ON MATRICES WHOSE EIGENVALUES ARE IN ARITHMETIC PROGRESSION

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1. *Introduction.* The following matrix problems are well known in quantum mechanics:

(*a*) *The one-dimensional harmonic oscillator.* Given*

$$H = P^2/2m + \tfrac{1}{2}m\omega^2 X^2, \quad [H, X] = hP/m, \quad [H, P] = -hm\omega^2 X, \tag{1}$$

determine the eigenvalues h_{jj} of H, and the matrix elements of X, P if H is diagonal. It is found (Wigner (4)) that

$$h_{jj} = h_{11} + (j-1)\,h\omega \quad (j = 1, 2, \ldots), \tag{2}$$

$$[P, X]_{jj} = \frac{2}{\omega}h_{11} \quad \text{if } j \text{ is odd} \quad \text{and} \quad \frac{2}{\omega}(h\omega - h_{11}) \quad \text{if } j \text{ is even.} \tag{3}$$

X, P, H, m represent respectively coordinate, linear momentum, energy and mass of the oscillator, $\omega = 2\pi$ times its frequency, and h is Planck's constant divided by 2π. If $[P, X] = hI$ is also specified, the special case $h_{11} = \frac{1}{2}h\omega$ is obtained. This represents the more usual treatment of the above problem.

(*b*) *The angular momentum operators.* Given

$$[M_1, M_3] = hM_2, \quad [M_2, M_1] = hM_3, \quad [M_3, M_2] = hM_1, \tag{4}$$

determine the matrix elements of these operators and of $M_2 = M_1^2 + M_2^2 + M_3^2$. It is found that the eigenvalues of M_1 are

$$(m_1)_{jj} = (-l + j - 1)\,h, \tag{5}$$

also

$$M^2 = l(l+1)\,h^2 I, \tag{6}$$

where l has one of the values $0, \frac{1}{2}, 1, \ldots$. The M_j's represent the Cartesian components of the angular momentum, M being the total angular momentum.

Both problems lead to eigenvalues (2) and (5) which are in arithmetic progression, and their solution should therefore be contained in a general theory of matrices having this property. As (*a*) has been discussed only very recently, the matrix-theoretical point of view which exhibits these problems in this way may be of interest. The relevant theorems are proved in § 2; here it should merely be noted that (1) and (4) imply the relations

$$[H, [H, X]] = -(h\omega)^2 X \quad \text{and} \quad [M_1, [M_1, M_3]] = -h^2 M_3 \tag{7}$$

respectively. An infinite representation of equations (4) is obtained in § 3.

* Capital letters denote square matrices, small letters are real or complex numbers, $[A, B] = i(AB - BA)$, $i = \sqrt{(-1)}$, and I is the identity. Matrices are Hermitian unless otherwise stated.

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2. *General theorems.* Relations (7) derive their importance from their occurrence in a theorem which will now be proved.

THEOREM 1. *Let W be a Hermitian matrix which has* (i) *an enumerable number of eigenvalues which are all different,* (ii) *a least eigenvalue. A necessary and sufficient condition for the eigenvalues of W to form an arithmetic progression is that there exists a matrix A such that* (iii) $[W, [W, A]] = cA$, *c being a non-zero real or complex number, and* (iv) *the matrix* $E = A^2 - c^{-1}[W, A]^2 + (-c)^{-\frac{1}{2}}[A, [W, A]]$ *shall have at most one eigenvalue which is zero.*

Comment. The first part of condition (i) is necessary, as we do not wish to admit continuous ranges to the eigenvalue spectrum of W. It must be understood from the formulation of (i) that W can have an enumerably infinite number of rows and columns, and that it can also be a finite matrix. As W is Hermitian a matrix representation exists in which W is diagonal with its eigenvalues arranged in ascending order. This representation will be referred to as the 'normal' representation. As its eigenvalues are all different, W can stand, in quantum-mechanical language, for a complete set of commuting observables. Hence it follows (Dirac (1)) that if another observable commutes with W, it is a function of W.

