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FORMS OF DIFFERING DEGREES OVER NUMBER FIELDS

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Abstract. Consider a system of polynomials in many variables over the ring of integers of a number field K. We prove an asymptotic formula for the number of integral zeros of this system in homogeneously expanding boxes. As a consequence, any smooth and geometrically integral variety $X \subseteq \mathbb{P}^m_K$ satisfies the Hasse principle, weak approximation, and the Manin–Peyre conjecture if only its dimension is large enough compared to its degree. This generalizes work of Skinner, who considered the case where all polynomials have the same degree, and recent work of Browning and Heath-Brown, who considered the case where $K = \mathbb{Q}$. Our main tool is Skinner's number field version of the Hardy–Littlewood circle method. As a by-product, we point out and correct an error in Skinner's treatment of the singular integral.

§1. *Introduction.*

1.1. *Main result.* Let K be a number field of degree n over $\mathbb{Q}$. We consider a system of polynomials in s variables over the ring of integers $\mathcal{O}_K$ of K and let D be the maximum of their degrees. We assume the polynomials to be ordered by their degrees, that is, for each $d \in \{1, \dots, D\}$, we are given polynomials

$$G_{d,1}, \dots, G_{d,t_d} \in \mathcal{O}_K[x_1, \dots, x_s]$$

of degree d, where $t_d \geqslant 0$ with $t_D \geqslant 1$. The total number of polynomials is $T := t_1 + \cdots + t_D$. We are interested in quantitative statements about the common zeros of these polynomials.

To this end, we fix an integral ideal $\mathfrak{n}$ of $\mathcal{O}_K$ and a $\mathbb{Z}$-basis $\omega_1, \dots, \omega_n$ of $\mathfrak{n}$. We will also consider $\omega_1, \dots, \omega_n$ as an $\mathbb{R}$-basis of $V := K \otimes_{\mathbb{Q}} \mathbb{R}$. By a *box* $\mathcal{B}$ *aligned to the basis*, we mean the set of all $\mathbf{x} = (x_1, \dots, x_s) \in V^s$, where each x_i has the form $x_{i,1}\omega_1 + \cdots + x_{i,n}\omega_n$, such that the coordinates $\mathbf{X} = (x_{i,j})_{i,j} \in \mathbb{R}^{ns}$ lie in a given box $B \subseteq [-1, 1]^{ns}$ with sides aligned to the coordinate axes of $\mathbb{R}^{ns}$. Given such a box $\mathcal{B}$, we study the asymptotics of the counting function

$$N(P) := \#\{\mathbf{x} \in \mathfrak{n}^s \cap P\mathcal{B} \,:\, G_{d,i}(\mathbf{x}) = 0 \text{ for all } 1 \leqslant d \leqslant D, 1 \leqslant i \leqslant t_d\}.$$

This counting function is a classical object of interest and has been investigated in many special cases. To name a few, Birch [1] considered forms over $\mathbb{Q}$ with all degrees equal. Skinner [11] generalized Birch's result to arbitrary number fields K. Schmidt [9] and Browning and Heath-Brown [2] considered forms over $\mathbb{Q}$ whose degrees might be different.

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The main purpose of this article is to generalize the work of Browning and Heath-Brown [2] to arbitrary number fields, just as Skinner did with Birch's work. In addition, we fix an error in Skinner's paper [11]; see §1.5. To state the main result, we need to introduce some further notation. Let

$$\Delta := \{1 \leqslant d \leqslant D \,:\, t_d \geqslant 1\}.$$

For each $1 \leqslant d \leqslant D$, $1 \leqslant i \leqslant t_d$, let $F_{d,i}$ be the leading form of the polynomial $G_{d,i}$, that is, the degree-d-part of $G_{d,i}$. For each degree $d \in \Delta$, we consider the $(t_d \times s)$-Jacobian matrix

$$J_d(\mathbf{x}) := \begin{pmatrix} \nabla F_{d,1}(\mathbf{x}) \\ \vdots \\ \nabla F_{d,t_d}(\mathbf{x}) \end{pmatrix}$$

corresponding to the leading forms of degree d and the affine variety $S_d \subset \mathbb{A}^s_K$ defined by the condition $\operatorname{rank}(J_d(\mathbf{x})) < t_d$. We define B_d to be the dimension of S_d, and $B_d := 0$ if $d \notin \Delta$. In the following, we always assume that $B_d < s$.

Let $\mathcal{D}_0 := 0$ and, for $1 \leqslant d \leqslant D$,

$$\mathcal{D}_d := t_1 + 2t_2 + \cdots + dt_d,$$

so that $\mathcal{D} := \mathcal{D}_D$ is the sum of the degrees of all our polynomials $G_{d,i}$. Write

$$s_d := \sum_{k=d}^{D} \frac{2^{k-1}(k-1)t_k}{s - B_k}. \tag{1.1}$$

Our main theorem is as follows.

THEOREM 1.1. *Assume that*

$$\mathcal{D}_d\left(\frac{2^{d-1}}{s - B_d} + s_{d+1}\right) + s_{d+1} + \sum_{j=d+1}^{D} s_j t_j < 1 \tag{1.2}$$

holds for all $d \in \Delta \cup \{0\}$. Then there is a positive δ such that

$$N(P) = \mathfrak{S}\mathfrak{J} \cdot P^{n(s-\mathcal{D})} + O(P^{n(s-\mathcal{D})-\delta})$$

for $P \to \infty$. Here $\mathfrak{S}$ is the usual singular series and $\mathfrak{J}$ is the usual singular integral. The implicit constant in the error term may depend on K, the polynomials $G_{d,i}$, $\mathfrak{n}$, the basis $\omega_1, \ldots, \omega_n$, and the box $\mathcal{B}$, but not on P, and the positive constant δ depends on K and the systems of forms.

More precisely, the positive constant δ in the exponent of the error term may be chosen depending only on n, T, and the margin by which the inequalities (1.2) are satisfied.

We describe later in this introduction what is meant by the usual singular series and the usual singular integral. Theorem 1.1 is a number field version of

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[2, Theorem 1.2]. One should note that our hypotheses (1.2) are exactly the ones used by Browning and Heath-Brown over $\mathbb{Q}$. In particular, they are independent of the degree n of K.

In the special case where all degrees are equal to D, our hypotheses (1.2) simplify to Skinner's condition

$$s - B_D > t_D(t_D + 1)(D - 1)2^{D-1} \tag{1.3}$$

from [11], which is the same one as Birch's [1].

Our proof relies on Skinner's number field version of the Hardy–Littlewood circle method [10, 11]. There are only very few other applications of the circle method over number fields whose results are even close to the best available results over $\mathbb{Q}$, and in particular do not depend on the degree n of K. See, for example, [3, 8, 11].

1.2. *Consequences.* Here we collect some consequences of Theorem 1.1. They are number field versions of the results stated in the introduction of [2]. We omit most of the proofs, since they are almost verbatim the same as in [2]. The following corollary provides simpler hypotheses that imply those of Theorem 1.1.

COROLLARY 1.2. *Let* $B_{\max} := \max\{B_d : d \in \Delta\}$ *and*

$$u_d := \sum_{k=d}^{D} 2^{k-1}(k-1)t_k \quad \text{for } 1 \leqslant d \leqslant D+1,$$

$$s_0(d) := \mathcal{D}_d(2^{d-1} + u_{d+1}) + u_{d+1} + \sum_{j=d+1}^{D} u_j t_j,$$

$$s_0 := \max\{s_0(d) : d \in \Delta \cup \{0\}\}.$$

Then the conclusion of Theorem 1.1 holds whenever $s > B_{\max} + s_0$.

Of course, the explicit bounds for s_0 in case of systems of quadratic forms and a form of higher degree, computed in [2, Corollary 1.4], and in case of systems consisting of two forms of differing degrees, computed in [2, Corollary 1.5], are still valid in our number field version. Moreover, [2, Theorem 1.6] provides us with the bounds

$$s_0 + T - 1 \leqslant \mathcal{D}^2 2^{D-1} \leqslant T^2 D^2 2^{D-1}$$

and

$$s_0 + T - 1 \leqslant (\mathcal{D} - 1)2^{\mathcal{D}}. \tag{1.4}$$

By the last inequality, the condition $s > B_{\max} + s_0$ in the corollary is implied by the original condition (1.3) of Birch and Skinner in case of a single form of degree D.

In the following, we will specialize our results to non-singular systems of leading forms. To the system of forms $F_{d,j}(\mathbf{x})$, $1 \leqslant d \leqslant D, 1 \leqslant j \leqslant t_d$, we

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associate the $(T \times s)$-Jacobian matrix $J(\mathbf{x})$ formed by the partial derivatives of all T forms $F_{d,j}$ with respect to all s variables x_i. We call the system $(F_{d,j})_{d,j}$ of forms *non-singular* if $\operatorname{rank} J(\mathbf{x}) = T$ for every non-zero $\mathbf{x} \in \overline{\mathbb{Q}}^s$ satisfying $F_{d,j}(\mathbf{x}) = 0$ for all d, j.

Following [2], we define two systems of forms $(F_{d,j})_{d,j}$ and $(\tilde{F}_{d,j})_{d,j}$, with $F_{d,j}, \tilde{F}_{d,j} \in \mathcal{O}_K[x_1, \dots, x_s]$ of degree d, to be *equivalent* if, for every pair (d, j), we have

$$\tilde{F}_{d,j} = F_{d,j} - \sum_{i<j} H_{d,i} F_{d,i} - \sum_{e<d} \sum_{i \leqslant t_e} H_{e,i} F_{e,i}, \tag{1.5}$$

with forms $H_{e,i} \in \mathcal{O}_K[x_1, \dots, x_s]$ of degree $d-e$. Moreover, we define a system $(F_{d,j})_{d,j}$ of forms to be *optimal* if, for any choice of (d, i), the sub-system

$$\{F_{d,j} \,:\, j \geqslant i\} \cup \{F_{e,j} \,:\, d < e \leqslant D,\, j \leqslant t_e\}$$

is non-singular. In [2, §3], it is shown that every non-singular system of forms is equivalent to an optimal system, and that every optimal system $(F_{d,i})_{d,i}$ satisfies

$$B_d \leqslant t_d + \cdots + t_D - 1 \quad \text{for all } 1 \leqslant d \leqslant D$$

and hence in particular that $B_{\max} \leqslant T - 1$.

Assume that we are given a system of polynomials $(G_{d,j})_{d,j}$, with $G_{d,j} \in \mathcal{O}_K[x_1, \dots, x_s]$ of degree d, with leading forms $(F_{d,j})_{d,j}$, and a system of forms $(\tilde{F}_{d,j})_{d,j}$ equivalent to $(F_{d,j})_{d,j}$. Applying the expression for $\tilde{F}_{d,j}$ in (1.5) to the polynomials $G_{d,j}$ instead of the forms $F_{d,j}$, we can easily write down a system of polynomials $(\tilde{G}_{d,j})_{d,j}$ with leading forms $(\tilde{F}_{d,j})_{d,j}$, such that the $\tilde{G}_{d,j}$ generate the same ideal of $\mathcal{O}_K[x_1, \dots, x_s]$ as the $G_{d,j}$. Hence, we may replace every system $(G_{d,j})_{d,j}$ with a non-singular system of leading forms $(F_{d,j})_{d,j}$ by a system $(\tilde{G}_{d,j})_{d,j}$ with an optimal system of leading forms $(\tilde{F}_{d,j})_{d,j}$, and having the same common zeros.

Together with (1.4), this allows us to deduce the following generalization of [2, Theorem 1.7].

THEOREM 1.3. *Suppose that our system of leading forms $(F_{d,j})_{d,j}$ is non-singular and satisfies $s > (\mathcal{D}-1)2^{\mathcal{D}}$. Then there is a positive δ such that*

$$N(P) = \mathfrak{S}\mathfrak{J} \cdot P^{n(s-\mathcal{D})} + O(P^{n(s-\mathcal{D})-\delta}).$$

Here $\mathfrak{J} > 0$ whenever the system of equations

$$F_{d,j}(\mathbf{x}) = 0 \quad \textit{for } 1 \leqslant d \leqslant D, 1 \leqslant j \leqslant t_d$$

has a non-zero solution in the interior of $\mathcal{B} \subset V$. Moreover, $\mathfrak{S} > 0$ whenever the system of equations

$$G_{d,j}(\mathbf{x}) = 0 \quad \textit{for } 1 \leqslant d \leqslant D, 1 \leqslant j \leqslant t_d$$

has a non-zero solution in the completion $n_{\mathfrak{p}}^s \subset (\mathcal{O}_K)_{\mathfrak{p}}^s$ for every prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$.

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The conditions under which $\mathfrak{J} > 0$ and $\mathfrak{S} > 0$ follow from well-known facts about $\mathfrak{J}$ and $\mathfrak{S}$ and from the fact that, under the hypotheses of Theorem 1.3, the system of leading forms $(F_{d,j})_{d,j}$ defines a smooth complete intersection (see [2, Lemma 3.2]). In particular, the singular series $\mathfrak{S}$ has the usual interpretation as a product of local densities.

Theorem 1.3 has far-reaching consequences for smooth projective complete intersections. In fact, every smooth complete intersection $X \subseteq \mathbb{P}^{s-1}_K$ is defined by a non-singular system of forms $(F_{d,i})_{d,i}$ and, if X is non-degenerate (not contained in a proper linear subspace of $\mathbb{P}^{s-1}_K$), then $\deg(X) \geqslant \mathcal{D}$.

It is known that an asymptotic formula as in Theorem 1.3 implies the Hasse principle and weak approximation for X; see [11]. We can also say something about the Manin–Peyre conjecture [4, 7]. Let Ω_K be the set of all places of K and, for each $v \in \Omega_K$, let $n_v := [K_v : \mathbb{Q}_v]$ be the local degree at v. Let $\|\cdot\|_v$ be any norm on K_v^s, coinciding with the usual max-norm if v is non-archimedean. Then

$$H((x_1 : \cdots : x_s)) := \prod_{v \in \Omega_K} \|(x_1, \dots, x_s)\|_v^{n_v(s-\mathcal{D})} \tag{1.6}$$

defines an anticanonical height function on the rational points $X(K)$. The proof of [6, Theorem 4.8] shows that the conclusion of Theorem 1.3 implies the Manin–Peyre conjecture for X with respect to the height H.

Thus, every smooth and non-degenerate complete intersection $X \subseteq \mathbb{P}^{s-1}_K$ with

$$s > (\deg(X) - 1)2^{\deg X}$$

satisfies the Hasse principle, weak approximation, and the Manin–Peyre conjecture for the anticanonical heights defined above.

