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EQUIVALENT CHARACTERIZATIONS ON ORDERINGS OF ORDER STATISTICS AND SAMPLE RANGES

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The purpose of this article is to present several equivalent characterizations of comparing the largest-order statistics and sample ranges of two sets of n independent exponential random variables with respect to different stochastic orders, where the random variables in one set are heterogeneous and the random variables in the other set are identically distributed. The main results complement and extend several known results in the literature. The geometric distribution can be regarded as the discrete counterpart of the exponential distribution. We also study the orderings of the largest-order statistics from geometric random variables and point out similarities and differences between orderings of the largest-order statistics from geometric variables and from exponential variables.

1. INTRODUCTION

Order statistics and statistics based on them play an important role in reliability and statistics; see Balakrishnan and Rao [1,2]. Let $X_1, \ldots, X_n$ be n random variables. The ith-order statistic, denoted by $X_{i:n}$, is the ith smallest one of the X_i's. In reliability, a k-out-of-n system is a system with n components that functions if and only if at least k of the components function. The survival function of a k-out-of-n system with lifetimes $X_1, \ldots, X_n$ of n components is the same as that of $(n-k+1)$st-order statistics, $X_{n-k+1:n}$. In particular, the lifetime of a parallel system (or 1-out-of-n system) is the same as the largest-order statistic. In statistics, sample range is a simple statistic for comparing variability of a distribution.

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In the past decades, there is a large amount of literature on stochastic comparisons on different aspects of order statistics. Among these, several researchers studied the problem of stochastically comparing the largest-order statistics and/or sample ranges of two sets of n exponential random variables, where the random variables in one set are independent and heterogeneous and the random variables in the other set are independent and identically distributed (i.i.d.)—for example, Bon and Păltănea [6], Dykstra, Kochar, and Rojo [8], Genest, Kochar, and Xu [9], Khaledi and Kochar [11–13], Kochar and Rojo [14], Kochar and Xu [16,17], and Zhao and Li [24]. Kochar and Xu [15] have given a comprehensive review on this topic up to 2007.

The purpose of this article is to reexamine the above problem of comparing the largest-order statistics and sample ranges of two sets of independent exponential random variables, where one set of random variables are i.i.d.. We present several equivalent characterizations on such comparisons with respect to different stochastic orders. The main results are given in Section 3, complementing and extending several known results in the literature. In Section 2 some definitions of majorization are recalled, and some recent developments on ordering order statistics and sample ranges are reviewed. The geometric distribution can be regarded as the discrete counterpart of the exponential distribution in the sense that both of them have constant hazard rates. In Section 4 we study the ordering of the largest-order statistics from geometric random variables. The main result and several examples illustrate similarities and differences between orderings of the largest-order statistics from geometric variables and from exponential variables.

Throughout, "increasing" and "decreasing" mean "nondecreasing" and "nonincreasing," respectively.

2. PRELIMINARIES

In this section we recall the definitions of majorization from Marshall and Olkin [18], and give some recent developments on stochastic comparisons of order statistics and sample ranges of independent exponential random variables (see also Khaledi and Kochar [12]). The formal definitions of some stochastic orders used in the sequel can be found in Shaked and Shanthikumar [22] amd Müller and Stoyan [19]. These stochastic orders include the usual stochastic order ($\leq_{\mathrm{st}}$), the hazard rate order ($\leq_{\mathrm{hr}}$), the reversed hazard rate order ($\leq_{\mathrm{rh}}$), the likelihood ratio order ($\leq_{\mathrm{lr}}$), the mean residual life order ($\leq_{\mathrm{mrl}}$), the increasing convex order ($\leq_{\mathrm{icx}}$), the dispersive order ($\leq_{\mathrm{disp}}$), and the excess wealth order ($\leq_{\mathrm{ew}}$).

2.1. Definitions of Majorization

The notion of majorization is quite useful in establishing various inequalities. Let $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ be the increasing arrangement of the components of the vector $\mathbf{x} = (x_1, \ldots, x_n)$.

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DEFINITION 2.1 (Marshall and Olkin [18]): *For* $\mathbf{x}, \mathbf{y} \in \Re^n$, $\mathbf{x}$ *is said to be the following:*

(a) *majorized by* $\mathbf{y}$, *denoted by* $\mathbf{x} \preceq_{\mathrm{m}} \mathbf{y}$, *if* $\sum_{i=1}^n x_{(i)} = \sum_{i=1}^n y_{(i)}$ *and*

$$\sum_{i=1}^{j} x_{(i)} \geq \sum_{i=1}^{j} y_{(i)} \quad \text{for } j = 1, \ldots, n-1;$$

(b) *weakly supmajorized by* $\mathbf{y}$, *denoted by* $\mathbf{x} \preceq^{\mathrm{w}} \mathbf{y}$, *if*

$$\sum_{i=1}^{j} x_{(i)} \geq \sum_{i=1}^{j} y_{(i)} \quad \text{for } j = 1, \ldots, n;$$

(c) *weakly submajorized by* $\mathbf{y}$, *denoted by* $\mathbf{x} \preceq_{\mathrm{w}} \mathbf{y}$, *if*

$$\sum_{i=j}^{n} x_{(i)} \leq \sum_{i=j}^{n} y_{(i)} \quad \text{for } j = 1, \ldots, n.$$

For a symmetric convex set $D \subseteq \Re^n$, a function $\psi : D \to \Re$ is said to be Schur-convex [Schur-concave] on D if $\mathbf{a} \preceq_{\mathrm{m}} \mathbf{b}$ and $\mathbf{a}, \mathbf{b} \in D$ imply that $\psi(\mathbf{a}) \leq [\geq]\, \psi(\mathbf{b})$.

The next two propositions will be used to prove the main results in the sequel.

PROPOSITION 2.2 (Marshall and Olkin [18, p .57]): *Let* $D \subseteq \Re^n$ *be a symmetric set and let* $\psi : D \to \Re$ *be continuously differentiable. Then* ψ *is Schur-convex on* D *if and only if* ψ *is symmetric on* D *and*

$$(x_i - x_j)\left(\frac{\partial \psi(\mathbf{x})}{\partial x_i} - \frac{\partial \psi(\mathbf{x})}{\partial x_j}\right) \geq 0 \quad \text{for all } i \neq j,\ \mathbf{x} \in D.$$

PROPOSITION 2.3 (Marshall and Olkin, [18, p. 59]): *A real-valued function* ψ *on the set* $D \subseteq \Re^n$ *satisfies*

$$\mathbf{y} \preceq^{\mathrm{w}} \mathbf{x} \Longrightarrow \psi(\mathbf{y}) \leq \psi(\mathbf{x})$$

if and only if ψ *is decreasing and Schur-convex on* D.

DEFINITION 2.4 (Bon and Păltănea [5]): *The vector* $\mathbf{x} \in \Re_+^n$ *is said to be p-smaller than another vector* $\mathbf{y} \in \Re_+^n$, *denoted by* $\mathbf{x} \preceq^p \mathbf{y}$, *if*

$$\prod_{i=1}^{j} x_{(i)} \geq \prod_{i=1}^{j} y_{(i)} \quad \text{for } j = 1, \ldots, n;$$

that is, $(\log x_1, \ldots, \log x_n) \preceq^w (\log y_1, \ldots, \log y_n)$.

