Econometric Theory, **25**, 2009, 1277–1288. Printed in the United States of America.
doi:10.1017/S0266466608090506

INTEGRATED MARKOV-SWITCHING GARCH PROCESS

JI-CHUN LIU
Xiamen University

This paper investigates stationarity of the so-called integrated Markov-switching generalized autoregressive conditionally heteroskedastic (GARCH) process, which is an important subclass of the Markov-switching GARCH process introduced by Francq, Roussignol, and Zakoïan (2001, *Journal of Time Series Analysis* 22, 197–220) and a Markov-switching version of the integrated GARCH (IGARCH) process. We show that, like the classical IGARCH process, a stationary solution with infinite variance for the integrated Markov-switching GARCH process may exist. To this purpose, an alternative condition for the existence of a strictly stationary solution of the Markov-switching GARCH process is presented, and some results obtained in Hennion (1997, *Annals of Probability* 25, 1545–1587) are employed. In addition, we also discuss conditions for the existence of a strictly stationary solution of the Markov-switching GARCH process with finite variance, which is a modification of Theorem 2 in Francq et al. (2001).

1. INTRODUCTION

To capture two important dynamic characteristics of time series, the regimes and conditional heteroskedasticity, Cai (1994) and Hamilton and Susmel (1994) proposed Markov-switching autoregressive conditionally heteroskedastic (ARCH) models. A generalization to Markov-switching generalized autoregressive conditionally heteroskedastic (GARCH) models was developed by Gray (1996) and subsequently modified by Klaassen (2002). In addition, an alternative Markov-switching GARCH model can be found in Haas, Mittnik, and Paolella (2004). Liu (2006) investigated the stationarity of the model considered by Haas et al. (2004). As a natural generalization of the Gray (1996) model, Francq, Roussignol, and Zakoïan (2001) defined the following d-regime Markov-switching GARCH(q, p) model, denoted in short by MSG(d, q, p):

$$\epsilon_t = \eta_t h_t^{1/2}, \tag{1}$$

I am grateful to the co-editor, Bruce Hansen, and three referees for very helpful comments and suggestions. Address correspondence to Ji-Chun Liu, School of Mathematical Science, Xiamen University, Xiamen, 361005, PR China; e-mail: liujichun65@126.com.

© 2009 Cambridge University Press 0266-4666/09 $15.00

https://doi.org/10.1017/S0266466608090506 Published online by Cambridge University Press

where the conditional variance process h_t is given by the GARCH(q, p) equation

$$h_t = \omega(\Delta_t) + \sum_{i=1}^{q} \alpha_i(\Delta_t)\epsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j(\Delta_t)h_{t-j} \tag{2}$$

and $\{\eta_t, t \in \mathbb{Z}\}$ is a sequence of independently and identically distributed (i.i.d.) random variables with mean zero and unit variance, $\{\Delta_t, t \in \mathbb{Z}\}$ is a stationary Markov chain with finite state space $S = \{1, \ldots, d\}$ and an irreducible and aperiodic $d \times d$ transition matrix, P, with typical element $p_{ij} = P(\Delta_t = j | \Delta_{t-1} = i)$, and the inequalities $\omega(k) > 0, \alpha_i(k) \geq 0, \beta_j(k) \geq 0, k = 1, \ldots, d, i = 1, \ldots, q \geq 1$, $j = 1, \ldots, p$, are assumed to hold, guaranteeing positivity of the variance process. Moreover, it is assumed that $\{\eta_t, t \in \mathbb{Z}\}$ and $\{\Delta_t, t \in \mathbb{Z}\}$ are independent. The stationary distribution of the Markov chain $\{\Delta_t, t \in \mathbb{Z}\}$ will be denoted by $\pi = (\pi_1, \pi_2, \ldots, \pi_d)^\tau$, where τ denotes the transpose of matrix. In addition, we always write $r = p + q$ throughout this paper.

It is one of the main results in Francq et al. (2001) that a sufficient and necessary condition for the existence of a strictly stationary solution of the MSG(d, q, p) process defined by (1) and (2) is given. Furthermore, conditions for the existence of a strictly stationary solution of the MSG(d, q, p) process with finite variance are also discussed. In addition, Francq and Zakoïan (2005) discussed the conditions for the existence of high-order moments of the MSG(d, q, p) process.

However, one important theoretical problem has not been treated by the previously mentioned authors, that is, the stationarity of the so-called integrated MSG(d, q, p) process. Precise definition of the integrated MSG(d, q, p) process will be given in the following section. Roughly speaking, the integrated MSG(d, q, p) process is an analogy of the integrated GARCH (IGARCH), which is also regarded as a Markov-switching version of the IGARCH process. The probabilistic structures and the various applications of the IGARCH process are considered by many authors; examples are Engle and Bollerslev (1986), Nelson (1990), Bougerol and Picard (1992a, 1992b), Lumsdaine (1995, 1996), Resnick (1997), Mikosch and Stărică (2000, 2004). In particular, Bougerol and Picard (1992a) discussed the stationarity of the IGARCH process using the limit theorem for products of positive i.i.d. random matrices in Kesten and Spitzer (1984).

Inspired by the approach of Bougerol and Picard (1992a), we investigate the stationarity of the integrated MSG(d, q, p) process utilizing the limit theorem for products of positive and ergodic random matrices in Hennion (1997). We show that, like the classical IGARCH model, a stationary solution with infinite variance for the integrated MSG(d, q, p) may exist. In addition, we also discuss conditions for the existence of the strictly stationary solution of the MSG(d, q, p) process with finite variance, which is an improvement of Theorem 2 in Francq et al. (2001).

