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The Vertex Degree Distribution of Passive Random Intersection Graph Models

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In a random passive intersection graph model the edges of the graph are decided by taking the union of a fixed number of cliques of random size. We give conditions for a random passive intersection graph model to have a limiting vertex degree distribution, in particular to have a Poisson limiting vertex degree distribution. We give related conditions which, in addition to implying a limiting vertex degree distribution, imply convergence of expectation.

1. Introduction

In a passive intersection model the edges of the graph are formed by taking the union of a fixed number of cliques, where the sizes of the cliques are random with some distribution and independent. In this paper we give necessary and sufficient conditions for the vertex degree of a fixed vertex to have a limiting distribution and to have a limiting Poisson distribution. Random passive intersection graph models were introduced in [2].

In defining the model of passive intersection graphs, it is helpful to first define a related model of bipartite graphs, which we denote by $\mathcal{BG}_{n,m}(n, \mathcal{P}_{(m)})$, where n and m are positive integers and $\mathcal{P}_{(m)} = (P_0, P_1, \ldots, P_m)$ is a probability distribution on the set of integers $[m] := \{0, 1, 2, \ldots, m\}$, *i.e.*, $P_0, P_1, \ldots, P_m$ are non-negative and $\sum_{j=0}^{m} P_j = 1$. In other papers [1, 3, 4, 7], the assumption is made that m grows like n^α for some constant $\alpha > 0$. We do not make that assumption here, but impose the weaker assumption that $m = m(n)$ satisfies

$$m \to \infty \quad \text{as } n \to \infty. \tag{1.1}$$

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The probabilities P_j implicitly depend on n. The independent sets of $\mathcal{BG}_{n,m}(n, \mathcal{P}_{(m)})$ are denoted by $\mathcal{V}$ and $\mathcal{W}$ and the cardinalities of $\mathcal{V}$ and $\mathcal{W}$ are given by $|\mathcal{V}| = n$ and $|\mathcal{W}| = m$, respectively.

In the model $\mathcal{BG}_{n,m}(n, \mathcal{P}_{(m)})$ a random graph is chosen according to the following procedure.

(1) The vertex degrees of the elements in the independent set $\mathcal{V}$ are determined by i.i.d. random variables $X(v)$, $v \in \mathcal{V}$ with common probability distribution $\mathcal{P}_{(m)} = (P_0, P_1, \ldots, P_m)$.

(2) Given the values of $X(v)$, the set of neighbours $\Gamma(v) \subseteq \mathcal{W}$ of v is determined by choosing a subset $\Gamma(v)$ in such a way that

$$\mathbb{P}(\Gamma(v) = \mathcal{S} \mid X(v) = k) = \binom{m}{k}^{-1}$$

for all $\mathcal{S} \subseteq [m]$ of cardinality $k = X(v)$. The subsets are chosen independently of one another.

Thus, given $\mathcal{S} \subseteq [m]$,

$$\mathbb{P}(\Gamma(v) = \mathcal{S}) = \frac{P_{|\mathcal{S}|}}{\binom{m}{|\mathcal{S}|}}$$

and the $\Gamma(v)$ are mutually independent random elements. This procedure determines $\mathcal{BG}_{n,m}(n, \mathcal{P}_{(m)})$ uniquely and, in particular, the neighbours of vertices in $\mathcal{W}$ are given by $\Gamma(w) = \{v \in \mathcal{V} : w \in \Gamma(v)\}$. By construction, we have $X(v) = |\Gamma(v)|$. Moreover, for $w \in \mathcal{W}$ we let $Y(w)$ denote $Y(w) = |\Gamma(w)|$.

The passive graph model corresponding to $\mathcal{BG}_{n,m}(n, \mathcal{P}_{(m)})$ defined in [2] has vertex set $\mathcal{W}$ and vertices $w_1, w_2 \in \mathcal{W}$ are adjacent if and only if $\Gamma(w_1) \cap \Gamma(w_2) \neq \emptyset$. We denote the passive random intersection graph model by $\mathcal{IG}_n^{\mathrm{p}}(\mathcal{P}_{(m)})$. A dual notion of intersection graphs, the active intersection graphs, was also defined in [2] and the vertex degree of a typical vertex in those models was studied in [3].