It is known that

$$\mathbf{x} \preceq_{\mathrm{m}} \mathbf{y} \Longrightarrow \mathbf{x} \preceq^p \mathbf{y},$$

where the implication is strict. For comprehensive discussion on the theory of majorization and its applications, one can refer to Marshall and Olkin [18].

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2.2. Some Recent Developments

Throughout this section, let $\{X_1, \ldots, X_n\}$ be a sequence of independent exponential random variables with respective hazard rates $\lambda_1, \ldots, \lambda_n$ and denote by $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ the order statistics of the X_i's. Let $\{Y_1, \ldots, Y_n\}$ and $\{Z_1, \ldots, Z_n\}$ be two sequences of i.i.d. exponential random variables with respective common hazard rates λ_{am} and λ_{gm}, where λ_{am} and λ_{gm} are the arithmetic and the geometric means of the λ_i's, respectively, given by

$$\lambda_{\text{am}} = \frac{1}{n}\sum_{i=1}^{n}\lambda_i, \qquad \lambda_{\text{gm}} = \left(\prod_{i=1}^{n}\lambda_i\right)^{1/n}.$$

Let $\{X_1^*, \ldots, X_n^*\}$ be another sequence of independent exponential random variables with respective hazard rates $\lambda_1^*, \ldots, \lambda_n^*$.

- Pledger and Proschan [20] proved that

$$(\lambda_1^*, \ldots, \lambda_n^*) \preceq_m (\lambda_1, \ldots, \lambda_n) \Longrightarrow X_{i:n}^* \le_{\text{st}} X_{i:n} \quad \text{for } i = 1, \ldots, n, \tag{2.1}$$

  which was strengthened from the univariate usual stochastic order to the multivariate stochastic order in Proschan and Sethuraman [21]. A stronger monotone coupling result was given by Hu [10] for the more general distributions F than the exponential distribution. A counterexample was given by Boland, El-Neweihi, and Proschan [4] to show that (2.1) cannot be strengthened from the usual stochastic order to the hazard rate order for $n \ge 3$. However, Dykstra et al. [8] showed that

$$(\lambda_1^*, \ldots, \lambda_n^*) \preceq_m (\lambda_1, \ldots, \lambda_n) \Longrightarrow X_{n:n}^* \le_{\text{rh}} X_{n:n}; \tag{2.2}$$

  Khaledi and Kochar [12] proved that

$$(\lambda_1^*, \ldots, \lambda_n^*) \preceq^p (\lambda_1, \ldots, \lambda_n) \Longrightarrow X_{n:n}^* \le_{\text{st}} X_{n:n}. \tag{2.3}$$

  A counterexample was given in Khaledi and Kochar [13] to show that (2.3) does not hold for other order statistics. Additionally, (2.3) cannot be strengthened from the usual stochastic order to the hazard rate order.

- Dykstra et al. [8] showed that $Y_{n:n} \le_{\text{hr}} X_{n:n}$ and $Y_{n:n} \le_{\text{disp}} X_{n:n}$. Kochar and Xu [16] strengthened this relationship to the likelihood ratio order:

$$Y_{n:n} \le_{\text{lr}} X_{n:n}. \tag{2.4}$$

  Khaledi and Kochar [12] proved that

$$Z_{n:n} \le_{\text{hr}} X_{n:n}, \qquad Z_{n:n} \le_{\text{disp}} X_{n:n}. \tag{2.5}$$

  Kochar and Xu [16] gave an example to show that $Z_{n:n} \not\le_{\text{rh}} X_{n:n}$ and $Z_{n:n} \not\le_{\text{lr}} X_{n:n}$. This example also means that $(\lambda_1^*, \ldots, \lambda_n^*) \preceq^p (\lambda_1, \ldots, \lambda_n)$ does not imply $X_{n:n}^* \le_{\text{rh}} X_{n:n}$.

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- Kochar and Rojo [14] showed that $Y_{j:n} - Y_{i:n} \leq_{\mathrm{st}} X_{j:n} - X_{i:n}$ whenever $i < j$ and, hence,

$$Y_{n:n} - Y_{1:n} \leq_{\mathrm{st}} X_{n:n} - X_{1:n}. \tag{2.6}$$

Kochar and Xu [16] strengthened the above result from the usual stochastic order to the reverse hazard rate order:

$$Y_{n:n} - Y_{1:n} \leq_{\mathrm{rh}} X_{n:n} - X_{1:n}. \tag{2.7}$$

Recently, Genest et al. [9] proved that

$$Y_{n:n} - Y_{1:n} \leq_{\mathrm{lr}} X_{n:n} - X_{1:n}, \tag{2.8}$$

$$Y_{n:n} - Y_{1:n} \leq_{\mathrm{disp}} X_{n:n} - X_{1:n}. \tag{2.9}$$

Khaledi and Kochar [11] proved that

$$Z_{n:n} - Z_{1:n} \leq_{\mathrm{st}} X_{n:n} - X_{1:n}. \tag{2.10}$$

- Suppose that $\lambda_1^* = \cdots = \lambda_n^* = \lambda^* = \lambda$. Bon and Păltănea [6] gave the necessary and sufficient condition on λ such that $X_{k:n}^* \leq_{\mathrm{st}} X_{k:n}$ for each k; that is,

$$X_{k:n}^* \leq_{\mathrm{st}} X_{k:n} \Longleftrightarrow \lambda \geq \lambda_k^{\mathrm{st}}, \tag{2.11}$$

where

$$\lambda_k^{\mathrm{st}} = \left(\frac{1}{\binom{n}{k}} \sum_{1 \leq j_1 < \cdots < j_k \leq n} \prod_{i=1}^{k} \lambda_{j_i} \right)^{1/k},$$

the kth symmetric mean of the λ_i's. Kochar and Xu [17] proved that

$$X_{n:n}^* \leq_{\mathrm{disp}} X_{n:n} \Longleftrightarrow \lambda \geq \lambda_{\mathrm{gm}}, \tag{2.12}$$

$$X_{n:n}^* \leq_{\mathrm{ew}} X_{n:n} \Longleftrightarrow \lambda \geq \lambda_{\mathrm{ew}}, \tag{2.13}$$

where

$$\lambda_{\mathrm{ew}} = \left(\sum_{i=1}^{n} \frac{1}{i} \right) \left[\sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \leq j_1 < \cdots < j_k \leq n} \frac{1}{\sum_{i=1}^{k} \lambda_{j_i}} \right]^{-1}.$$

Zhao and Li [24] obtained that

$$X_{n:n}^* - X_{1:n}^* \leq_{\mathrm{st}} X_{n:n} - X_{1:n} \Longleftrightarrow \lambda \geq \lambda_{\mathrm{range-st}}, \tag{2.14}$$

where

$$\lambda_{\mathrm{range-st}} = \left(\frac{\prod_{i=1}^{n} \lambda_i}{\lambda_{\mathrm{am}}} \right)^{1/(n-1)},$$

which implies (2.10) since $\lambda_{\mathrm{gm}} \geq \lambda_{\mathrm{range-st}}$.