The organization of this paper is as follows. In Section 2, we first discuss conditions for the existence of the strictly stationary solution of the MSG(d, q, p) process with finite variance. In the remainder of this section, the stationarity of the

https://doi.org/10.1017/S0266466608090506 Published online by Cambridge University Press

integrated MSG(d, q, p) process is investigated. Section 3 concludes. All proofs are given in the Appendix.

2. MAIN RESULTS

We shall first consider the conditions for the existence of a strictly stationary solution of the MSG(d, q, p) process with finite variance. We need some notation. Define

$$C_t(\Delta_t) = \begin{bmatrix} \alpha_{1:q-1}(\Delta_t)\eta_t^2 & \alpha_q(\Delta_t)\eta_t^2 & \beta_{1:p-1}(\Delta_t)\eta_t^2 & \beta_p(\Delta_t)\eta_t^2 \\ I_{q-1} & 0 & 0 & 0 \\ \alpha_{1:q-1}(\Delta_t) & \alpha_q(\Delta_t) & \beta_{1:p-1}(\Delta_t) & \beta_p(\Delta_t) \\ 0 & 0 & I_{p-1} & 0 \end{bmatrix} \tag{3}$$

and $\underline{d}_t(\Delta_t) = (\omega(\Delta_t)\eta_t^2, 0, \ldots, 0, \omega(\Delta_t), 0, \ldots, 0)^\tau$, $\underline{\xi}_t = (\epsilon_t^2, \ldots, \epsilon_{t-q+1}^2, h_t, \ldots, h_{t-p+1})^\tau$, where $\alpha_{1:q-1}(\Delta_t) = (\alpha_1(\Delta_t), \ldots, \alpha_{q-1}(\Delta_t))$, $\beta_{1:p-1}(\Delta_t) = (\beta_1(\Delta_t), \ldots, \beta_{p-1}(\Delta_t))$. Then (1) and (2) can be rewritten in vectorial form as

$$\underline{\xi}_t = \underline{d}_t(\Delta_t) + C_t(\Delta_t)\underline{\xi}_{t-1}. \tag{4}$$

Remark 1. When $p = 0$, obvious modifications have to be done as follows:

$$C_t(\Delta_t) = \begin{bmatrix} \alpha_{1:q-1}(\Delta_t)\eta_t^2 & \alpha_q(\Delta_t)\eta_t^2 \\ I_{q-1} & 0 \end{bmatrix},$$

and $\underline{d}_t(\Delta_t) = (\omega(\Delta_t)\eta_t^2, 0, \ldots, 0)^\tau$, $\underline{\xi}_t = (\epsilon_t^2, \ldots, \epsilon_{t-q+1}^2)^\tau$.

Again, let

$$A_t = \begin{bmatrix} \mathbf{1}_{\{\Delta_t=1,\Delta_{t-1}=1\}}C_t(1) & \mathbf{1}_{\{\Delta_t=1,\Delta_{t-1}=2\}}C_t(1) & \cdots & \mathbf{1}_{\{\Delta_t=1,\Delta_{t-1}=d\}}C_t(1) \\ \mathbf{1}_{\{\Delta_t=2,\Delta_{t-1}=1\}}C_t(2) & \mathbf{1}_{\{\Delta_t=2,\Delta_{t-1}=2\}}C_t(2) & \cdots & \mathbf{1}_{\{\Delta_t=2,\Delta_{t-1}=d\}}C_t(2) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{1}_{\{\Delta_t=d,\Delta_{t-1}=1\}}C_t(d) & \mathbf{1}_{\{\Delta_t=d,\Delta_{t-1}=2\}}C_t(d) & \cdots & \mathbf{1}_{\{\Delta_t=d,\Delta_{t-1}=d\}}C_t(d) \end{bmatrix}, \tag{5}$$

$$A = \begin{bmatrix} p_{11}\tilde{A}(1) & p_{21}\tilde{A}(1) & \cdots & p_{d1}\tilde{A}(1) \\ p_{12}\tilde{A}(2) & p_{22}\tilde{A}(2) & \cdots & p_{d2}\tilde{A}(2) \\ \vdots & \vdots & \ddots & \vdots \\ p_{1d}\tilde{A}(d) & p_{2d}\tilde{A}(d) & \cdots & p_{dd}\tilde{A}(d) \end{bmatrix}, \tag{6}$$

https://doi.org/10.1017/S0266466608090506 Published online by Cambridge University Press

where $\tilde{A}(k)$ is the $r \times r$ matrix obtained by replacing η_t^2 by 1 in $C_t(k)$, $k = 1, \ldots, d$, and $\mathbf{1}_{\{\cdot\}}$ denotes an indicator function. Moreover, write $\underline{\zeta}_t = \mathbf{1}_{\Delta_t} \otimes \underline{\xi}_t$ and $\underline{b}_t = \mathbf{1}_{\Delta_t} \otimes \underline{d}_t(\Delta_t)$, where $\mathbf{1}_{\Delta_t} = (\mathbf{1}_{\{\Delta_t=1\}}, \mathbf{1}_{\{\Delta_t=2\}}, \ldots, \mathbf{1}_{\{\Delta_t=d\}})^\tau$ and $\otimes$ denotes the Kronecker product of matrices. In view of (4), we can obtain the following random difference equation:

$$\underline{\zeta}_t = \underline{b}_t + A_t \underline{\zeta}_{t-1}. \tag{7}$$

Using the random recurrence equation (7), we can now state the following result.