Consider condition (ii) next. If W has an infinity of eigenvalues, arranged in ascending order, these can range from $-\infty$ to $+\infty$, but in general W will have a smallest or a greatest eigenvalue. If W has a smallest eigenvalue it fulfils the condition (ii) of the theorem. If it has a greatest eigenvalue, then $W' \equiv -W$ has a smallest eigenvalue and condition (ii) of the theorem will apply to it. In this sense the theorem is somewhat more general than the wording suggests.

Condition (iii) has the trivial solution $A = 0$, whatever W, so that it can impose a restriction on W only if some further condition, which entails $A \neq 0$, is added. This is expressed by (iv). (iv) can be shown to be equivalent to the requirement that all matrix elements of A which are of the form $a_{j,j-1}$ or $a_{j,j+1}$ shall be non-zero in the normal representation. Condition (iii) is sufficient to show that all the other matrix element of A vanish. While the truth of these last two statements follows from equations established during the process of proving Theorem 3 (below), they show that the interpretation of condition (iv) in terms of matrix elements is simple.

Proof. Condition (iii) implies that

$$2WAW - AW^2 - W^2A - cA = 0. \tag{8}$$

In virtue of (i), the rows and columns of the matrices may be put into one-one correspondence with the integers and hence distinguished by suffixes j, $k = 1, 2, \ldots$. In the normal representation equation (8) becomes

$$[(w_j - w_k)^2 + c]\, a_{jk} = 0 \quad (j, k = 1, 2, \ldots). \tag{9}$$

Here c is a negative (real) number, since if it were complex or positive the expression in square brackets could never vanish, thus implying $A = 0$ in contradiction with condition (iv). Put

$$c = -d^2 \quad (d > 0). \tag{10}$$

With the notation

$$X \equiv A - \frac{i}{d}[W, A], \quad Y \equiv A + \frac{i}{d}[W, A], \tag{11}$$

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it is easily verified from (iii) that

$$WX = X(W+dI), \quad WY = Y(W-dI), \tag{12}$$

and that the matrix E of condition (iv) satisfies

$$E = XY = A^2 + \frac{1}{d^2}[W,A]^2 + \frac{1}{d}[A,[W,A]].$$

E commutes with W since, by (12),

$$WXY - XYW = X(W+dI)\,Y - X(W+dI)\,Y = 0.$$

Similarly, let

$$B \equiv \frac{1}{d}[W,A] \tag{13}$$

in the Jacobian identity

$$[W,[A,B]] + [A,[B,W]] + [B,[W,A]] = 0,$$

and use condition (iii) in the second term. It follows that the commutator of A and B also commutes with W. Hence the following relations hold:

$$[W,(A^2+B^2)] = [W,[A,B]] = 0, \quad E = XY = A^2 + B^2 + [A,B]. \tag{14}$$

Let now w_j be any eigenvalue of W and α_j the corresponding eigenfunction. One then obtains from (12)

$$WY\alpha_j = Y(W-dI)\,\alpha_j = Y(w_j - d)\,Y\alpha_j. \tag{15}$$

Thus, unless $Y\alpha_j$ vanishes, $Y\alpha_j$ is another eigenfunction of W belonging to the eigenvalue $w_j - d$. If, however, one chooses $j = 1$, in the argument (15), w_1 being the least eigenvalue, an eigenvalue $w_1 - d$ does not exist, so that $Y\alpha_1 = 0$, and hence

$$XY\alpha_1 = E\alpha_1 = 0. \tag{16}$$

By equation (14) an eigenfunction of W is also one of E, so that (16) shows that the eigenvalue zero belongs to the eigenfunction α_1 of E. It now follows from condition (iv) that, since E has no other eigenvalues which are zero, $Y\alpha_j \neq 0$ for $j \neq 1$. Accordingly, one can infer from the argument (15) that the eigenvalues of W are given by

$$w_j = w_1 + (j-1)\,d \quad (j = 1, 2, \ldots).$$

If W has r rows and columns this sequence ceases for $j = r$. If W has an infinite number of eigenvalues the sequence never ceases. This proves the sufficiency of our conditions (iii) and (iv).

