occurs. If the events $A_1, \dots, A_n$ are independent, then

$$\mathbb{P}\left(\bigcap_{i=1}^n \overline{A}_i\right) = \prod_{i=1}^n (1 - \mathbb{P}(A_i)), \tag{1.2}$$

so to compute the left-hand side of (1.2) exactly, we only need to know the probabilities $\mathbb{P}(A_i)$ themselves.

We show that if the dependencies are controlled and the probabilities $\mathbb{P}(A_i)$ are sufficiently small, then the value of (1.1) can be efficiently approximated within an arbitrarily small relative error $0 < \epsilon < 1$ from the probabilities

$$\mathbb{P}(A_{i_1} \cap \ldots \cap A_{i_k}) \quad \text{where} \quad k = O(\ln(n/\epsilon))$$

of the intersections of logarithmically many events A_i.

This research is partially supported by NSF Grant DMS 1855428.

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https://doi.org/10.1017/S0963548326100492 Published online by Cambridge University Press

Let $\Omega_1, \ldots, \Omega_m$ be probability spaces and let $\Omega = \Omega_1 \times \cdots \times \Omega_m$ be their product. We write a point $x \in \Omega$ as $x = (\xi_1, \ldots, \xi_m)$, where $\xi_j \in \Omega_j$, and call ξ_j *coordinates* of x. We consider complex-valued random variables $f : \Omega \longrightarrow \mathbb{C}$. For such an f, we define the *set of coordinates that f depends on*, as a subset $\{\xi_j : j \in J_f\}$ where $J_f \subset \{1, \ldots, m\}$, such that for any two $x', x'' \in \Omega$, where $x' = (\xi_1', \ldots, \xi_m')$ and $x'' = (\xi_1'', \ldots, \xi_m'')$, we have $f(x') = f(x'')$ whenever $\xi_j' = \xi_j''$ for all $j \in J_f$ and there is no subset $J' \subsetneq J_f$ with that property. We say that f *depends on at most r coordinates* if $|J_f| \leq r$. We say that two random variables $f, g : \Omega \longrightarrow \mathbb{C}$ *share a coordinate* if $J_f \cap J_g \neq \emptyset$, and any coordinate ξ_j with $j \in J_f \cap J_g$ is called a *shared coordinate* of f and g.

Let $A \subset \Omega$ be an event. The *indicator* of A is the random variable $[A] : \Omega \longrightarrow \{0, 1\}$ defined by

$$[A](x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases} \tag{1.3}$$

Similarly, we say that A *depends on at most r coordinates* if $[A]$ depends on at most r coordinates, and that two events $A, B \subset \Omega$ *share a coordinate* if $[A]$ and $[B]$ share a coordinate.

By $\overline{A}$ we denote the complement of an event $A \subset \Omega$, so that $[\overline{A}] = 1 - [A]$. We also note that $[A][B] = [A \cap B]$ for events $A, B \subset \Omega$.

We can now state the main result.

Theorem 1. *Let $\Omega_1, \ldots, \Omega_m$ be probability spaces and let $\Omega = \Omega_1 \times \cdots \times \Omega_m$ be their product. Let $A_1, \ldots, A_n \subset \Omega$ be events such that each event A_i depends on at most r_i coordinates and for each event A_i at most Δ_i other events A_j share a coordinate with A_i. Let*

$$\Delta = \max\{5, \Delta_i : i = 1, \ldots, n\}$$

and let

$$\mu_i = \min\{r_i,\ \Delta_i + 1\} \quad \textit{for} \quad i = 1, \ldots, n.$$

Then for any $0 < \epsilon < 1$ there is

$$K = e^{O(\Delta)} \ln(n/\epsilon)$$

such that whenever

$$\mathbb{P}(A_i) < (3\Delta)^{-3\mu_i} \quad \textit{for} \quad i = 1, \ldots, n, \tag{1.4}$$

the value of $\mathbb{P}\left(\bigcap_{i=1}^n \overline{A}_i\right)$, up to relative error ϵ, is determined by the probabilities of k-wise intersections

$$\mathbb{P}(A_{i_1} \cap \ldots \cap A_{i_k}), \quad \textit{where} \quad k \leq K. \tag{1.5}$$

More precisely, for $1 \leq k \leq K$, let

$$\sigma_k = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \mathbb{P}(A_{i_1} \cap \ldots \cap A_{i_k}) \tag{1.6}$$

Then for given Δ, n and $0 < \epsilon < 1$, there is a polynomial $Q = Q_{\Delta,n,\epsilon}$ in K variables and with deg $Q \leq K$, *such that*

$$\left| \ln \mathbb{P}\left(\bigcap_{i=1}^n \overline{A}_i \right) - Q(\sigma_1, \ldots, \sigma_K) \right| \leq \epsilon,$$

provided (1.4) holds.

The implied constant in the 'O' notation is absolute.

Note that an additive ϵ-approximation of $\ln \mathbb{P}\left(\bigcap_{i=1}^n \overline{A}_i\right)$ translates into a relative ϵ-approximation of $\mathbb{P}\left(\bigcap_{i=1}^n \overline{A}_i\right)$. We are interested in the regime where r_i and Δ are small (fixed in advance), while n, m and $1/\epsilon$ are allowed to grow.

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The polynomial Q has rational coefficients, and given $\sigma_1, \ldots, \sigma_K$, its value $Q(\sigma_1, \ldots, \sigma_K)$ can be computed efficiently, in polynomial in K time, so the main difficulty is in computing σ_k. Computing probabilities (1.5) reduces to operating with $R = r_{i_1} + \ldots + r_{i_k}$ coordinates and often can be done roughly in exponential in R time, which leads to an approximation algorithm of a quasi-polynomial $n^{O(\ln(n/\epsilon))}$ complexity, if Δ and r_i are bounded above by constants, fixed in advance. Moreover, in some situations the combinatorial methods of [25] and [20] may sharpen it to a genuinely polynomial time algorithm (more on that is in Section 2.2 below).

We would like to argue that the form of the approximation for $\ln \mathbb{P}\left(\bigcap_{i=1}^n \overline{A}_i\right)$ obtained in Theorem 1, that is, a polynomial in σ_k, should not look completely unexpected. When the events $A_1, \ldots, A_n$ are independent, from (1.2), we can write

$$\ln \mathbb{P}\left(\bigcap_{i=1}^n \overline{A}_i\right) = \sum_{i=1}^n \ln\left(1 - \mathbb{P}(A_i)\right).$$

Assuming that $\mathbb{P}(A_i) < \alpha$ for some fixed $\alpha < 1$, we can approximate the logarithms by their Taylor polynomials:

$$\ln\left(1 - \mathbb{P}(A_i)\right) \approx -\sum_{k=1}^K \frac{1}{k}\left(\mathbb{P}(A_i)\right)^k \quad \text{and} \quad \ln \mathbb{P}\left(\bigcap_{i=1}^n \overline{A}_i\right) \approx -\sum_{k=1}^K \frac{1}{k}\left(\sum_{i=1}^n \left(\mathbb{P}(A_i)\right)^k\right).$$

The Newton identities allow us to write the inner sums $\sum_{i=1}^n \left(\mathbb{P}(A_i)\right)^k$ as polynomials in elementary symmetric functions

$$\sum_{1 \le i_1 < \ldots < i_k \le n} \mathbb{P}(A_{i_1}) \cdots \mathbb{P}(A_{i_k})$$

and hence as polynomials in σ_k, as we have in Theorem 1.

We note that one can always assume that $r_i \le 2^{\Delta_i}$ (we will not use it). Indeed, let us choose an event, without loss of generality, A_n. Let $I \subset \{1, \ldots, n-1\}$ be the set of indices i such that A_i shares a coordinate with A_n, so that $|I| \le \Delta_n$. Let $J \subset \{1, \ldots, m\}$ be the set of indices j such that A_n depends on the coordinate ξ_j, so that $|J| \le r_n$. With each $j \in J$ we associate a subset $I_j \subset I$ consisting of all indices $i \in I$ such that A_i depends on ξ_j. Now, all coordinates ξ_j with the same associated subset I_j can be 'lumped together' into a single coordinate. For example, if A_n does not share a coordinate with any other event A_i, then all the coordinates that A_n depends on can be considered as a single coordinate in the probability product space, and if A_n shares a coordinate with exactly one other event, say A_1, then all the coordinates shared by A_n and A_1 can be considered as a single coordinate in the product space, while all remaining coordinates, if any, that A_n depends on can be considered as another coordinate. Since there are $2^{|I|} \le 2^{\Delta_n}$ subsets of I, in the end we obtain at most 2^{Δ_n} coordinates for A_n.

The bound (1.4) is unlikely to be optimal, though some upper bound in terms of Δ is definitely needed, since otherwise even deciding whether $\mathbb{P}\left(\bigcap_{i=1}^n \overline{A}_i\right) > 0$ can be quite hard. It would be interesting to find out whether there has to be an exponential dependence on μ_i.

We now give a simple example of how Theorem 1 can be applied.