THEOREM 1. *The MSG(d, q, p) process defined by (1) and (2) has a unique strictly stationary and ergodic solution with finite variance if and only if* $\rho(A) < 1$, *where* $\rho(A)$ *denotes the spectral radius of A defined by (6).*

In addition, note that, in Francq et al. (2001), a different form of A is defined as follows:

$$\mathbf{P} = \begin{bmatrix} p_{11}\bar{A}(1) & p_{21}\bar{A}(1) & \cdots & p_{d1}\bar{A}(1) \\ p_{12}\bar{A}(2) & p_{22}\bar{A}(2) & \cdots & p_{d2}\bar{A}(2) \\ \vdots & \vdots & \ddots & \vdots \\ p_{1d}\bar{A}(d) & p_{2d}\bar{A}(d) & \cdots & p_{dd}\bar{A}(d) \end{bmatrix}, \tag{8}$$

where

$$\bar{A}(k) = \begin{bmatrix} \gamma_{1:s-1}(k) & \gamma_s(k) \\ I_{s-1} & 0 \end{bmatrix},$$

$s = \max(q, p)$, $\gamma_{1:s-1}(k) = (\gamma_1(k), \ldots, \gamma_{s-1}(k))$, $\gamma_l(k) = \alpha_l(k) + \beta_l(k)$, $l = 1, \ldots, s$, and $\alpha_i(k) = 0$ and $\beta_j(k) = 0$ for $i > q$ and $j > p$, $k = 1, \ldots, d$. However, according to Theorem 1 of the present paper, Theorem 2 in Francq et al. (2001) can be modified as follows.

COROLLARY 1. *The MSG(d,q,p) process defined by (1) and (2) has a unique strictly stationary and ergodic solution with finite variance if and only if* $\rho(\mathbf{P}) < 1$, *where* $\mathbf{P}$ *is defined as in (8).*

Remark 2. It is evident that Corollary 1 is an improvement of Theorem 2 in Francq et al. (2001), because $\mathbf{P}$ in the second part of Theorem 2 in Francq et al. (2001) is assumed to be irreducible. The following example shows that the irreducibility of $\mathbf{P}$ does not follow from an irreducible and aperiodic $d \times d$ transition matrix $P = [p_{ij}]$.

https://doi.org/10.1017/S0266466608090506 Published online by Cambridge University Press

Example 1

For $d = 2$, let the transition matrix

$$P = \begin{bmatrix} 0 & 1 \\ 0.5 & 0.5 \end{bmatrix},$$

$$\bar{A}(1) = \begin{bmatrix} \gamma_1(1) & \gamma_2(1) & \gamma_3(1) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \bar{A}(2) = \begin{bmatrix} \gamma_1(2) & \gamma_2(2) & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

where $\gamma_3(1) > 0$ and $\gamma_1(2) + \gamma_2(2) > 0$. Then

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 0 & \gamma_1(1) & \gamma_2(1) & \gamma_3(1) \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0.5\gamma_1(2) & 0.5\gamma_2(2) & 0 & 0.5\gamma_1(2) & 0.5\gamma_2(2) & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 & 0 \end{bmatrix}.$$

It is obvious that the transition matrix P is irreducible and aperiodic. However, it is easy to check that $\mathbf{P}$ is reducible.

Based on Theorem 1, an analogy of the IGARCH process, the so-called integrated MSG(d, q, p) process is defined as follows.

DEFINITION 1. *The MSG(d, q, p) process defined as in (1) and (2) is called an integrated MSG(d, q, p) process if $\rho(A) = 1$, where A is defined by (6).*

We show the main result of the present paper.

THEOREM 2. *Assume that $\rho(A) = 1$ and $\alpha_i(k) > 0$ and $\beta_j(k) > 0$, $i = 1, \ldots, q$, $j = 1, \ldots, p$, $k = 1, \ldots, d$, $\eta_t \neq 0$ almost surely (a.s.). Then the MSG(d,q,p) process defined by (1) and (2) has a unique strictly stationary and ergodic solution with infinite variance.*

Remark 3. Obviously, Theorem 2 is a generalization of Corollary 2.2 in Bougerol and Picard (1992a). Indeed, when $d = 1$, $\rho(A) = \sum_{i=1}^{q} \alpha_i + \sum_{j=1}^{p} \beta_j$, and the conditions in Theorem 2 reduce to $\sum_{i=1}^{q} \alpha_i + \sum_{j=1}^{p} \beta_j = 1$, $\alpha_i > 0$ and $\beta_j > 0$, $i = 1, \ldots, q$, $j = 1, \ldots, p$, and $\eta_t \neq 0$, a.s., which are the same as the conditions of Corollary 2.2 in Bougerol and Picard (1992a).

https://doi.org/10.1017/S0266466608090506 Published online by Cambridge University Press

3. CONCLUSIONS

This paper provides some explicit results for the structural properties of the Markov-switching GARCH (MSG(d, q, p)) process introduced by Francq et al. (2001). First, we discuss conditions for the existence of the strictly stationary solution of the MSG(d, q, p) process with finite variance, which is an improvement of Theorem 2 in Francq et al. (2001). Moreover, the main result of this paper, the stationarity of the integrated MSG(d, q, p) process is proved under some mild conditions.

REFERENCES

Bougerol, P. & N. Picard (1992a) Stationarity of GARCH processes and of some nonnegative time series. *Journal of Econometrics* 52, 115–127.

Bougerol, P. & N. Picard (1992b) Strict stationarity of generalized autoregressive process. *Annals of Probability* 20, 1714–1730.

Cai, J. (1994) A Markov model of switching-regime ARCH. *Journal of Business & Economic Statistics* 12, 309–316.