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3. EQUIVALENT CHARACTERIZATIONS

Let $X_1, \ldots, X_n$ be independent exponential random variables with respective hazard rates $\lambda_1, \ldots, \lambda_n$ and let $X_1^*, \ldots, X_n^*$ be i.i.d. exponential random variables with a common hazard rate λ. Equations (2.11)–(2.13) present equivalent characterizations of $X_{n:n}^* \leq_{\text{st}} X_{n:n}$, $X_{n:n}^* \leq_{\text{disp}} X_{n:n}$, and $X_{n:n}^* \leq_{\text{ew}} X_{n:n}$, respectively. The next two theorems give further equivalent characterizations of the orders $\leq_{\text{lr}}$, $\leq_{\text{hr}}$, $\leq_{\text{rh}}$, $\leq_{\text{st}}$, $\leq_{\text{mrl}}$, $\leq_{\text{icx}}$, and $\leq_{\text{disp}}$ between $X_{n:n}^*$ and $X_{n:n}$.

THEOREM 3.1: *Let $X_1, \ldots, X_n$ be independent exponential random variables with respective hazard rates $\lambda_1, \ldots, \lambda_n$ and let $X_1^*, \ldots, X_n^*$ be i.i.d. exponential random variables with a common hazard rate λ. Then*

(a) $X_{n:n}^* \leq_{\text{lr}} X_{n:n} \iff X_{n:n}^* \leq_{\text{rh}} X_{n:n} \iff \lambda \geq \lambda_{\text{am}}$;

(b) $X_{n:n}^* \leq_{\text{hr}} X_{n:n} \iff X_{n:n}^* \leq_{\text{st}} X_{n:n} \iff X_{n:n}^* \leq_{\text{disp}} X_{n:n} \iff \lambda \geq \lambda_{\text{gm}}$;

(c) $X_{n:n}^* \leq_{\text{mrl}} X_{n:n} \iff X_{n:n}^* \leq_{\text{icx}} X_{n:n} \iff X_{n:n}^* \leq_{\text{ew}} X_{n:n} \iff \lambda \geq \lambda_{\text{ew}}$.

PROOF:

(a) First, observe that $X_{n:n}^*$ is stochastically decreasing in λ with respect to the likelihood ratio order. In terms of (2.4), if $\lambda \geq \lambda_{\text{am}}$, then $X_{n:n}^* \leq_{\text{lr}} X_{n:n}$ and, hence, $X_{n:n}^* \leq_{\text{rh}} X_{n:n}$. To prove part (a), we have to prove that $X_{n:n}^* \leq_{\text{rh}} X_{n:n}$ implies $\lambda \geq \lambda_{\text{am}}$.

The distribution function of $X_{n:n}^*$ is given by

$$F_{n,n}(x) = \prod_{i=1}^{n} (1 - e^{-\lambda_i x}), \qquad x > 0,$$

with density function

$$f_{n,n}(x) = F_{n,n}(x) \sum_{i=1}^{n} \frac{\lambda_i e^{-\lambda_i x}}{1 - e^{-\lambda_i x}}, \qquad x > 0,$$

and, hence, its reversed hazard rate function is

$$\eta_{\lambda_1,\ldots,\lambda_n}(x) = \sum_{i=1}^{n} \frac{\lambda_i e^{-\lambda_i x}}{1 - e^{-\lambda_i x}}, \qquad x > 0.$$

Similarly, the reversed hazard rate function of $X_{n:n}^*$ is given by

$$\eta_{\lambda,\ldots,\lambda}(x) = \frac{n\lambda e^{-\lambda x}}{1 - e^{-\lambda x}}, \quad x \geq 0.$$

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By Taylor's expansion of the functions $\eta_{\lambda_1,\ldots,\lambda_n}(x)$ and $\eta_{\lambda,\ldots,\lambda}(x)$ at the origin, we have

$$\eta_{\lambda_1,\ldots,\lambda_n}(x) = \frac{n}{x} - \frac{1}{2}\sum_{i=1}^{n}\lambda_i + o(1),$$

$$\eta_{\lambda,\ldots,\lambda}(x) = \frac{n}{x} - \frac{n\lambda}{2} + o(1).$$

If $X^*_{n:n} \leq_{\text{rh}} X_{n:n}$, then $\eta_{\lambda_1,\ldots,\lambda_n}(x) \geq \eta_{\lambda,\ldots,\lambda}(x)$ for all $x \geq 0$ and, hence, $\lambda \geq \lambda_{\text{am}}$.

(b) Since $\lambda_n^{\text{st}} = \lambda_{\text{gm}}$, it follows from (2.11) that $X^*_{n:n} \leq_{\text{st}} X_{n:n}$ if and only if $\lambda \geq \lambda_{\text{gm}}$. By (2.5), if $\lambda \geq \lambda_{\text{gm}}$, then $X^*_{n:n} \leq_{\text{lr}} Z_{n:n} \leq_{\text{hr}} X_{n:n}$, which implies $X^*_{n:n} \leq_{\text{st}} X_{n:n}$. This proves part (b).

(c) Since the order $\leq_{\text{mrl}}$ is stronger than the order $\leq_{\text{icx}}$ (see Theorem 4.A.26 in Shaked and Shanthikumar [23]), $X^*_{n:n} \leq_{\text{mrl}} X_{n:n}$ implies $X^*_{n:n} \leq_{\text{icx}} X_{n:n}$, which, in turn, implies $\mathsf{E}X^*_{n:n} \leq \mathsf{E}X_{n:n}$. By Kochar and Xu [17], $\mathsf{E}X^*_{n:n} \leq \mathsf{E}X_{n:n}$ implies $X^*_{n:n} \leq_{\text{ew}} X_{n:n}$.

Conversely, assume that $X^*_{n:n} \leq_{\text{ew}} X_{n:n}$. Shaked and Shanthikumar [22] proved that for two nonnegative random variables U and V, if $U \leq_{\text{ew}} V$ and either U or V is DMRL (decreasing mean residual life), then $U \leq_{\text{mrl}} V$. It follows from Theorem 5.8 in Barlow and Proschan [3] that $X^*_{n:n}$ is IFR (increasing failure rate) and, hence, DMRL. Using this, we conclude that $X^*_{n:n} \leq_{\text{mrl}} X_{n:n}$. This completes the proof of the theorem.

■

THEOREM 3.2: *Under the assumptions of Theorem 3.1, we have*

$$X_{n:n} \leq_{\text{order}} X^*_{n:n} \iff \lambda \leq \lambda_{\min} \equiv \min_{1\leq i\leq n}\lambda_i, \tag{3.1}$$

where $\leq_{\text{order}}$ *is any one of the orders* $\leq_{\text{lr}}$, $\leq_{\text{hr}}$, $\leq_{\text{rh}}$, $\leq_{\text{st}}$, *and* $\leq_{\text{disp}}$.