1.2 Example: Counting integer points

Let $\ell_1, \ldots, \ell_n : \mathbb{R}^m \longrightarrow \mathbb{R}$ be polynomials. In the cube $C = [0, c]^m \subset \mathbb{R}^m$, where c is a positive integer, we consider the set

$$S = \left\{x \in C \cap \mathbb{Z}^m : \quad \ell_i(x) \le 0 \quad \text{for} \quad i = 1, \ldots, n\right\} \tag{1.7}$$

of integer points, satisfying a system of polynomial inequalities.

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Suppose further that each polynomial ℓ_i depends on at most r coordinates of $x \in \mathbb{R}^m$, $x = (\xi_1, \ldots, \xi_m)$, and that for each polynomial ℓ_i there are not more than Δ of other polynomials ℓ_j that depend on some of the same coordinates. We assume that r and Δ are fixed in advance, while n, m, and c can vary. We want to approximate the cardinality $|S|$ of S.

For $i = 1, \ldots, m$, we interpret the set $\mathbb{Z} \cap [0, c]$ of integer points in an interval as a probability space Ω_i with the uniform measure, and let $\Omega = \Omega_1 \times \cdots \times \Omega_m$. We further introduce events

$$A_i = \left\{ x \in C \cap \mathbb{Z}^m : \quad \ell_i(x) > 0 \right\} \quad \text{for} \quad i = 1, \ldots, n$$

and see that

$$|S| = (c+1)^m \mathbb{P}\left(\bigcap_{i=1}^{n} \overline{A}_i \right).$$

Suppose now that

$$\mathbb{P}(A_i) = |A_i|(c+1)^{-m} \ < \ (3\Delta)^{-3r} \quad \text{for} \quad i = 1, \ldots, n.$$

Theorem 1 then asserts that up to a relative error $0 < \epsilon < 1$, the cardinality of S is determined by the cardinalities

$$A_{i_1} \cap \ldots \cap A_{i_k} = \left\{ x \in C \cap \mathbb{Z}^m : \quad \ell_{i_1}(x) > 0, \ldots, \ell_{i_k}(x) > 0 \right\} \tag{1.8}$$

for $k = O\left(\ln (n/\epsilon)\right)$. We note that the inequalities in (1.8) involve at most kr coordinates ξ_j, so that we can compute the probability of each intersection (1.8) in the most straightforward way by enumerating $(c+1)^{rk}$ integer points, which in the end results in an approximation algorithm to count points in (1.7) of $(m(c+1))^{O(\ln (n/\epsilon))}$ complexity.

If $\deg \ell_i \le 1$ for $i = 1, \ldots, n$, we can do better: in this case, A_i is the set of integer points in a semi-open polyhedron, and the exact enumeration in (1.8) can be done with

$$((1 + \ln c)(kr))^{O(kr \ln (kr))}$$

complexity, see [1], which leads to a quasi-polynomial algorithm for approximating the number of integer points in (1.7).

Similar algorithms can be designed for computing volumes, in which case the probability spaces Ω_i are identified with intervals and hence become infinite, see also [10] for the complexity of (deterministic) volume computation in small dimensions.

1.3 Connections

The Lovász Local Lemma [8] provides a sufficient criterion for the event $\bigcap_{i=1}^{n} \overline{A}_i$ to have positive probability. Namely, for each $i = 1, \ldots, n$, let $\Gamma_i \subset \{1, \ldots, n\}$ be the set of indices $j \neq i$ such that A_j shares a coordinate with A_i. Suppose that there are numbers $0 < s_i < 1$ for $i = 1, \ldots, n$, such that

$$\mathbb{P}(A_i) \ \le \ s_i \prod_{j \in \Gamma_i} (1 - s_j) \quad \text{for} \quad i = 1, \ldots, n. \tag{1.9}$$

Then

$$\mathbb{P}\left(\bigcap_{i=1}^{n} \overline{A}_i \right) \ \ge \ \prod_{j=1}^{n} (1 - s_i) \ > \ 0. \tag{1.10}$$

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As in Theorem 1, let Δ be an upper bound on the number of events A_j that share a coordinate with any particular event A_i. Let us choose $s_i = \frac{1}{\Delta}$ for $i = 1, \ldots, n$ in (1.9). Then for the right-hand side of (1.9), we have

$$s_i \prod_{j \in \Gamma_i} (1 - s_j) \geq \frac{1}{\Delta} \left(1 - \frac{1}{\Delta}\right)^{\Delta} \sim \frac{1}{e\Delta},$$

and hence our condition (1.4) is much more restrictive than (1.9). The constraints (1.9) and the lower bound (1.10) are, in fact, sharp [28] for the intersection $\bigcap_{i=1}^n \overline{A}_i$ to have positive probability, see also [27] and [14]. We, however, are interested in further approximating that probability arbitrarily closely.

There has been a lot of work on how to approximate $\mathbb{P}\left(\bigcap_{i=1}^n \overline{A}_i\right)$ under some conditions, similar to (1.9). This includes randomised algorithms, see [15] for a survey of the *partial rejection sampling* approach inspired by the algorithmic proof [24] of the Lovász Local Lemma, as well as deterministic, see [23, 13, 12, 29] and references therein. It appears that they all operate under some additional assumptions, either restricting the intersection pattern of A_i to 'extremal' where $A_i \cap A_j = \emptyset$ whenever A_i and A_j share a coordinate [15], or restricting the structure of the sets A_i, such as to those coming from CNF Boolean formulas [23] or atomic clauses with one violation per clause [29], or assuming that each Ω_i is a finite set with uniform measure and an upper bound on the number of elements in Ω_i [13, 12].

Thus Theorem 1 establishes a new regime, where approximate counting of satisfying assignments in the constraint satisfaction problem can be done efficiently, in quasi-polynomial time. For each formula, we need to bound from above the probability that it is not satisfied and also bound the degree of the dependency graph of the formulas, but we do not need to assume anything about the combinatorics of the set A_i of assignments where a particular formula is not satisfied, see also Section 4.2.

Bencs and Regts [6] and then Guo and N [11] used an approach somewhat similar to ours to approximate volumes of some combinatorially defined polytopes in deterministic polynomial time (we talk more about the similarities and differences of the approaches in Section 2.2 and Section 4.1). After the first version of this paper appeared as a preprint, Mann and Waite [22] discussed approximating (1.1), among other questions. They applied the method of *cluster expansion*, which we discuss in Section 4.1.

Linial and Nisan [19] and then Kahn, Linial, and Samorodnitsky [16] considered the problem of approximating $\mathbb{P}\left(\bigcup_{i=1}^n A_i\right)$ from the probabilities $\mathbb{P}(A_{i_1} \cap \ldots \cap A_{i_k})$ of k-wise intersections, without assuming anything about the dependencies of A_i. In [19], Linial and Nisan showed in particular that for $k = \Omega(\sqrt{n})$, the probability of the union can be approximated from the probabilities of k-wise intersections within a relative error of $O\left(e^{-2k/\sqrt{n}}\right)$ and that for $k = O(\sqrt{n})$ the probability of the union cannot be approximated better than within a factor of $O(n/k^2)$. Then in [16], Kahn, Linial, and Samorodnitsky proved that for any k, the probability of the union can be approximated from the probability of the k-wise intersections within an additive error of $e^{-\Omega\left(\frac{k^2}{n \ln n}\right)}$ and that the result is sharp, up to a logarithmic factor in the exponent. Since in our case $\mathbb{P}\left(\bigcap_{i=1}^n \overline{A}_i\right)$ can be exponentially small in n, an attempt to combine the estimates of [19] and [16] and the obvious formula (1.1) can produce a huge relative error in estimating the probability of the intersection.

The methods of [19] and [16] on the one hand, and the method of this paper on the other hand, seem to be quite different. In particular, the final answer is given in different forms: in [19] and [16], the value of $\mathbb{P}\left(\bigcup_{i=1}^n A_i\right)$ is approximated by a linear function in the quantities σ_k of (1.6), whereas our Theorem 1 approximates $\ln \mathbb{P}\left(\bigcap_{i=1}^n \overline{A}_i\right)$ by a polynomial in σ_k. Nevertheless, it feels like the approaches must be somehow related. In particular, the wording of one of the results

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of [16] comes tantalisingly close to the wording of our Theorem 1: it is proved in [16] that the number, say s, of satisfying assignments of a Boolean DNF formula in n variables is determined exactly by the set of numbers of satisfying assignments for all conjunctions of $k \leq \log_2 n + 1$ terms of the formula, although it is not clear how to compute that s from that set of numbers.