Conversely, if the eigenvalues of W form an arithmetic progression, a matrix A, fulfilling conditions (iii) and (iv) of the theorem, always exists. The proof is easily effected by choosing the normal representation and showing that any matrix A of the form

$$a_{jk} = a_{j,j-1}\,\delta_{j-1,k} + a_{j,j+1}\,\delta_{j+1,k}, \quad a_{j,j-1}, a_{j,j+1} \neq 0 \quad (j,k = 1, 2, \ldots),$$

satisfies the required conditions.

Here, and in the sequel, it is assumed that if a set of matrix elements $a_{j,j-1}$ occur, the elements which are obtained for $j = 1$ are zero. Similarly, if a finite number, r, of rows and columns is contemplated, elements involving a suffix larger than r, such as $a_{j,j+1}$ for $j = r$, must be taken to vanish.

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THEOREM 2. (i) *The matrix B fulfils all conditions stipulated for A in Theorem* 1. (ii) $[A, B]$ *and* A^2+B^2 *can both be expressed as functions of* W.

Proof. (i) follows if $\frac{1}{d}[W, A]$ is substituted for A in conditions (iii) and (iv) of Theorem 1. (ii) follows immediately from (14) and the Comments on Theorem 1.

THEOREM 3. *In the normal matrix representation the following relations hold*:

(i) $[B, A]_{jj} = 2\sum_{k=1}^{k=j}[(-1)^{j+k}(A^2+B^2)_{kk}] - (A^2+B^2)_{jj}$.

(ii) $(A^2+B^2)_{jj} = 2\sum_{k=1}^{k=j}[B, A]_{kk} - [B, A]_{jj}$.

Also, if $[B, A]$ *is a finite matrix, then*

(iii) $\operatorname{tr}([B, A]) = 0$.

Proof. By (9) all elements of A which have not either of the two forms $a_{j,j-1}$, $a_{j,j+1}$ vanish. From the definition of B in (13) it now follows that

$$b_{jk} = i(a_{j,j-1}\delta_{j-1,k} - a_{j,j+1}\delta_{j+1,k}). \tag{17}$$

Forming $A \pm iB$ and using (11), one finds

$$x_{jk} = 2a_{j,j-1}\delta_{j-1,k}, \quad y_{jk} = 2a_{j,j+1}\delta_{j+1,k}.$$

Hence
$$(YX)_{jk} = (A^2+B^2+[B, A])_{jk} = 4a_{j,j+1}a_{j+1,j}\delta_{jk}. \tag{18}$$

Similarly, $(XY)_{jk} = 4a_{j,j-1}a_{j-1,j}\delta_{jk}$. From this relation, together with condition (iv) of Theorem 1 and equation (16), it follows that the elements $a_{j,j+1}$, $a_{j+1,j}$ $(j = 1, 2, \ldots)$ of A (the only ones which have not yet been proved to vanish), must in fact be non-zero.

Using these results and (17), we find, on taking matrix elements, that

$$[B, A]_{jj} = 2(a_{j,j+1}a_{j+1,j} - a_{j-1,j}a_{j,j-1}).$$

which has the solution
$$2a_{j,j+1}a_{j+1,j} = \sum_{k=1}^{k=j}[B, A]_{kk}. \tag{19a}$$

Similarly, it is found that

$$(A^2+B^2)_{jj} = 2(a_{j,j+1}a_{j+1,j} + a_{j-1,j}a_{j,j-1}).$$

which has the solution

$$2a_{j,j+1}a_{j+1,j} = \sum_{k=1}^{k=j}(-1)^{j+k}(A^2+B^2)_{kk}. \tag{19b}$$

Now combining (18) and (19) one finds

$$(A^2+B^2+[B, A])_{jj} = 2\sum_{k=1}^{k=j}[B, A]_{kk} = 2\sum_{k=1}^{k=j}(-1)^{j+k}(A^2+B^2)_{kk}.$$

The equations (i) and (ii) of the theorem are easily obtained from this relation. The result (iii) follows immediately from equation (19a). Since the trace of a matrix is a class-function the result (iii) holds for all matrix representations if it holds for one representation.