Engle, R.F. & T. Bollerslev (1986) Modeling the persistence of conditional variances. *Econometric Reviews* 5, 1–50.

Francq, C., M. Roussignol, & J.-M. Zakoïan (2001) Conditional heteroskedasticity driven by hidden Markov chains. *Journal of Time Series Analysis* 22, 197–220.

Francq, C. & J.-M. Zakoïan (2005) The L^2-structures of standard and switching-regime GARCH models. *Stochastic Processes and Their Applications* 115, 1157–1582.

Gray, S.F. (1996) Modeling the conditional distribution of interest rates as a regime-swithing process. *Journal of Financial Economics* 42, 27–62.

Haas, M., S. Mittnik, & M.S. Paolella (2004) A new approach to Markov-switching GARCH models. *Journal of Financial Econometrics* 2, 493–530.

Hamilton, J.D. & R. Susmel (1994) Autoregressive conditional heteroskedasticity and changes in regime. *Journal of Econometrics* 64, 307–333.

Hennion, H. (1997) Limit theorems for products of positive random matrices. *Annals of Probability* 25, 1545–1587.

Kesten, H. & F. Spitzer (1984) Convergence in distribution for products of random matrices. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* 67, 363–386.

Kingman, J.F.C. (1973) Subadditive ergodic theory. *Annals of Probability* 1, 883–899.

Klaassen, F. (2002) Improving GARCH volatility forecasts with regime-switching GARCH. *Empirical Economics* 27, 363–394.

Liu, J.-C. (2006) Stationarity of a Markov-switching GARCH model. *Journal of Financial Econometrics* 4, 573–593.

Lumsdaine, R.L. (1995) Finite-sample properties of the maximum likelihood estimator in GARCH(1,1) and IGARCH(1,1) models: A Monte Carlo investigation. *Journal of Business & Economic Statistics* 13, 1–10.

Lumsdaine, R.L. (1996) Consistency and asymptotic normality of the quasi-maximum likelihood estimator in IGARCH(1,1) and covariance stationary GARCH(1,1) models. *Econometrica* 64, 575–596.

Mikosch, T. & C. Stărică (2000) Limit theory for the sample autocorrelations and extremes of a GARCH(1,1) process. *Annals of Statistics* 28, 1427–1451.

Mikosch, T. & C. Stărică (2004) Non-stationarities in financial time series, the long range dependence and the IGARCH effects. *Reviews of Economics and Statistics* 86, 378–390.

Nelson, D.B. (1990) Stationarity and persistence in the Garch(1,1) model. *Econometric Theory* 6, 318–334.

https://doi.org/10.1017/S0266466608090506 Published online by Cambridge University Press

Resnick, S.I. (1997) Heavy tail modeling and teletraffic data, with discussion and a rejoinder by the author. *Annals of Statistics* 25, 1805–1869.
Seneta, E. (1981) *Nonnegative Matrices and Markov Chains*. Springer-Verlag.

APPENDIX

The proof of Theorem 1 is based on the following result.

LEMMA 1. *$\rho(\mathbf{P}) = \rho(A)$, where A and $\mathbf{P}$ are defined as in (6) and (8), respectively.*

Proof. Indeed, because $\mathbf{P}$ is a nonnegative matrix, $\rho(\mathbf{P})$ is an eigenvalue of $\mathbf{P}$. Now, assume that $x = (x_1^\tau, \ldots, x_d^\tau)^\tau$ associated to the eigenvalue $\rho(\mathbf{P})$ is the eigenvector of $\mathbf{P}$, that is, $\mathbf{P}x = \rho(\mathbf{P})x$, where $x_k = (x_{k1}, \ldots, x_{ks})^\tau$, $k = 1, \ldots, d$. Let $y = (y_{11}^\tau, y_{12}^\tau, \ldots, y_{d1}^\tau, y_{d2}^\tau)^\tau$, where $y_{k1} = (x_{k1}, \ldots, x_{kq})^\tau$ and $y_{k2} = (x_{k1}, \ldots, x_{kp})^\tau$, $k = 1, \ldots, d$. It is easy to check that $Ay = \rho(\mathbf{P})y$. This shows that $\rho(\mathbf{P}) \le \rho(A)$.

Conversely, assume that $y^* = (y_1^*, \ldots, y_{dr}^*)^\tau$ associated to the eigenvalue $\rho(A)$ such that $Ay^* = \rho(A)y^*$. Then, it is easy to prove that $y_{kd+i}^* = y_{kd+q+i}^*$, where $i = 1, \ldots, s^*$, $s^* = \min(q, p)$, and $k = 0, \ldots, d-1$. Therefore, let $x_{ki}^* = y_{(k-1)d+i}^*$, $i = 1, \ldots, s$, when $s = q$, or $x_{ki}^* = y_{(k-1)d+q+i}^*$, $i = 1, \ldots, s$, when $s = p$, where $k = 1, \ldots, d$; we easily obtain that $\mathbf{P}x^* = \rho(A)x^*$, where $x^* = (x_1^{*\tau}, \ldots, x_d^{*\tau})^\tau$ and $x_k^* = (x_{k1}^*, \ldots, x_{ks}^*)^\tau$, $k = 1, \ldots, d$. This shows that $\rho(A) \le \rho(\mathbf{P})$. Thus, $\rho(\mathbf{P}) = \rho(A)$. ■

LEMMA 2. *For any $n \ge 1$,*

$$\mathrm{E}\left[A_0 A_{-1} \ldots A_{t-n+1} \middle| \Delta_{t-n}\right] = A^n Q_{t-n},$$

where $Q_t = \mathrm{diag}\,(\mathbf{1}_{\{\Delta_t=1\}} I_r, \mathbf{1}_{\{\Delta_t=2\}} I_r, \ldots, \mathbf{1}_{\{\Delta_t=d\}} I_r)$ and $\{A_t, t \in \mathbb{Z}\}$ and A are defined as in (5) and (6), respectively.