2. The method of the proof of Theorem 1 and preliminaries

2.1 The method

Let Ω be the probability space and let $A_1, \ldots, A_n \subset \Omega$ be events as in Theorem 1. We define a univariate polynomial of a complex variable z by

$$p(z) = \int_\Omega \prod_{i=1}^{n} (1 - z[A_i])\, dx, \tag{2.1}$$

where $[A_i]$ is the indicator of A_i defined by (1.3) and dx stands for the integration against the product probability measure in Ω. Then

$$p(1) = \int_\Omega \prod_{i=1}^{n} (1 - [A_i])\, dx = \mathbb{P}\left(\bigcap_{i=1}^{n} \overline{A}_i\right).$$

On the other hand, $p(0) = 1$ and for the k-th derivative of p at 0 we have

$$\begin{aligned} p^{(k)}(0) &= (-1)^k k! \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} \int_\Omega [A_{i_1}] \cdots [A_{i_k}]\, dx \\ &= (-1)^k k! \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} \mathbb{P}\left(A_{i_1} \cap \ldots \cap A_{i_k}\right) \\ &= (-1)^k k! \sigma_k. \end{aligned} \tag{2.2}$$

Clearly, $\deg p \leq n$. Thus our goal is to approximate $p(1)$ within relative error ϵ from $p^{(k)}(0)$ with $k = O(\ln(n/\epsilon))$.

2.2 Approximating $p(1)$ from $p^{(k)}(0)$

It turns out that for our purposes it suffices to show that $p(z) \neq 0$ in some neighbourhood of the interval $[0, 1]$ in the complex plane $\mathbb{C}$. More precisely, if a polynomial $g(z)$ with $\deg g \leq n$ does not have zeros in a δ-neighbourhood of $[0, 1] \subset \mathbb{C}$ for some $0 < \delta < 1$, then for any $0 < \epsilon < 1$, the value of $g(1)$, up to relative error ϵ (properly defined if $g(1)$ is a non-zero complex number), is determined by $g^{(k)}(0)$ for $k \leq K = e^{O(1/\delta)} \ln(n/\epsilon)$. Given the values $g(0), g'(0), \ldots, g^{(K)}(0)$, the approximation can be computed in $O(K^3)$ time. Furthermore, if $g(0) = 1$ then an approximation of $\ln g(1)$ within additive error ϵ can be written as a polynomial Q in $g^{(k)}(0)$ with $k \leq K$ and $\deg Q \leq K$. Below we briefly describe how it is done, see Section 2.2 of [2] and also [3] for detail.

Suppose we have a univariate polynomial $g : \mathbb{C} \longrightarrow \mathbb{C}$ of degree $n \geq 1$ such that $g(z) \neq 0$ in the disc $|z| < \beta$ of some radius $\beta > 1$. We can choose a branch of $f(z) = \ln g(z)$ in the disc and consider the Taylor polynomial of $T_m(z)$ of degree $m \geq 1$ of $f(z)$, computed at $z = 0$:

$$T_m(z) = f(0) + \sum_{k=1}^{m} \frac{f^{(k)}(0)}{k!} z^k.$$

It turns out that $T_m(1)$ approximates $f(1)$ within an additive error that decreases exponentially

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with m:

$$\left|f(z) - T_m(z)\right| \leq \frac{n}{(m+1)\beta^m(\beta-1)} \quad \text{provided} \quad |z| \leq 1.$$

Hence to approximate $f(1)$ by $T_m(1)$, respectively $g(1)$ by $\exp\{T_m(1)\}$, within an additive error, respectively relative error, of $0 < \epsilon < 1$, one can choose $m = O_\beta(\ln(n/\epsilon))$. Moreover, the derivatives $f^{(k)}(0)$ for $k = 1, \ldots, m$ are computed from the derivatives $g^{(k)}(0)$ for $k = 1, \ldots, m$ by solving an $m \times m$ system of linear equations with an invertible triangular matrix having $g(0) \neq 0$ on the diagonal.

Suppose now the polynomial g satisfies a weaker condition or being non-zero in a δ-neighbourhood U of the interval $[0,1] \subset \mathbb{C}$. Then one can reduce it to the already considered case of a polynomial that is non-zero in a disc, by replacing $g(z)$ with the composition $h(z) = g(\phi(z))$, where $\phi : \mathbb{C} \longrightarrow \mathbb{C}$ is an explicit polynomial such that $\phi(0) = 0$, $\phi(1) = 1$, and ϕ maps the disc $\{z : |z| \leq \beta\}$ with some $\beta = 1 + e^{-O(1/\delta)}$ inside U. We have $\deg h = \deg g + \deg \phi$ and since $\phi(0) = 0$, the derivatives $h^{(k)}(0)$ for $k = 1, \ldots, m$ are computed from the derivatives $g^{(k)}(0)$ for $k = 1, \ldots, m$.

We show that for some $\delta = O(1/\Delta)$ and $p(z)$ defined by (2.1), we indeed have $p(z) \neq 0$ in the δ-neighbourhood of $[0,1] \subset \mathbb{C}$, assuming the hypothesis of Theorem 1. Assuming that the values of σ_k defined by (1.6) can be computed in $n^{O(k)}$ time, the straightforward approach of [2] and [3] allows one to approximate $p(1) = \mathbb{P}\left(\bigcap_{i=1}^n \overline{A}_i\right)$ within relative error ϵ in quasi-polynomial $n^{O(\ln(n/\epsilon))}$ time, provided we consider Δ to be fixed in advance.

If we assume that $\mathbb{P}\left(A_{i_1} \cap \ldots \cap A_{i_k}\right)$ can be computed in $2^{O(k)}$ time, there is a possibility of a speed-up to a genuinely polynomial time, using the approach of Patel and Regts [25], see also [20]. Here, we very briefly sketch how. With events $A_1, \ldots, A_n$ we associate a hypergraph $H = (V, E)$ with set $V = \{1, \ldots, m\}$ of vertices and set $E \subset 2^V$ of edges, where each event A_i is represented by an edge $J_i \subset \{1, \ldots, m\}$, consisting of the vertices j such that A_i depends on the coordinate ξ_j. From the technique of [25] and [20], it follows that to compute (2.2), it suffices to compute the probabilities when the associated hypergraph is connected (note that for an arbitrary hypergraph, the required probability is the product of the probabilities over its connected components). Then it turns out that for a fixed Δ, and for $k = O_\Delta \ln(n/\epsilon)$ there are only $(n/\epsilon)^{O_\Delta(1)}$ connected subhypergraphs of H with k edges. Very recently, a different way to accelerate computations was described in [7].

We note that a particular version of the polynomial (2.1) was used earlier by Bencs and Regts [6] to approximate the volume of a truncated independence polytope of a graph and then by Guo and N [11] to approximate the volume of a truncated fractional matching polytope.

Thus our goal is to prove that the polynomial $p(z)$ of (2.1) does not have zeros in a neighbourhood of the interval $[0,1] \subset \mathbb{C}$. We let

$$U = \{z \in \mathbb{C} : \operatorname{dist}(z, [0,1]) \leq \delta\} \quad \text{for} \quad \delta = \frac{1}{6\Delta},$$

where dist is the Euclidean distance in the complex plane $\mathbb{C} = \mathbb{R}^2$.

Let us fix a $z \in U$ and let us define $f_i : \Omega \longrightarrow \mathbb{C}$ by

$$f_i = 1 - z[A_i] \qquad \text{for} \quad i = 1, \ldots, n.$$

Then

$$|f_i(x)| \leq 1 + \frac{1}{6\Delta} \quad \text{for all} \quad x \in \Omega \quad \text{and} \quad i = 1, \ldots, n$$

and

$$\mathbb{P}\left(f_i(x) \neq 1\right) < (3\Delta)^{-3\mu_i} \quad \text{for} \quad i = 1, \ldots, n.$$

Hence we will be proving the following reformulation of Theorem 1.

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Theorem 2. *Let $\Omega_1, \ldots, \Omega_m$ be probability spaces and let $\Omega = \Omega_1 \times \cdots \times \Omega_m$ be their product. Let $f_1, \ldots, f_n : \Omega \longrightarrow \mathbb{C}$ be random variables, such that each f_i depends on not more than r_i coordinates and shares a coordinate with at most Δ_i other functions f_j. Let*

$$\Delta = \max\{5,\ \Delta_i : i = 1, \ldots, n\}$$

and let

$$\mu_i = \min\{r_i,\ \Delta_i + 1\} \quad \textit{for} \quad i = 1, \ldots, n.$$

Suppose further that

$$\left|f_i(x)\right| \ \leq\ 1 + \frac{1}{6\Delta} \quad \textit{for all} \quad x \in \Omega \quad \textit{and} \quad i = 1, \ldots, n \tag{2.3}$$

and that

$$\mathbb{P}\left(f_i(x) \neq 1\right) \ <\ (3\Delta)^{-3\mu_i} \quad \textit{for} \quad i = 1, \ldots, n. \tag{2.4}$$

Then

$$\int_\Omega \prod_{i=1}^{n} f_i(x)\, dx \neq 0.$$

We will prove Theorem 2 by proving a stronger statement by induction on the number n of functions.

We will use some simple geometric facts.