We will give conditions under which the vertex degree of a typical vertex in $\mathcal{IG}_n^{\mathrm{p}}(\mathcal{P}_{(m)})$ has a limiting distribution. Fix a vertex $w \in \mathcal{W}$. We would like to know about the distribution of the degree of w, which we denote by ψ. In the classic $\mathcal{G}(n, p)$ model of random graphs where each edge occurs independently and with probability p, when $p = c/n$ for a constant $c > 0$, the degree of a typical vertex is asymptotically Poisson(c). It is natural to ask when ψ has limiting Poisson distribution in $\mathcal{IG}_n^{\mathrm{p}}(\mathcal{P}_{(m)})$. A start in this direction was made in [3].

Theorem 1.1. *Given a real number $c > 0$, the probability density function $\mathbb{P}(\psi = k)$ converges to $\frac{e^{-c}c^k}{k!}$ as $n \to \infty$ for each $k \geqslant 0$ if the probability distribution $\mathcal{P}_{(m)}$ satisfies*

(i) $\frac{n}{m} \sum_{j=3}^{m} j^2 P_j = o(1)$,

(ii) $P_2 \sim \frac{cm}{2n}$ for some constant $c > 0$.

The idea behind the theorem is that, under the assumptions on the P_j, most of the cliques are of size at most 2. The cliques of size 2 contribute exactly one edge to $\mathcal{IG}_n^{\mathrm{p}}(\mathcal{P}_{(m)})$ while the cliques of size 0 or 1 do not contribute any edges. The result is that edges appear in an approximately independent manner and the assumption on P_2 ensures that the limiting degree is Poisson.

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In this paper we determine conditions under which the limiting distribution of ψ is Poisson and whether other limiting distributions for ψ can occur. Theorem 1.2 addresses the convergence of ψ to a Poisson distribution with no assumption of convergence of $\mathbb{E}\psi$.

Theorem 1.2. *Given a real number $c > 0$, the probability density function $\mathbb{P}(\psi = k)$ converges to $\frac{e^{-c}c^k}{k!}$ as $n \to \infty$ for each $k \geqslant 0$ if and only if the probability distribution $\mathcal{P}_{(m)}$ satisfies*

(i) $\frac{n}{m}\sum_{j=3}^{m} jP_j = o(1)$,

(ii) $P_2 \sim \frac{cm}{2n}$.

The next theorem gives conditions for the convergence of ψ to a Poisson distribution and the associated convergence of $\mathbb{E}\psi$.

Theorem 1.3. *Given a real number $c > 0$, $\lim_{n\to\infty} \mathbb{E}\psi = c$ and the probability density function $\mathbb{P}(\psi = k)$ converges to $\frac{e^{-c}c^k}{k!}$ as $n \to \infty$ for each $k \geqslant 0$ if and only if the probability distribution $\mathcal{P}_{(m)}$ satisfies*

(i) $\frac{n}{m}\sum_{j=3}^{m} j^2P_j = o(1)$,

(ii) $P_2 \sim \frac{cm}{2n}$.

Note that the conditions of Theorem 1.3 implying convergence in distribution and expectation to Poisson are exactly those of Theorem 1.1.

We give conditions for the convergence of $\mathbb{P}(\psi = k)$ for each integer k in the following theorem.

Theorem 1.4. *The probability density function $\mathbb{P}(\psi = k)$ converges as $n \to \infty$ for each $k \geqslant 0$ and $\lim_{n\to\infty} \mathbb{P}(\psi = 0) > 0$ if and only if the probability distribution $\mathcal{P}_{(m)}$ satisfies*

(i) $\frac{n}{m}\sum_{j=2}^{m} jP_j$ *converges as* $n \to \infty$,

(ii) $\frac{n}{m}P_j$ *converges for each* $j \geqslant 2$.