PROOF:

Sufficiency: Suppose that $\lambda \leq \lambda_{\min}$. Then $X_i \leq_{\text{lr}} X^*_j$ for all pairs (i, j). By Corollary 1.C.34 in Shaked and Shanthikumar [23], we have $X_{n:n} \leq_{\text{lr}} X^*_{n:n}$ and, hence, $X_{n:n} \leq_{\text{order}} X^*_{n:n}$, where $\leq_{\text{order}}$ is any one of the orders $\leq_{\text{hr}}$, $\leq_{\text{rh}}$, and $\leq_{\text{st}}$. It remains to prove that $\lambda \leq \lambda_{\min}$ implies $X_{n:n} \leq_{\text{disp}} X^*_{n:n}$.

Let $f_{n,n}, g_{n,n}$ and $F_{n,n}, G_{n,n}$ denote the density and the distribution functions of $X_{n:n}$ and $X^*_{n:n}$, respectively. Then

$$G_{n,n}(x) = (1 - e^{-\lambda x})^n, \qquad x > 0,$$

$$G^{-1}_{n,n}F_{n,n}(x) = -\frac{1}{\lambda}\ln\left[1 - \prod_{i=1}^{n}(1 - e^{-\lambda_i x})^{1/n}\right], \qquad x > 0.$$

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To prove $X_{n:n} \le_{\text{disp}} X^*_{n:n}$, we use a similar argument to that in the proof of Theorem 2.1 in Dykstra et al. [8]. It suffices to show that

$$f_{n,n}(x) \ge g_{n,n}(G^{-1}_{n,n}F_{n,n}(x)), \qquad x > 0,$$

that is, for $x > 0$,

$$\left[\prod_{i=1}^{n}(1-e^{-\lambda_i x})\right]\sum_{i=1}^{n}\frac{\lambda_i e^{-\lambda_i x}}{1-e^{-\lambda_i x}} \ge n\lambda\left[1-\prod_{i=1}^{n}(1-e^{-\lambda_i x})^{1/n}\right] \times \prod_{i=1}^{n}(1-e^{-\lambda_i x})^{(n-1)/n}$$

or, equivalently,

$$\frac{1}{n}\sum_{i=1}^{n}\lambda_i\left(\frac{1}{1-e^{-\lambda_i x}}-1\right) \ge \lambda\left[\prod_{i=1}^{n}\frac{1}{(1-e^{-\lambda_i x})^{1/n}}-1\right].$$

This is true by applying the arithmetic–geometric inequality and using $\lambda \le \lambda_i$ for each i.

Necessity: By Theorem 3.B.13(a) in Shaked and Shanthikumar [23], $X_{n:n} \le_{\text{disp}} X^*_{n:n}$ implies $X_{n:n} \le_{\text{st}} X^*_{n:n}$. Thus, it suffices to check that $X_{n:n} \le_{\text{st}} X^*_{n:n}$ implies $\lambda \le \lambda_{\min}$. Assume that $\lambda_{\min} < \lambda$. Then

$$\lim_{x\to\infty}\frac{\mathsf{P}(X^*_{n:n} > x)}{\mathsf{P}(X_{n:n} > x)} = \lim_{x\to\infty}\frac{1-(1-e^{-\lambda x})^n}{1-\prod_{i=1}^{n}(1-e^{-\lambda_i x})} = \lim_{x\to\infty}\frac{ne^{-\lambda x}+o(e^{-\lambda x})}{\sum_{i=1}^{n}e^{-\lambda_i x}+o(e^{-\lambda_{\min} x})} = 0,$$

which is a contradiction. Therefore, $\lambda \le \lambda_{\min}$. This completes the proof.

■

For sample ranges, (2.14) presents an equivalent characterization of the usual stochastic order $\le_{\text{st}}$. The next two results, Theorems 3.3 and 3.4, give further equivalent characterizations of the orders $\le_{\text{lr}}$, $\le_{\text{rh}}$, $\le_{\text{st}}$, and/or $\le_{\text{hr}}$ between sample ranges $R^*_X = X^*_{n:n} - X^*_{1:n}$ and $R_X = X_{n:n} - X_{1:n}$.

THEOREM 3.3: *Under the assumptions of Theorem 3.1, we have*

$$X^*_{n:n} - X^*_{1:n} \le_{\text{lr}} X_{n:n} - X_{1:n} \Longleftrightarrow X^*_{n:n} - X^*_{1:n} \le_{\text{rh}} X_{n:n} - X_{1:n} \Longleftrightarrow \lambda \ge \lambda_{\text{am}}. \tag{3.2}$$

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PROOF: Note that

$$X^*_{n:n} - X^*_{1:n} = \sum_{k=2}^{n} (X^*_{k:n} - X^*_{k-1:n}), \tag{3.3}$$

the sum of $n-1$ independent random variables, where $X^*_{k:n} - X^*_{k-1:n}$ has the exponential distribution with hazard rate $(n-k+1)\lambda$ for each k. Based on this, it is easy to see that $X^*_{n:n} - X^*_{1:n}$ is stochastically decreasing in λ with respect to the likelihood ratio order. In terms of (2.7) and (2.8), to prove (3.2), it requires one to show that $X^*_{n:n} - X^*_{1:n} \leq_{\text{rh}} X_{n:n} - X_{1:n}$ implies $\lambda \geq \lambda_{\text{am}}$.

From David and Nagaraja [7, p. 26], the distribution and the density functions of the sample range $R_X = X_{n:n} - X_{1:n}$ are given by

$$F_{R_X}(x) = \left(\frac{\prod_{i=1}^{n} (1 - e^{-\lambda_i x})}{\sum_{i=1}^{n} \lambda_i} \right) \sum_{i=1}^{n} \frac{\lambda_i}{1 - e^{-\lambda_i x}}, \qquad x > 0,$$

and

$$f_{R_X}(x) = \left(\frac{\prod_{i=1}^{n} (1 - e^{-\lambda_i x})}{\sum_{i=1}^{n} \lambda_i} \right) \sum_{i=1}^{n} \frac{\lambda_i}{1 - e^{-\lambda_i x}} \sum_{j=1, j\neq i}^{n} \frac{\lambda_j e^{-\lambda_j x}}{1 - e^{-\lambda_j x}}, \qquad x > 0.$$

Thus, the reversed hazard rate function of R_X is

$$\eta_{R_X}(x) = \sum_{i=1}^{n} \frac{\lambda_i}{1 - e^{-\lambda_i x}} \sum_{j=1, j\neq i}^{n} \frac{\lambda_j e^{-\lambda_j x}}{1 - e^{-\lambda_j x}} \left(\sum_{i=1}^{n} \frac{\lambda_i}{1 - e^{-\lambda_i x}} \right)^{-1}, \qquad x > 0;$$

also see Kochar and Xu [16]. Similarly, the distribution and the reversed hazard rate functions of sample range $R^*_X = X^*_{n:n} - X^*_{1:n}$ are given by

$$F_{R^*_X}(x) = (1 - e^{-\lambda x})^{n-1}, \qquad x > 0,$$

$$\eta_{R^*_X}(x) = \frac{(n-1)\lambda e^{-\lambda x}}{1 - e^{-\lambda x}}, \qquad x > 0,$$

respectively. By Taylor's expansion of $\eta_{R_X}(x)$ and $\eta_{R^*_X}(x)$ at the origin, we have

$$\eta_{R_X}(x) = \frac{n-1}{x} - (n-1)\lambda_{\text{am}} + o(1),$$

$$\eta_{R^*_X}(x) = \frac{n-1}{x} - (n-1)\lambda + o(1).$$

If $X^*_{n:n} - X^*_{1:n} \leq_{\text{rh}} X_{n:n} - X_{1:n}$, then $\eta_{R^*_X}(x) \leq \eta_{R_X}(x)$ for all $x > 0$, implying $\lambda \geq \lambda_{\text{am}}$. This proves the theorem. ■