Proof. The proof of Lemma 2 can be easily established along the same lines as Lemma 1 in Liu (2006). ■

Proof of Theorem 1. Assume that $\rho(A) < 1$. Then, by Lemma 1, $\rho(\mathbf{P}) = \rho(A) < 1$. Therefore, the "if" part of the theorem follows from Theorem 2 in Francq et al. (2001).

Conversely, suppose that there exists a strictly stationary solution $\{\epsilon_t, t \in \mathbb{Z}\}$ of the MSG(d, q, p) process defined by (1) and (2) with finite variance. Iterating (7), we have, for any $m > 0$, that

$$\underline{\zeta}_0 = \underline{b}_0 + \sum_{n=1}^{m} A_0 A_{-1} \ldots A_{-n+1} \underline{b}_{-n} + A_0 A_{-1} \ldots A_{-m} \underline{\zeta}_{-m-1}.$$

Notice that all entries of $\underline{\zeta}_t$, A_t, and $\underline{b}_t$ are nonnegative. Therefore, for any $m > 0$,

$$\sum_{n=1}^{m} A_0 A_{-1} \ldots A_{-n+1} \underline{b}_{-n} \le \underline{\zeta}_0, \quad \text{a.s.},$$

https://doi.org/10.1017/S0266466608090506 Published online by Cambridge University Press

where the inequality is assumed to hold element-wise. This shows that the series

$$\sum_{n=1}^{m} \mathrm{E}\left[A_0 A_{-1} \dots A_{-n+1} \underline{b}_{-n}\right]$$

converges. Lemma 2 together with $Q_t \underline{b}_t = \underline{b}_t$ implies that

$$\mathrm{E}\left[A_0 A_{-1} \dots A_{-n+1} \underline{b}_{-n}\right] = A^n \underline{b},$$

where $\underline{b} = \mathrm{E}[\underline{b}_t] = (b_1^\tau, b_2^\tau, \dots, b_d^\tau)^\tau$, $b_i = \pi_i \omega(i)(1, 0, \dots, 0, 1, 0, \dots, 0)^\tau$, $i = 1, \dots, d$. It follows that

$$\lim_{n\to\infty} A^n \underline{b} = 0. \tag{A.1}$$

We will prove that, for $1 \le i \le dr$,

$$\lim_{n\to\infty} A^n e_i = 0, \tag{A.2}$$

where $\{e_i\}_{i=1}^{dr}$ is the canonical, orthonormal basis on $\mathbb{R}^{dr}$. Indeed, note that $\pi_i \omega(i) > 0$, $i = 1, \dots, d$. By (A.1), it is evident that, for $i = 1 + mr$ and $i = q + 1 + mr$, $m = 0, 1, \dots, d-1$,

$$\lim_{n\to\infty} A^n e_i = 0.$$

Thus, because, for $m = 0, 1, \dots, d-1$, $Ae_{q+mr} = \sum_{i=1}^{d} p_{m+1i} \alpha_q(i)(e_{1+(i-1)r} + e_{q+1+(i-1)r})$, we have

$$\lim_{n\to\infty} A^n e_{q+mr} = \sum_{i=1}^{d} p_{m+1i} \alpha_q(i) \lim_{n\to\infty} A^{n-1}(e_{1+(i-1)r} + e_{q+1+(i-1)r}) = 0.$$

Now, if (A.2) holds for all $i = i_m = l + mr$, $2 < l \le q$, $m = 0, 1, \dots, d-1$, then, using

$$Ae_{i_m - 1} = \sum_{j=1}^{d} p_{m+1j} \left[\alpha_{l-1}(j)(e_{1+(j-1)r} + e_{q+1+(j-1)r}) + e_{l+(j-1)r}\right],$$

we may get that

$$\lim_{n\to\infty} A^n e_{i_m - 1} = 0.$$

Thus, by a backward recursion, (A.2) is true for $1 + mr \le i \le q + mr$, $m = 0, 1, \dots, d-1$. In the same way, we can obtain that (A.2) holds for $q + 1 + mr \le i \le r + mr$, $m = 0, 1, \dots, d-1$. This shows that (A.2) is true for all $1 \le i \le dr$. Moreover, (A.2) yields

$$\lim_{n\to\infty} A^n = 0.$$

Then $\rho(A) < 1$. This completes the proof of the theorem. ■

Proof of Corollary 1. The assertion follows immediately from Lemma 1 and Theorem 1. ■

To prove the stationarity of the integrated MSG(d, q, p) process, we need some more notation defined as in Hennion (1997) and some lemmas. Consider the linear space $\mathbb{R}^m$

https://doi.org/10.1017/S0266466608090506 Published online by Cambridge University Press

endowed with its canonical basis $\{e_i, i = 1, \ldots, m\}$ with the scalar product $\langle \cdot, \cdot \rangle$ for which this basis is orthonormal and with the norm defined by

$$x \in \mathbb{R}^m, \qquad \|x\| = \sum_{i=1}^{m} |\langle x, e_i \rangle|.$$

Set

$$C = \{x : x \in \mathbb{R}^m,\ \langle x, e_i \rangle > 0,\ i = 1, \ldots, m\},$$

$$\overline{C} = \{x : x \in \mathbb{R}^m,\ \langle x, e_i \rangle \geq 0,\ i = 1, \ldots, m\};$$

$$B = C \cap \{x : x \in \mathbb{R}^m,\ \|x\| = 1\},\ \overline{B} = \overline{C} \cap \{x : x \in \mathbb{R}^m,\ \|x\| = 1\};$$

$$\|M\| = \sup\{\|Mx\| : x \in \overline{C},\ \|x\| = 1\}, \qquad \upsilon(M) = \inf\{\|Mx\| : x \in \overline{C},\ \|x\| = 1\},$$

$$M \cdot x = \frac{Mx}{\|Mx\|},$$

where M is an $m \times m$ matrix and $x \in \mathbb{R}^m$.