It is possible that $\mathbb{P}(\psi = k)$ converges but not to a probability distribution. Consider $\mathcal{P}_{(m)}$ defined for a sequence of positive integers $d = d(n)$ by

$$P_j = \begin{cases} 1 - \frac{m}{nd} & \text{if } j = 0, \\ \frac{m}{nd} & \text{if } j = d, \\ 0 & \text{if } j \notin \{0, d\}. \end{cases} \tag{1.2}$$

The conditions of Theorem 1.4 are satisfied for this choice of $\mathcal{P}_{(m)}$. Either $w \notin \Gamma(v)$ for all $v \in \mathcal{V}$, in which case $\psi = 0$, or $w \in \Gamma(v)$ for at least one $v \in \mathcal{V}$, in which case $\psi \geqslant d(n)$. If $d = m$, then

$$\mathbb{P}(\psi = 0) = P_0^n = \left(1 - n^{-1}\right)^n \to e^{-1}$$

and so

$$\mathbb{P}(\psi = m) \to 1 - e^{-1},$$

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hence, for each $k \geqslant 0$

$$\mathbb{P}(\psi = k) \to \begin{cases} e^{-1} & \text{if } k = 0, \\ 0 & \text{if } k > 0. \end{cases}$$

If we consider (1.2) with d fixed, then we get a variant of Poisson convergence. For a given vertex $w \in \mathcal{W}$, let Z be the number of vertices $u \in \mathcal{W} \setminus \{w\}$ such that there exist vertices v_1 and v_2 from $\mathcal{V}$ and w, u are both in $\Gamma(v_1) \cap \Gamma(v_2)$. Then

$$\mathbb{P}(Z > 0) \leqslant \mathbb{E}Z \leqslant (m-1)\binom{n}{2}\left(P_d \frac{\binom{m-2}{d-2}}{\binom{m}{d}}\right)^2 = O\left(\frac{1}{m}\right) = o(1).$$

Therefore, with probability tending to one the sets $\{\Gamma(v) \setminus \{w\} : w \in \Gamma(v)\}$ are mutually disjoint and $\mathbb{P}(\psi = k) \to 0$ if k is not of the form $k = j(d-1)$ for some non-negative integer j. Moreover, we have

$$\begin{aligned} \mathbb{P}(\psi = j(d-1)) &\sim \binom{n}{j}\left(P_d \frac{\binom{m-1}{d-1}}{\binom{m}{d}}\right)^j \left(1 - P_d \frac{\binom{m-1}{d-1}}{\binom{m}{d}}\right)^{n-j} \\ &\sim \frac{n^j}{j!}\left(\frac{c}{n}\right)^j \left(1 - \frac{c}{n}\right)^{n-j} \\ &\to e^{-c}\frac{c^j}{j!}. \end{aligned}$$

To ensure that ψ converges to a probability distribution, it suffices that the $\mathbb{E}\psi$ converges as well as $\mathbb{P}(\psi = k)$ due to tightness. We are able to prove this companion result to Theorem 1.4.

Theorem 1.5. *The probability density function $\mathbb{P}(\psi = k)$ converges for all $k \geqslant 0$, the expectation $\mathbb{E}\psi$ converges as $n \to \infty$, and $\lim_{n\to\infty} \mathbb{P}(\psi = 0) > 0$ if and only if the probability distribution $\mathcal{P}_{(m)}$ satisfies*

(i) $\frac{n}{m}\sum_{j=2}^{m} jP_j$ *converges as* $n \to \infty$,

(ii) $\frac{n}{m}\sum_{j=2}^{m} j^2P_j$ *converges as* $n \to \infty$,

(iii) $\frac{n}{m}P_j$ *converges for each* $j \geqslant 2$.

Note that if condition (iii) is satisfied with $\frac{n}{m}P_j$ converging to constants C_j and condition (ii) is satisfied with $\frac{n}{m}\sum_{j=2}^{m} j^2P_j$ converging to $\sum_{j=2}^{m} j^2C_j$, then condition (i) is automatically satisfied with $\frac{n}{m}\sum_{j=2}^{m} jP_j$ converging to $\sum_{j=2}^{m} jC_j$ by the general dominated convergence theorem (p. 232 of [5]).

It would be interesting to have a classification of the possible limiting distributions of ψ in Theorems 1.4 and 1.5.

In view of how the exponent of j in our various theorems changes according to whether $\mathbb{E}\psi$ is required to converge or not, we propose the following conjectures.