It is still an open problem whether there is an equivalent characterization of the hazard rate ordering between sample ranges R_X and R^*_X in Theorem 3.3; that is, there exists a positive constant $\lambda_{\text{range-hr}}$ such that

$$X^*_{n:n} - X^*_{1:n} \leq_{\text{hr}} X_{n:n} - X_{1:n} \Longleftrightarrow \lambda \geq \lambda_{\text{range-hr}}.$$

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THEOREM 3.4: *Under the assumptions of Theorem 3.1, we have*

$$X_{n:n} - X_{1:n} \leq_{\text{order}} X^*_{n:n} - X^*_{1:n} \Longleftrightarrow \lambda \leq \lambda_{\min},$$

where $\leq_{\text{order}}$ *is any one of the orders* $\leq_{\text{lr}}$, $\leq_{\text{hr}}$, $\leq_{\text{rh}}$, *and* $\leq_{\text{st}}$.

PROOF:

Sufficiency: Suppose that $\lambda \leq \lambda_{\min}$. We only need to prove that $R_X \leq_{\text{lr}} R^*_X$ that is, $f_{R_X}(x)/f_{R^*_X}(x)$ is decreasing in $x \in \Re_+$, where f_{R_X} and $f_{R^*_X}$ are the density functions of sample ranges R_X and R^*_X, respectively. Note that

$$\frac{f_{R_X}(x)}{f_{R^*_X}(x)} = \frac{1}{(n-1)\lambda \sum_{i=1}^n \lambda_i} \sum_{1 \leq i \neq j \leq n} \lambda_i \lambda_j e^{-(\lambda_j - \lambda)x} \prod_{k \neq i,j} \left(\frac{1 - e^{-\lambda_k x}}{1 - e^{-\lambda x}} \right).$$

Since $\lambda \leq \lambda_{\min}$, it is easy to see that both $e^{-(\lambda_j - \lambda)x}$ and $(1 - e^{-\lambda_k x})/(1 - e^{-\lambda x})$ are decreasing in $x \in \Re_+$. Therefore, $f_{R_X}(x)/f_{R^*_X}(x)$ is decreasing in x.

Necessity: It suffices to prove the necessity for the order $\leq_{\text{st}}$. By Taylor's expansion of the respective survival functions $\overline{F}_{R_X}(x)$ and $\overline{F}_{R^*_X}$ of R_X and R^*_X at the origin, we have

$$\begin{aligned}
\overline{F}_{R_X}(x) &= 1 - \frac{1}{n\lambda_{\text{am}}} \left[1 - \sum_{i=1}^n e^{-\lambda_i x} + o(e^{-\lambda_{\min} x}) \right] \sum_{i=1}^n \lambda_i (1 + e^{-\lambda_i x} \\
&\quad + o(e^{-\lambda_{\min} x})) \\
&= \sum_{i=1}^n e^{-\lambda_i x} - \frac{1}{n\lambda_{\text{am}}} \sum_{i=1}^n \lambda_i e^{-\lambda_i x} + o(e^{-\lambda_{\min} x}) \\
&= \frac{1}{n\lambda_{\text{am}}} \sum_{i=1}^n \Lambda_i e^{-\lambda_i x} + o(e^{-\lambda_{\min} x}), \qquad \textbf{(3.4)}
\end{aligned}$$

$$\overline{F}_{R^*_X}(x) = 1 - (1 - e^{-\lambda x})^{n-1} = (n-1)e^{-\lambda x} + o(e^{-\lambda x}), \qquad \textbf{(3.5)}$$

where $\Lambda_i = n\lambda_{\text{am}} - \lambda_i$ for each i. If $X_{n:n} - X_{1:n} \leq_{\text{st}} X^*_{n:n} - X^*_{1:n}$, then $\overline{F}_{R_X}(x) \leq \overline{F}_{R^*_X}(x)$ for all $x > 0$. From (3.4) and (3.5), it follows that $\lambda \leq \lambda_{\min}$.

■

Remark 3.5: Under the assumptions of Theorem 3.4, we have

$$X_{n:n} - X_{1:n} \leq_{\text{disp}} X^*_{n:n} - X^*_{1:n} \Longrightarrow \lambda \leq \lambda_{\min}. \qquad \textbf{(3.6)}$$

To see it, by Theorem 3.B.13(a) in Shaked and Shanthikumar [23], $R_X \leq_{\text{disp}} R^*_X$ implies $R_X \leq_{\text{st}} R^*_X$. Applying Theorem 3.4 yields $\lambda \leq \lambda_{\min}$.

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Using the same notations as those in the proof of Theorem 3.4, $X_{n:n} - X_{1:n} \leq_{\text{disp}} X^*_{n:n} - X^*_{1:n}$ is equivalent to

$$f_{R^*_X}\left(F^{-1}_{R^*_X}(F_{R_X}(x))\right) \leq f_{R_X}(x) \quad \text{for } x > 0,$$

which can be written as

$$(n-1)\lambda\left(\sum_{i=1}^{n}\frac{\lambda_i}{1-e^{-\lambda_i x}}\right)\left[\left(\frac{\sum_{i=1}^{n}\lambda_i}{\sum_{i=1}^{n}\lambda_i\prod_{j\neq i}(1-e^{-\lambda_j x})}\right)^{1/(n-1)} - 1\right]$$

$$\leq \sum_{1\leq i\neq j\leq n}\frac{\lambda_i\lambda_j e^{-\lambda_j x}}{(1-e^{-\lambda_i x})(1-e^{-\lambda_j x})}. \tag{3.7}$$

It is easy to see that (3.7) holds for $n = 2$ when $\lambda \leq \lambda_{\min}$; that is,

$$\lambda \leq \min\{\lambda_1, \lambda_2\} \Longrightarrow X_{2:2} - X_{1:2} \leq_{\text{disp}} X^*_{2:2} - X^*_{1:2}.$$

However, it is still unknown whether implication (3.6) can be reversed for $n \geq 3$.

An immediate consequence of Theorem 3.1 is the following corollary.