Lemma 3. *Let Ω be a probability space and let $f : \Omega \longrightarrow \mathbb{C}$ be an integrable random variable. Suppose that for every $\omega \in \Omega$ we have $f(\omega) \neq 0$ and that for any $\omega_1, \omega_2 \in \Omega$ the angle between $f(\omega_1) \neq 0$ and $f(\omega_2) \neq 0$, considered as vectors in $\mathbb{R}^2 = \mathbb{C}$, does not exceed θ for some $0 \leq \theta < 2\pi/3$. Then*

$$\left|\int_\Omega f(x)\, dx\right| \ \geq\ \left(\cos\frac{\theta}{2}\right) \int_\Omega |f(x)|\, dx.$$

Proof. See Lemma 3.3 of [4] and also Lemma 3.6.3 of [2]. □

Lemma 4. *Let $v \in \mathbb{C} \setminus \{0\}$ be a non-zero complex number and let $u \in \mathbb{C}$ be another complex number such that*

$$|u| \ \leq\ (\sin\theta)|v| \quad \textit{for some} \quad 0 \leq \theta < \pi/2.$$

Then the angle between v and $v + u$, considered as vectors in $\mathbb{R}^2 = \mathbb{C}$, does not exceed θ.

Proof. See Lemma 3.6.4 of [2]. □

We will also use the classical Markov inequality in the following situation.

Lemma 5. *Let Ω_1 and Ω_2 be probability spaces with respective probability measures $\mathbb{P}_1$ and $\mathbb{P}_2$, let $\Omega = \Omega_1 \times \Omega_2$ be their product with probability measure $\mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2$, and let $A \subset \Omega$ be an event. For $\omega_1 \in \Omega_1$ we define an event $A_{\omega_1} \subset \Omega_2$ by*

$$A_{\omega_1} = \{\omega_2 \in \Omega_2 : \quad (\omega_1, \omega_2) \in A\}.$$

Then for any $\tau \geq 1$, we have

$$\mathbb{P}_1(\omega_1 : \mathbb{P}_2\left(A_{\omega_1}\right) \ \geq\ \tau\mathbb{P}(A)) \ \leq\ \tau^{-1}.$$

Proof. We define a random variable $f : \Omega_1 \longrightarrow [0, 1]$ by

$$f(\omega_1) = \mathbb{P}_2(A_{\omega_1}) \quad \text{for} \quad \omega_1 \in \Omega_1.$$

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Then

$$\int_{\Omega_1} f(\omega_1)\, d\omega_1 = \mathbb{P}(A),$$

where $d\omega_1$ stands for the integration against the probability measure $\mathbb{P}_1$. Since f is non-negative, the result follows by applying the standard Markov inequality to f. □

3. Proof of Theorem 2

Our goal is to prove Theorem 2. We do that by proving a stronger result by induction on the number n of functions f_i.

3.1 The induction hypothesis

Let

$$\theta = \frac{1}{\Delta^2} \quad \text{and} \quad \beta = 1 + \frac{1}{\Delta}.$$

The induction hypothesis consists of the following two claims.

Claim 1. *Let $f_i : \Omega \longrightarrow \mathbb{C}$, $i = 1, \ldots, n$, be random variables as in Theorem 2. Then*

$$\int_\Omega \prod_{i=1}^{n} f_i(x)\, dx \neq 0, \quad \int_\Omega \prod_{i=1}^{n-1} f_i(x)\, dx \neq 0 \tag{3.1}$$

and the angle between two non-zero complex numbers (3.1), considered as vectors in $\mathbb{R}^2 = \mathbb{C}$, does not exceed θ. For $n = 1$, we agree that the second number is 1, following the general convention that the product of the numbers from an empty set is 1, while the sum of the numbers from an empty set is 0. □

Claim 2. *Let $f_i : \Omega \longrightarrow \mathbb{C}$, $i = 1, \ldots, n$, be random variables as in Claim 1, and let $k \leq n$. Let $\widehat{f}_{n-k+1}, \ldots, \widehat{f}_n : \Omega \longrightarrow \mathbb{C}$ be random variables such that*

$$\left|\widehat{f}_i(x)\right| \leq 1 + \frac{1}{6\Delta} \quad \text{for all} \quad x \in \Omega \quad \text{and} \quad i = n-k+1, \ldots, n, \tag{3.2}$$

and such that each $\widehat{f}_i$ depends on a subset of the coordinates that f_i depends on (note, however, that $\widehat{f}_i$ do not have to satisfy (2.4)). Then

$$\left| \int_\Omega \left(\prod_{i=1}^{n-k} f_i(x) \prod_{i=n-k+1}^{n} \widehat{f}_i(x) \right) dx \right| \leq \beta^k \left| \int_\Omega \prod_{i=1}^{n} f_i(x)\, dx \right|.$$

In words: if we replace any k of the functions f_i by some functions $\widehat{f}_i$, depending on the same or a smaller set of coordinates, and still satisfying (2.3) but not necessarily (2.4), then the absolute value of the integral of the product can increase by a factor β^k at most. □

Remark 1. Suppose that Claim 1 holds, and let $\widehat{f}_n : \Omega \longrightarrow \mathbb{C}$ be yet another random variable that depends on a subset of coordinates that f_n depends on and such that

$$\left|\widehat{f}_n(x)\right| \leq 1 + \frac{1}{6\Delta} \quad \text{and} \quad \mathbb{P}\left(\widehat{f}_n(x) \neq 1\right) < (3\Delta)^{-3\mu_n}.$$

Then the angle between two complex numbers

$$\int_\Omega \prod_{i=1}^{n} f_i(x)\, dx \neq 0 \quad \text{and} \quad \int_\Omega \widehat{f}_n(x) \prod_{i=1}^{n-1} f_i(x)\, dx \neq 0 \tag{3.3}$$

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does not exceed $2\theta = \dfrac{2}{\Delta^2}$. Indeed, both numbers (3.3) form an angle of at most θ with

$$\int_\Omega \prod_{i=1}^{n-1} f_i(x)\, dx \neq 0.$$

Iterating, we conclude that if $\widehat{f}_{n-k+1}, \ldots, \widehat{f}_n : \Omega \longrightarrow \mathbb{C}$ are random variables satisfying (3.2) and also such that

$$\mathbb{P}\left(\widehat{f}_i(x) \neq 1\right) \;<\; (3\Delta)^{-3\mu_i} \quad \text{for} \quad i = n-k+1, \ldots, n$$

then the angle between two complex numbers

$$\int_\Omega \prod_{i=1}^{n} f_i(x)\, dx \neq 0 \quad \text{and} \quad \int_\Omega \left(\prod_{i=1}^{n-k} f_i(x) \prod_{i=n-k+1}^{n} \widehat{f}_i(x) \right) dx \neq 0$$

does not exceed $2k\theta = 2k/\Delta^2$.

In words: if we replace any k of the random variables f_i satisfying (2.3) and (2.4) by some other random variables on the same or smaller sets of coordinates satisfying the same conditions, then the value of the integral rotates by at most an angle of $2k/\Delta^2$.

Finally, for the sake of transparency, in the proof below we replace condition (2.4) by

$$\mathbb{P}(f_i(x) \neq 1) \;<\; (\gamma\Delta)^{-3\mu_i}, \tag{3.4}$$

where $\gamma > 0$ is some absolute constant. In the course of the proof it then becomes clear that the induction can be carried through for a sufficiently large Δ and γ, and numerical computations show that we can indeed choose $\Delta \geq 5$ and $\gamma = 3$, as claimed.

3.2 Base n = 1

In this case, we have just one function f_1 in some $r \leq r_1$ coordinates, satisfying (2.3) and (3.4).

If $r = 0$ then $\mu_1 = 0$. Since $r = 0$, the function f_1 is a constant and by (3.4), we must have $f_1 \equiv 1$, so that

$$\int_\Omega f_1(x)\, dx = 1.$$

Claim 1 then holds tautologically. By (3.2),

$$\left| \int_\Omega \widehat{f}_1(x)\, dx \right| \;\leq\; 1 + \frac{1}{6\Delta} \;\leq\; \beta \left| \int_\Omega f_1(x)\, dx \right|,$$

and hence Claim 2 also holds.

Suppose now that $r \geq 1$, so that $\mu_1 = 1$. Let

$$B = \{x \in \Omega : \quad f_1(x) = 1\}, \quad \text{so that} \quad \mathbb{P}(\overline{B}) \;<\; (\gamma\Delta)^{-3}.$$

From (2.3) and (3.4),

$$\left| 1 - \int_\Omega f_1(x)\, dx \right| = \left| \int_{\overline{B}} (1 - f_1(x))\, dx \right| \;\leq\; \left(2 + \frac{1}{6\Delta} \right) (\gamma\Delta)^{-3}. \tag{3.5}$$

It is clear now that for $\gamma \geq 1$ sufficiently large, we have

$$\left(2 + \frac{1}{6\Delta} \right) (\gamma\Delta)^{-3} \;\leq\; \sin \frac{1}{\Delta^2} = \sin\theta, \tag{3.6}$$

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so that by Lemma 4 the angle between complex numbers 1 and

$$\int_\Omega f_1(x)\,dx \neq 0$$

does not exceed θ and Claim 1 holds. Numerical computations show that (3.6) indeed holds for $\Delta \geq 3$ and $\gamma \geq 1$.