Conjecture. Given a real number $c > 0$, $\mathbb{E}\psi^r$ converges to the rth moment of a Poisson(c) distributed random variable for each r, $1 \leqslant r \leqslant N$, $N \geqslant 1$, and the probability density function $\mathbb{P}(\psi = k)$ converges to $\frac{e^{-c}c^k}{k!}$ as $n \to \infty$ for each $k \geqslant 0$ if and only if the probability distribution

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$\mathcal{P}_{(m)}$ satisfies

(i) $\frac{n}{m}\sum_{j=3}^{m} j^{N+1}P_j = o(1)$,

(ii) $P_2 \sim \frac{cm}{2n}$.

Conjecture. The rth moment $\mathbb{E}\psi^r$ converges for each r, $1 \leqslant r \leqslant N$, $N \geqslant 1$, and the probability density function $\mathbb{P}(\psi = k)$ converges for each $k \geqslant 0$, and $\lim_{n\to\infty} \mathbb{P}(\psi = 0) > 0$, if and only if the probability distribution $\mathcal{P}_{(m)}$ satisfies

(i) $\frac{n}{m}\sum_{j=3}^{m} j^{r+1}P_j$ converges for each r in the range $1 \leqslant r \leqslant N$,

(ii) $\frac{n}{m}P_2$ converges.

In this paper we have assumed that $\mathbb{P}(\psi = 0)$ converges to a positive number, in which case any limiting distributions have positive total mass. We do not know whether it is possible for $\lim_{n\to\infty} \mathbb{P}(\psi = 0) = 0$ and also to have ψ converge to a distribution having positive total mass. It would also be interesting to have a further characterization of the possible limiting distributions in Theorem 1.4. These questions may be addressed in future research.

We present some preliminary results in Section 2. We then prove Theorems 1.2 and 1.3 in Section 3 and prove Theorems 1.4 and 1.5 in Section 4.

2. Preliminaries

We define μ to be

$$\mu = \sum_{j=1}^{m} jP_j. \tag{2.1}$$

The starting point of our analysis is the following lemma from [3], which was essentially stated in [2].

Lemma 2.1. *Let ψ be a random variable with the distribution of the vertex degree of a typical vertex w in $\mathcal{IG}_n^{\mathrm{p}}(\mathcal{P}_{(m)})$. Then*

$$\mathbb{P}(\psi = k) = \binom{m-1}{k}\sum_{s=0}^{k}\binom{k}{s}(-1)^s\left(1 - \frac{\mu}{m} + \sum_{j=0}^{k-s+1} jP_j\frac{(m-j)_{m-k+s-1}}{(m)_{m-k+s}}\right)^n. \tag{2.2}$$

Moreover, the expected vertex degree is

$$\mathbb{E}\psi = (m-1)\left[1 - \left(1 - \sum_{j=0}^{m} P_j\frac{j(j-1)}{m(m-1)}\right)^n\right]. \tag{2.3}$$

Essentially, we will expand the internal sum in (2.2) in a power series and use the following lemma to eliminate most of the terms.

Lemma 2.2. *For non-negative integers k and r, we have*

$$\sum_{s=0}^{k}(-1)^s\binom{k}{s}s^r = \begin{cases} (-1)^k k! & \text{if } r = k, \\ 0 & \text{if } r < k. \end{cases}$$

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Proof. It is well known that

$$\sum_{s=0}^{k}(-1)^s\binom{k}{s}s^r=(-1)^k k!S_{r,k},$$

where $S_{r,k}$ are the Stirling numbers of the second kind. Using $S_{k,k}=1$ and the fact that for $r<k$, $S_{r,k}=0$ proves the lemma. □

3. Convergence of vertex degree to Poisson

In this section we prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. First assume that $\mathcal{P}_{(m)}$ satisfies conditions (i) and (ii) of the theorem. Fixing k, since

$$\frac{(m-j)_{m-k+s-1}}{(m)_{m-k+s}}=\frac{(m-j)\cdot\ldots\cdot(k-s+1)(k-s)\cdot\ldots\cdot(k-s+2-j)}{m\cdot\ldots\cdot(m-j+1)(m-j)\cdot\ldots\cdot(k-s+1)}=\frac{(k-s)_{j-1}}{(m)_j},$$