Assume that A defined as in (6) is irreducible and $\rho(A) = 1$. Then, according to the Perron–Frobenius theorem of the irreducible nonnegative matrix (see Seneta, 1981), there exists some $y_0 = (y_{11}^0, \ldots, y_{1r}^0, \ldots, y_{d1}^0, \ldots, y_{dr}^0)^\tau \in B$ such that

$$y_0^\tau A = y_0^\tau. \tag{A.3}$$

Let $T = \operatorname{diag}(y_{11}^0, \ldots, y_{1r}^0, \ldots, y_{d1}^0, \ldots, y_{dr}^0)^\tau$,

$$Y_n = \mathbf{1}_d^\tau \otimes I_r (T A_{-n} T^{-1}) \mathbf{1}_d \otimes I_r, \tag{A.4}$$

and $\mathcal{F}_n = \sigma(\Delta_n, \eta_n, \Delta_{n-1}, \eta_{n-1}, \ldots)$, where $\mathbf{1}_d$ is a d-dimensional column of ones, that is, $\mathbf{1}_d = (1, \ldots, 1)^\tau$, and A_{-n} is defined by (5), $n \in \mathbb{Z}$. On the other hand, for $y \in \overline{B}$, $m, n \in \mathbb{Z}$, $m \leq n$, write

$$Z_{n+1,n}^y = y, \qquad Z_{m,n}^y = (Y_m \ldots Y_n) \cdot y. \tag{A.5}$$

Using the same ideas as in the proof of Lemma 4 in Liu (2006), Lemma 2 and (A.3) lead to the following result.

LEMMA 3. *Assume that A defined as in (6) is irreducible and $\rho(A) = 1$. Then, for $n \geq 2$,*

$$\mathrm{E}[\|Y_1 Z_{2,n}^y\|] = 1,$$

where Y_1 and $Z_{2,n}^y$ are defined by (A.4) and (A.5), respectively.

In what follows, a sufficient condition is provided to ensure the irreducibility of A.

LEMMA 4. *Assume that $\alpha_i(k) > 0$ and $\beta_j(k) > 0$, $i = 1, \ldots, q$, $j = 1, \ldots, p$, $k = 1, \ldots, d$. Then A defined by (6) is irreducible.*

https://doi.org/10.1017/S0266466608090506 Published online by Cambridge University Press

Proof. Because $\alpha_i(k) > 0$ and $\beta_j(k) > 0$, $i = 1, \ldots, q$, $j = 1, \ldots, p$, $k = 1, \ldots, d$, from the definition of $\tilde{A}(k) = [a_{ij}(k)]$, we know that

$$a_{11}(k)a_{12}(k)\ldots a_{1r}(k) > 0 \quad \text{and} \quad a_{21}(k)a_{32}(k)\ldots a_{r,r-1}(k) > 0, \tag{A.6}$$

where $k = 1, \ldots, d$. Furthermore, note that the transition matrix $P = [p_{ij}]$ is irreducible and aperiodic. Therefore, for any $1 \le i_0, j_0 \le d$, there exist $i_{01}, \ldots, i_{0m}$ such that

$$p_{i_0 i_{01}} p_{i_{01} i_{02}} \cdots p_{i_{0m} j_0} > 0, \tag{A.7}$$

where $m := m(i_0, j_0)$. To prove the irreducibility of $A = [a_{ij}]$, it is sufficient that for any $1 \le i, j \le dr$, there exist $i_1, \ldots, i_n$ such that

$$a_{i i_1} a_{i_1 i_2} \ldots a_{i_n j} > 0, \tag{A.8}$$

where $n := n(i, j)$. Let $i = (j_0 - 1)r + l_1$ and $j = (i_0 - 1)r + l_2$, where $1 \le i_0, j_0 \le d$, and $1 \le l_1, l_2 \le r$.

We first consider the case of $l_1 = 1$, that is, $i = (j_0 - 1)r + 1$. In this case, by (A.6) and (A.7), it is easy to see that

$$a_{i,(i_{0m}-1)r+1} = p_{i_{0m} j_0} a_{11}(j_0) > 0,$$

$$a_{(i_{0m}-1)r+1,(i_{0,m-1}-1)r+1} = p_{i_{0,m-1} i_{0m}} a_{11}(i_{0m}) > 0, \ldots,$$

$$a_{(i_{02}-1)r+1,(i_{01}-1)r+1} = p_{i_{01} i_{02}} a_{11}(i_{02}) > 0, \qquad a_{(i_{01}-1)r+1, j} = p_{i_0 i_{01}} a_{1 l_2}(i_{01}) > 0.$$

This shows that (A.8) holds for the case of $i = (j_0 - 1)r + 1$, where $1 \le j_0 \le d$ and any $1 \le j \le dr$. Next, assume that (A.8) is true for $l_1 = k$, that is, $i = (j_0 - 1)r + k$ and any $1 \le j \le dr$ and $1 \le j_0 \le d$, where $1 \le k \le r - 1$. Then, because

$$a_{(j_0-1)r+k+1,(i_{0m}-1)r+k} = p_{i_{0m} j_0} a_{k+1,k}(j_0) > 0 \quad \text{and} \quad a_{(i_{0m}-1)r+k,i_1} a_{i_1 i_2} \ldots a_{i_n j} > 0,$$

we know that (A.8) also is true for $l_1 = k + 1$, that is, $i = (j_0 - 1)r + k + 1$, and any $1 \le j \le dr$, where $1 \le j_0 \le d$. This completes the proof of the lemma. ■

To apply the limit theorem in Hennion (1997), we also need the following lemma.