To check Claim 2, from (3.5), we have

$$\left|\int_\Omega f_1(x)\,dx\right| \geq 1 - \left(2 + \frac{1}{6\Delta}\right)(\gamma\Delta)^{-3}. \tag{3.7}$$

From (3.2), we have

$$\left|\int_\Omega \widehat{f}_1(x)\,dx\right| \leq 1 + \frac{1}{6\Delta}. \tag{3.8}$$

It is now clear that for $\gamma > 1$ sufficiently large, we have

$$1 + \frac{1}{6\Delta} \leq \left(1 + \frac{1}{\Delta}\right)\left(1 - \left(2 + \frac{1}{6\Delta}\right)(\gamma\Delta)^{-3}\right), \tag{3.9}$$

and so by (3.7) and (3.8),

$$\left|\int_\Omega \widehat{f}_1(x)\,dx\right| \leq \beta\left|\int_\Omega f_1(x)\,dx\right|,$$

and Claim 2 holds too.

Numerical computations show that (3.9) holds provided $\Delta \geq 3$ and $\gamma \geq 1$.

3.3 *Induction step $n-1 \Longrightarrow n$, special case*

Thus we assume that $n \geq 2$. Suppose that f_n depends on the coordinates $\xi_j : j \in J$, so $|J| = r \leq r_n$. If $r = 0$, then f_n is a constant, and by (3.4), we have $f_n \equiv 1$. Then

$$\prod_{i=1}^{n} f_i(x) = \prod_{i=1}^{n-1} f_i(x)$$

and Claim 1 holds trivially.

Since $\widehat{f}_n$ depends on a subset of the set of coordinates that f_n depends on, the function $\widehat{f}_n$ is also a constant. Then by (3.2)

$$\begin{aligned}
&\left|\int_\Omega \left(\prod_{i=1}^{n-k} f_i(x) \prod_{i=n-k+1}^{n} \widehat{f}_i(x)\right) dx\right| \\
&\leq \left(1 + \frac{1}{6\Delta}\right)\left|\int_\Omega \left(\prod_{i=1}^{n-k} f_i(x) \prod_{i=n-k+1}^{n-1} \widehat{f}_i(x)\right) dx\right| \\
&\leq \beta^k \left|\int_\Omega \prod_{i=1}^{n} f_i(x)\,dx\right|,
\end{aligned}$$

where the last inequality follows by the induction hypothesis. Thus Claim 2 also holds.

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3.4 Induction step $n-1 \Longrightarrow n$, preparations for the general case

Without loss of generality, we assume now that the set $\{\xi_j : j \in J\}$ of coordinates that f_n depends on is non-empty, so f_n depends on some r coordinates, $1 \le r \le r_n$. We form two product probability spaces

$$\Omega_1 = \times_{j\in J}\Omega_j \quad \text{and} \quad \Omega_2 = \times_{j\notin J}\Omega_j, \tag{3.10}$$

so that

$$\Omega = \Omega_1 \times \Omega_2.$$

We then write $x \in \Omega$ as $x = (u, y)$, where $u \in \Omega_1$ and $y \in \Omega_2$. We denote the probability measure in Ω by $\mathbb{P}$, in Ω_1 by $\mathbb{P}_1$ and in Ω_2 by $\mathbb{P}_2$. We use the notation dx, du, and dy to denote integration against $\mathbb{P}$, $\mathbb{P}_1$, and $\mathbb{P}_2$, respectively.

Let $I \subset \{1, \dots, n-1\}$ be the set of indices i such that $f_i(x)$ shares a coordinate with $f_n(x)$. Then

$$|I| \le \Delta. \tag{3.11}$$

For $i \in I$ and $u \in \Omega_1$, we define a random variable $h_i(\cdot\,|u) : \Omega_2 \longrightarrow \mathbb{C}$ by

$$h_i(y|u) = f_i(u, y) \quad \text{for} \quad u \in \Omega_1, \quad y \in \Omega_2 \quad \text{and} \quad i \in I.$$

Similarly, for $i \in I$ such that $i > n-k$ and random variables $\widehat{f_i}$ as in Claim 2, we define a random variable $\widehat{h}_i(\cdot\,|u) : \Omega_2 \longrightarrow \mathbb{C}$ by

$$\widehat{h}_i(y|u) = \widehat{f}_i(u, y) \quad \text{for} \quad u \in \Omega_1, \quad y \in \Omega_2 \quad \text{and} \quad i \in I.$$

We note that each $h_i(y|u)$ depends on at most $r_i - 1$ coordinates and shares a coordinate with at most $\Delta_i - 1$ of functions $h_j(y|u)$ for $j \in I \setminus \{i\}$ and $f_j(y)$ for $j \notin I \cup \{n\}$.

We further define functions $\phi, \widehat{\phi} : \Omega_1 \longrightarrow \mathbb{C}$ by

$$\begin{aligned}
\phi(u) &= \int_{\Omega_2} \left(\prod_{i\in I} h_i(y|u) \prod_{\substack{i\notin I \\ i<n}} f_i(y) \right) dy \quad \text{and} \\
\widehat{\phi}(u) &= \int_{\Omega_2} \left(\prod_{\substack{i\in I \\ i\le n-k}} h_i(y|u) \prod_{\substack{i\notin I \\ i\le n-k}} f_i(y) \prod_{\substack{i\in I \\ i>n-k}} \widehat{h}_i(y|u) \prod_{\substack{i\notin I \\ n-k<i<n}} \widehat{f}_i(y) \right) dy.
\end{aligned} \tag{3.12}$$

Then by Fubini's Theorem

$$\begin{aligned}
\int_\Omega \prod_{i=1}^{n} f_i(x)\, dx &= \int_{\Omega_1} f_n(u)\phi(u)\, du \\
\int_\Omega \prod_{i=1}^{n-1} f_i(x)\, dx &= \int_{\Omega_1} \phi(u)\, du \quad \text{and} \\
\int_\Omega \left(\prod_{i=1}^{n-k} f_i(x) \prod_{i=n-k+1}^{n} \widehat{f}_i(x) \right) dx &= \int_{\Omega_1} \widehat{f}_n(u)\widehat{\phi}(u)\, du.
\end{aligned} \tag{3.13}$$

Applying Lemma 5 with $\tau = (\gamma\Delta)^3$, from (3.4), we conclude that

$$\mathbb{P}_1\Big(u \in \Omega_1 : \mathbb{P}_2(h_i(\cdot\,|u) \ne 1) \ \ge\ (\gamma\Delta)^{-3(\mu_i-1)}\Big) \ \le\ (\gamma\Delta)^{-3} \quad \text{for} \quad i \in I. \tag{3.14}$$

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We define an event $A \subset \Omega_1$ by

$$A = \Big\{ u \in \Omega_1 : \mathbb{P}_2(h_i(\cdot\,|u) \neq 1) \ < \ (\gamma\Delta)^{-3(\mu_i - 1)} \quad \text{for all} \quad i \in I \Big\}. \tag{3.15}$$

Then by (3.11), (3.14), and a union bound,

$$\mathbb{P}_1(\overline{A}) \ \leq \ \frac{1}{\gamma^3\Delta^2}. \tag{3.16}$$

We define an event $B \subset \Omega_1$ by

$$B = \{u \in \Omega_1 : f_n(u) = 1\}. \tag{3.17}$$

so that by (3.4),

$$\mathbb{P}_1(\overline{B}) \ < \ (\gamma\Delta)^{-3\mu_n} \ \leq \ (\gamma\Delta)^{-3}. \tag{3.18}$$

Combining (3.16) and (3.18), we conclude that

$$\begin{aligned} &\mathbb{P}_1\left(\overline{A} \cup \overline{B}\right) \ \leq \ \frac{2}{\gamma^3\Delta^2}, \\ &\mathbb{P}_1\left(A \cap B\right) = 1 - \mathbb{P}_1\left(\overline{A} \cup \overline{B}\right) \ \geq \ \frac{\gamma^3\Delta^2 - 2}{\gamma^3\Delta^2} \quad \text{and} \\ &\frac{\mathbb{P}_1\left(\overline{A} \cup \overline{B}\right)}{\mathbb{P}_1\left(A \cup B\right)} \ \leq \ \frac{2}{\gamma^3\Delta^2 - 2}. \end{aligned} \tag{3.19}$$

It is the last inequality that we will be using.