we can rewrite (2.2) as

$$\begin{aligned}\mathbb{P}(\psi=k)&=\binom{m-1}{k}\sum_{s=0}^{k}\binom{k}{s}(-1)^s\left(1-\frac{\mu}{m}+\sum_{j=0}^{k-s+1}jP_j\frac{(k-s)_{j-1}}{(m)_j}\right)^n\\&=\binom{m-1}{k}\sum_{s=0}^{k}\binom{k}{s}(-1)^s\left(1+\frac{P_1-\mu}{m}+\sum_{j=2}^{k+1}jP_j\frac{(k-s)_{j-1}}{(m)_j}\right)^n,\end{aligned}$$

where we have used the fact that $(k-s)_{j-1}=0$ if $j>k-s+1$ in the last step. By expanding $\left(1+\frac{P_1-\mu}{m}+\sum_{j=2}^{k-s+1}jP_j\frac{(k-s)_{j-1}}{(m)_j}\right)^n$ in a Taylor series of order k about $1+\frac{P_1-\mu}{m}$, we get

$$\mathbb{P}(\psi=k)=\binom{m-1}{k}\sum_{s=0}^{k}\binom{k}{s}(-1)^s\sum_{i=0}^{k}\binom{n}{i}\left(1+\frac{P_1-\mu}{m}\right)^{n-i}\left(\sum_{j=2}^{k+1}jP_j\frac{(k-s)_{j-1}}{(m)_j}\right)^i+\Delta,\tag{3.1}$$

where

$$\Delta=\binom{m-1}{k}\sum_{s=0}^{k}\binom{k}{s}(-1)^s\binom{n}{k+1}\left(1+\frac{P_1-\mu}{m}\right)^{n-k-1}\xi_s^{k+1}$$

for some ξ_s in the range

$$0\leqslant\xi_s\leqslant\sum_{j=2}^{k+1}jP_j\frac{(k-s)_{j-1}}{(m)_j}.$$

By our assumptions on $\mathcal{P}_{(m)}$, we know that $P_j=O(m/n)$ for all $j\geqslant 1$, and therefore

$$\left(1+\frac{P_1-\mu}{m}\right)^{n-k-1}=O(1)$$

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and

$$\sum_{j=2}^{k+1} jP_j \frac{(k-s)_{j-1}}{(m)_j} = O\left(\frac{1}{mn}\right)$$

uniformly for all s in the range $0 \leqslant s \leqslant k$. We therefore have

$$\Delta = O\left(m^k n^{k+1} (mn)^{-k-1}\right) = O\left(m^{-1}\right) = o(1),$$

where the last step holds because of (1.1).

Now we expand the expression to the power i in (3.1) in a multinomial series, obtaining

$$\begin{aligned}
\mathbb{P}(\psi = k) &= \binom{m-1}{k} \sum_{s=0}^{k} \binom{k}{s} (-1)^s \sum_{i=0}^{k} \binom{n}{i} \left(1 + \frac{P_1 - \mu}{m}\right)^{n-i} \\
&\quad \times \left\{ \sum_{\substack{(\eta_2,\ldots,\eta_{k+1}):\\ \eta_2+\cdots+\eta_{k+1}=i}} \binom{i}{\eta_2,\ldots,\eta_{k+1}} \prod_{j=2}^{k+1} \left(jP_j \frac{(k-s)_{j-1}}{(m)_j} \right)^{\eta_j} \right\} + o(1) \\
&= \binom{m-1}{k} \sum_{i=0}^{k} \binom{n}{i} \left(1 + \frac{P_1 - \mu}{m}\right)^{n-i} \\
&\quad \times \left\{ \sum_{\substack{(\eta_2,\ldots,\eta_{k+1}):\\ \eta_2+\cdots+\eta_{k+1}=i}} \binom{i}{\eta_2,\ldots,\eta_{k+1}} \sum_{s=0}^{k} \binom{k}{s} (-1)^s \prod_{j=2}^{k+1} \left(jP_j \frac{(k-s)_{j-1}}{(m)_j} \right)^{\eta_j} \right\} + o(1).
\end{aligned} \tag{3.2}$$

Notice that all sums and products in (3.2) are finite.