It is evident from Theorem 3 that if either of the two matrices $[B, A]$, A^2+B^2 is given, the other can be determined. The next assumption of physical importance is

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the case which arises if either of these two matrices has itself eigenvalues which are in arithmetic progression. The calculation is based on Theorem 3. We merely state the result.

THEOREM 4. *If the eigenvalues of* $[B, A]$ or A^2+B^2 *form an ascending arithmetic series with difference* λ, *then the relations*

(i) $(A^2+B^2)_{jj} = (2j-1)[B,A]_{11}+(j-1)^2\lambda$, *or*

(ii) $[A,B]_{11} = [A,B]_{33} = \ldots, [A,B]_{22} = [A,B]_{44} = \ldots, [A,B]_{11}+[A,B]_{22} = \lambda$

hold respectively.

3. *Comments on the theorems.* With the aid of Table 1 problems (*a*) and (*b*) can be exhibited as special cases of the general theory. As W has in both cases a classical analogue, transition to classical variables can be accomplished by letting d tend to zero. Hence, by the correspondence principle, d must be a function of Planck's constant, $d = d(h)$, such that $d(0) = 0$, and this can be confirmed from Table 1.

Table 1. *Key to the various notations*

§ 2	W	A	B	d	λ
Problem (*a*)	H	X	$P/m\omega$	$h\omega$	$2h/m\omega$
Problem (*b*)	M_1	M_3	M_2	h	$-h^2$

Theorem 2 (ii) states that $[A,B]$ and A^2+B^2 can both be expressed as functions of W, and Theorem 3 that either function is known if the other is given. Now the physics involved in the problems of § 1 is such that in (*a*) A^2+B^2, in (*b*) $[B,A]$, is a known multiple of W, leading to a definite value of λ in Theorem 4. Inspection of the matrix elements which this theorem furnishes leads to the additional results given in the main diagonal of Table 2. The equation for $[B,A]$ in problem (*a*) (3, 4) is equivalent to equation (3).

Table 2. *Special cases of Theorem* 2 (ii)

Problem (*a*)	Problem (*b*)
$\{m\omega^2[B,A]-h\omega\}^2 = \{2w_1-h\omega\}^2$	$[B,A] = -hW$
$A^2+B^2 = 2W/m\omega^2$	$A^2+B^2 = w_1(w_1-h)\,I-W^2$

The known realization of the commutation rules (4) and of

$$M_1^2+M_2^2+M_3^2 = l(l+1)\,I, \quad \text{where} \quad l = -(M_1)_{11}/h, \tag{20}$$

occurs when the matrices are all finite. They are then also Hermitian. If the usual 'plane' limiting process (Landsberg (2)) is employed, an infinite representation, which is not Hermitian*, is also possible. In either case the matrix elements are formed according to the laws

$$(M_1)_{jj} = (-l+j-1)\,h, \quad (M_2)_{j,j-1} = i\,(M_3)_{j,j-1}, \quad (M_2)_{j,j+1} = -i\,(M_3)_{j,j+1}, \tag{21}$$
$$2(M_3)_{j,j+1}\,(M_3)_{j+1,j} = h^2 j[l-\tfrac{1}{2}(j-1)].$$

* I am indebted to Dr C. Strachan for drawing attention to this point.

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All other elements vanish, and $j = 1, 2, \ldots$. If the number of rows and columns has the finite value r, say, Theorem 3 (iii) yields the relation $r = 2l + 1$, and the usual properties of the angular momentum operators follow.

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