Browning and Heath-Brown showed in [2] that every smooth and geometrically integral variety $X \subseteq \mathbb{P}^m_{\mathbb{Q}}$ satisfying

$$\dim(X) \geqslant (\deg(X) - 1)2^{\deg(X)} - 1 \tag{1.7}$$

is already a complete intersection. Their arguments hold as well over arbitrary number fields, which provides us with the following nice consequence of Theorem 1.3, generalizing [2, Theorem 1.1].

THEOREM 1.4. *Let $X \subseteq \mathbb{P}^m_K$ be a smooth and geometrically integral variety satisfying (1.7). Then X satisfies the Hasse principle, weak approximation, and the Manin–Peyre conjecture with respect to the height functions defined in (1.6).*

1.3. *The circle method over number fields.* Our proof of Theorem 1.1 relies on the Hardy–Littlewood circle method, implemented over the number field K by Skinner [11]. We start by reviewing some notation from [11].

Let Ω_K, Ω_∞, and Ω_0 denote the sets of all places, archimedean places, and non-archimedean places of K, and write K_v for the completion of K at the place v. If $v \in \Omega_\infty$, then we identify K_v with the field $\mathbb{R}$ or $\mathbb{C}$ in the usual way.

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We identify $V = K \otimes_{\mathbb{Q}} \mathbb{R}$ with $\prod_{v\in\Omega_\infty} K_v$. This allows us to naturally define the conjugates $x^{(v)} \in K_v$ of $x \in V$ via projection and to extend the norm and trace of K to functions $N : V \to \mathbb{R}$ and $\mathrm{Tr} : V \to \mathbb{R}$. On V, we moreover have an $\mathbb{R}$-vector norm defined by

$$|x| := \max\{|x_1|, \dots, |x_n|\} \quad \text{for } x = x_1\omega_1 + \cdots + x_n\omega_n$$

that satisfies $|x| \asymp \max_{v\in\Omega_\infty}\{|x|_v\}$, where $|x|_v$ is the usual absolute value on $K_v \in \{\mathbb{R}, \mathbb{C}\}$. We extend the norm to V^s via $|\mathbf{x}| := \max_{j=1,\dots,s}\{|x_j|\}$ for $\mathbf{x} = (x_1, \dots, x_s)$.

Let

$$R := \{x_1\omega_1 + \cdots + x_n\omega_n \,:\, x_i \in [0, 1)\} \subset V.$$

We normalize the Haar measure on V by $\mathrm{vol}(R) = 1$. Elements of $V^T = \prod_{d=1}^{D} V^{t_d}$ are written with double indices $\boldsymbol{\alpha} = (\alpha_{d,i})_{\substack{1\leqslant d\leqslant D\\ 1\leqslant i\leqslant t_d}}$. We write $e(x) = \mathrm{e}^{2\pi \mathrm{i} x}$ for $x \in \mathbb{R}$ and $\Phi(x) = e(\mathrm{Tr}(x))$ for $x \in V$. The circle method is based on the identity

$$N(P) = \int_{\boldsymbol{\alpha}\in R^T} S(\boldsymbol{\alpha})\, d\boldsymbol{\alpha}, \tag{1.8}$$

where

$$S(\boldsymbol{\alpha}) := \sum_{\mathbf{x}\in\mathfrak{n}^s\cap P\mathcal{B}} \Phi\Big(\sum_{d=1}^{D}\sum_{i=1}^{t_D} \alpha_{d,i} G_{d,i}(\mathbf{x})\Big). \tag{1.9}$$

We divide R^T into major and minor arcs as follows. Let $\varpi \in (0, 1/3)$ be a fixed constant to be specified in §5. For $\gamma \in K$, we have the denominator ideal $\mathfrak{a}_\gamma := \{\beta \in \mathcal{O}_K \,:\, \beta\gamma \in \mathfrak{n}\}$. For $\boldsymbol{\gamma} = (\gamma_{d,i})_{d,i} \in (R \cap K)^T$, let $\mathfrak{a}_{\boldsymbol{\gamma}} := \bigcap_{d,i} \mathfrak{a}_{\gamma_{d,i}}$. The major arc corresponding to $\boldsymbol{\gamma}$ is

$$\mathfrak{M}_{\boldsymbol{\gamma}} := \{\boldsymbol{\alpha} \in R^T \,:\, |\alpha_{d,i} - \gamma_{d,i}| \leqslant P^{-d+\varpi} \text{ for all } 1 \leqslant d \leqslant D, 1 \leqslant i \leqslant t_d\},$$

where the distance $|\alpha_{d,i} - \gamma_{d,i}|$ is to be understood modulo $\mathfrak{n}$. We define the major arcs

$$\mathfrak{M} := \bigcup_{\substack{\boldsymbol{\gamma}\in(R\cap K)^T\\ \mathfrak{N}\mathfrak{a}_{\boldsymbol{\gamma}}\leqslant P^{\varpi}}} \mathfrak{M}_{\boldsymbol{\gamma}}$$

and the minor arcs

$$\mathfrak{m} := R^T \setminus \mathfrak{M}.$$

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In §4, we show that, under the hypotheses of Theorem 1.1, the contribution of the minor arcs $\mathfrak{m}$ to the integral in (1.8) is absorbed by the error term. In §§5 and 6, we evaluate the contribution of the major arcs $\mathfrak{M}$ as $\mathfrak{S}\mathfrak{J}P^{n(s-\mathcal{D})} + O(P^{n(s-\mathcal{D})-\delta})$, with the singular series

$$\mathfrak{S} = \prod_{\mathfrak{p}} \sum_{j=0}^{\infty} \frac{1}{\mathfrak{N}\mathfrak{p}^{js}} \sum_{\substack{\boldsymbol{\gamma} \in (R\cap K)^T \\ \mathfrak{a}_{\boldsymbol{\gamma}} = \mathfrak{p}^j}} \sum_{\mathbf{x} \in (\mathfrak{n}/\mathfrak{p}^j\mathfrak{n})^s} \Phi\Bigg(\sum_{d=1}^{D}\sum_{i=1}^{t_d} \gamma_{d,i} G_{d,i}(\mathbf{x})\Bigg)$$

and the singular integral

$$\mathfrak{J} = \int_{\boldsymbol{\gamma} \in V^T} \int_{\mathbf{x} \in \mathcal{B}} \Phi\Bigg(\sum_{d=1}^{D}\sum_{i=1}^{t_d} \gamma_{d,i} F_{d,i}(\mathbf{x})\Bigg)\, d\mathbf{x}\, d\boldsymbol{\gamma}.$$

In §§2 and 3, we prove the main tool to be used in our estimations, an iterative Weyl-type lemma for the exponential sum $S(\boldsymbol{\alpha})$ that generalizes the version from [2] to number fields.

1.4. *Further notation.* It is sometimes useful to identify V with $\mathbb{R}^n$ via the basis $\omega_1, \dots, \omega_n$. Then $\mathbf{x} \in V^s$ with $x_i = x_{i,1}\omega_1 + \cdots + x_{i,n}\omega_n$ is identified with $\mathbf{X} = (x_{i,j})_{i,j} \in \mathbb{R}^{ns}$. The volume on V becomes the usual Lebesgue measure on $\mathbb{R}^n$, and the norm $|\cdot|$ becomes the usual max-norm on $\mathbb{R}^n$, which we will also denote by $|\cdot|$. To each polynomial $G \in V[x_1, \dots, x_s]$, we associate the polynomial

$$G^*(\mathbf{X}) := \mathrm{Tr}(G(\mathbf{x})) \in \mathbb{R}[\mathbf{X}]$$

and the system $G_j^*(\mathbf{X})$, $1 \leqslant j \leqslant n$, defined via

$$G_j^*(\mathbf{X}) := \mathrm{Tr}(\omega_j G(\mathbf{x})) \in \mathbb{R}[\mathbf{X}].$$

Then any $\mathbf{x}$ in V^s satisfies $G_{d,i}(\mathbf{x}) = 0$ for all d, i if and only if $G_{d,i,j}^*(\mathbf{X}) = 0$ for all d, i, j, and our system of T polynomials in s variables over $\mathcal{O}_K$ is equivalent to a system of nT polynomials in ns variables over $\mathbb{Z}$. In fact, the affine variety defined over $\mathbb{Q}$ by the polynomials $G_{d,i,j}^*(\mathbf{X})$ is the Weil restriction of the K-variety defined by the polynomials $G_{d,i}(\mathbf{x})$. These identifications allow us to write $S(\boldsymbol{\alpha})$ as an exponential sum over $\mathbb{Z}^{ns}$:

$$S(\boldsymbol{\alpha}) = \sum_{\mathbf{X} \in \mathbb{Z}^{ns} \cap PB} e\Bigg(\sum_{d=1}^{D}\sum_{i=1}^{t_d}\sum_{j=1}^{n} \alpha_{d,i,j} G_{d,i,j}^*(\mathbf{X})\Bigg),$$

where $\alpha_{d,i} = \alpha_{d,i,1}\omega_1 + \cdots + \alpha_{d,i,n}\omega_n$.

We denote the standard basis of the free V-module V^s by $\mathbf{v}_1, \dots, \mathbf{v}_s$, and the standard basis of $\mathbb{R}^{ns}$ by $\mathbf{E}_{ij}$ ($i = 1, \dots, s$ and $j = 1, \dots, n$). By our identification, we obtain $\mathbf{E}_{ij} = \omega_j \mathbf{v}_i$.

For $\beta \in \mathbb{R}$, we write $\|\beta\|$ for the distance of β to the nearest integer.

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For any form $F \in V[x_1, \dots, x_s]$ of degree d, we write $F(\mathbf{x}_1|\cdots|\mathbf{x}_d)$ for the corresponding polar d-multilinear form, normalized by $d!F(\mathbf{x}) = F(\mathbf{x}|\cdots|\mathbf{x})$. Similarly, $F^*(\mathbf{X}_1|\cdots|\mathbf{X}_d)$ is the polar d-multilinear form corresponding to F^*. Observe that $F^*(\mathbf{X}_1|\cdots|\mathbf{X}_d) = \mathrm{Tr}(F(\mathbf{x}_1|\cdots|\mathbf{x}_d))$.

1.5. *The singular integral.* Our main task in §6 is to show that the integral

$$\mathfrak{J}(H) := \int_{\substack{\boldsymbol{\gamma}\in V^T \\ |\boldsymbol{\gamma}|\leqslant H}} \int_{\mathbf{x}\in\mathcal{B}} \Phi\Bigg(\sum_{d=1}^{D}\sum_{i=1}^{t_d} \gamma_{d,i} F_{d,i}(\mathbf{x})\Bigg)\, d\mathbf{x}\, d\boldsymbol{\gamma} \tag{1.10}$$

converges absolutely as $H \to \infty$, and that

$$\mathfrak{J}(H) - \mathfrak{J} \ll H^{-\delta}$$

for some positive δ; see Lemma 6.8. In the case where all degrees are equal, i.e. $t_d = 0$ for all $d \neq D$, this is done by Skinner in [**11**, Lemma 9]. For the proof, Skinner suggests to think of our forms $F_{D,i}$ as the forms $F^*_{D,i,j}$ over $\mathbb{Z}$ and to apply Schmidt's Lemma 8.1 from [**9**]. Schmidt's lemma is a formalization of the classical indirect approach to the singular integral, already used by Birch [**1**], where the Weyl-type lemma used in the treatment of the minor arcs is applied once more to bound the inner integral in (1.10). Hence, it depends on a certain hypothesis, called the *restricted hypothesis* by Schmidt. Applied to our forms $F^*_{D,i,j}$, it requires that at least one of the following alternatives holds for some $\Omega > nt_D + 1$ and each $\Delta \in (0, 1]$: every $\boldsymbol{\alpha} \in R^{t_D}$ satisfies:

(i) $|S(\boldsymbol{\alpha})| \leqslant P^{ns-\Delta\Omega}$; or

(ii) there is $q \in \mathbb{N}$, $q \leqslant P^{\Delta}$ with $\|q\alpha_{D,i,j}\| \leqslant P^{-D+\Delta}$ for all $1 \leqslant i \leqslant t_D$ and $1 \leqslant j \leqslant n$.

Skinner gives no argument why this hypothesis would hold. In fact, we were not able to deduce it from either Skinner's Weyl-type lemma [**11**, Lemma 2] or Birch's Weyl-type lemma [**1**, Lemma 2.5] applied to the $F^*_{D,i,j}$, without replacing the lower bound (1.3) on the number of variables s by the stronger bound

$$s - B_D > t_D(nt_D + 1)(D - 1)2^{D-1}, \tag{1.11}$$

and we see no reason why it should hold otherwise. Let us note that with the stronger assumption (1.11) instead of (1.3), the main theorem of [**11**] would follow directly from the techniques of [**1**] applied to the $G^*_{D,i,j}$.

In §6, we apply genuine number field arguments to our treatment of the singular integral, culminating in Lemma 6.8. When all degrees are equal to D, the hypothesis (5.6) of Lemma 6.8 is exactly Skinner's hypothesis (1.3), so we prove [**11**, Lemma 9] as a special case of Lemma 6.8.

§2. *Exponential sums.* For a function $f : V^s \to \mathbb{R}$ and $\mathbf{h} \in V^s$, we write $\Delta_{\mathbf{h}}(f)(\mathbf{x}) := f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x})$ for the usual forward difference operator. The

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following lemma is a number field analogue of van der Corput's variant of Weyl differencing.

LEMMA 2.1. *Let $q \in \mathcal{O}_K \setminus \{0\}$ and $H \in \mathbb{Z}$ with $1 \leqslant H \ll P/|q|$. Let $f : V^s \to \mathbb{R}$ and $\mathcal{I}$ be a box aligned to the basis $\omega_1, \dots, \omega_n$. Then*

$$\left|\sum_{\mathbf{x}\in\mathfrak{n}^s\cap P\mathcal{I}} \Phi(f(\mathbf{x}))\right|^2 \ll \left(\frac{P}{H}\right)^{ns} \sum_{\substack{\mathbf{h}\in\mathfrak{n}^s\\ |\mathbf{h}|<H}} \left|\sum_{\mathbf{x}\in\mathfrak{n}^s\cap P\mathcal{I}'(\mathbf{h})} \Phi(\Delta_{q\mathbf{h}}(f)(\mathbf{x}))\right|,$$

where $\mathcal{I}'(\mathbf{h})$ is a box aligned to the basis $\omega_1, \dots, \omega_n$ that depends on $\mathbf{h}$. The implicit constant depends only on K and s.