Consider $\boldsymbol{\eta} := (\eta_1, \ldots, \eta_k)$ such that $\sum_{j=2}^{k+1} (j-1)\eta_j < k$. By expanding $(k-s)_{j-1}$ in polynomials in the variable s of order $j-1$ and using Lemma 2.2 it follows that the contribution to $\mathbb{P}(\psi = k)$ from such $\boldsymbol{\eta}$ is exactly zero.

Consider $\boldsymbol{\eta} := (\eta_1, \ldots, \eta_k)$ such that $\sum_{j=2}^{k+1} (j-1)\eta_j > k$. By our assumptions on $\mathcal{P}_{(m)}$, $P_j = O(m/n)$ if $j \geqslant 2$. For such $\boldsymbol{\eta}$, the product in (3.2) is of order

$$\begin{aligned}
\prod_{j=2}^{k+1} \left(jP_j \frac{(k-s)_{j-1}}{(m)_j} \right)^{\eta_j} &= O\left(\prod_{j=2}^{k+1} \left(nm^{j-1} \right)^{-\eta_j} \right) \\
&= O\left(n^{-i} m^{-\sum_{j=2}^{k+1} (j-1)\eta_j} \right) \\
&= O\left(n^{-i} m^{-k-1} \right).
\end{aligned}$$

Such $\boldsymbol{\eta}$ contribute terms of order $O(m^{-1}) = o(1)$ to $\mathbb{P}(\psi = k)$.

Consider $\boldsymbol{\eta} := (\eta_1, \ldots, \eta_k)$ such that $\sum_{j=2}^{k+1} (j-1)\eta_j = k$ and such that $\eta_j > 0$ for some $j \geqslant 3$. By our assumptions on $\mathcal{P}_{(m)}$, $P_j = o(m/n)$ if $j \geqslant 3$. For such $\boldsymbol{\eta}$, the product in (3.2) is of order

$$o\left(n^{-i} m^{-\sum_{j=2}^{k+1} (j-1)\eta_j} \right) = o\left(n^{-i} m^{-k} \right).$$

Such $\boldsymbol{\eta}$ contribute terms of order $o(1)$ to $\mathbb{P}(\psi = k)$.

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Finally, consider $\boldsymbol{\eta}$ for which $\eta_2 = k$ and $\eta_j = 0$ for $j \geqslant 3$. For such $\boldsymbol{\eta}$, we have $i = k$, and the contribution of this term to $\mathbb{P}(\psi = k)$ is, by Lemma 2.2,

$$\begin{aligned}
&\binom{m-1}{k}\binom{n}{k}\left(1+\frac{P_1-\mu}{m}\right)^{n-k}\sum_{s=0}^{k}\binom{k}{s}(-1)^s\left(\frac{2P_2(k-s)}{m(m-1)}\right)^k \\
&\quad= \binom{m-1}{k}\binom{n}{k}\left(1+\frac{P_1-\mu}{m}\right)^{n-k}\left(\frac{2P_2}{m(m-1)}\right)^k k! \\
&\quad\sim \frac{m^k n^k}{(k!)^2}e^{-c}\frac{c^k}{m^k n^k}k! \\
&\quad\sim \frac{e^{-c}c^k}{k!}.
\end{aligned}$$

We have shown that $\lim_{n\to\infty}\mathbb{P}(\psi = k) = e^{-c}c^k/k!$.

Now assume that $\lim_{n\to\infty}\mathbb{P}(\psi = k) = e^{-c}c^k/k!$ for all $k \geqslant 0$. Taking $k = 0$, we have

$$\mathbb{P}(\psi = 0) = \left(1-\frac{\mu-P_1}{m}\right)^n \to e^{-c} \quad \text{as } n\to\infty, \tag{3.3}$$

hence

$$\frac{n}{m}(\mu - P_1) \to c \quad \text{as } n\to\infty. \tag{3.4}$$

Taking $k = 1$ gives

$$\begin{aligned}
\mathbb{P}(\psi = 1) &= (m-1)\left[\left(1-\frac{\mu-P_1}{m}+\frac{2P_2}{m(m-1)}\right)^n - \left(1-\frac{\mu-P_1}{m}\right)^n\right] \\
&= (m-1)n\left(1-\frac{\mu-P_1}{m}+\xi\right)^{n-1}\frac{2P_2}{m(m-1)},
\end{aligned}$$

where ξ is a number in the range $0 \leqslant \xi \leqslant 2P_2/m(m-1)$. It follows that

$$\lim_{n\to\infty}\frac{2P_2 n}{m}e^{-c} = ce^{-c},$$

giving condition (ii) of the theorem. Finally, from condition (ii) and (3.4) we deduce condition (i). □