Since each function $h_i(\cdot\,|u)$ depends on at most $r_i - 1$ coordinates and shares a coordinate with at most $\Delta_i - 1$ functions $h_j(\cdot\,|u)$ for $j \in I \setminus \{i\}$ and $f_j(y)$ for $j \in \{1, \ldots, n-1\} \setminus I$, by (3.15) for all $u \in A$ the $n-1$ functions $h_i(\cdot\,|u)$ for $i \in I$ and f_i for $i \in \{1, \ldots, n-1\} \setminus I$ satisfy Claim 1 of the induction hypothesis. Let us choose any two $u_1, u_2 \in A$. When we switch from $u = u_1$ to $u = u_2$ in the integral (3.12) for ϕ, we change the $|I|$ functions $h_i(\cdot\,|u_1)$ to $h_i(\cdot\,|u_2)$, and hence applying Claim 1 of the induction hypothesis and Remark 1, we conclude that $\phi(u_1) \neq 0, \phi(u_2) \neq 0$ and the angle between these two non-zero complex numbers is at most $2|I|\theta \ \leq \ 2/\Delta$ by (3.11). Hence by Lemma 3,

$$\left| \int_{A \cap B} \phi(u)\, du \right| \ \geq \ \left(\cos \frac{1}{\Delta} \right) \int_{A \cap B} |\phi(u)|\, du \ > \ 0. \tag{3.20}$$

Let us now pick any $u_1 \in A$ and any $u_2 \in \Omega_1$. If in the formula (3.12) for $\phi(u)$, we switch $u = u_1$ to $u = u_2$, we switch at most $|I| \leq \Delta$ of functions $h_i(\cdot\,|u_1)$ to $h_i(\cdot\,|u_2)$. Applying Claim 2 of the induction hypothesis, we conclude that

$$\begin{gathered} |\phi(u_2)| \ \leq \beta^{|I|} |\phi(u_1)| \ \leq \ \left(1 + \frac{1}{\Delta}\right)^{\Delta} |\phi(u_1)| \ \leq \ 3|\phi(u_1)| \\ \text{for all} \quad u_1 \in A, \ u_2 \in \Omega_1. \end{gathered} \tag{3.21}$$

It follows then that

$$\int_{\overline{A} \cup \overline{B}} |\phi(u)|\, du \ \leq \ \frac{3\mathbb{P}_1\left(\overline{A} \cup \overline{B}\right)}{\mathbb{P}_1\left(A \cap B\right)} \int_{A \cap B} |\phi(u)|\, du. \tag{3.22}$$

Indeed, (3.22) obviously holds if $\mathbb{P}_1\left(\overline{A} \cup \overline{B}\right) = 0$, and if $\mathbb{P}_1\left(\overline{A} \cup \overline{B}\right) > 0$ then (3.22) states that the average value of $|\phi(u)|$ on $\overline{A} \cup \overline{B}$ does not exceed three times the average value of $|\phi(u)|$ on $A \cap B$, which follows from (3.21).

Similarly, to pass from $\phi(u)$ to $\widehat{\phi}(u)$ for $u \in A$, we change at most $k-1$ in total functions $h_i(\cdot\,|u)$ to $\widehat{h}_i(\cdot\,|u)$ and $f_i(y)$ to $\widehat{f}_i(y)$ in (3.12), and hence applying Claim 2 of the induction hypothesis,

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we obtain

$$|\widehat{\phi}(u)| \;\le\; \beta^{k-1}|\phi(u)| \quad \text{for all} \quad u \in A$$

and hence

$$\int_{A\cap B} |\widehat{\phi}(u)|\,du \;\le\; \beta^{k-1}\int_{A\cap B} |\phi(u)|\,du. \tag{3.23}$$

On the other hand, in $\widehat{\phi}(u_2)$ for $u_2 \in \Omega_1$ compared to $\phi(u_1)$ for $u_1 \in A$, we replace additionally up to $|I| \le \Delta$ functions $h_i(\cdot\,|u_1)$ by $h_i(\cdot\,|u_2)$, and so from Claim 2 of the induction hypothesis, we get

$$\begin{gathered}|\widehat{\phi}(u_2)| \;\le \beta^{k-1}\beta^{|I|}|\phi(u_1)| \;\le\; 3\beta^{k-1}|\phi(u_1)| \\ \text{for all} \quad u_1 \in A,\; u_2 \in \Omega_1.\end{gathered} \tag{3.24}$$

It follows then that

$$\int_{\overline{A}\cup\overline{B}} |\widehat{\phi}(u)|\,du \;\le\; 3\beta^{k-1}\frac{\mathbb{P}_1(\overline{A}\cup\overline{B})}{\mathbb{P}_1(A\cap B)}\int_{A\cap B} |\phi(u)|\,du. \tag{3.25}$$

Indeed, (3.25) obviously holds if $\mathbb{P}_1(\overline{A}\cup\overline{B}) = 0$, and if $\mathbb{P}_1\left(\overline{A}\cup\overline{B}\right) > 0$ then (3.25) states that the average value of $|\widehat{\phi}(u)|$ on $\overline{A}\cup\overline{B}$ does not exceed $3\beta^{k-1}$ times the average value of $|\phi(u)|$ on $A\cap B$, which follows from (3.24).

3.5 Induction step $n-1 \Longrightarrow n$, Claim 1

We compare the first two integrals in (3.13) and argue that the bulk of both integrals come from integrating over the event $A\cap B$, for A defined by (3.15) and B defined by (3.17), where the integrals obviously coincide. Indeed, by (3.17) we have

$$\int_{A\cap B} f_n(u)\phi(u)\,du = \int_{A\cap B} \phi(u)\,du.$$

From (3.13),

$$\left|\int_\Omega \prod_{i=1}^{n} f_i(x)\,dx - \int_{A\cap B}\phi(u)\,du\right| \;\le\; \int_{\overline{A}\cup\overline{B}} |f_n(u)\phi(u)|\,du \tag{3.26}$$

and similarly,

$$\left|\int_\Omega \prod_{i=1}^{n-1} f_i(x)\,dx - \int_{A\cap B}\phi(u)\,du\right| \;\le\; \int_{\overline{A}\cup\overline{B}} |\phi(u)|\,du. \tag{3.27}$$

Using (3.2), (3.22), (3.19), and (3.20) in that order, we conclude that

$$\begin{aligned}\int_{\overline{A}\cup\overline{B}} |f_n(u)\phi(u)|\,du \;&\le\; \left(1+\frac{1}{6\Delta}\right)\int_{\overline{A}\cup\overline{B}} |\phi(u)|\,du \\ &\le \left(1+\frac{1}{6\Delta}\right)\frac{3\mathbb{P}_1(\overline{A}\cup\overline{B})}{\mathbb{P}_1(A\cap B)}\int_{A\cap B} |\phi(u)|\,du \\ &\le \left(1+\frac{1}{6\Delta}\right)\left(\frac{6}{\gamma^3\Delta^2-2}\right)\int_{A\cap B} |\phi(u)|\,du \\ &\le \left(1+\frac{1}{6\Delta}\right)\left(\frac{6}{\gamma^3\Delta^2-2}\right)\left(\frac{1}{\cos(1/\Delta)}\right)\left|\int_{A\cap B}\phi(u)\,du\right|.\end{aligned} \tag{3.28}$$

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and, similarly,

$$\int_{\overline{A}\cup\overline{B}} |\phi(u)|\,du \le \left(\frac{6}{\gamma^3\Delta^2-2}\right)\left(\frac{1}{\cos(1/\Delta)}\right)\left|\int_{A\cap B}\phi(u)\,du\right|. \tag{3.29}$$

In view of (3.26)–(3.29) and Lemma 4, to make sure that

$$\int_\Omega \prod_{i=1}^n f_i(x)\,dx \neq 0, \quad \int_\Omega \prod_{i=1}^{n-1} f_i(x)\,dx \neq 0$$

and that the angles between each of these two integrals and

$$\int_{A\cap B}\phi(u)\,du \neq 0,$$

cf. (3.20), do not exceed $\theta/2 = \frac{1}{2\Delta^2}$ each, we should make sure that

$$\left(1+\frac{1}{6\Delta}\right)\left(\frac{6}{\gamma^3\Delta^2-2}\right)\left(\frac{1}{\cos(1/\Delta)}\right) \le \sin\frac{1}{2\Delta^2}. \tag{3.30}$$

Clearly, (3.30) holds for a sufficiently large $\gamma > 1$. Computations show that (3.30) holds when $\Delta \ge 3$ and $\gamma \ge 3$, so Claim 1 holds.

3.6 *Induction step $n-1 \Longrightarrow n$, Claim 2*

Here we compare the first and the last integrals in (3.13), and once again argue that the main contribution to both come from the integral over the event $A \cap B$, for A defined by (3.15) and B defined by (3.17). From (3.13) and (3.2), we have

$$\left|\int_\Omega \left(\prod_{i=1}^{n-k} f_i(x) \prod_{i=n-k+1}^{n} \widehat{f}_i(x)\right) dx\right| = \left|\int_{\Omega_1} \widehat{f}_n(u)\widehat{\phi}(u)\,du\right|$$
$$\le \left(1+\frac{1}{6\Delta}\right)\int_{\Omega_1} |\widehat{\phi}(u)|\,du.$$

As before, the last integral we split into two cases: over $A\cap B$ and over $\overline{A}\cup\overline{B}$. Applying (3.2), (3.23), (3.25), and (3.20) in that order, we get

$$\left|\int_{\Omega_1} \widehat{f}_n(u)\widehat{\phi}(u)\,du\right| \le \left(1+\frac{1}{6\Delta}\right)\left(\int_{A\cap B}|\widehat{\phi}(u)|\,du + \int_{\overline{A}\cup\overline{B}}|\widehat{\phi}(u)|\,du\right)$$
$$\le \beta^{k-1}\left(1+\frac{1}{6\Delta}\right)\left(1+3\frac{\mathbb{P}_1(\overline{A}\cup\overline{B})}{\mathbb{P}_1(A\cap B)}\right)\int_{A\cap B}|\phi(u)|\,du$$
$$\le \beta^{k-1}\left(1+\frac{1}{6\Delta}\right)\left(1+\frac{6}{\gamma^3\Delta^2-2}\right)\left(\frac{1}{\cos(1/\Delta)}\right)\left|\int_{A\cap B}\phi(u)\,du\right|.$$