COROLLARY 3.6: *Let $X_1, \ldots, X_n$ be independent exponential random variables with respective hazard rates $\lambda_1, \ldots, \lambda_n$ and let $W_1, \ldots, W_n$ be i.i.d. exponential random variables with a common hazard rate λ_{hm}, where λ_{hm} is the harmonic mean of the λ_i's, given by*

$$\lambda_{\text{hm}} = \frac{n}{\sum_{i=1}^{n}\lambda_i^{-1}}.$$

Then

$$W_{n:n} \leq_{\text{ew}} X_{n:n}, \qquad W_{n:n} \leq_{\text{mrl}} X_{n:n}, \qquad W_{n:n} \leq_{\text{icx}} X_{n:n}.$$

However, for unequal λ_i's,

$$W_{n:n} - W_{1:n} \not\leq_{\text{st}} X_{n:n} - X_{1:n}.$$

PROOF: The desired results follows from (2.13), (2.14), Lemmas 3.7 and 3.8, and the same argument as that in the proof of Theorem 3.1(c). ■

LEMMA 3.7: *For all $(\lambda_1, \ldots, \lambda_n) \in \Re^n_+$, we have $\lambda_{\text{hm}} \geq \lambda_{\text{ew}}$, where the equality holds if and only if $\lambda_1 = \cdots = \lambda_n$.*

LEMMA 3.8: *For all $(\lambda_1, \ldots, \lambda_n) \in \Re^n_+$, we have $\lambda_{\text{range}-\text{st}} \geq \lambda_{\text{hm}}$, where the equality holds if and only if $\lambda_1 = \cdots = \lambda_n$.*

The proofs of the above two lemmas are given in the Appendix.

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To end this section, we make the following conjecture.

Conjecture: Under the assumptions of Corollary 3.6, we have

$$W_{n:n} - W_{1:n} \leq_{\text{ew}} X_{n:n} - X_{1:n}$$

and, hence, $W_{n:n} - W_{1:n} \leq_{\text{icx}} X_{n:n} - X_{1:n}$ and $W_{n:n} - W_{1:n} \leq_{\text{mrl}} X_{n:n} - X_{1:n}$.

4. ORDERING THE LARGEST-ORDER STATISTICS OF GEOMETRIC VARIABLES

Let $X_1, \ldots, X_n$ be a sequence of independent geometric random variables with parameters $p_1, \ldots, p_n$, respectively, where

$$\mathsf{P}(X_i = k) = p_i(1 - p_i)^k, \qquad k = 0, 1, \ldots,$$

with $p_i \in (0, 1)$ for each i, and let $X_1^*, \ldots, X_n^*$ be another sequence of independent geometric random variables with parameters $p_1^*, \ldots, p_n^*$, respectively.

THEOREM 4.1: *If* $(p_1^*, \ldots, p_n^*) \preceq^p (p_1, \ldots, p_n)$, *then* $X_{n:n}^* \leq_{\text{st}} X_{n:n}$.

PROOF: The distribution functions of $X_{n:n}$ and $X_{n:n}^*$ are given by

$$F_{n,n}(k) = \prod_{i=1}^{n}(1 - [1 - p_i]^{k+1}), \qquad k = 0, 1, \ldots,$$

and

$$F_{n,n}^*(k) = \prod_{i=1}^{n}(1 - [1 - p_i^*]^{k+1}), \qquad k = 0, 1, \ldots,$$

respectively. By Proposition 2.3, to prove $F_{n,n}(k) \geq F_{n,n}^*(k)$ for each fixed k, it requires one to prove that

$$\psi(\mathbf{u}) = \prod_{i=1}^{n}(1 - [1 - e^{u_i}]^{k+1})$$

is increasing and Schur-concave in $\mathbf{u} \in \Re_-^n \equiv (-\infty, 0)^n$. The monotonicity of $\psi(\mathbf{u})$ is trivial. We have to prove the Schur-concavity of $\psi(\mathbf{u})$. Setting $p_i = e^{u_i}$ for each i,

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it follows that

$$\frac{\partial \psi(\mathbf{u})}{\partial u_i} = (k+1)\psi(\mathbf{u})\frac{p_i(1-p_i)^k}{1-(1-p_i)^{k+1}},$$

$$\frac{\partial \psi(\mathbf{u})}{\partial u_i} - \frac{\partial \psi(\mathbf{u})}{\partial u_j} = (k+1)\psi(\mathbf{u})\left[\frac{p_i(1-p_i)^k}{1-(1-p_i)^{k+1}} - \frac{p_j(1-p_j)^k}{1-(1-p_j)^{k+1}}\right].$$

Define $h(p) = (1-p)^k p/[1-(1-p)^{k+1}]$. Since $h'(p) \stackrel{\text{sgn}}{=} 1-(k+1)p-(1-p)^{k+1} \le 0$ for $p \in (0,1)$, $h(p)$ is decreasing in $p \in (0,1)$. Thus,

$$(u_i - u_j)\left(\frac{\partial \psi(\mathbf{u})}{\partial u_i} - \frac{\partial \psi(\mathbf{u})}{\partial u_j}\right) \le 0, \quad \forall\, i \ne j,$$

which, by Proposition 2.2, implies that $\psi(\mathbf{u})$ is Schur-concave in $\mathbf{u} \in \Re_-^n$. This completes the proof. ■

Theorem 4.1 might not hold for other order statistics, as shown by the following example.

Example 4.2: Let (X_1, X_2, X_3) be a vector of independent geometric variables with parameter vector $(p_1, p_2, p_3) = (0.01, 0.1, 0.9)$ and let (X_1^*, X_2^*, X_3^*) be another vector of independent geometric variables with parameter vector $(p_1^*, p_2^*, p_3^*) = (0.1, 0.2, 0.5)$. Clearly, $(p_1^*, p_2^*, p_3^*) \preceq^p (p_1, p_2, p_3)$. Since $\mathsf{P}(X_{1:3} \ge k) = \prod_{i=1}^3 (1-p_i)^k$ for each k, we have

$$\mathsf{P}(X_{1:3} \ge 1) = 0.081 < 0.360 = \mathsf{P}(X_{1:3}^* \ge 1),$$

implying $X_{1:3}^* \not\le_{\text{st}} X_{1:3}$.

The next example shows that an analogue of (2.2) does not hold for geometric random variables; that is,

$$(p_1^*, \ldots, p_n^*) \preceq_m (p_1, \ldots, p_n) \Longrightarrow X_{n:n}^* \le_{\text{rh}} X_{n:n}.$$

Example 4.3: Let (X_1, X_2, X_3) be a vector of independent geometric variables with parameter vector $(p_1, p_2, p_3) = (0.2, 0.4, 0.8)$ and let (X_1^*, X_2^*, X_3^*) be another vector of independent geometric variables with parameter vector $(p_1^*, p_2^*, p_3^*) = (0.4, 0.4, 0.6)$. Clearly, $(p_1^*, p_2^*, p_3^*) \preceq_m (p_1, p_2, p_3)$. However,

$$\frac{F_{3,3}(1)}{F_{3,3}(0)} = 3.456 < 3.584 = \frac{F_{3,3}^*(1)}{F_{3,3}^*(0)}$$

and

$$\frac{F_{3,3}(3)}{F_{3,3}(2)} = 1.3518 > 1.2831 = \frac{F_{3,3}^*(3)}{F_{3,3}^*(2)},$$

which imply $X_{3:3}^* \not\le_{\text{rh}} X_{3:3}$.

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THEOREM 4.4: *Let $X_1, \ldots, X_n$ be independent geometric random variables with respective parameters $p_1, \ldots, p_n$ and let $V_1, \ldots, V_n$ be i.i.d. geometric random variables with a common parameter p. Denote by p_{cg} the complementary geometric mean of the p_i's, given by*

$$p_{\mathrm{cg}} = 1 - \left(\prod_{i=1}^{n} (1 - p_i) \right)^{1/n}.$$

If $p \geq p_{\mathrm{cg}}$, then $V_{n:n} \leq_{\mathrm{lr}} X_{n:n}$.