Proof. The proof is analogous to the version over $\mathbb{Q}$ (see the proof of [2, Lemma 4.1]). Let $\chi_{P\mathcal{I}}$ be the indicator function of $P\mathcal{I}$. Let $R^* \subset V$ be the set of all $\mathbf{u} = u_1\omega_1 + \cdots + u_n\omega_n \in V$ with $1 \leqslant u_j \leqslant H$ for all $1 \leqslant j \leqslant n$. Then

$$\begin{aligned} H^{ns} \sum_{\mathbf{x}\in\mathfrak{n}^s\cap P\mathcal{I}} \Phi(f(\mathbf{x})) &= \sum_{\mathbf{u}\in(\mathfrak{n}\cap R^*)^s} \sum_{\mathbf{x}\in\mathfrak{n}^s} \Phi(f(\mathbf{x}+q\mathbf{u}))\chi_{P\mathcal{I}}(\mathbf{x}+q\mathbf{u}) \\ &= \sum_{\substack{\mathbf{x}\in\mathfrak{n}^s\\ |\mathbf{x}|\ll P}} \sum_{\mathbf{u}\in(\mathfrak{n}\cap R^*)^s} \Phi\left(f(\mathbf{x}+q\mathbf{u})\chi_{P\mathcal{I}}(\mathbf{x}+q\mathbf{u}).\right. \end{aligned}$$

Here we used that $\chi_{P\mathcal{I}}(\mathbf{x}+q\mathbf{u}) \neq 0$ implies that $|\mathbf{x}| \leqslant P + |q\mathbf{u}| \ll P + |q||\mathbf{u}| \ll P$. By Cauchy's inequality,

$$\begin{aligned} H^{2ns}\left|\sum_{\mathbf{x}\in\mathfrak{n}^s\cap P\mathcal{I}} \Phi(f(\mathbf{x}))\right|^2 &\ll P^{ns} \sum_{\substack{\mathbf{x}\in\mathfrak{n}^s\\ |\mathbf{x}|\ll P}} \left|\sum_{\mathbf{u}\in(\mathfrak{n}\cap R^*)^s} \Phi(f(\mathbf{x}+q\mathbf{u}))\chi_{P\mathcal{I}}(\mathbf{x}+q\mathbf{u})\right|^2 \\ &= P^{ns} \sum_{\substack{\mathbf{h}\in\mathfrak{n}^s\\ |\mathbf{h}|<H}} n(\mathbf{h}) \sum_{y\in\mathfrak{n}^s} \Phi(f(\mathbf{y}+q\mathbf{h}))\chi_{P\mathcal{I}}(\mathbf{y}+q\mathbf{h}) \\ &\quad \cdot \overline{\Phi(f(\mathbf{y}))}\chi_{P\mathcal{I}}(\mathbf{y}), \end{aligned}$$

where

$$n(\mathbf{h}) = \#\{(\mathbf{u}, \mathbf{v}) \in (\mathfrak{n}\cap R^*)^{2s} \,:\, \mathbf{h} = \mathbf{v} - \mathbf{u}\} \leqslant H^{ns}.$$

Thus,

$$\left|\sum_{\mathbf{x}\in\mathfrak{n}^s\cap P\mathcal{I}} \Phi(f(\mathbf{x}))\right|^2 \ll \left(\frac{P}{H}\right)^{ns} \sum_{\substack{\mathbf{h}\in\mathfrak{n}^s\\ |\mathbf{h}|<H}} \left|\sum_{\mathbf{y}\in\mathfrak{n}^s\cap P\mathcal{I}'(\mathbf{h})} \Phi(f(\mathbf{y}+q\mathbf{h}))\overline{\Phi(f(\mathbf{h}))}\right|,$$

where $\mathcal{I}'(\mathbf{h}) \subseteq \mathcal{I}$ is a box aligned to the basis $\omega_1, \dots, \omega_n$, depending on $\mathbf{h}$. □

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Let $f, g \in V[x_1, \dots, x_s]$ with $\deg f \leqslant d$. Suppose that $qg = g_1 + g_2$ with $q \in \mathfrak{n} \setminus \{0\}$, $g_1 \in \mathcal{O}_K[x_1, \dots, x_s]$, and $g_2 \in V[x_1, \dots, x_s]$ such that all coefficients a_j of g_2 of each degree j satisfy

$$|a_j| \ll_j \varphi P^{-j} \tag{2.1}$$

for some $\varphi \geqslant |q|$.

Let $\mathcal{B}'$ be a box aligned to the basis $\omega_1, \dots, \omega_n$. We are interested in the estimation of the exponential sum

$$\Sigma := \sum_{\mathbf{x} \in \mathfrak{n}^s \cap P\mathcal{B}'} \Phi(f(\mathbf{x}) + g(\mathbf{x})).$$

In the process, f will be the dominant polynomial whereas g originates from the higher exponents, which are already well approximable. We aim for an estimate of the form

$$|\Sigma| = P^{ns} L$$

with small L. Let F be the homogeneous part of f of degree d, and recall that $(qF)^*(\mathbf{X}_1 | \cdots | \mathbf{X}_d)$ is the d-multilinear polar form corresponding to the form $(qF)^*(\mathbf{X})$.

LEMMA 2.2. *For $M \geqslant 1$, we have*

$$L^{2^{d-1}} \ll P^{-(d-1)ns} (\varphi M)^{(d-1)ns} (\log P)^{ns} \mathcal{M}, \tag{2.2}$$

where $\mathcal{M}$ is the number of all $(\mathbf{x}_1, \dots, \mathbf{x}_{d-1}) \in (\mathfrak{n}^s)^{d-1}$ satisfying

$$|\mathbf{x}_i| \leqslant \frac{P}{\varphi M} \quad \text{for all } 1 \leqslant i < d \text{ and}$$
$$\|(qF)^*(\mathbf{X}_1 | \cdots | \mathbf{X}_{d-1} | \mathbf{E}_{i,j})\| \leqslant \frac{1}{P \varphi^{d-2} M^{d-1}} \quad \text{for all } 1 \leqslant i \leqslant s, 1 \leqslant j \leqslant n.$$

Proof. This is mostly analogous to the proof of [2, Lemma 4.1]. The lemma holds trivially if $\varphi \geqslant P$. Hence, we assume that $\varphi \leqslant P$.

We start by $d - 2$ Weyl differencing steps, that is, $d - 2$ applications of Lemma 2.1 with $q := 1$, $H := P$, linked by Cauchy's inequality. This yields

$$L^{2^{d-2}} \ll P^{-ns(d-1)} \sum_{\substack{\mathbf{h}_1 \in \mathfrak{n}^s \\ |\mathbf{h}_1| < P}} \cdots \sum_{\substack{\mathbf{h}_{d-2} \in \mathfrak{n}^s \\ |\mathbf{h}_{d-2}| < P}} \left| \sum_{\mathbf{x} \in \mathfrak{n}^s \cap P\mathcal{I}} \Psi(\mathbf{x}) \right|, \tag{2.3}$$

where

$$\Psi(\mathbf{x}) := \Phi(\Delta_{\mathbf{h}_1, \dots, \mathbf{h}_{d-2}}(f + g)(\mathbf{x}))$$

and $\mathcal{I} \subset \mathcal{B}'$ is a box aligned to the basis $\omega_1, \dots, \omega_n$ that depends on $\mathbf{h}_1, \dots, \mathbf{h}_{d-2}$.

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For the $(d-1)$st differencing step, we choose q as in the setup before our lemma and

$$H := \left\lfloor \frac{P}{\varphi} \right\rfloor \geqslant 1.$$

By Lemma 2.1,

$$\left| \sum_{\mathbf{x} \in \mathfrak{n}^s \cap P\mathcal{I}} \Psi(\mathbf{x}) \right|^2 \ll \varphi^{ns} \sum_{\substack{\mathbf{w} \in \mathfrak{n}^s \\ |\mathbf{w}| < H}} \left| \sum_{\mathbf{y} \in \mathfrak{n}^s \cap P\mathcal{I}'} \Psi(\mathbf{y} + q\mathbf{w}) \overline{\Psi(\mathbf{y})} \right|,$$

where $\mathcal{I}' \subseteq \mathcal{I}$ is a box aligned to the basis $\omega_1, \dots, \omega_n$, depending on $\mathbf{w}$. Together with Cauchy's inequality and (2.3), this yields

$$L^{2^{d-1}} \ll P^{-nsd} \varphi^{ns} \sum_{\substack{\mathbf{h}_1 \in \mathfrak{n}^s \\ |\mathbf{h}_1| < P}} \cdots \sum_{\substack{\mathbf{h}_{d-2} \in \mathfrak{n}^s \\ |\mathbf{h}_{d-2}| < P}} \sum_{\substack{\mathbf{w} \in \mathfrak{n}^s \\ |\mathbf{w}| < H}} \left| \sum_{\mathbf{y} \in \mathfrak{n}^s \cap P\mathcal{I}'} \Psi(\mathbf{y} + q\mathbf{w}) \overline{\Psi(\mathbf{y})} \right|.$$

Note that Ψ implicitly depends on $\mathbf{h}_1, \dots, \mathbf{h}_{d-2}$, and $\mathcal{I}'$ depends on $\mathbf{h}_1, \dots, \mathbf{h}_{d-2}$, and $\mathbf{w}$.

Now we take a closer look at the summands of the innermost sum:

$$\Psi(\mathbf{y} + q\mathbf{w}) \overline{\Psi(\mathbf{y})} = \Phi(\Delta_{\mathbf{h}_1, \dots, \mathbf{h}_{d-2}, q\mathbf{w}}(f + g)(\mathbf{y})).$$

Recall that F is the leading form of f of degree d. Then, by linearity of $F(\mathbf{x}_1 | \cdots | \mathbf{x}_d)$, we have that

$$\Delta_{\mathbf{h}_1, \dots, \mathbf{h}_{d-2}, q\mathbf{w}}(f)(\mathbf{y}) = qF(\mathbf{h}_1 | \cdots | \mathbf{h}_{d-2} | \mathbf{w} | \mathbf{y}) + C.$$

Now we focus on g. Since all coefficients of $\Delta_{\mathbf{h}_1, \dots, \mathbf{h}_{d-2}, q\mathbf{w}}(g_1)$ are $\mathcal{O}_K$-multiples of q, we have

$$\Phi(\Delta_{\mathbf{h}_1, \dots, \mathbf{h}_{d-2}, q\mathbf{w}}(q^{-1} g_1)(\mathbf{y})) = 1$$

for all $\mathbf{y} \in \mathcal{O}_K$. Since $|\mathbf{h}_i| \leqslant P$ for all $1 \leqslant i \leqslant d-2$, it follows from (2.1) that each coefficient b_j of $\Delta_{\mathbf{h}_1, \dots, \mathbf{h}_{d-2}}(g_2)$ of any degree j satisfies

$$|b_j| \ll_j |\mathbf{h}_1| \cdots |\mathbf{h}_{d-2}| \varphi P^{-j-(d-2)}.$$

Since $|q\mathbf{w}| \ll |q| H \leqslant \varphi H \ll P$, the coefficients c_j of $\Delta_{\mathbf{h}_1, \dots, \mathbf{h}_{d-2}, q\mathbf{w}}(q^{-1} g_2)$ of degree j are bounded by

$$|c_j| \ll |q^{-1} b_{j+1} q w_\ell| \ll |b_{j+1}| |\mathbf{w}| \ll_j |\mathbf{h}_1| \cdots |\mathbf{h}_{d-2}| \varphi P^{-j-(d-1)} H \ll P^{-j},$$

where b_{j+1} is a coefficient of $\Delta_{\mathbf{h}_1, \dots, \mathbf{h}_{d-2}}(g_2)$ of degree $j+1$ and w_ℓ is a component of $\mathbf{w}$.

Write $\mathbf{u} := q\mathbf{w}$. With the tuples $\mathbf{H}_i, \mathbf{U} \in \mathbb{Z}^{sn}$ corresponding to $\mathbf{h}_i, \mathbf{u} \in \mathfrak{n}^s$, we have shown that any jth-order partial derivative of $\Delta_{\mathbf{H}_1, \dots, \mathbf{H}_{d-2}, \mathbf{U}}((q^{-1} g_2)^*)$ is $\ll_j P^{-j}$ uniformly on $[-P, P]^{ns}$.

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Let $I' \subseteq [-1,1]^{sn}$ be the box in $\mathbb{R}^{sn}$ corresponding to $\mathcal{I}'$. Using the above computations,

$$\begin{aligned}\sum_{\mathbf{y}\in\mathfrak{n}^s\cap P\mathcal{I}'} &\Psi(\mathbf{y}+q\mathbf{w})\overline{\Psi(\mathbf{y})}\\ &= \sum_{\mathbf{Y}\in\mathbb{Z}^{ns}\cap PI'} e((qF)^*(\mathbf{H}_1|\cdots|\mathbf{H}_{d-2}|\mathbf{W}|\mathbf{Y}) + \mathrm{Tr}(C)\\ &\quad + \Delta_{\mathbf{H}_1,\dots,\mathbf{H}_{d-2},\mathbf{U}}((q^{-1}g_2)^*)(\mathbf{Y}))\,.\end{aligned}$$

In the same manner as Browning and Heath-Brown, we apply multidimensional partial summation and our uniform bounds for the partial derivatives of $\Delta_{\mathbf{H}_1,\dots,\mathbf{H}_{d-2},\mathbf{U}}((q^{-1}g_2)^*)$ to obtain

$$\sum_{\mathbf{y}\in\mathfrak{n}^s\cap\mathcal{I}'} \Psi(\mathbf{y}+q\mathbf{w})\overline{\Psi(\mathbf{y})} \ll \left|\sum_{\mathbf{Y}\in\mathbb{Z}^{ns}\cap I''} e((qF)^*(\mathbf{H}_1|\dots|\mathbf{H}_{d-2}|\mathbf{W}|\mathbf{Y}))\right|,$$

with a box $I'' \subseteq I'$ aligned to the coordinate axes. We are now in exactly the same situation as in the proof of [2, Lemma 4.1], just in dimension sn instead of n and with φ instead of $q\varphi$. What remains of the proof is identical to the arguments of [2] starting at (4.5), just with tuples $\mathbf{Y}$, $\mathbf{H}_j$, $\mathbf{W}$ of sn variables instead of tuples $\mathbf{y}$, $\mathbf{x}_j$, $\mathbf{w}$ of n variables. □

§3. *The iterative argument.* Our aim in this section is to find a Weyl-type estimate for the exponential sum $S(\alpha)$ defined in (1.9). To this end, we write

$$|S(\alpha)| = P^{ns}L.$$

We define $Q_{D+1} := 1$ and, for $d \in \Delta$,

$$Q_d := (\log P)^{e(d)}L^{-s_d/n}, \tag{3.1}$$

where $e(d)$ is an explicit but irrelevant exponent which could be computed from the arguments in the proof of Lemma 3.1. For those $1 \leqslant d \leqslant D$ with $d \notin \Delta$, we set $Q_d := Q_k$, where k is the smallest integer bigger than d in Δ. Similarly, we can extend the definition of the exponents $e(d)$ to these values.