It is clear now that for sufficiently large $\Delta \ge 1$ and $\gamma > 1$, we have

$$\left(1+\frac{1}{6\Delta}\right)\left(1+\frac{6}{\gamma^3\Delta^2-2}\right)\left(\frac{1}{\cos(1/\Delta)}\right) \le 1+\frac{1}{3\Delta} \tag{3.31}$$

and computations show that (3.31) holds for $\Delta \ge 5$ and $\gamma \ge 3$. Hence,

$$\left|\int_{\Omega_1} \widehat{f}_n(u)\widehat{\phi}(u)\,du\right| \le \beta^{k-1}\left(1+\frac{1}{3\Delta}\right)\left|\int_{A\cap B}\phi(u)\,du\right| \tag{3.32}$$
$$\text{provided} \quad \Delta \ge 5 \quad \text{and} \quad \gamma \ge 3.$$

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On the other hand, from (3.13) and (3.28),

$$\begin{aligned}\left|\int_\Omega \prod_{i=1}^n f_i(x)\,dx\right| &= \left|\int_{\Omega_1} f_n(u)\phi(u)\,du\right| \\ &\geq \left|\int_{A\cap B}\phi(u)\,du\right| - \left|\int_{\overline{A}\cup\overline{B}} f_n(u)\phi(u)\,du\right| \\ &\geq \left(1-\left(1+\frac{1}{6\Delta}\right)\left(\frac{6}{\gamma^3\Delta^2-2}\right)\left(\frac{1}{\cos(1/\Delta)}\right)\right)\left|\int_{A\cap B}\phi(u)\,du\right|.\end{aligned}$$

It is now clear that if $\Delta \geq 1$ and $\gamma \geq 1$ are sufficiently large, then

$$\left(1+\frac{1}{6\Delta}\right)\left(\frac{6}{\gamma^3\Delta^2-2}\right)\left(\frac{1}{\cos(1/\Delta)}\right) \leq \frac{1}{15\Delta}. \tag{3.33}$$

Computations show that (3.33) indeed holds for $\Delta \geq 5$ and $\gamma \geq 3$, so we have

$$\left|\int_{\Omega_1} f_n(u)\phi(u)\,du\right| \geq \left(1-\frac{1}{15\Delta}\right)\left|\int_{A\cap B}\phi(u)\,du\right| \quad \text{provided} \quad \Delta\geq 5 \quad \text{and} \quad \gamma\geq 3. \tag{3.34}$$

Since for $\Delta \geq 5$, we have

$$\left(1+\frac{1}{3\Delta}\right)\left(1-\frac{1}{15\Delta}\right)^{-1} < 1+\frac{1}{\Delta}=\beta,$$

combining (3.13), (3.32), and (3.34), we conclude that Claim 2 holds.

4. Concluding remarks

4.1 Comparison with the cluster expansion method

One can prove a version of Theorem 1 using the method of *cluster expansion*, see [14] for a survey. This method was also used by Bencs and Regts [6] in the particular case of approximating the volume of a truncated independence polytope of a graph and then by Guo and N [11] to approximate the volume of a truncated matching polytope of a hypergraph. We briefly sketch how cluster expansion can work in the general framework of Theorem 1.

Given events $A_1,\dots,A_n$ as in Theorem 1, we still consider the polynomial

$$\begin{aligned}p(z) &= \int_\Omega \prod_{i=1}^n (1-z[A_i])\,dx \\ &= \sum_{k=0}^n (-1)^k z^k \sum_{1\leq i_1<\ldots<i_k\leq n} \mathbb{P}(A_{i_1}\cap\ldots\cap A_{i_k})\end{aligned} \tag{4.1}$$

and aim to prove that $p(z)\neq 0$ in a neighbourhood of $[0,1]\subset\mathbb{C}$, see Sections 2.1 and 2.2. We construct a graph $G=(V,E)$ with set $V=\{1,\dots,n\}$ of vertices, and vertices i and j spanning an edge whenever A_i and A_j share a coordinate. The expansion (4.1) allows one to apply the *polymer model* here: for each collection $1\leq i_1<\ldots<i_k\leq n$, we consider the subgraph $H=H_{i_1,\dots,i_k}$ of G induced by $i_1,\dots,i_k$ and define its (complex) weight by

$$w(H_{i_1,\dots,i_k}) = (-1)^k z^k \mathbb{P}(A_{i_1}\cap\ldots\cap A_{i_k}).$$

Then (4.1) is the sum of $w(H)$ over all induced subgraphs H of G, including the empty subgraph with weight 1.

We say that two connected induced subgraphs H_1 and H_2 of G are *compatible*, denoted $H_1\sim H_2$, if they are vertex-disjoint and there are no vertices v_1 of H_1 and v_2 of H_2 spanning

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an edge of G, and *incompatible*, denoted $H_1 \not\sim H_2$, otherwise. The crucial property that makes the polymer model work is that if the induced subgraph H is a union of pairwise compatible connected subgraphs, say $H^1, \ldots, H^s$, then

$$w(H) = w(H^1) \cdots w(H^s).$$

The Kotecký–Preiss condition [17], see also [14] for a survey, then provides a sufficient condition for $p(z)$ to be non-zero: if one can assign non-negative real weights $\psi(H) \geq 0$ to the induced subgraphs H such that for every connected induced subgraph F of G, we have

$$\sum_{H:\ H \not\sim F} |w(H)| e^{\psi(H)} \ \leq \ \psi(F), \tag{4.2}$$

where the sum is taken over all connected induced subgraphs of G that are incompatible with F, including F itself. One can bound $|w(H)|$ from above by arguing that the graph H with $|H|$ vertices contains a subset of size at least $|H|/(\Delta+1)$, corresponding to mutually independent events. The condition (4.2) allows one to establish that $p(z) \neq 0$ in a disc

$$\big\{ z \in \mathbb{C}: \quad |z| < \rho \big\} \tag{4.3}$$

of some radius $\rho > 0$ (one should choose $\psi(H)$ to be proportional to the number $|H|$ of vertices in H). The bound one gets from this approach is

$$\mathbb{P}(A_i) < \Delta^{-\gamma\Delta}, \tag{4.4}$$

for an absolute constant $\gamma > 0$, which is more restrictive than the bound of Theorem 1. After the first version of this paper was posted as a preprint, Mann and Waite [22], among other results, obtained a similar to (4.4) bound, only they bounded $|w(H)|$ in terms of the chromatic number of H.

One reason why our inductive proof of Theorems 1 and 2 achieve less restrictive conditions than (4.4) is that we aim to prove that $p(z)$ has no zeros in some fixed, but otherwise arbitrarily small neighbourhood of $[0,1] \subset \mathbb{C}$, which is still sufficient to deduce Theorem 1 from, while the cluster expansion conditions ensure a stronger claim of $p(z)$ having no zeros in a disc. In our case, for the disc (4.3) to contain a neighbourhood of $[0,1]$, we need to have $\rho = 1+\delta$ for some fixed $\delta > 0$. If we modify our proof so that it works also for the disc (4.3), we would also arrive to (4.4), basically because the condition (2.3) that $|f_i(x)| \leq 1 + \frac{1}{6\Delta}$ would have to be replaced by $|f_i(x)| \leq 2 + \delta$, which would require the values of $\mathbb{P}(A_i)$ to be much smaller. On the other hand, one advantage of having the polynomial $p(z)$ to be non-zero in the disc (4.3) containing $[0,1]$ is that it allows one to improve the bound for K in Theorem 1 to $K = O(\ln(n/\epsilon))$, where the implicit constant in the 'O' notation is absolute.

In their approximation of the volume of the truncated hypergraph matching polytope, Guo and N proposed in [11] a different polymer model, where the underlying graph is bipartite, with the coordinates on one side matched to the events on the other side. It would be interesting to find out if the model extends to the general case treated in this paper, and if it does, what bounds on the probabilities of A_i it gives.

4.2 Optimality

It is unclear if the upper bound for $\mathbb{P}(A_i)$ in Theorem 1 needs to be exponentially small in μ_i. From the proof of Theorem 2, the exponential dependence arises because of the following phenomenon: even when an event $A \subset \Omega_1 \times \Omega_2$ has small probability, for some $\omega_1 \in \Omega_1$, its section

$$A_{\omega_1} = \big\{ \omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in A \big\} \tag{4.5}$$

may still have a large probability in Ω_2. Since by the Markov inequality, the probability to hit such an $\omega_1 \in \Omega_1$ cannot be too large, our inductive proof carries through, but the very existence of such ω_1 creates the exponential in μ_i dependence in our proof.