Proof of Theorem 1.3. Assume that conditions (i) and (ii) of the theorem hold. Then, the conditions of Theorem 1.2 are satisfied and $\lim_{n\to\infty}\mathbb{P}(\psi = k) = e^{-c}\frac{c^k}{k!}$ for all $k \geqslant 0$. From (2.3), we have

$$\mathbb{E}\psi = (m-1)\left[1-\left(1-\frac{(P_1+2P_2-\mu)+2P_2}{m(m-1)}-\frac{1}{m(m-1)}\sum_{j=3}^{m}j^2P_j\right)^n\right]. \tag{3.5}$$

Since $\mu - P_1 - 2P_2 = \sum_{j=3}^{m} jP_j = o\left(\frac{m}{n}\right)$, we conclude that $\lim_{n\to\infty}\mathbb{E}\psi = c$.

Now suppose that $\lim_{n\to\infty}\mathbb{E}\psi = c$ and that $\lim_{n\to\infty}\mathbb{P}(\psi = k) = e^{-c}\frac{c^k}{k!}$ for all $k \geqslant 0$. We know from Theorem 1.2 that $\mu - P_1 - 2P_2 = o\left(\frac{m}{n}\right)$ and that $P_2 \sim \frac{cm}{2n}$ (condition (ii)). It now follows from (3.5) that condition (i) also holds. □

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4. Convergence of vertex degree

In this section we prove Theorems 1.4 and 1.5.

Proof of Theorem 1.4. Suppose that conditions (i) and (ii) of the theorem hold. Let λ and C_j, $j \geqslant 2$ be non-negative constants such that

$$\lambda = \lim_{n\to\infty} \frac{n}{m} \sum_{j=2}^{m} jP_j$$

and

$$C_j = \lim_{n\to\infty} \frac{n}{m} P_j.$$

As $P_j = O(m/n)$ for all $j \geqslant 1$, the argument in the proof of Theorem 1.2 applies and (3.2) is valid. Terms with $\boldsymbol{\eta}$ for which $\sum_{j=2}^{k+1}(j-1)\eta_j < k$ contribute 0 to $\mathbb{P}(\psi = k)$, as in Theorem 1.2. Terms with $\boldsymbol{\eta}$ for which $\sum_{j=2}^{k+1}(j-1)\eta_j > k$ contribute $o(1)$ to $\mathbb{P}(\psi = k)$, as in Theorem 1.2. Now consider $\boldsymbol{\eta}$ for which $\sum_{j=2}^{k+1}(j-1)\eta_j = k$ and let $i = \sum_{j=2}^{k+1}\eta_j$. Such terms contribute

$$\begin{aligned}
&\binom{m-1}{k}\binom{n}{i}\left(1+\frac{P_1-\mu}{m}\right)^{n-i}\binom{i}{\eta_2,\dots,\eta_{k+1}}\sum_{s=0}^{k}\binom{k}{s}(-1)^s\prod_{j=2}^{k+1}\left(jP_j\frac{(k-s)_{j-1}}{(m)_j}\right)^{\eta_j}\\
&\quad\sim \frac{m^k n^i e^{-\lambda}}{k!\,i!}\binom{i}{\eta_2,\dots,\eta_{k+1}}\sum_{s=0}^{k}\binom{k}{s}(-1)^s\prod_{j=2}^{k+1}\left(jC_j\frac{(k-s)_{j-1}}{nm^{j-1}}\right)^{\eta_j}\\
&\quad\sim \frac{e^{-\lambda}}{k!\,i!}\binom{i}{\eta_2,\dots,\eta_{k+1}}\sum_{s=0}^{k}\binom{k}{s}(-1)^s\prod_{j=2}^{k+1}\left(jC_j(k-s)_{j-1}\right)^{\eta_j}
\end{aligned}$$

to $\mathbb{P}(\psi = k)$. The contributions from terms corresponding to all possible $\boldsymbol{\eta}$ are convergent. Therefore $\mathbb{P}(\psi = k)$ converges for each $k > 0$. Moreover, $\lim_{n\to\infty}\mathbb{P}(\psi = 0) \to e^{-\lambda} > 0$ because of (3.3).