For $j \in \Delta$, we consider upper bounds

$$L^{2^{j-1}} \leqslant \left(\frac{Q_{j+1}}{P}\right)^{n(s-B_j)}(\log P)^{ns+1}. \tag{3.2}$$

Let $I_d^{(1)}$ be the set of all $\boldsymbol{\alpha} \in R^T$ such that (3.2) holds for $j = d$ but fails for every $j > d$. Moreover, let $I^{(2)}$ be the set of all $\boldsymbol{\alpha} \in R^T$ such that (3.2) fails for all $j \in \Delta$. We are going to prove the following number field analogue of [2, Lemma 6.2].

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LEMMA 3.1. *Let $d \in \Delta$ and $P \gg 1$. If $\boldsymbol{\alpha} \in I_d^{(1)}$, then*

$$L^{2^{d-1}+(s-B_d)s_{d+1}} \ll P^{-n(s-B_d)+\epsilon}. \tag{3.3}$$

Moreover, there are $q_j \in \mathfrak{n}$, $\boldsymbol{\nu}_j \in \mathfrak{n}^{t_j}$ for all $d < j \leqslant D$, $j \in \Delta$ satisfying

$$q_k \mid q_j \qquad \text{for all } k > j, k \in \Delta, \tag{3.4}$$

$$|q_j| \leqslant Q_j, \tag{3.5}$$

$$|q_j \alpha_{j,i} - \nu_{j,i}| \leqslant Q_j P^{-j} \quad \text{for all } 1 \leqslant i \leqslant t_j. \tag{3.6}$$

If $\boldsymbol{\alpha} \in I^{(2)}$, then there are $q \in \mathfrak{n}$, $\boldsymbol{\nu}_j \in \mathfrak{n}^{t_j}$ for all $j \in \Delta$ satisfying (3.4)–(3.6).

The idea is to iteratively apply Lemma 2.2. Recall that $\mathbf{v}_j$ denotes the jth element of the standard basis of V^s. For $d \leqslant D$, we define the matrix

$$\widehat{J}_d(\mathbf{x}_1, \dots, \mathbf{x}_{d-1}) := (F_{d,i}(\mathbf{x}_1 | \cdots | \mathbf{x}_{d-1} | \mathbf{v}_j))_{\substack{1 \leqslant i \leqslant t_d \\ 1 \leqslant j \leqslant s}}$$

and the corresponding affine variety $\widehat{S}_d \subseteq (\mathbb{A}_K^s)^{d-1}$ defined by the condition

$$\operatorname{rank}(\widehat{J}_d(\mathbf{x}_1, \dots, \mathbf{x}_{d-1})) < t_d.$$

We need an estimate for the number of integral points on $\widehat{S}_d$ of bounded norm. Let

$$\mathcal{M}_0(P) := \#\{(\mathbf{x}_1, \dots, \mathbf{x}_{d-1}) \in \widehat{S}_d(K) \cap (\mathfrak{n}^s)^{d-1} : |\mathbf{x}_i| \leqslant P \text{ for all } 1 \leqslant i < d\}.$$

LEMMA 3.2. *For $P \geqslant 1$, we have*

$$\mathcal{M}_0(P) \ll P^{n(B_d + s(d-2))}.$$

Proof. As in [2, Lemma 5.1], using [11, Lemma 3] instead of [1, Lemma 3.1]. □

The main tool for the proof of Lemma 3.1 is the following iterative argument.

LEMMA 3.3. *Let $|S(\boldsymbol{\alpha})| = P^{ns}L$ and $d \in \Delta$. Furthermore, suppose that:*

- *either $d = D$, $q = 1$, and $Q = 1$;*
- *or $d < D$, and there exist $Q \geqslant 1$ and $q \in \mathfrak{n}$ with $|q| \leqslant Q$, and $\boldsymbol{\nu}_j \in \mathfrak{n}^{t_j}$ such that*

$$|q\alpha_{j,i} - \nu_{j,i}| \leqslant QP^{-j} \quad \text{for } d < j \leqslant D \text{ and } 1 \leqslant i \leqslant t_j.$$

Then, for P sufficiently large, either

$$L^{2^{d-1}} \leqslant \left(\frac{Q}{P}\right)^{n(s-B_d)} (\log P)^{ns+1}$$

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or there exists $q^* \in \mathfrak{n}$ *with*

$$|q^*| \leqslant Q^* := \left(\frac{(\log P)^{ns+1}}{L^{2^{d-1}}}\right)^{t_d(d-1)/n(s-B_d)} (\log P)$$

and $\boldsymbol{\nu}_d \in \mathfrak{n}^{t_d}$ *such that*

$$|q^* q\alpha_{d,i} - \nu_{d,i}| \leqslant QQ^*P^{-d} \quad \textit{for } 1 \leqslant i \leqslant t_d.$$

Proof. The key tool is Lemma 2.2. We distinguish the two cases $d = D$ and $d < D$:

- $d = D$: in this case, we choose $\varphi := 1$, $g = g_1 = g_2 := 0$, and

$$f(\mathbf{x}) := \sum_{j=1}^{D}\sum_{i=1}^{t_j} \alpha_{j,i} G_{j,i}(\mathbf{x});$$

- $d < D$: then we let

$$f(\mathbf{x}) := \sum_{j=1}^{d}\sum_{i=1}^{t_j} \alpha_{j,i} G_{j,i}(\mathbf{x}) \quad \text{and} \quad g(\mathbf{x}) := \sum_{j=d+1}^{D}\sum_{i=1}^{t_j} \alpha_{j,i} G_{j,i}(\mathbf{x}).$$

By hypothesis, we have $q\alpha_{j,i} = \nu_{j,i} + \theta_{j,i}$ for $d < j \leqslant D$ and $1 \leqslant i \leqslant t_j$, with $|\theta_{j,i}| \leqslant QP^{-j}$. We write $qg = g_1 + g_2$, where

$$g_1 := \sum_{j=d+1}^{D}\sum_{i=1}^{t_j} \nu_{j,i} G_{j,i}(\mathbf{x}) \quad \text{and} \quad g_2 := \sum_{j=d+1}^{D}\sum_{i=1}^{t_j} \theta_{j,i} G_{j,i}(\mathbf{x}).$$

This allows us to choose $\varphi := Q$.

Now we apply Lemma 2.2 with $M := \max\{1, M_1\}$, where

$$M_1 := \frac{P}{Q}\left(\frac{L^{2^{d-1}}}{(\log P)^{ns+1}}\right)^{1/n(s-B_d)}.$$

If $M_1 \leqslant 1$, then

$$\frac{L^{2^{d-1}}}{(\log P)^{ns+1}} \leqslant \left(\frac{Q}{P}\right)^{n(s-B_d)},$$

as required by the first alternative in the conclusion of the lemma.

Therefore, we may suppose that $M = M_1 > 1$ and consider two cases according to whether all points $(\mathbf{x}_1, \ldots, \mathbf{x}_{d-1}) \in (\mathfrak{n}^s)^{d-1}$ counted by $\mathcal{M}$ from Lemma 2.2 are in the affine variety $\widehat{S}_d$ or not.

If they all lie in $\widehat{S}_d$, then an application of Lemma 3.2 implies that

$$\mathcal{M} \leqslant \mathcal{M}_0\left(\frac{P}{QM}\right) \ll \left(\frac{P}{QM}\right)^{nB_d + ns(d-2)}.$$

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Hence, we have

$$\begin{aligned} L^{2^{d-1}} &\ll P^{-(d-1)sn}(QM)^{(d-1)ns}(\log P)^{ns}\left(\frac{P}{QM}\right)^{nB_d+ns(d-2)} \\ &\ll (QM)^{ns-nB_d}P^{nB_d-ns}(\log P)^{ns} \\ &= \left(\frac{QM}{P}\right)^{n(s-B_d)}(\log P)^{ns}. \end{aligned}$$

Substituting M yields

$$L^{2^{d-1}} \ll L^{2^{d-1}}(\log P)^{-1},$$

which is a contradiction for P sufficiently large.

In the remaining case, we are given a point $(\mathbf{x}_1, \dots, \mathbf{x}_{d-1}) \in (\mathfrak{n}^s)^{d-1}$ with:

- $|\mathbf{x}_i| \leqslant P/QM$ for all $1 \leqslant i < d$;
- $\|(qF)^*(\mathbf{X}_1|\cdots|\mathbf{X}_{d-1}|\mathbf{E}_{p,\ell})\| \leqslant 1/PQ^{d-2}M^{d-1}$ for all $1 \leqslant p \leqslant s$, $1 \leqslant \ell \leqslant n$; and
- $\operatorname{rank}(\widehat{J}_d(\mathbf{x}_1, \dots, \mathbf{x}_{d-1})) = t_d$.

Without loss of generality, we assume that the matrix W consisting of the first t_d columns of $\widehat{J}_d(\mathbf{x}_1, \dots, \mathbf{x}_{d-1})$ has full rank. Let $\widetilde{q}^* := \det W$. Then $\widetilde{q}^* \in \mathfrak{n}$ and

$$|\widetilde{q}^*| \ll |\mathbf{x}_1|^{t_d}\cdots|\mathbf{x}_{d-1}|^{t_d} \ll \left(\frac{P}{QM}\right)^{t_d(d-1)}.$$

We set

$$\beta_p^* = \sum_{k=1}^{n}\beta_{p,k}^*\omega_k := qF(\mathbf{x}_1|\mathbf{x}_2|\cdots|\mathbf{x}_{d-1}|\mathbf{v}_p) = \sum_{i=1}^{t_d}\alpha_{d,i}qF_{d,i}(\mathbf{x}_1|\cdots|\mathbf{x}_{d-1}|\mathbf{v}_p).$$

Then

$$\operatorname{Tr}(\omega_\ell\beta_p^*) = \operatorname{Tr}(\omega_\ell qF(\mathbf{x}_1|\cdots|\mathbf{x}_{d-1}|\mathbf{v}_p)) = (qF)^*(\mathbf{X}_1|\cdots|\mathbf{X}_{d-1}|\mathbf{E}_{p,\ell}),$$

so

$$\|\operatorname{Tr}(\omega_\ell\beta_p^*)\| \leqslant \frac{1}{PQ^{d-2}M^{d-1}}.$$

Thus, we can write

$$\sum_{k=1}^{n}\beta_{p,k}^*\operatorname{Tr}(\omega_\ell\omega_k) = \operatorname{Tr}(\omega_\ell\beta_p^*) = a_{p,\ell} + d_{p,\ell}, \tag{3.7}$$

with $a_{p,\ell} \in \mathbb{Z}$ and $|d_{p,\ell}| \leqslant (PQ^{d-2}M^{d-1})^{-1}$. With

$$\begin{aligned} \Omega &:= (\operatorname{Tr}(\omega_\ell\omega_k))_{k,\ell=1,\dots,n}, & B_p^* &:= (\beta_{p,1}^*, \dots, \beta_{p,n}^*), \\ A_p &:= (a_{p,1}, \dots, a_{p,n}), & D_p &:= (d_{p,1}, \dots, d_{p,n}), \end{aligned}$$

we can write (3.7) as

$$B_p^*\Omega = A_p + D_p.$$

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Therefore,

$$B_p^* = A_p\Omega^{-1} + D_p\Omega^{-1} =: A'_p + D'_p.$$

Write $d'_p := d'_{p,1}\omega_1 + \cdots + d'_{p,n}\omega_n$, where $D'_p = (d'_{p,1}, \ldots, d'_{p,n})$, and define a'_p analogously. Then

$$|d'_p| \ll \max_k\{|d_{p,k}|\} \leqslant \frac{1}{PQ^{d-2}M^{d-1}}.$$

Let $\boldsymbol{\alpha}_d := (\alpha_{d,1}, \ldots, \alpha_{d,t_d})$. On the one hand, by our definition of β_p^*, we see that

$$\boldsymbol{\alpha}_d \cdot qW = (\beta_p^*)_{1\leqslant p\leqslant t_d}.$$

On the other hand, since W has full rank, there exists $\boldsymbol{\nu}_d \in \mathfrak{n}^{t_d}$ such that

$$\boldsymbol{\nu}_d \cdot W = \det(\Omega)\widetilde{q}^*(a'_p)_{1\leqslant p\leqslant t_d}.$$

Subtracting one from the other yields

$$(\det(\Omega)\widetilde{q}^* q\boldsymbol{\alpha}_d - \boldsymbol{\nu}_d) \cdot W = \det(\Omega)\widetilde{q}^*(d'_p)_{1\leqslant p\leqslant t_d}.$$

We let $q^* := \widetilde{q}^* \det(\Omega)$ and obtain

$$\begin{aligned} q^*q\boldsymbol{\alpha}_d - \boldsymbol{\nu}_d &= q^*(d'_p)_{1\leqslant p\leqslant t_d}W^{-1} \\ &\ll |\mathbf{x}_1|^{t_d-1}\cdots|\mathbf{x}_{d-1}|^{t_d-1}\max_p\{|d'_p|\} \\ &\ll \left(\frac{P}{QM}\right)^{(t_d-1)(d-1)}\frac{1}{PQ^{d-2}M^{d-1}}. \end{aligned}$$

Furthermore, we have

$$|q^*| \ll |\widetilde{q}^*| \ll \left(\frac{P}{QM}\right)^{t_d(d-1)}$$

and thus

$$|q^*| \leqslant Q^* = \left(\frac{P}{QM}\right)^{t_d(d-1)}(\log P)$$

for large enough P. □

Now we are ready to prove Lemma 3.1.

Proof of Lemma 3.1. We iteratively apply the preceding lemma in order to reduce the degree of f in every step. In the first step with $d = D$, we see that either

$$L^{2^{D-1}} \leqslant P^{n(B_d-s)}(\log P)^{ns+1},$$

and hence $\boldsymbol{\alpha} \in I_D^{(1)}$, or there are a $q_D \leqslant Q_D$, with

$$Q_D = ((\log P)^{ns+1}L^{-2^{D-1}})^{(D-1)t_D/n(s-B_D)}\log P,$$

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and a $\boldsymbol{v}_D \in \mathfrak{n}^{t_D}$ such that

$$|q\alpha_{D,i} - v_{D,i}| \ll QP^{-D} \quad (1 \leqslant i \leqslant t_D).$$

In the second case, we apply Lemma 3.3 with $d := \max\{\Delta \setminus \{D\}\}$. Then either

$$L^{2^{d-1}} \leqslant \left(\frac{Q_D}{P}\right)^{n(s-B_d)} (\log P)^{ns+1},$$

and thus $\boldsymbol{\alpha} \in I_d^{(1)}$, or there are a $q_d := q_D q^* \leqslant Q_d := Q_D Q^*$ with

$$Q^* = \left(\frac{(\log P)^{ns+1}}{L^{2^{d-1}}}\right)^{t_d(d-1)/n(s-B_d)} \log P,$$

and a $\boldsymbol{v}_d \in \mathfrak{n}^{t_d}$ such that

$$|q_d\alpha_{d,i} - v_{d,i}| \leqslant Q_d P^{-d} \quad (1 \leqslant i \leqslant t_d).$$

Since we also have

$$|q_d\alpha_{D,i} - q^* v_{D,i}| \leqslant Q^* Q_D P^{-D} = Q_d P^{-D},$$

we may apply Lemma 3.3 again with the next lower value of d. Iterating this process, we get sequences of q_d and Q_d for decreasing values of $d \in \Delta$. The set of $\boldsymbol{\alpha}$ such that for all $d \in \Delta$ the second case of Lemma 3.3 holds is exactly $I^{(2)}$. □

§4. *Minor arcs.* First, let us consider the integral of $S(\boldsymbol{\alpha})$ over $I_D^{(1)}$.