PROOF: Since $V_{n:n}$ is stochastically decreasing in p with respect to the likelihood ratio order, without loss of generality assume that $p = p_{\mathrm{cg}}$. Let $S_1, \ldots, S_n$ be independent exponential random variables with respective hazard rates $\lambda_1, \ldots, \lambda_n$ and let $T_1, \ldots, T_n$ be i.i.d. exponential random variables with a common hazard rate λ_{am}, where $\lambda_i = -\ln(1 - p_i)$ for each i. Then $1 - p_i = e^{-\lambda_i}$, $i = 1, \ldots, n$, and $1 - p_{\mathrm{cg}} = e^{-\lambda_{\mathrm{am}}}$. Denote by $f_{n,n}$ and $g_{n,n}$ the density functions of $T_{n:n}$ and $S_{n:n}$, respectively. Then

$$\begin{aligned} g_{n,n}(x) &= \prod_{i=1}^{n} (1 - e^{-\lambda_i x}) \sum_{j=1}^{n} \frac{\lambda_j e^{-\lambda_j x}}{1 - e^{-\lambda_j x}} \\ &= -\prod_{i=1}^{n} (1 - [1 - p_i]^x) \sum_{j=1}^{n} \frac{(1 - p_j)^x \ln(1 - p_j)}{1 - (1 - p_j)^x} \\ &= \frac{\partial}{\partial x} \prod_{i=1}^{n} (1 - [1 - p_i]^x), \qquad x \in \Re_+. \end{aligned}$$

Similarly,

$$f_{n,n}(x) = \frac{\partial}{\partial x} (1 - [1 - p_{\mathrm{cg}}]^x)^n, \qquad x \in \Re_+.$$

So we have

$$\begin{aligned} \frac{\mathsf{P}(X_{n:n} = k)}{\mathsf{P}(V_{n:n} = k)} &= \frac{\prod_{i=1}^{n} (1 - [1 - p_i]^{k+1}) - \prod_{i=1}^{n} (1 - [1 - p_i]^k)}{(1 - [1 - p_{\mathrm{cg}}]^{k+1})^n - (1 - [1 - p_{\mathrm{cg}}]^k)^n} \\ &= \frac{\int_k^{k+1} g_{n,n}(u)\, du}{\int_k^{k+1} f_{n,n}(u)\, du}, \qquad k = 0, 1, \ldots. \end{aligned}$$

By (2.4), $T_{n:n} \leq_{\mathrm{lr}} S_{n:n}$; that is,

$$g_{n,n}(u) f_{n,n}(v) \leq g_{n,n}(v) f_{n,n}(u), \quad \forall\, v \geq u > 0.$$

Integrating both sides with respect to u and v, $(u, v) \in [k, k+1] \times [k+1, k+2]$, we have

$$\int_k^{k+1} g_{n,n}(u)\, du \int_{k+1}^{k+2} f_{n,n}(v)\, dv \leq \int_{k+1}^{k+2} g_{n,n}(v)\, dv \int_k^{k+1} f_{n,n}(u)\, du, \quad \forall\, k \geq 0,$$

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implying that $\mathsf{P}(X_{n:n} = k)/\mathsf{P}(V_{n:n} = k)$ is increasing in k. The desired result now follows. ■

Now, let $X_1, \ldots, X_n$ be independent geometric variables with parameters $p_1, \ldots, p_n$, respectively, and let $Y_1, \ldots, Y_n$ be i.i.d. geometric variables with a common parameter $\bar{p} = n^{-1} \sum_{i=1}^{n} p_i$. Since the order $\preceq_m$ is stronger than the order $\preceq^p$, it follows from Theorem 4.1 that $Y_{n:n} \leq_{\mathrm{st}} X_{n:n}$. Clearly, $\bar{p} < p_{\mathrm{cg}}$ for unequal p_i's. One might wonder whether the likelihood ratio ordering between $Y_{n:n}$ and $X_{n:n}$ still holds. The answer in negative; that is, for unequal p_i's,

$$Y_{n:n} \not\leq_{\mathrm{rh}} X_{n:n} \quad \text{and} \quad Y_{n:n} \not\leq_{\mathrm{lr}} X_{n:n}. \tag{4.1}$$

This means that an analogue of (2.4) does not hold for geometric random variables. To see the first assertion of (4.1), note that

$$Y_{n:n} \leq_{\mathrm{rh}} X_{n:n} \Longleftrightarrow \prod_{i=1}^{n} \frac{1-(1-p_i)^{k+1}}{1-(1-\bar{p})^{k+1}} \geq \prod_{i=1}^{n} \frac{1-(1-p_i)^{k}}{1-(1-\bar{p})^{k}}, \quad \forall\, k = 1, 2, \ldots,$$

$$\Longleftrightarrow \sum_{i=1}^{n} \ln \frac{1-(1-p_i)^{k+1}}{1-(1-p_i)^{k}} \geq \sum_{i=1}^{n} \ln \frac{1-(1-\bar{p})^{k+1}}{1-(1-\bar{p})^{k}}, \quad \forall\, k = 1, 2, \ldots,$$

where the last inequality does not hold for $k = 1$.

It is still an open problem whether $Y_{n:n} \leq_{\mathrm{hr}} X_{n:n}$ holds.

The geometric distribution can be regarded as the discrete counterpart of the exponential distribution in the sense that both of them have constant hazard rates. They have several similarities in many respects. However, the above examples illustrate that there are some differences between orderings of the largest-order statistics from geometric variables and from exponential variables.

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APPENDIX

PROOF OF LEMMA 3.7: First note that $\lambda_{\text{hm}} \geq \lambda_{\text{ew}}$ is equivalent to

$$\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\lambda_i}\right)\left(\sum_{i=1}^{n}\frac{1}{i}\right) \leq \sum_{k=1}^{n}(-1)^{k+1}\sum_{1\leq j_1<\cdots<j_k\leq n}\frac{1}{\sum_{i=1}^{k}\lambda_{j_i}},$$

which can be written as

$$\left(\frac{1}{n}\sum_{i=1}^{n}u_i\right)\left(\sum_{i=1}^{n}\frac{1}{i}\right) \leq \sum_{k=1}^{n}(-1)^{k+1}\sum_{1\leq j_1<\cdots<j_k\leq n}\frac{1}{\sum_{i=1}^{k}u_{j_i}^{-1}} \tag{A.1}$$

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by setting $u_i \equiv u_i(\lambda_i) = 1/\lambda_i$ for each i. Denote by $\psi(\mathbf{u})$ the right-hand side of (A.1), and let $T_1, \ldots, T_n$ be i.i.d. exponential random variables with a common hazard rate $1/\bar{u}$, where $\bar{u} = n^{-1} \sum_{i=1}^n u_i$. Then it follows from the proof of Proposition 4.2 in Kochar and Xu [17] that

$$\psi(\bar{u}, \ldots, \bar{u}) = \mathsf{E}[T_{n:n}] = \bar{u} \sum_{i=1}^n \frac{1}{i}.$$