Now suppose that $\mathbb{P}(\psi = k)$ converges for each $k \geqslant 0$ and let $\psi_0 := \lim_{n\to\infty}\mathbb{P}(\psi = 0) > 0$. Then (3.3) gives us

$$\frac{n}{m}(\mu - P_1) \to -\log\psi_0 \quad \text{as } n \to \infty$$

showing that condition (i) holds. Suppose now that condition (ii) does not hold and let j_0 be the least integer greater than 1 such that $\frac{n}{m}P_{j_0}$ does not converge. We will derive a contradiction.

Condition (i) implies that $P_j = O(m/n)$ for all $j \geqslant 2$, and so the expansion (3.2) with k chosen to be $k = j_0 - 1$ is valid. Terms with $\boldsymbol{\eta}$ for which $\sum_{j=2}^{j_0}(j-1)\eta_j < j_0 - 1$ contribute 0 to $\mathbb{P}(\psi = j_0 - 1)$ (which converges by assumption) as in Theorem 1.2. Terms with $\boldsymbol{\eta}$ for which $\sum_{j=2}^{j_0}(j-1)\eta_j > j_0 - 1$ contribute $o(1)$ to $\mathbb{P}(\psi = j_0 - 1)$, as in Theorem 1.2.

Consider terms corresponding to $\boldsymbol{\eta}$ for which

$$\sum_{j=2}^{j_0}(j-1)\eta_j = j_0 - 1. \tag{4.1}$$

If $\eta_{j_0} = 0$, then (3.2) does not involve P_j for $j \geqslant j_0$ and so by the definition of j_0 the contributions to $\mathbb{P}(\psi = j_0)$ for those η_{j_0} are convergent. Therefore, we may assume that $\eta_{j_0} > 0$, which implies

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that $\eta_{j_0} = 1$ and $\eta_j = 0$ for all other $j \geqslant 2$. This is the unique $\boldsymbol{\eta}$ whose contribution to $\mathbb{P}(\psi = j_0)$ involves P_{j_0}. Call it $\boldsymbol{\eta}^*$. For $\boldsymbol{\eta}^*$, $i = \sum_{j=2}^{j_0} \eta_j = 1$.

Now consider the contributions to $\mathbb{P}(\psi = j_0)$ from terms corresponding to $\boldsymbol{\eta}^*$ with the various s fixed. If $s > 0$, then the term corresponding to $\boldsymbol{\eta}$ and s does not involve P_j for $j \geqslant j_0$ because $(j_0 - 1 - s)_{j_0 - 1} = 0$.

Therefore, only the contribution corresponding to $\boldsymbol{\eta}^*$ and $s = 0$ can possibly be divergent. The contribution from this term is

$$\binom{m-1}{j_0-1} n \left(1 - \frac{\mu - P_1}{m}\right)^{n-1} \left(\frac{j_0! P_{j_0}}{(m)_{j_0}}\right) \sim j_0 \psi_0 \frac{n}{m} P_{j_0}.$$

The convergence of $\mathbb{P}(\psi = j_0 - 1)$ now implies that nP_{j_0}/m converges, contradicting the definition of j_0. We have shown that condition (ii) must hold. □

Proof of Theorem 1.5. Assume that conditions (i), (ii) and (iii) of the theorem hold. Then, the conditions of Theorem 1.2 are satisfied as well and $\lim_{n\to\infty} \mathbb{P}(\psi = k)$ exists for all $k \geqslant 0$. Using (3.5) we find that $\lim_{n\to\infty} \mathbb{E}\psi$ also exists.

Now suppose that $\lim_{n\to\infty} \mathbb{E}\psi$ exists, that $\lim_{n\to\infty} \mathbb{P}(\psi = k)$ exists for all $k \geqslant 0$ and that $\lim_{n\to\infty} \mathbb{P}(\psi = 0) > 0$. We know from Theorem 1.2 that condition (i) holds and condition (iii) holds. In particular, $\frac{n}{m} P_2$ converges. It now follows from (3.5) that condition (ii) also holds. □

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