LEMMA 4.1. *If*

$$\mathcal{D}\frac{2^{D-1}}{s - B_D} < 1, \tag{4.1}$$

then

$$\int_{I_D^{(1)}} |S(\boldsymbol{\alpha})|\, d\boldsymbol{\alpha} \ll P^{n(s-\mathcal{D})-\delta}$$

for some $\delta > 0$.

Proof. For $\boldsymbol{\alpha} \in I_D^{(1)}$, we have

$$L^{2^{D-1}} \leqslant P^{-n(s-B_D)}(\log P)^{ns+1} \leqslant P^{-n(s-B_D)+\varepsilon}.$$

Therefore, the integral can be estimated by

$$\begin{aligned}\int_{I_D^{(1)}} |S(\boldsymbol{\alpha})|\, d\boldsymbol{\alpha} &\ll \operatorname{vol}(I_D^{(1)}) \sup_{\boldsymbol{\alpha} \in I_D^{(1)}} |S(\boldsymbol{\alpha})| \ll 1 \cdot P^{ns} P^{-n(s-B_D)/2^{D-1}+\epsilon} \\ &= P^{n(s-(s-B_D)/2^{D-1})+\varepsilon} \ll P^{n(s-\mathcal{D})-\delta}\end{aligned}$$

for a suitable $\delta > 0$, using (4.1), provided that ϵ was chosen small enough. □

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We split R^T into dyadic sets as follows. For any $L_0 > 0$, let

$$\mathcal{A}(L_0) := \{\boldsymbol{\alpha} \in R^T : |S(\boldsymbol{\alpha})| = P^{ns}L \text{ with } L_0 < L \leqslant 2L_0\}.$$

For $I = I_d^{(1)}$, $d < D$, or $I = I^{(2)}$, we write $\mathcal{A}(L_0; I) := I \cap \mathcal{A}(L_0) \cap \mathfrak{m}$ and estimate the integral

$$T(L_0; I) := \int_{\mathcal{A}(L_0;I)} |S(\boldsymbol{\alpha})|\, d\boldsymbol{\alpha}.$$

We will make use of the following facts.

LEMMA 4.2. *Let $\epsilon > 0$ and $a \in \mathcal{O}_K$ with $N(a) \leqslant H$. Then the number of $b \in \mathcal{O}_K$ with $b \mid a$ and $|b| \leqslant H$ is $\ll_\epsilon H^\epsilon$.*

Proof. There are at most $\ll_\epsilon H^{\epsilon/2}$ ideals of $\mathcal{O}_K$ dividing the principal ideal $a\mathcal{O}_K$. Let $\mathfrak{b}$ be any principal ideal among these divisors. The number of generators of $\mathfrak{b}$ with all conjugates bounded by $\ll H$ is $\ll_\epsilon H^{\epsilon/2}$, which one can see by counting units with bounded conjugates (for example, as in the proof of [5, Lemma 7.2]). □

LEMMA 4.3. *There are positive constants e_0, e_1 such that all $\boldsymbol{\alpha} \in R^T \smallsetminus I_D^{(1)}$ satisfy $L \gg P^{-e_0}$ and $Q_j \ll P^{e_1}$ for all j.*

Proof. The lower bound for L follows directly from the definition of $I_D^{(1)}$. The upper bound for Q_j is an immediate consequence of this. □

LEMMA 4.4. *Let $d \in \Delta$, $d < D$. If*

$$\mathcal{D}_d\left(\frac{2^{d-1}}{s - B_d} + s_{d+1}\right) + s_{d+1} \sum_{j=d+1}^{D} s_j t_j < 1, \tag{4.2}$$

then $T(L_0; I_d^{(1)}) \ll P^{n(s-\mathcal{D})-\delta}$ for some $\delta > 0$.

Proof. By Lemma 3.1, every $\boldsymbol{\alpha} \in I_d^{(1)}$ satisfies

$$L^{2^{d-1}/n(s-B_d)+s_{d+1}/n} \ll P^{-1+\varepsilon},$$

and there are $q_j \in \mathfrak{n} \smallsetminus \{0\}$, $\boldsymbol{\nu}_j \in \mathfrak{n}^{t_j}$ for all $j \in \Delta$, $j > d$, satisfying (3.4)–(3.6).

Since $q_j R$ is a fundamental domain for the ideal lattice $q_j\mathfrak{n} \subseteq \mathfrak{n}$ in V, there are exactly $|N(q_j)|$ points of $\mathfrak{n}$ in $q_j R$. Hence, for any given q_j, it is enough to consider $\ll |N(q_j)|^{t_j}$ elements $\boldsymbol{\nu}_j$.

Let us estimate the volume of the set of all $(\alpha_{j,i})_{j>d, 1\leqslant i\leqslant t_d}$ belonging to a given $q_j, \boldsymbol{\nu}_j$. By (3.6), we see that every coordinate $q_j\alpha_{j,i}$ takes values in a set of volume $\leqslant Q_j^n P^{-jn}$. Since multiplication by q_j is an $\mathbb{R}$-linear transformation on V of determinant $\asymp N(q_j)$, each $\alpha_{i,j}$ belongs

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to a set of volume $\ll N(q_j)^{-1}Q_j^n P^{-jn}$ and hence the total volume is $\ll \prod_{j=d+1}^{D}(|N(q_j)|^{-1}Q_j^n P^{-jn})^{t_j}$. Due to (3.4) and Lemmas 4.2 and 4.3, each choice of q_{d+1} defines $\ll_\epsilon P^\epsilon$ values of $q_{d+2},\dots,q_D$. Summing over all these q_j and all the corresponding $\boldsymbol{\nu}_j$, we see that

$$\begin{aligned}\operatorname{vol}\mathcal{A}(L_0;I_d^{(1)}) &\ll P^{\varepsilon}Q_{d+1}^n\prod_{j=d+1}^{D}(Q_jP^{-j})^{nt_j}\\ &\ll P^{2\varepsilon-n\sum_{j=d+1}^{D}jt_j}L_0^{-s_{d+1}-\sum_{j=d+1}^{D}s_jt_j}.\end{aligned}$$

Therefore,

$$T(L_0;I_d^{(1)}) \ll P^{n(s-\mathcal{D}+\mathcal{D}_d)+2\varepsilon}L_0^{1-s_{d+1}-\sum_{j=d+1}^{D}s_jt_j} \ll P^{n(s-\mathcal{D})-\delta},$$

as long as (4.2) holds and ϵ is small enough. □

Finally, we concentrate on the integral over $\mathcal{A}(L_0;I^{(2)})$. In particular, we will make use of the fact that $\mathcal{A}(L_0;I^{(2)})\subseteq\mathfrak{m}$.

LEMMA 4.5. *Let $d\in\Delta$, $d<D$. If*

$$s_1+\sum_{j=1}^{D}s_jt_j<1, \tag{4.3}$$

then $T(L_0;I^{(2)})\ll P^{n(s-\mathcal{D})-\delta}$ for some $\delta>0$.

Proof. For each $\boldsymbol{\alpha}\in I^{(2)}$, we have $q_d\in\mathfrak{n}\setminus\{0\}$ and $\boldsymbol{\nu}_d\in\mathfrak{n}^{t_d}$, $d\in\Delta$, with (3.4)–(3.6) and, as in the previous lemma, it suffices to consider $\ll|N(q_d)|^{t_d}$ tuples $\boldsymbol{\nu}_d$ for each q_d.

Let $\boldsymbol{\gamma}:=(q_d^{-1}\nu_{d,i})_{d,i}$. Then it is not hard to see that

$$\begin{aligned}|\alpha_{d,i}-\gamma_{d,i}| &\ll Q_d^nP^{-d} \quad \text{for all } j,d \text{ and}\\ \mathfrak{N}\mathfrak{a}_{\boldsymbol{\gamma}} &\ll Q_1^n.\end{aligned}$$

With $e_{\max}:=\max_d\{e(d)\}$, we have

$$Q_d^n\ll L^{-s_1}(\log P)^{ne_{\max}}.$$

Let ϖ be as in the definition of the major arcs, and suppose that $L\geqslant P^{-\varpi/(2s_1)}$. If P is large enough, we deduce that

$$\begin{aligned}|\alpha_{d,i}-\gamma_{d,i}| &\ll P^{-d+\varpi} \quad \text{for all } j,d \text{ and}\\ \mathfrak{N}\mathfrak{a}_{\boldsymbol{\gamma}} &\ll P^{\varpi}\end{aligned}$$

and hence $\boldsymbol{\alpha}\in\mathfrak{M}$. We conclude that $T(L_0;I^{(2)})=0$ unless

$$L_0\ll P^{-\varpi/(2s_1)}. \tag{4.4}$$

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Let us assume that (4.4) holds. As in the proof of Lemma 4.4, we see that

$$\mathrm{vol}(\mathcal{A}(L_0, I^{(2)})) \ll P^{\varepsilon} Q_1^n \prod_{j=1}^{D} (Q_j P^{-j})^{nt_j} \ll P^{2\varepsilon - n\sum_{j=1}^{D} jt_j} L_0^{-s_1 - \sum_{j=1}^{D} s_j t_j}.$$

This implies that

$$T(L_0; I^{(2)}) \ll P^{n(s-\mathcal{D})+2\varepsilon} L_0^{1-s_1-\sum_{j=1}^{D} s_j t_j} \ll P^{n(s-\mathcal{D})-\delta},$$

provided that (4.3) holds and ϵ is small enough. □

The previous lemmas allow us to estimate the integral of $|S(\boldsymbol{\alpha})|$ over $\boldsymbol{\alpha} \in \mathfrak{m}$. Lemma 4.1 gives a sufficient bound for the integral over $\mathfrak{m} \cap I_D^{(1)}$. For $\boldsymbol{\alpha} \in I_d^{(1)}$, $d < D$, or $\boldsymbol{\alpha} \in I^{(2)}$, we have $c_0 P^{-e_0} \leqslant L \leqslant c_1$, with constants c_0, c_1 independent from P. We split this interval in dyadic parts and obtain

$$\int_{\mathfrak{m} \cap I_d^{(1)}} |S(\boldsymbol{\alpha})|\, d\boldsymbol{\alpha} \ll \sum_{j=0}^{\lceil \log_2(c_0^{-1} c_1 P^{e_0}) \rceil} T(2^j c_0 P^{-e_0}, I_d^{(1)}) \ll P^{n(s-\mathcal{D})-\delta} (\log P)$$

by Lemma 4.4. An analogous argument using Lemma 4.5 bounds the integral over $\mathfrak{m} \cap I^{(2)}$.

§5. *Major arcs: singular series.* We now choose the parameter ϖ in the definition of the major arcs by

$$\varpi := \frac{1}{4 + (n+1)T}.$$

Furthermore, recall that $\mathfrak{B} \subset V^s$ is a box aligned to the basis and $B \subseteq [-1, 1]^{ns}$ the corresponding box in R^{ns}.

We start by showing that the major arcs are disjoint in pairs provided that P is large enough.

LEMMA 5.1. *Let $\boldsymbol{\gamma}_1 \neq \boldsymbol{\gamma}_2 \in (R \cap K)^T$ with $\mathfrak{N}\mathfrak{a}_{\boldsymbol{\gamma}_j} \leqslant P^{\varpi}$ for $j \in \{1, 2\}$. For $P \gg 1$, we have $\mathfrak{M}_{\boldsymbol{\gamma}_1} \cap \mathfrak{M}_{\boldsymbol{\gamma}_2} = \emptyset$.*

Proof. If $\boldsymbol{\alpha} \in \mathfrak{M}_{\boldsymbol{\gamma}_1} \cap \mathfrak{M}_{\boldsymbol{\gamma}_2}$, then, writing $\boldsymbol{\gamma}_j = (\gamma_{j,d,i})_{d,i}$,

$$|\gamma_{1,d,i} - \gamma_{2,d,i}| \leqslant |\gamma_{1,d,i} - \alpha_{d,i}| + |\alpha_{d,i} - \gamma_{2,d,i}| \ll P^{-d+\varpi} \leqslant P^{-1+\varpi}$$

holds for all $1 \leqslant d \leqslant D$, $1 \leqslant i \leqslant t_d$. By Minkowski's convex body theorem, there is a non-zero $q \in \mathfrak{a}_{\boldsymbol{\gamma}_1} \cap \mathfrak{a}_{\boldsymbol{\gamma}_2}$ with $|q| \ll P^{2\varpi/n}$. Hence, $q(\gamma_{1,d,i} - \gamma_{2,d,i}) \in \mathfrak{n}$ and $|q(\gamma_{1,d,i} - \gamma_{2,d,i})| \ll P^{-1+(1+2/n)\varpi}$ for all d, i. Since $\varpi < 1/3$, this implies that $\boldsymbol{\gamma}_1 = \boldsymbol{\gamma}_2$ whenever P is large enough. □

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For $\boldsymbol{\gamma} \in (R \cap K)^T$, we define

$$\Sigma(\boldsymbol{\gamma}) := \sum_{\mathbf{x} \in (\mathfrak{n}/\mathfrak{a}_{\boldsymbol{\gamma}}\mathfrak{n})^s} \Phi\Bigg(\sum_{d=1}^{D}\sum_{i=1}^{t_d} \gamma_{d,i} G_{d,i}(\mathbf{x})\Bigg)$$

and, for $\boldsymbol{\gamma} \in V^T$, let

$$J(\boldsymbol{\gamma}) := \int_{\mathcal{B}} \Phi\Bigg(\sum_{d=1}^{D}\sum_{i=1}^{t_d} \gamma_{d,i} F_{d,i}(\mathbf{x})\Bigg)\, d\mathbf{x}.$$