Thus, (A.1) reduces to

$$\psi(\bar{u}, \ldots, \bar{u}) \le \psi(\mathbf{u}) \quad \text{for all } \mathbf{u} = (u_1, \ldots, u_n) \in \Re_+^n. \tag{A.2}$$

To prove (A.2), it is enough to check that $\psi(\mathbf{u})$ is Schur-convex in $\mathbf{u} \in \Re_+^n$. By Proposition 2.2, it requires one to verify that

$$(u_i - u_j)\left(\frac{\partial \psi(\mathbf{u})}{\partial u_i} - \frac{\partial \psi(\mathbf{u})}{\partial u_j}\right) \ge 0, \quad \forall\, i \ne j, \tag{A.3}$$

for all $\mathbf{u} \in \Re_+^n$. Without loss of generality, fix $(i,\ j) = (1, 2)$ and assume that $u_1 \ge u_2$ (i.e., $\lambda_1 \le \lambda_2$). Note that

$$\begin{aligned}\frac{\partial \psi(\mathbf{u})}{\partial u_1} &= 1 + \frac{1}{u_1^2} \sum_{k=1}^{n-1} (-1)^k \sum_{|J|=k, J \subseteq E\setminus\{1\}} \left(\frac{1}{u_1^{-1} + \sum_{\ell \in J} u_\ell^{-1}}\right)^2 \\ &= 1 - \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{|J|=k, J \subseteq E\setminus\{1\}} \left(\frac{\lambda_1}{\lambda_1 + \sum_{\ell \in J} \lambda_\ell}\right)^2.\end{aligned}$$

To prove (A.3) or $\partial \psi(\mathbf{u})/\partial u_1 \ge \partial \psi(\mathbf{u})/\partial u_2$, we have to prove that

$$\begin{aligned}&\sum_{k=1}^{n-1} (-1)^{k+1} \sum_{|J|=k, J \subseteq E\setminus\{1\}} \left(\frac{\lambda_1}{\lambda_1 + \sum_{\ell \in J} \lambda_\ell}\right)^2 \\ &\qquad \le \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{|J|=k, J \subseteq E\setminus\{2\}} \left(\frac{\lambda_2}{\lambda_2 + \sum_{\ell \in J} \lambda_\ell}\right)^2.\end{aligned} \tag{A.4}$$

Observe that the left-hand side of (A.4) can be written as

$$\begin{aligned}\text{LHS} &= \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{|J|=k, J \subseteq E\setminus\{1\}} \left(\frac{1}{1 + \sum_{\ell \in J} \lambda_\ell/\lambda_1}\right)^2 \\ &= \int_0^\infty \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{|J|=k, J \subseteq E\setminus\{1\}} \frac{\exp\left\{-[1 + \sum_{\ell \in J} \lambda_\ell/\lambda_1]x\right\}}{1 + \sum_{\ell \in J} \lambda_\ell/\lambda_1}\, dx \\ &= \int_0^\infty \int_x^\infty \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{|J|=k, J \subseteq E\setminus\{1\}} \exp\left\{-\left[1 + \sum_{\ell \in J} \frac{\lambda_\ell}{\lambda_1}\right] t\right\} dt\, dx \\ &= \int_0^\infty \int_x^\infty \left[1 - \left(1 - e^{-\lambda_2 t/\lambda_1}\right) \prod_{k=3}^n \left(1 - e^{-\lambda_k t/\lambda_1}\right)\right] e^{-t}\, dt\, dx,\end{aligned}$$

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which is a decreasing function of $(\lambda_2/\lambda_1, \lambda_3/\lambda_1, \ldots, \lambda_n/\lambda_1)$. Similarly, the right-hand side of (A.4) can be written as

$$\text{RHS} = \int_0^\infty \int_x^\infty \left[1 - \left(1 - e^{-\lambda_1 t/\lambda_2}\right) \prod_{k=3}^{n} \left(1 - e^{-\lambda_k t/\lambda_2}\right)\right] e^{-t}\, dt\, dx.$$

Since $(\lambda_2/\lambda_1, \lambda_3/\lambda_1, \ldots, \lambda_n/\lambda_1) \geq (\lambda_1/\lambda_2, \lambda_3/\lambda_2, \ldots, \lambda_n/\lambda_2)$, it follows that LHS $\leq$ RHS, (i.e., (A.4)) holds. This completes the proof of the lemma. ■

PROOF OF LEMMA 3.8: First note that $\lambda_{\text{range}-\text{st}} \geq \lambda_{\text{hm}}$ is equivalent to

$$\lambda_{\text{am}} \left(\prod_{i=1}^{n} \lambda_i\right)^{-1} \leq \left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{\lambda_i}\right)^{n-1},$$

which can be written as

$$\left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{u_i}\right) \prod_{i=1}^{n} u_i \leq \left(\frac{1}{n} \sum_{i=1}^{n} u_i\right)^{n-1} \tag{A.5}$$

by setting $u_i = 1/\lambda_i$ for $i = 1, \ldots, n$. Define

$$\psi(\mathbf{u}) = \left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{u_i}\right) \prod_{i=1}^{n} u_i, \quad \mathbf{u} = (u_1, \ldots, u_n) \in \Re_+^n.$$

Then (A.5) reduces to

$$\psi(\mathbf{u}) \leq \psi(\bar{\mathbf{u}}),$$

where $\bar{\mathbf{u}} = (\bar{u}, \ldots, \bar{u})$ and $\bar{u} = n^{-1} \sum_{i=1}^{n} u_i$. To prove $\psi(\mathbf{u}) \leq \psi(\bar{\mathbf{u}})$, we have to check that $\psi(\mathbf{u})$ is Schur-concave on $\Re_+^n$.

For a given vector $\mathbf{u} = (u_1, \ldots, u_n)$ and for each $1 \leq i \leq n$, denote by $\mathbf{u}^{(i)}$ the vector obtained from $\mathbf{u}$ by deleting its ith component. Then

$$\frac{\partial \psi(\mathbf{u})}{\partial u_i} = \frac{1}{n} \sum_{k=1, k\neq i}^{n} \left(\prod_{\ell \neq i,k} u_\ell\right) = \frac{1}{n} S_{n-2}(\mathbf{u}^{(i)}),$$

where $S_r(\mathbf{u})$ is the rth elementary symmetric function of $\mathbf{u}$. Similarly,

$$\frac{\partial \psi(\mathbf{u})}{\partial u_j} = \frac{1}{n} S_{n-2}(\mathbf{u}^{(j)}).$$

Thus,

$$(u_i - u_j)\left[\frac{\partial \psi(\mathbf{u})}{\partial u_i} - \frac{\partial \psi(\mathbf{u})}{\partial u_j}\right] = \frac{1}{n}(u_i - u_j)\left[S_{n-2}(\mathbf{u}^{(i)}) - S_{n-2}(\mathbf{u}^{(j)})\right] \leq 0.$$

Therefore, by Proposition 2.2, we conclude that $\psi(\mathbf{u})$ is Schur-concave in $\mathbf{u} \in \Re_+^n$. The rest of the proof is trivial. This completes the proof. ■

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