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When we have a better control over the probability of A_{ω_1} defined by (4.5), the bounds for $\mathbb{P}(A_i)$ improve. In particular, this happens if there is a non-trivial upper bound on $\mathbb{P}(A_{\omega_1})$ as long as $A_{\omega_1} \neq \Omega_2$. This is the case, for example, when each Ω_j is a finite subset of some field $\mathbb{F}$ and each event A_i is defined by a polynomial equation. The analysis also becomes simpler, we sketch it in Section 4.3. We also note that in counting solutions to CNF Boolean formulas, hypergraph colourings or independent sets in hypergraphs, where better bounds are known, each event $A_i \subset \Omega$ has the structure of a direct product $A_i = A_i^1 \times \cdots \times A_i^m$ with $A_i^j \subset \Omega_j$, cf. [5], or a union of a small number of such products [9, 12, 13, 21, 23].

On the other hand, the bounds obtained in [11] for 'bad events' A_i in the problem of approximating the volumes of some combinatorially defined polytopes indeed appear to be exponentially small in μ_i. More precisely, the bounds of [11], when stated in terms of our Theorem 1, appear to be $\mathbb{P}(A_i) = \Delta^{-O(r)} r^{-O(r)}$ for $r = \max_{i=1,\ldots,n} r_i$.

4.3 A simpler case

Let $\Omega_1, \ldots, \Omega_m$ be probability spaces, let $\Omega = \Omega_1 \times \ldots \times \Omega_m$ be their product, and let $A_1, \ldots, A_n \subset \Omega$. As in Section 2.2, we fix a δ-neighbourhood U of the interval $[0,1] \subset \mathbb{C}$ and with a $z \in U$, we associate a random variable $f_i : \Omega \longrightarrow \mathbb{C}$ defined by $f_i(x) = 1 - z[A_i]$, so that

$$|f_i(x)| \;\leq\; 1 + \delta \quad \text{for all} \quad x \in \Omega. \tag{4.6}$$

As before, we assume that each function f_i shares a coordinate with at most Δ other functions f_k.

We also make a stronger assumption as follows. As in Section 3.4, let us write $\Omega = \Omega_1 \times \Omega_2$ and $\mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2$, write $x \in \Omega$ as $x = (u, y)$, where $u \in \Omega_1$ and $y \in \Omega_2$, and for f_i and $u \in \Omega_1$, consider the restriction $h_i(\cdot\,|u) : \Omega_2 \longrightarrow \mathbb{C}$. Now, we assume that for some $0 < \rho < 1/(\Delta + 1)$, each f_i and every $u \in \Omega_1$, we have

$$\begin{aligned} &\mathbb{P}(x \in \Omega :\, f_i(x) \neq 1) \;\leq\; \rho \quad \text{and} \\ &\mathbb{P}_2(y \in \Omega_2 :\, h_i(y|u) \neq 1\,) \;\leq\; \rho, \quad \text{unless } h_i(\cdot\,|u) \text{ is constant on } \Omega_2. \end{aligned} \tag{4.7}$$

We assume that (4.7) holds with every decomposition $\Omega = \Omega_1 \times \Omega_2$.

One particular example where (4.7) is satisfied is as follows. Suppose that $\Omega_1 = \ldots = \Omega_m$ is a subset with q elements of some field $\mathbb{F}$, each Ω_j endowed with the uniform probability measure, and suppose that each A_i is the set of solutions $(\xi_1, \ldots, \xi_m) \in \Omega$ to a polynomial equation of some degree $d \geq 1$ over $\mathbb{F}$. In this case, we can choose $\rho = d/q$ in (4.7), which is known as the Schwartz–Zippel lemma [26], see also Theorem 6.13 in [18].

We claim that if we choose $\delta = 1/2\Delta$ in (4.6) then with $\rho = \gamma/\Delta^2$ in (4.7) for some absolute constant $\gamma > 0$, we have

$$\int_\Omega \prod_{i=1}^n f_i(x)\, dx \neq 0 \quad \text{for all} \quad z \in U.$$

We sketch the argument below, leaving full detail to the reader, since the proof is very similar to that of Section 3, only that with (4.7) it simplifies a lot. We proceed by induction on n, proving that

$$\int_\Omega \prod_{i=1}^n f_i(x)\, dx \neq 0, \qquad \int_\Omega \prod_{i=1}^{n-1} f_i(x)\, dx \neq 0, \tag{4.8}$$

the angle between the two complex numbers in (4.8), considered as non-zero vectors in $\mathbb{R}^2 = \mathbb{C}$ does not exceed some $\theta \leq \pi/2\Delta$, while the ratio of their absolute values (in either order) does not exceed some $\beta \geq 1 + \delta$.

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One can easily check that the base of the induction ($n = 1$) holds when

$$\beta \ \geq \ \frac{1}{1-(2+\delta)\rho} \quad \text{and} \quad \sin\theta \ \geq \ (2+\delta)\rho.$$

Suppose now that $n > 1$. Let $\left\{\xi_j : j \in J\right\}$ be the set of variables that f_n depends on. If $J = \emptyset$, function f_n is a constant, necessarily equal to 1, and the assertion holds with any $\beta \geq 1$ and any $\theta \geq 0$. Assuming that $J \neq \emptyset$, we split $\Omega = \Omega_1 \times \Omega_2$ by (3.10), write $x = (u, y)$ and define $\phi : \Omega_1 \longrightarrow \mathbb{C}$ by the first equation in (3.12), so that the first two equations in (3.13) hold. Next, similarly to Section 3.4, we define two events $A, B \subset \Omega_1$ by

$$A = \{u \in \Omega_1 : \quad \mathbb{P}_2\left(h_i(\cdot\,|u) \neq 1\right) \ \leq \ \rho \quad \text{for all} \quad i \in I\} \quad \text{and}$$
$$B = \{u \in \Omega_1 : \quad f_n(u) = 1\}.$$

As in Section 3.4, we argue that the bulk in both integrals (4.8) are supplied by the integral

$$\int_{A\cap B} f_n(u)\phi(u)\,du = \int_{A\cap B} \phi(u)\,du.$$

Clearly,

$$\mathbb{P}_1(\overline{B}) \ \leq \ \rho. \tag{4.9}$$

To bound $\mathbb{P}_1(\overline{A})$, we observe that we can have $\mathbb{P}_2(h_i(\cdot\,|u) \neq 1) > \rho$ only if $h_i(\cdot\,|u)$ is a constant other than 1 on Ω_2, in which case $\mathbb{P}_2(h_i(\cdot\,|u) \neq 1) = 1$. Since $\mathbb{P}(f_i(x) \neq 1) \leq \rho$, we must have

$$\mathbb{P}_1\{u \in \Omega_1 : \mathbb{P}_2(h_i(\cdot\,|u) \neq 1) > \rho\} \ \leq \ \rho \quad \text{for all} \quad i \in I$$

and hence

$$\mathbb{P}_1\left(\overline{A}\right) \ \leq \ \Delta\rho. \tag{4.10}$$

Again, since all functions $h_i(\cdot\,|u)$ satisfying $\mathbb{P}_2(h_i(\cdot\,|u) \neq 1) > \rho$ are necessarily constants, in the absolute value not exceeding $1 + \delta$, we can factor them out from the integral in (3.12) for $\phi(u)$ and conclude by the induction hypothesis that

$$|\phi(u_1)| \ \leq \ \beta^{2\Delta}|\phi(u_2)| \quad \text{for all} \quad u_1 \in \Omega_1 \quad \text{and all} \quad u_2 \in A. \tag{4.11}$$

As in Section 3.4, from the induction hypothesis we conclude that for any $u_1, u_2 \in A$, the angle between any two complex numbers $\phi(u_1) \neq 0$ and $\phi(u_2) \neq 0$ does not exceed $2\Delta\theta$ and hence

$$\left|\int_{A\cap B} \phi(u)\,du\right| \ \geq \ (\cos\Delta\theta)\int_{A\cap B} |\phi(u)|\,du. \tag{4.12}$$

Finally, as in Section 3, combining (3.13) and (4.9)–(4.12), we conclude that the induction can proceed provided

$$\frac{\beta-1}{\beta+1} \ \geq \ \frac{\rho(\Delta+1)\beta^{2\Delta+1}}{(1-\rho(\Delta+1))\cos(\Delta\theta)} \quad \text{and} \quad \sin\frac{\theta}{2} \ \geq \ \frac{\rho(\Delta+1)\beta^{2\Delta+1}}{(1-\rho(\Delta+1))\cos(\Delta\theta)},$$

(we leave the details to the reader).

Let us choose

$$\theta = \delta = \frac{1}{2\Delta} \quad \text{and} \quad \beta = 1 + \frac{1}{\Delta}.$$

It follows then that the induction can proceed if in (4.7), we have $\rho = \gamma/\Delta^2$ for some absolute constant $\gamma > 0$.

Acknowledgements

The author is grateful to the anonymous referees for their careful reading of the paper and several suggestions to improve the presentation, and, in particular, for the suggestion to look for a special structure of the events A_i that may improve general bounds.

https://doi.org/10.1017/S0963548326100492 Published online by Cambridge University Press

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Cite this article: Barvinok A (2026). Computing the probability of intersection. *Combinatorics, Probability and Computing* **35**, 714–733. https://doi.org/10.1017/S0963548326100492

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