LEMMA 5.2. *For $\boldsymbol{\gamma} \in (R \cap K)^T$ with $\mathfrak{N}\mathfrak{a}_{\boldsymbol{\gamma}} \leqslant P^{\varpi}$, let $\boldsymbol{\alpha} \in \mathfrak{M}_{\boldsymbol{\gamma}}$ and write $\alpha_{d,i} = \gamma_{d,i} + \theta_{d,i}$ for all $1 \leqslant d \leqslant D$ and $1 \leqslant i \leqslant t_d$. Then*

$$S(\boldsymbol{\alpha}) = \mathfrak{N}\mathfrak{a}_{\boldsymbol{\gamma}}^{-s} P^{ns} \Sigma(\boldsymbol{\gamma}) J((\theta_{d,i} P^d)_{d,i}) + O\Bigg(\mathfrak{N}\mathfrak{a}_{\boldsymbol{\gamma}} \sum_{d=1}^{D}\sum_{i=1}^{t_d} |\theta_{d,i}| P^{ns+d-1} + \mathfrak{N}\mathfrak{a}_{\boldsymbol{\gamma}} P^{ns-1}\Bigg).$$

Proof. Whenever d, i, j appear as indices of a sum, the sum runs over $1 \leqslant d \leqslant D$, $1 \leqslant i \leqslant t_d$, and $1 \leqslant j \leqslant n$. As usual, we write $\gamma_{d,i} = \gamma_{d,i,1}\omega_1 + \cdots + \gamma_{d,i,n}\omega_n$ and similarly $\theta_{d,i} = \theta_{d,i,1}\omega_1 + \cdots + \theta_{d,i,n}\omega_n$. With these conventions, we have

$$S(\boldsymbol{\alpha}) = \sum_{\mathbf{X} \in \mathbb{Z}^{sn} \cap PB} e\Bigg(\sum_{d,i,j} (\theta_{d,i,j} + \gamma_{d,i,j}) G^*_{d,i,j}(\mathbf{X})\Bigg).$$

Let $N := \mathfrak{N}\mathfrak{a}_{\boldsymbol{\gamma}} \in (\mathbb{N} \cap \mathfrak{a}_{\boldsymbol{\gamma}})$. Then $N\boldsymbol{\gamma} \in \mathfrak{n}^T$, so in particular $N\gamma_{d,i,j} \in \mathbb{Z}$ for all d, i, j. Applying the standard argument over $\mathbb{Q}$, we see that $S(\boldsymbol{\alpha})$ is the sum of

$$\frac{1}{N^{ns}} \sum_{\mathbf{Y} \in ([0, N-1] \cap \mathbb{Z})^{sn}} e\Bigg(\sum_{d,i,j} \gamma_{d,i,j} G^*_{d,i,j}(\mathbf{Y})\Bigg) \cdot J((\theta_{d,i} P^d)_{d,i}) \cdot P^{ns}$$

and an error term as in the lemma. It remains to show that

$$\frac{1}{N^{ns}} \sum_{\mathbf{Y} \in ([0, N-1] \cap \mathbb{Z})^{sn}} e\Bigg(\sum_{d,i,j} \gamma_{d,i,j} G^*_{d,i,j}(\mathbf{Y})\Bigg) = \frac{1}{N^s} \Sigma(\boldsymbol{\gamma}).$$

This follows from the following observations. Write $\mathbf{y} = (y_1, \ldots, y_s)$ with, as usual, $y_j = y_{j,1}\omega_1 + \cdots + y_{j,n}\omega_n$. If $\mathbf{Y}$ runs through $([0, N-1] \cap \mathbb{Z})^{ns}$, then $\mathbf{y}$ runs through a set of representatives of $(\mathfrak{n}/(N\mathfrak{n}))^s$. Moreover,

$$e\Bigg(\sum_{d,i,j} \gamma_{d,i,j} G^*_{d,i,j}(\mathbf{Y})\Bigg) = \Phi\Bigg(\sum_{d,i} \gamma_{d,i} G_{d,i}(\mathbf{y})\Bigg)$$

depends only on $\mathbf{y}$ modulo $\mathfrak{a}_{\boldsymbol{\gamma}}\mathfrak{n}$, and each coset of $(\mathfrak{n}/(N\mathfrak{n}))^s$ modulo $\mathfrak{a}_{\boldsymbol{\gamma}}\mathfrak{n}$ has $N^{(n-1)s}$ elements. □

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For $H > 0$, let

$$\mathfrak{S}(H) := \sum_{\substack{\boldsymbol{\gamma} \in (R \cap K)^T \\ \mathfrak{N}\mathfrak{a}_{\boldsymbol{\gamma}} \leqslant H}} \frac{\Sigma(\boldsymbol{\gamma})}{\mathfrak{N}\mathfrak{a}_{\boldsymbol{\gamma}}^s}$$

and

$$\mathfrak{J}(H) := \int_{\substack{\boldsymbol{\gamma} \in V^T \\ |\boldsymbol{\gamma}| \leqslant H}} J(\boldsymbol{\gamma})\, d\boldsymbol{\gamma}.$$

LEMMA 5.3. *There is a positive constant δ such that, for large enough P,*

$$\int_{\mathfrak{M}} S(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} = \mathfrak{S}(P^{\varpi})\mathfrak{J}(P^{\varpi})P^{n(s-\mathcal{D})} + O(P^{n(s-\mathcal{D})-\delta}).$$

Proof. By Lemma 5.1, we have

$$\int_{\mathfrak{M}} S(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} = \sum_{\substack{\boldsymbol{\gamma} \in (R \cap K)^T \\ \mathfrak{N}\mathfrak{a}_{\boldsymbol{\gamma}} \leqslant P^{\varpi}}} \int_{\mathfrak{M}_{\boldsymbol{\gamma}}} S(\boldsymbol{\alpha})\, d\boldsymbol{\alpha}.$$

Using Lemma 5.2 and the obvious fact that $\mathrm{vol}(\mathfrak{M}_{\boldsymbol{\gamma}}) = P^{-n\mathcal{D}+nT\varpi}$, it follows that

$$\int_{\mathfrak{M}_{\boldsymbol{\gamma}}} S(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} = \frac{1}{\mathfrak{N}\mathfrak{a}_{\boldsymbol{\gamma}}^s} P^{ns}\Sigma(\boldsymbol{\gamma}) \int_{|\theta_{d,i}| \leqslant P^{-d+\varpi}} J((\theta_{d,i}P^d)_{d,i})\, d\boldsymbol{\theta} + O(P^{n(s-\mathcal{D})-1+(nT+2)\varpi}).$$

After a coordinate change in the integral over $\boldsymbol{\theta}$ and summing over all $\boldsymbol{\gamma}$, we obtain

$$\int_{\mathfrak{M}} S(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} = \mathfrak{S}(P^{\varpi})\mathfrak{J}(P^{\varpi})P^{n(s-\mathcal{D})} + O(m(P^{\varpi}) \cdot P^{n(s-\mathcal{D})-1+(nT+2)\varpi}),$$

where

$$m(P^{\varpi}) := |\{\boldsymbol{\gamma} \in (R \cap K)^T \mid \mathfrak{N}\mathfrak{a}_{\boldsymbol{\gamma}} \leqslant P^{\varpi}\}| \ll P^{\varpi(T+1)}.$$

Hence, we obtain an error term

$$O(P^{n(s-\mathcal{D})-1+\varpi((n+1)T+3)}) = O(P^{n(s-\mathcal{D})-\delta}). \qquad \square$$

Whenever the respective limit exists, we let

$$\mathfrak{S} := \lim_{H \to \infty} \mathfrak{S}(H) \quad \text{and} \quad \mathfrak{J} := \lim_{H \to \infty} \mathfrak{J}(H).$$

The rest of this section is devoted to the absolute and fast convergence of the singular series $\mathfrak{S}$. The singular integral $\mathfrak{J}$ will be treated in the next section.

We start with an estimate for $\Sigma(\boldsymbol{\gamma})$. Let $\boldsymbol{\gamma} \in (R \cap K)^T$. By definition of $\mathfrak{a}_{\boldsymbol{\gamma}}$, we can write $\gamma_{d,i}\mathcal{O}_K = \mathfrak{n}\mathfrak{a}_{d,i}/\mathfrak{a}_{\boldsymbol{\gamma}}$, where $\mathfrak{a}_{d,i}$ is an ideal of $\mathcal{O}_K$ and $\mathfrak{a}_{\boldsymbol{\gamma}} + \sum_{d=1}^{D}\sum_{i=1}^{t_d} \mathfrak{a}_{d,i} = \mathcal{O}_K$.

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LEMMA 5.4. *Write $\gamma_{d,i}\mathcal{O}_K = \mathfrak{n}\mathfrak{a}_{d,i}/\mathfrak{a}_{\boldsymbol{\gamma}}$ as above. Then, for $\epsilon > 0$,*

$$\Sigma(\boldsymbol{\gamma}) \ll \mathfrak{N}\mathfrak{a}_{\boldsymbol{\gamma}}^{s+\epsilon} \min_{j\in\Delta}\left\{\frac{\mathfrak{N}(\mathfrak{a}_{\boldsymbol{\gamma}} + \sum_{d=j}^{D}\sum_{i=1}^{t_d}\mathfrak{a}_{d,i})}{\mathfrak{N}\mathfrak{a}_{\boldsymbol{\gamma}}}\right\}^{1/s_j}.$$

Proof. This generalizes [2, Lemma 8.2]. The proof is essentially the same. We choose $\boldsymbol{\alpha} = \boldsymbol{\gamma}, \boldsymbol{\theta} = \mathbf{0}$ in Lemma 5.2. Clearly, $\boldsymbol{\alpha} = \boldsymbol{\gamma}$ is in $\mathfrak{M}_{\boldsymbol{\gamma}}$ for $P = \mathfrak{N}\mathfrak{a}_{\boldsymbol{\gamma}}^{A}$, with any large fixed value of A. Since $J(\mathbf{0}) \gg 1$, we obtain

$$\Sigma(\boldsymbol{\gamma}) \ll \frac{\mathfrak{N}\mathfrak{a}_{\boldsymbol{\gamma}}^{s}\cdot|S(\boldsymbol{\gamma})|}{P^{ns}} + \frac{\mathfrak{N}\mathfrak{a}_{\boldsymbol{\gamma}}^{s+1}}{P}.$$

We choose $A > s+1$ to obtain

$$\Sigma(\boldsymbol{\gamma}) \ll 1 + \mathfrak{N}\mathfrak{a}_{\boldsymbol{\gamma}}^{s}L, \tag{5.1}$$

with L defined via $|S(\boldsymbol{\alpha})| = P^{ns}L$ as earlier in the paper. Let us apply Lemma 3.1 to estimate L. If $\boldsymbol{\gamma} \in I_d^{(1)}$ for some $d \in \Delta$, then (3.3) and (5.1) show that

$$\Sigma(\boldsymbol{\gamma}) \ll 1 + \frac{N\mathfrak{a}_{\boldsymbol{\gamma}}^{s}P^{\epsilon}}{P^{n(s-B_d)/(2^{d-1}+(s-B_d)s_{d+1})}} \ll 1$$

if A is chosen large enough. Now let us assume that $\boldsymbol{\alpha} = \boldsymbol{\gamma} \in I^{(2)}$. In this case, Lemma 3.1 yields $q_j \in \mathfrak{n}$ and $\boldsymbol{\nu}_j \in \mathfrak{n}^{t_j}$ for $j \in \Delta$ satisfying (3.4)–(3.6), with Q_j given in (3.1). Assume first that $q_j\gamma_{j,i} \neq \nu_{j,i}$ for some j, i. By Minkowski's convex body theorem, there is a $q \in \mathfrak{a}_{\boldsymbol{\gamma}} \setminus \{0\}$ with $|q| \ll \mathfrak{N}\mathfrak{a}_{\gamma}^{1/n}$. Then $q(q_j\gamma_{j,i} - \nu_{j,i}) \in \mathfrak{n}$, and so

$$1 \ll |q(q_j\gamma_{j,i} - \nu_{j,i})| \ll \mathfrak{N}\mathfrak{a}_{\gamma}^{1/n}Q_jP^{-j} \ll \mathfrak{N}\mathfrak{a}_{\boldsymbol{\gamma}}^{1/n}L^{-s_j/n}P^{-j+\epsilon}.$$

This gives an upper bound for L, and substituting this bound in (5.1) shows that $\Sigma(\boldsymbol{\gamma}) \ll 1$ as long as we have chosen A big enough. Hence, we are left with the case where

$$q_j\gamma_{j,i} = \nu_{j,i} \quad \text{for all } j \in \Delta, 1 \leqslant i \leqslant t_j. \tag{5.2}$$

Since $\nu_{j,i} \in \mathfrak{n}$, we find integral ideals $\mathfrak{b}_{j,i}$ such that $\nu_{j,i}\mathcal{O}_K = \mathfrak{n}\mathfrak{b}_{j,i}$. After cancellation, (5.2) gives

$$q_j\mathfrak{a}_{j,i} = \mathfrak{a}_{\boldsymbol{\gamma}}\mathfrak{b}_{j,i} \quad \text{for all } j \in \Delta, 1 \leqslant i \leqslant t_j.$$

In the following arguments, we write $\mathfrak{a}^{(j)} := \mathfrak{a}_{j,1} + \cdots + \mathfrak{a}_{j,t_j}$ and $\mathfrak{b}^{(j)} := \mathfrak{b}_{j,1} + \cdots + \mathfrak{b}_{j,t_j}$. Then $q_j\mathfrak{a}^{(j)} = \mathfrak{a}_{\gamma}\mathfrak{b}^{(j)}$ holds for all $j \in \Delta$. With

$$\mathfrak{d}_j := \mathfrak{a}_{\boldsymbol{\gamma}} + \mathfrak{a}^{(j)} \quad \text{and} \quad \mathfrak{f}_j := q_j\mathcal{O}_K + \mathfrak{b}^{(j)},$$

we have thus

$$\frac{q_j\mathcal{O}_K}{\mathfrak{f}_j}\cdot\frac{\mathfrak{a}^{(j)}}{\mathfrak{d}_j} = \frac{\mathfrak{a}_{\boldsymbol{\gamma}}}{\mathfrak{d}_j}\cdot\frac{\mathfrak{b}^{(j)}}{\mathfrak{f}_j}.$$

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We claim that

$$\frac{\mathfrak{a}_{\boldsymbol{\gamma}}}{\sum_{d>j}\mathfrak{d}_d} \text{ divides } q_j\mathcal{O}_K \quad \text{for all } j \in \Delta. \tag{5.3}$$