LEMMA 5. *Assume that the conditions in Lemma 4 are satisfied and $\eta_t \neq 0$ a.s. Then there exists n_0 such that, for any $n \ge n_0$,*

$$P(Y^{(n)} \in S^{\circ}) = 1, \tag{A.9}$$

where $Y^{(n)} = Y_1 Y_2 \ldots Y_n$ and S° denotes the set of $dr \times dr$ matrices all entries of which are strictly positive.

Proof. Note that

$$Y_n = \mathbf{1}_d^{\tau} \otimes I_r (T A_{-n} T^{-1}) \mathbf{1}_d \otimes I_r = \sum_{i=1}^{d} \sum_{j=1}^{d} \mathbf{1}_{\{\Delta_{-n}=i, \Delta_{-n-1}=j\}} T(i) C_{-n}(i) T^{-1}(j)$$

$$= T(\Delta_{-n}) C_{-n}(\Delta_{-n}) T^{-1}(\Delta_{-n-1}), \tag{A.10}$$

https://doi.org/10.1017/S0266466608090506 Published online by Cambridge University Press

where $T(k) = \operatorname{diag}(y_{k1}^0, \ldots, y_{kr}^0)$, $k = 1, \ldots, d$. Thus, by the definition of $C_{-n}(k)$, it is not hard to verify by straightforward calculations that, with probability one,

$$(Y_n)_{1l} = \eta_{-n}^2 \sum_{i=1}^{d} \sum_{j=1}^{d} \mathbf{1}_{\{\Delta_{-n}=i, \Delta_{-n-1}=j\}} y_{i1}^0 \alpha_l(i) (y_{jl}^0)^{-1} > 0, \qquad l = 1, \ldots, q,$$

$$(Y_n)_{1m} = \eta_{-n}^2 \sum_{i=1}^{d} \sum_{j=1}^{d} \mathbf{1}_{\{\Delta_{-n}=i, \Delta_{-n-1}=j\}} y_{i1}^0 \beta_{m-q}(i) (y_{jm}^0)^{-1} > 0, \qquad m = q+1, \ldots, r,$$

and $(Y_n)_{i+1,i} > 0$, $i = 1, \ldots, r-1$. This shows that all elements of the first row in $Y^{(1)} = Y_1$ are strictly positive with probability one and $(Y^{(1)})_{i+1,i} > 0$, a.s., $i = 1, \ldots, r-1$. Now, assume that $(Y^{(m)})_{ij} > 0$, a.s., $i = 1, \ldots, m$, $j = 1, \ldots, r$, and $(Y^{(m)})_{k+1,k} > 0$, a.s., $k = m, \ldots, r-1$, where $1 \le m \le r-1$. Because $Y^{(m+1)} = Y^{(m)} Y_{m+1}$, it is easy to obtain that $(Y^{(m+1)})_{ij} > 0$, a.s., $i = 1, \ldots, m+1$, $j = 1, \ldots, r$, and $(Y^{(m+1)})_{k+1,k} > 0$, a.s., $k = m+1, \ldots, r-1$, where $1 \le m \le r-1$. Therefore, taking $n_0 = r$, we have that (A.9) holds for $n \ge n_0$. ■

Now recall that the top Lyapunov exponent associated to the sequence $\{C_t(\Delta_t), t \in \mathbb{Z}\}$ defined by (3) of strictly stationary and ergodic random matrices is defined by

$$\gamma = \inf \left\{ \mathrm{E} \left[\frac{1}{n+1} \log \| C_0(\Delta_0) C_{-1}(\Delta_{-1}) \ldots C_{-n}(\Delta_{-n}) \| \right], n \in \mathbb{N} \right\},$$

when $\mathrm{E}\left[\log^+ \| C_0(\Delta_0) \|\right]$ is finite. Moreover, it is known that a.s.,

$$\gamma = \lim_{n \to \infty} \frac{1}{n} \log \| C_0(\Delta_0) C_{-1}(\Delta_{-1}) \ldots C_{-n}(\Delta_{-n}) \|$$

(see Kingman, 1973, Thm. 6). This shows that γ is independent of the chosen norm.

Along the lines of the work done by Bougerol and Picard (1992a) for standard GARCH models, using (4), we may obtain the following lemma.