Indeed, since $q_j\mathcal{O}_K/\mathfrak{f}_j + \mathfrak{b}^{(j)}/\mathfrak{f}_j = \mathcal{O}_K$, we see that $(q_j\mathcal{O}_K/\mathfrak{f}_j) \mid (\mathfrak{a}_{\boldsymbol{\gamma}}/\mathfrak{d}_j)$. The opposite divisibility follows analogously and hence

$$\frac{q_j\mathcal{O}_K}{\mathfrak{f}_j} = \frac{\mathfrak{a}_{\boldsymbol{\gamma}}}{\mathfrak{d}_j}. \tag{5.4}$$

Let $k \in \Delta$, $k > j$. Since (5.4) holds for k as well as for j, we see that $q_j\mathfrak{d}_j\mathfrak{f}_k = q_k\mathfrak{d}_k\mathfrak{f}_j$. By (3.4), we can write $q_j = q_k\hat{q_j}$ with $\hat{q_j} \in \mathcal{O}_K$. Substituting this in the above equality, cancelling $q_k\mathcal{O}_K$, and dividing both sides by $\mathfrak{d}_j + \mathfrak{f}_j$ shows that

$$\hat{q_j}\mathfrak{f}_k\frac{\mathfrak{d}_j}{\mathfrak{d}_j+\mathfrak{f}_j} = \mathfrak{d}_k\frac{\mathfrak{f}_j}{\mathfrak{d}_j+\mathfrak{f}_j}$$

and in particular

$$\frac{\mathfrak{d}_j}{\mathfrak{d}_j+\mathfrak{f}_j} \text{ divides } \mathfrak{d}_k \quad \text{for all } k > j. \tag{5.5}$$

Let $\delta_j := \sum_{d>j}\mathfrak{d}_d$. By (5.4),

$$q_j\mathcal{O}_K = \frac{\mathfrak{a}_{\boldsymbol{\gamma}}}{\delta_j}\cdot\frac{\mathfrak{f}_j\delta_j}{\mathfrak{d}_j} = \frac{\mathfrak{a}_{\boldsymbol{\gamma}}}{\delta_j}\cdot\frac{\mathfrak{f}_j}{\mathfrak{d}_j+\mathfrak{f}_j}\cdot\frac{\delta_j}{\mathfrak{d}_j(\mathfrak{d}_j+\mathfrak{f}_j)^{-1}}.$$

By (5.5), the second and the third factors on the right-hand side are integral ideals, and hence (5.3) holds as claimed. Hence,

$$\begin{aligned}\frac{\mathfrak{N}\mathfrak{a}_{\boldsymbol{\gamma}}}{\mathfrak{N}(\mathfrak{a}_{\boldsymbol{\gamma}} + \sum_{d=j}^{D}\sum_{i=1}^{t_d}\mathfrak{a}_{d,i})} &= \frac{\mathfrak{N}\mathfrak{a}_{\boldsymbol{\gamma}}}{\mathfrak{N}\delta_j} \leqslant \mathfrak{N}(q_j\mathcal{O}_K) \ll |q_j|^n \\ &\ll Q_j^n \ll L^{-s_j}(\log P)^{ne(j)}.\end{aligned}$$

This gives an upper bound for L, which, once substituted into (5.1), proves the lemma. □

LEMMA 5.5. *Let $\mathfrak{a}$ be a fractional ideal of K. Then*

$$|R \cap \mathfrak{a}| \ll \frac{1}{\mathfrak{N}\mathfrak{a}} + 1.$$

Proof. There is a constant c depending only on K and our basis $\omega_1, \ldots, \omega_n$ such that $R \cap \mathfrak{a} \subseteq \{x \in \mathfrak{a} \mid |x^{(j)}| \leqslant c \text{ for all } v\}$. Here the $x^{(v)}$ are all the (real and complex) conjugates of x. The result then follows from [**5**, Lemma 7.1]. □

We are now ready to treat our singular series under the hypotheses of Theorem 1.1.

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LEMMA 5.6. *Assume that*

$$s_1 + \sum_{j=1}^{D} s_j t_j < 1. \tag{5.6}$$

Then the series defining $\mathfrak{S}$ *converges absolutely and there is a positive constant* δ *such that*

$$\mathfrak{S} - \mathfrak{S}(H) \ll H^{-\delta}$$

holds for large enough H.

Proof. We write

$$A(\mathfrak{a}) := \sum_{\substack{\boldsymbol{\gamma} \in (R \cap K)^T \\ \mathfrak{a}_{\boldsymbol{\gamma}} = \mathfrak{a}}} |\Sigma(\boldsymbol{\gamma})|.$$

For each $\boldsymbol{\gamma} \in (R \cap K)^T$ with $\mathfrak{a}_{\boldsymbol{\gamma}} = \mathfrak{a}$, we write again $\gamma_{d,j}\mathcal{O}_K = \mathfrak{n}\mathfrak{a}_{d,j}/\mathfrak{a}$ and define

$$\mathfrak{d}_j = \mathfrak{a} + \sum_{d=j}^{D} \sum_{i=1}^{t_d} \mathfrak{a}_{d,i}. \tag{5.7}$$

Then, for $j_0 := \min \Delta$, we have $\mathfrak{d}_{j_0} = \mathcal{O}_K$. By Lemma 5.4, we see that

$$\Sigma(\boldsymbol{\gamma}) \ll \mathfrak{N}\mathfrak{a}^{s+\epsilon/2} \min_{j \in \Delta} \left\{ \frac{\mathfrak{N}\mathfrak{d}_j}{\mathfrak{N}\mathfrak{a}} \right\}^{1/s_j} \leqslant \mathfrak{N}\mathfrak{a}^{s+\epsilon/2} \prod_{j \in \Delta} \left(\frac{\mathfrak{N}\mathfrak{d}_j}{\mathfrak{N}\mathfrak{a}} \right)^{\lambda_j/s_j}$$

for any $\lambda_j \geqslant 0$ with $\sum_{j \in \Delta} \lambda_j = 1$. As in [2, §8], we choose

$$\lambda_j := \begin{cases} \theta + t_{j_0} s_{j_0} & \text{if } j = j_0, \\ t_j s_j & \text{if } j \in \Delta \setminus \{j_0\}, \end{cases} \tag{5.8}$$

where $\theta = 1 - \sum_{j \in \Delta} t_j s_j \in (s_1, 1)$. Hence,

$$A(\mathfrak{a}) \ll \mathfrak{N}\mathfrak{a}^{s+\epsilon/2} \sum_{\substack{\mathfrak{d}_j \mid \mathfrak{a} \\ j \in \Delta}} m((\mathfrak{d}_j)_{j \in \Delta}, \mathfrak{a}) \left(\frac{1}{\mathfrak{N}\mathfrak{a}} \right)^{\theta/s_{j_0}} \prod_{j \in \Delta} \left(\frac{\mathfrak{N}\mathfrak{d}_j}{\mathfrak{N}\mathfrak{a}} \right)^{t_j},$$

where $m((\mathfrak{d}_j)_{j \in \Delta}, \mathfrak{a})$ is the number of all $\boldsymbol{\gamma} \in (R \cap K)^T$ with $\mathfrak{a}_{\boldsymbol{\gamma}} = \mathfrak{a}$ and (5.7). Clearly, any $\boldsymbol{\gamma}$ with $\mathfrak{a}_{\boldsymbol{\gamma}} = \mathfrak{a}$ and (5.7) satisfies

$$\gamma_{j,i} \in \frac{\mathfrak{n}\mathfrak{d}_j}{\mathfrak{a}} \quad \text{for all } j \in \Delta, 1 \leqslant i \leqslant t_j.$$

Using this and Lemma 5.5,

$$m((\mathfrak{d}_j)_{j \in \Delta}, \mathfrak{a}) \leqslant \prod_{j \in \Delta} \left| R \cap \frac{\mathfrak{n}\mathfrak{d}_j}{\mathfrak{a}} \right|^{r_j} \ll \prod_{j \in \Delta} \left(\frac{\mathfrak{N}\mathfrak{a}}{\mathfrak{N}\mathfrak{d}_j} \right)^{t_j}.$$

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Hence,

$$A(\mathfrak{a}) \ll \mathfrak{N}\mathfrak{a}^{s+\epsilon/2-\theta/s_{j_0}} \sum_{\substack{\mathfrak{d}_j \mid \mathfrak{a} \\ j\in\Delta}} 1 \ll \mathfrak{N}\mathfrak{a}^{s-\theta/s_{j_0}+\epsilon}.$$

Since $\theta > s_1 = s_{j_0}$, this shows that $\mathfrak{S}$ converges absolutely and that $\mathfrak{S}-\mathfrak{S}(H) \ll H^{\delta}$ for some appropriate delta. □

§6. *Major arcs: singular integral.* Throughout this section, we will assume (5.6). For $\boldsymbol{\gamma} = (\gamma_{d,i})_{d,i} \in V^T$, we write $\boldsymbol{\gamma}_d := (\gamma_{d,1}, \dots, \gamma_{d,t_d})$.

LEMMA 6.1. *Let $a \in [0, n]$ and $\epsilon > 0$. For any $\boldsymbol{\gamma} \in V^T$, we have*

$$J(\boldsymbol{\gamma}) \ll 1. \tag{6.1}$$

Assume that $|\boldsymbol{\gamma}| \geqslant 1$ and let $d \in \Delta$ with $\boldsymbol{\gamma}_d \neq 0$. Then there exists a unit $u_d \in \mathcal{O}_K^{\times}$ such that

(1) $|u_d| \ll |\boldsymbol{\gamma}|^{a/n}$;
(2) $J(\boldsymbol{\gamma}) \ll |\boldsymbol{\gamma}|^{\epsilon}|u_d\boldsymbol{\gamma}_d|^{-a/s_d}$.

Moreover, we have

$$J(\boldsymbol{\gamma}) \ll |\boldsymbol{\gamma}|^{\epsilon}|\boldsymbol{\gamma}_d|^{-1/s_d}. \tag{6.2}$$

Proof. It is clear that $J(\boldsymbol{\gamma}) \ll 1$ holds for all $\boldsymbol{\gamma} \in V^T$. Let $d \in \Delta$ and assume that $|\boldsymbol{\gamma}| \geqslant 1$ and $\boldsymbol{\gamma}_d \neq 0$. We apply Lemma 5.2 with $\boldsymbol{\alpha} := (P^{-j}\gamma_{j,i})_{j,i}$ and $P := |\boldsymbol{\gamma}|^A$ for fixed large A. Clearly, $\boldsymbol{\alpha} \in \mathfrak{M}_{\mathbf{0}}$ as soon as $A \geqslant 1/\varpi$. Since $\Sigma(\mathbf{0}) = 1$, we obtain

$$J(\boldsymbol{\gamma}) \ll L + |\boldsymbol{\gamma}|P^{-1} \ll L + |\boldsymbol{\gamma}|^{-a/s_d} \tag{6.3}$$

if A was chosen big enough.

If $L \leqslant |\boldsymbol{\gamma}|^{-a/s_d+\epsilon}$, then (6.3) yields

$$J(\boldsymbol{\gamma}) \ll |\boldsymbol{\gamma}|^{\epsilon}|\boldsymbol{\gamma}|^{-a/s_d} \leqslant |\boldsymbol{\gamma}|^{\epsilon}|\boldsymbol{\gamma}_d|^{-a/s_d},$$

and we can choose $u_d = 1$. Therefore, we may assume from now on that

$$L \geqslant |\boldsymbol{\gamma}|^{-a/s_d+\epsilon}, \tag{6.4}$$

so that

$$J(\boldsymbol{\gamma}) \ll L. \tag{6.5}$$

The remainder of this proof is devoted to the deduction of suitable upper bounds for L. Let us first assume that $\boldsymbol{\alpha} \in I_j^{(1)}$ for some $j \in \Delta$. Then the definition of $I_j^{(1)}$, see (3.2), yields an upper bound

$$L \ll |\boldsymbol{\gamma}|^{-a/s_d+\epsilon} \leqslant |\boldsymbol{\gamma}|^{\epsilon}|\boldsymbol{\gamma}_d|^{-a/s_d},$$

provided that we have chosen A big enough to ensure that $A(s-B_j) > a(2^{j-1} + s_{j+1}(s-B_j))/(ns_d)$.

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If $\boldsymbol{\alpha} \in I^{(2)}$, then Lemma 3.1 yields $q_d \in \mathfrak{n}$, and $\boldsymbol{v}_d \in \mathfrak{n}^{t_d}$ satisfying

$$|q_d| \leqslant Q_d, \tag{6.6}$$
$$|q_d\alpha_{d,i} - v_{d,i}| \leqslant Q_d P^{-d} \quad \text{for all } 1 \leqslant i \leqslant t_d, \tag{6.7}$$

with Q_d defined by (3.1).

Suppose that $v_{d,i} \neq 0$ for some $1 \leqslant i \leqslant t_d$. Then

$$1 \leqslant |v_{d,i}| \ll Q_d P^{-d}|\gamma_{d,i}| + Q_d P^{-d} \ll Q_d P^{-d}|\boldsymbol{\gamma}| = L^{-s_d/n}(\log P)^{e(d)} P^{-d}|\boldsymbol{\gamma}|.$$

This yields an upper bound

$$\begin{aligned} L &\ll (\log P)^{ne(d)/s_d} P^{-nd/s_d} |\boldsymbol{\gamma}|^{n/s_d} \ll |\boldsymbol{\gamma}|^{\epsilon} |\boldsymbol{\gamma}|^{(n/s_d)(1-dA)} \\ &\ll |\boldsymbol{\gamma}|^{\epsilon} |\boldsymbol{\gamma}|^{-a/s_d} \ll |\boldsymbol{\gamma}|^{\epsilon} |\boldsymbol{\gamma}_d|^{-a/s_d} \end{aligned}$$

if A is chosen big enough. We are left with the case where $\boldsymbol{v}_d = \mathbf{0}$. Let t be a generator of the principal ideal $q_d \mathcal{O}_K$ with the property that

$$N(q_d)^{1/n} \ll |t|_v \ll N(q_d)^{1/n} \quad \text{for all } v \in \Omega_\infty, \tag{6.8}$$

and let $u_d := q_d/t \in \mathcal{O}_K^\times$. Thanks to (6.8), (6.6), and (6.4), we obtain

$$|u_d| \asymp N(q_d)^{-1/n}|q_d| \leqslant Q_d = L^{-s_d/n}(\log P)^{e(d)} \ll |\boldsymbol{\gamma}|^{a/n}.$$

Moreover, due to (6.7), for all $1 \leqslant i \leqslant t_d$, we have

$$P^{-d}|u_d \boldsymbol{\gamma}_d| = |u_d \boldsymbol{\alpha}_d| \asymp N(q_d)^{-1}|q_d \boldsymbol{\alpha}_d| \leqslant Q_d P^{-d} = L^{-s_d/n}(\log P)^{e(d)} P^{-d}, \tag{6.9}$$

so

$$L \ll |\boldsymbol{\gamma}|^{\epsilon} |u_d \boldsymbol{\gamma}_d|^{-n/s_d}.$$

The estimate $J(\boldsymbol{\gamma}) \ll |\boldsymbol{\gamma}|^{\epsilon} |u_d \boldsymbol{\gamma}_d|^{-a/s_d}$ follows immediately from this and (6.5) if $|u_d \boldsymbol{\gamma}_d| \geqslant 1$, and from the trivial estimate $J(\boldsymbol{\gamma}) \ll 1$ if $|u_d \boldsymbol{\gamma}_d| \leqslant 1$.

For the proof of (6.2), we proceed as above with $a = 1$, until (6.9). Here we conclude that

$$P^{-d}|\boldsymbol{\gamma}_d| = |\boldsymbol{\alpha}_d| \ll |q_d^{-1}||q_d \boldsymbol{\alpha}_d| \ll Q_d^{n-1} \cdot Q_d P^{-d} = L^{-s_d}(\log P)^{ne(d)} P^{-d}$$

and thus

$$L \ll |\boldsymbol{\gamma}|^{\epsilon} |\boldsymbol{\gamma}_d|^{-1/s_d}. \qquad \square$$

We write

$$N(\boldsymbol{\gamma}_d) := \prod_{v \in \Omega_\infty} |\boldsymbol{\gamma}_d|_v^{n_v},$$

where $|\boldsymbol{\gamma}_d|_v := \max\{|\gamma_{d,1}|_v, \dots, |\gamma_{d,t_d}|_v\}$ and $n_v := [K_v : \mathbb{R}]$ is the local degree. Let $b \in (0, 1)$ be a constant to be specified later and, for $H \geqslant 1$, let

$$\begin{aligned} M(H) &:= \{\boldsymbol{\gamma} \in V^T : |\boldsymbol{\gamma}| \gg H\}, \\ M_>(H) &:= \{\boldsymbol{\gamma} \in V^T : |\boldsymbol{\gamma}| \in (H, 2H] \text{ and there exists } d \text{ with } N(\boldsymbol{\gamma}_d) \geqslant H^b\}, \\ M_<(H) &:= \{\boldsymbol{\gamma} \in V^T : |\boldsymbol{\gamma}| \in (H, 2H] \text{ and, for all } d, \text{ we have } N(\boldsymbol{\gamma}_d) \leqslant H^b\}. \end{aligned}$$

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Define the integral

$$I(H) := \int_{M(H)} \max_{d\in\Delta}\{|\boldsymbol{\gamma}_d|^{n/s_d}\}^{-1}\, d\boldsymbol{\gamma}.$$

LEMMA 6.2. *There is $\delta > 0$ such that, for $H \geqslant 1$,*

$$I(H) \ll H^{-\delta}.$$

Proof. We identify V^T with $\mathbb{R}^{nT}$ using the basis $\omega_1, \dots, \omega_n$ of V. The exponent n/s_d in the definition $I(H)$ is good enough for the arguments given after [2, Lemma 8.3] to apply. □

For $\mathbf{u} = (u_d)_{d\in\Delta}$ with $u_d \in \mathcal{O}_K^\times$ for all $d \in \Delta$, let

$$I_>(\mathbf{u}, H) := \int_{M_>(H)} \max_{d\in\Delta}\{|u_d\boldsymbol{\gamma}_d|^{n/s_d}\}^{-1}\, d\boldsymbol{\gamma}.$$

LEMMA 6.3. *Let δ be as in Lemma 6.2. Then, for $H \geqslant 1$,*

$$I_>(\mathbf{u}, H) \ll H^{-\delta b/n}.$$

Proof. Let $\phi : V^T \to V^T$ be the $\mathbb{R}$-linear transformation $(\gamma_{d,i})_{d,i} \mapsto (u_d\gamma_{d,i})_{d,i}$. Since the u_d are all units, we have $\det\phi \asymp 1$. Moreover, let $\boldsymbol{\gamma} \in M_>(H)$ and d be such that $N(u_d\boldsymbol{\gamma}_d) = N(\boldsymbol{\gamma}_d) \geqslant H^b$. Then, in particular,

$$\max_{v\in\Omega_\infty}\{|u_d\boldsymbol{\gamma}_d|_v\} \geqslant H^{b/n}$$

and thus $|\phi(\boldsymbol{\gamma})| \gg H^{b/n}$. We have shown that $\phi(M_>(H)) \subseteq M(H^{b/n})$. By Lemma 6.2, we obtain

$$\begin{aligned} I_>(\mathbf{u}, H) \asymp \int_{\phi(M_>(H))} \max_{d\in\Delta}\{|\boldsymbol{v}_d|^{n/s_d}\}^{-1}\, d\boldsymbol{v} &\ll \int_{M(H^{b/n})} \max_{d\in\Delta}\{|\boldsymbol{v}_d|^{n/s_d}\}^{-1}\, d\boldsymbol{v} \\ &= I(H^{b/n}) \ll H^{-b\delta/n}. \end{aligned}$$ □

Let

$$I_<(H) := \int_{M_<(H)} \max_{d\in\Delta}\{|\boldsymbol{\gamma}_d|^{1/s_d}\}^{-1}\, d\boldsymbol{\gamma}.$$

We will prove that $I_<(H) \ll H^{-\delta}$ for some $\delta > 0$.

LEMMA 6.4. *For $A_1, \dots, A_n, B \in (0, \infty)$, let*

$$I(A_1, \dots, A_n, B) := \int_{\substack{0\leqslant x_i\leqslant A_i \\ x_1\cdots x_n\leqslant B}} dx_1 \cdots dx_n.$$

Then

$$I(A_1, \dots, A_n, B) \ll B \log\left(\frac{A_1\cdots A_n}{B} + 2\right)^{n-1}.$$

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Proof. Elementary computations using induction. □

LEMMA 6.5. *Let $\theta > 0$. For any small $\epsilon > 0$, and $H \geqslant 1$, we have*

$$\int_{\substack{\gamma \in V \\ N(\gamma) \leqslant H^b \\ |\gamma| \geqslant H}} \frac{1}{(1+|\gamma|)^{1+\theta}}\, d\gamma \ll H^{-1-\theta+\epsilon+b}.$$

Proof. For $v \in \Omega_\infty$, let $t_v := |\gamma|_v^{n_v}$, and assume that $\max_v |\gamma|_v = |\gamma|_v$. Passing to polar coordinates at the complex places, we see that the integral in the lemma is

$$\ll \int_{t_w \gg H^{n_w}} \frac{1}{t_w^{(1+\theta)/n_w}} \left(\int_{\substack{t_v, v \neq w \\ t_v \leqslant t_w^{n_v/n_w} \\ \prod_{v \neq w} t_v \leqslant H^b/t_w}} \prod_{v \neq w} dt_v \right) dt_w.$$

Using the notation of Lemma 6.4, the inner integral is just

$$I((t_w^{n_v/n_w})_{v \neq w}, H^b/t_w) \ll \frac{H^b}{t_w^{1-\epsilon}}$$

and thus the integral in the lemma is

$$\ll H^b \int_{t_w \gg H^{n_w}} \frac{1}{t_w^{(1+\theta)/n_w+1-\epsilon}}\, dt_w \ll H^{b-1-\theta+n_w\epsilon}.$$ □

LEMMA 6.6. *Let $\theta > 0$. For any small $\epsilon > 0$, we have*

$$\int_{\substack{\gamma \in V \\ N(\gamma) \leqslant H^b}} \frac{1}{(1+|\gamma|)^{1+\theta}}\, d\gamma \ll H^b.$$

Proof. We start as in the proof of Lemma 6.5 and see that the integral is

$$\begin{aligned}
&\ll \int_{t_w=0}^{\infty} \frac{1}{(1+t_w^{1/n_w})^{(1+\theta)}} \cdot I((t_w^{n_v/n_w})_{v \neq w}, H^b/t_w)\, dt_w \\
&\ll \int_{t_w=0}^{H^{bn_w/n}} t_w^{\sum_{v \neq w} n_v/n_w}\, dt_w + H^b \int_{t_w=H^{bn_w/n}}^{\infty} \frac{1}{t_w^{(1+\theta)/n_w+1-\epsilon}}\, dt_w \\
&\ll H^b + H^{b(1-1/n-\theta/n+\epsilon n_w/n)} \ll H^b.
\end{aligned}$$ □

LEMMA 6.7. *Assume that $b \leqslant 1/T$. Then there is $\delta > 0$ such that, for $H \geqslant 1$,*

$$I_<(H) \ll H^{-\delta}.$$

Proof. Let $d_0 \in \Delta$ and assume that $|\boldsymbol{\gamma}| = |\boldsymbol{\gamma}_{d_0}| \in (H, 2H]$. Since $|\boldsymbol{\gamma}| > H \geqslant 1$, we have

$$\max_{d \in \Delta} \{|\boldsymbol{\gamma}_d|^{1/s_d}\} \gg \max_{d \in \Delta} \{(1+|\boldsymbol{\gamma}_d|)^{1/s_d}\} \geqslant \prod_{d \in \Delta} (1+|\boldsymbol{\gamma}_d|)^{\lambda_d/s_d}$$

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for any choice of $0 \leqslant \lambda_d \leqslant 1$ with $\sum_{d\in\Delta} \lambda_d = 1$. We choose

$$\lambda_d := s_d t_d + \theta/|\Delta|,$$

where $\theta := 1 - \sum_{d\in\Delta} s_d t_d \in (0,1)$. With $\theta_d := \theta/(|\Delta| s_d t_d)$, this gives

$$I_<(H) \ll \int_{\substack{\boldsymbol{\gamma}_{d_0}\in V^{t_{d_0}} \\ N(\boldsymbol{\gamma}_{d_0})\leqslant H^b \\ |\boldsymbol{\gamma}_{d_0}|\geqslant H}} \frac{1}{(1+|\boldsymbol{\gamma}_{d_0}|)^{t_{d_0}(1+\theta_{d_0})}}\, d\boldsymbol{\gamma}_{d_0} \cdot \prod_{\substack{d\in\Delta \\ d\neq d_0}} \int_{\substack{\boldsymbol{\gamma}_d\in V^{t_d} \\ N(\boldsymbol{\gamma}_d)\leqslant H^b}} \frac{1}{(1+|\boldsymbol{\gamma}_d|)^{t_d(1+\theta_d)}}\, d\boldsymbol{\gamma}_d.$$

Further assuming that $|\boldsymbol{\gamma}_{d_0}| = |\boldsymbol{\gamma}_{d_0,i_0}|$, we get

$$I_<(H) \ll \int_{\substack{\gamma_{d_0,i_0}\in V \\ N(\gamma_{d_0,i_0})\leqslant H^b \\ |\gamma_{d_0,i_0}|\geqslant H}} \frac{1}{(1+|\gamma_{d_0,i_0}|)^{1+\theta_{d_0}}}\, d\gamma_{d_0,i_0} \cdot \prod_{\substack{d\in\Delta \\ 1\leqslant i\leqslant t_d \\ (d,i)\neq(d_0,i_0)}} \int_{\substack{\gamma_{d,i}\in V \\ N(\gamma_{d,i})\leqslant H^b}} \frac{1}{(1+|\gamma_{d,i}|)^{1+\theta_d}}\, d\gamma_{d,i}.$$

By Lemmas 6.5 and 6.6, this product is $\ll H^{-1-\theta_{d_0}+\epsilon+Tb} \ll H^{-\theta_{d_0}/2}$ if we choose ϵ small enough. □

With all our auxiliary results in place, we can now proceed to our main task, the estimation of the singular integral $\mathfrak{J}$.

LEMMA 6.8. *Assume (5.6). Then the integral defining $\mathfrak{J}$ converges absolutely and there is a positive constant δ such that*

$$\mathfrak{J} - \mathfrak{J}(H) \ll H^{-\delta}$$

holds for all large enough H.

Proof. Let us fix $b := 1/T$. We have

$$\mathfrak{J} - \mathfrak{J}(H) \ll \int_{|\boldsymbol{\gamma}|>H} |J(\boldsymbol{\gamma})|\, d\boldsymbol{\gamma} = \sum_{j=0}^{\infty} \int_{2^j H<|\boldsymbol{\gamma}|\leqslant 2^{j+1}H} |J(\boldsymbol{\gamma})|\, d\boldsymbol{\gamma}$$
$$\ll \sum_{j=0}^{\infty} \left(\int_{M_<(2^j H)} |J(\boldsymbol{\gamma})|\, d\boldsymbol{\gamma} + \int_{M_>(2^j H)} |J(\boldsymbol{\gamma})|\, d\boldsymbol{\gamma} \right).$$

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We first consider the integrals over $M_<(2^jH)$. Here we estimate $|J(\boldsymbol{\gamma})|$ by (6.2) and obtain

$$\sum_{j=0}^{\infty}\int_{M_<(2^jH)}|J(\boldsymbol{\gamma})|\,d\boldsymbol{\gamma}\ll\sum_{j=0}^{\infty}(2^jH)^{\epsilon}I_<(2^jH)\ll H^{\epsilon-\delta}\sum_{j=0}^{\infty}2^{j(\epsilon-\delta)}\ll H^{-\delta/2},$$

by Lemma 6.7, if ϵ was chosen small enough.

For the integrals over $M_>(2^jH)$, we use the estimates from (1) and (2) in Lemma 6.1 with $a=n$. We obtain

$$\sum_{j=0}^{\infty}\int_{M_>(2^jH)}|J(\boldsymbol{\gamma})|\,d\boldsymbol{\gamma}\ll\sum_{j=0}^{\infty}(2^jH)^{\epsilon}\sum_{\substack{\mathbf{u}=(u_d)_{d\in\Delta}\\ u_d\in\mathcal{O}_K^{\times}\\ |u_d|\ll 2^jH}}I_>(\mathbf{u},2^jH).$$

By Lemma 6.3, we have $I_>(\mathbf{u},2^j)\ll(2^jH)^{-\delta}$. Moreover, it is well known that the number of units $u\in\mathcal{O}_K^{\times}$ with $|u|\ll 2^jH$ is $\ll\log(2^jH)^{|\Omega_\infty|-1}$. Hence, the inner sum in the above expression has $\ll(2^jH)^{\epsilon}$ summands. Altogether, we see that

$$\sum_{j=0}^{\infty}\int_{M_>(2^jH)}|J(\boldsymbol{\gamma})|\,d\boldsymbol{\gamma}\ll H^{2\epsilon-\delta}\sum_{j=0}^{\infty}2^{j(2\epsilon-\delta')}\ll H^{-\delta/2}$$

if ϵ was chosen small enough. □

Our Theorem 1.1 is now an immediate consequence of the estimation of the minor arcs in §4 and the treatment of the major arcs in Lemmas 5.3, 5.6, and 6.8.

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