LEMMA 6. *The MSG(d, q, p) process defined by (1) and (2) has a unique strictly stationary and ergodic solution if and only if the top Lyapunov exponent γ associated to the random matrices $\{C_t(\Delta_t), t \in \mathbb{Z}\}$ is strictly negative, where $\{C_t(\Delta_t), t \in \mathbb{Z}\}$ is defined as in (3).*

Proof of Theorem 2. Let $X_n = Y_n^\tau$; then $X^{(n)} = X_n X_{n-1} \ldots X_1 = Y^{(n)\tau}$, $n \ge 1$, where Y_n is defined by (A.4). Then, using (A.10),

$$\begin{aligned}
\gamma_1 &= \lim_{n \to \infty} \frac{1}{n} \log \| X^{(n)} \|_E = \lim_{n \to \infty} \frac{1}{n} \log \| Y^{(n)} \|_E \\
&= \lim_{n \to \infty} \frac{1}{n} \log \| T(\Delta_{-1}) C_{-1}(\Delta_{-1}) T^{-1}(\Delta_{-2}) \ldots T(\Delta_{-n}) C_{-n}(\Delta_{-n}) T^{-1}(\Delta_{-n-1}) \|_E \\
&= \lim_{n \to \infty} \frac{1}{n} \log \| T(\Delta_{-1}) C_{-1}(\Delta_{-1}) C_{-2}(\Delta_{-2}) \ldots C_{-n}(\Delta_{-n}) T^{-1}(\Delta_{-n-1}) \|_E \\
&= \lim_{n \to \infty} \frac{1}{n} \log \| C_{-1}(\Delta_{-1}) C_{-2}(\Delta_{-2}) \ldots C_{-n}(\Delta_{-n}) \|_E = \gamma,
\end{aligned}$$

https://doi.org/10.1017/S0266466608090506 Published online by Cambridge University Press

where the operator norm $\|\cdot\|_E$ on the set of $r \times r$ real matrices is associated with the euclidean norm on $\mathbb{R}^r$. Let us now examine the conditions in Hennion (1997). By Lemma 5, we know that the condition $(\mathcal{C})$ in Hennion (1997) is satisfied and

$$P\left(T(\omega) = \inf\{n : n \geq 1, X^{(n)} \in S^{\circ}\} \leq r\right) = 1.$$

And, note that

$$\delta = \min_{i,j}\{y^0_{i,q+1}\alpha_l(i)(y^0_{jl})^{-1}, y^0_{i,q+1}\beta_{m-q}(i)(y^0_{jm})^{-1}, \qquad l = 1, \ldots, q,$$

$$m = q+1, \ldots, r\} > 0$$

and $\sum_{j=1}^r (X_0)_{ij} \geq \delta$, $i = 1, \ldots, r$. Thus,

$$\begin{aligned} m_1 &= \mathrm{E}\left[|\log\|X_0\||\right] + \mathrm{E}\left[|\log v(X_0)|\right] \\ &= \mathrm{E}\left[|\log\|X_0\||\,\mathbf{1}_{\{\|X_0\|\leq 1\}}\right] + \mathrm{E}\left[|\log\|X_0\||\,\mathbf{1}_{\{\|X_0\|>1\}}\right] \\ &\quad + \mathrm{E}\left[|\log v(X_0)|\,\mathbf{1}_{\{v(X_0)\leq 1\}}\right] + \mathrm{E}\left[|\log v(X_0)|\,\mathbf{1}_{\{v(X_0)>1\}}\right] \\ &\leq 2|\log\delta| + 2\mathrm{E}\left[|\|X_0\| - 1|\right] < \infty. \end{aligned}$$

From Lemma 3.3 in Hennion (1997), there exists a stationary and ergodic sequence $\{Z_m, m \in \mathbb{Z}\}$ of random elements of B such that $Z_m \in \mathcal{F}_{-m}$,

$$Z_1 = \lim_{n\to\infty}(Y_1 \ldots Y_n)\cdot y \quad \text{and} \quad Z_m = \lim_{n\to\infty}(Y_m \ldots Y_n)\cdot y = Y_m \cdot Z_{m+1}, \tag{A.11}$$

where $y \in \overline{B}$. Therefore, (A.11) implies that the sequence $\{Z^y_{m+1,n}\}$ converges to the random element Z_{m+1} with probability one as $n \to \infty$, where $\{Z^y_{m+1,n}\}$ is defined as in (A.5).

We can write

$$\log\|Y^{(n)}y\| = \log\|Y^{(n-1)}(Y_n \cdot y)\| + \log\|Y_n y\| = \sum_{k=1}^n \log\|Y_k Z^y_{k+1,n}\|.$$

Therefore, by Theorem 2 and Lemma 5.1 in Hennion (1997),

$$\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^n \log\|Y_m Z^y_{m+1,n}\| = \lim_{n\to\infty}\frac{1}{n}\log\|Y^{(n)}y\| = \gamma_1.$$

Thus,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^n \log\|Y_m Z_{m+1}\| = \mathrm{E}\left[\log\|Y_1 Z_2\|\right] = \gamma_1. \tag{A.12}$$

But $\mathrm{E}\left[\|Y_1 Z_2\|\right] < \infty$ follows from $\|Y_1 Z_2\| \leq \|Y_1\|\|Z_2\| = \|Y_1\|$. Applying the dominated convergence theorem together with Lemma 3, we can obtain that

$$\mathrm{E}\left[\|Y_1 Z_2\|\right] = \lim_{n\to\infty}\mathrm{E}\left[\|Y_1 Z^y_{2,n}\|\right] = 1.$$

If $\|Y_1 Z_2\| = 1$ a.s., it is easy to prove that $\eta^2_{-1} \in \sigma(\Delta_{-1}) \vee \mathcal{F}_{-2}$, contradicting the definition of η_t. Therefore, $P(\|Y_1 Z_2\| = 1) < 1$. Thus, by (A.12),

$$\gamma = \gamma_1 = \mathrm{E}\left[\log\|Y_1 Z_2\|\right] < \log\mathrm{E}\left[\|Y_1 Z_2\|\right] = 0.$$

Lemma 6 implies the theorem. This completes the proof of the theorem. ■

https://doi.org/10.1017/S0266466608090506 Published online by Cambridge University Press