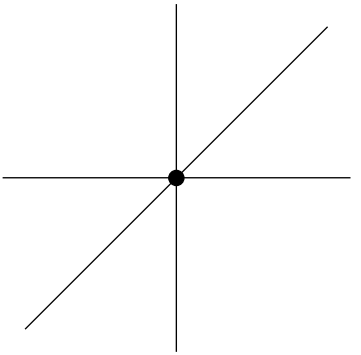
Teaching Note

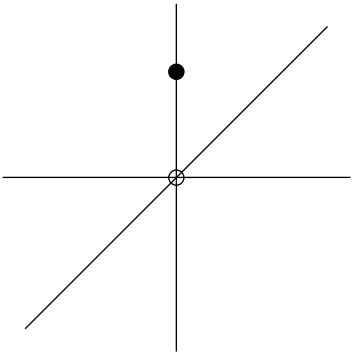
Geometric Discontinuities

Intuitively, a continuous real function $f(x)$ is one whose graph can be drawn without lifting the pen from the paper. More precisely, it is a function whose value at any point $x = a$, $f(a)$, is equal to the limit of $f(x)$ as a approaches a. Many of the functions students come across are continuous, so much so that they may come to regard discontinuous functions as rare or even unnatural. In fact, nothing could be further from the truth, but on the other hand, naturally occurring discontinuous real functions are not easy to find. In this note we present some easy and – we hope – convincing examples from geometry and trigonometry.



As a start, consider the function $f(x) = x^2/x$ for $x \neq 0$. Now, this formula does not define $f(0)$, so let's define $f(0)$ to be 0. This gives a continuous function $\mathbb{R} \to \mathbb{R}$, because $\lim_{x \to a} f(x) = f(a)$, for all $x \in \mathbb{R}$. Its graph looks like this:

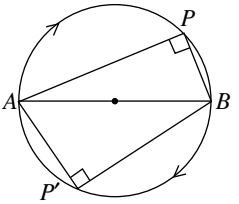
Now let's define another function $g(x) = x^2/x$, and $g(0) = 1$ (we are quite entitled to do this!). The graph of $g(x)$ looks like this:



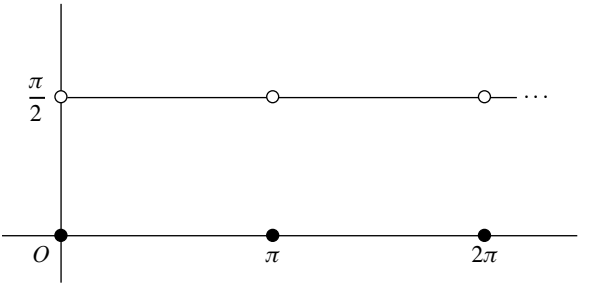
We say that $g(x)$ has a 'jump' discontinuity at $x = 0$. [A solid circle stands for the function value and a blank circle stands for a gap.]

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"But," a student may object, "you're just making it up as you go along; show me a naturally occurring example." – a fair point.

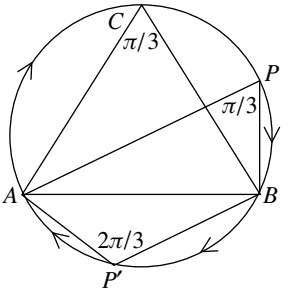


Consider a circle with radius 1 and a fixed diameter AB. A point P moves along the circumference of the circle in a clockwise direction, starting at A. We plot the size of the angle $\angle APB$ against the distance travelled by the point P. Since the angle in a semi-circle is a right angle, the graph looks like this:



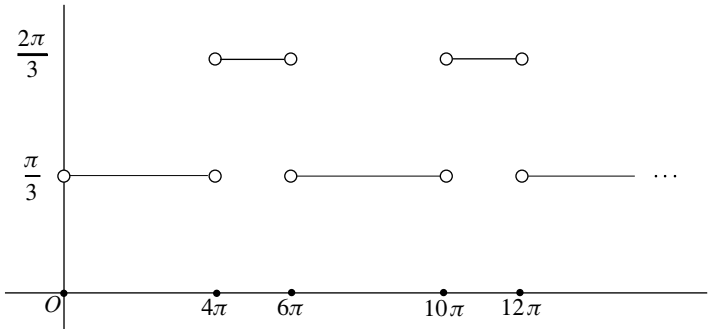
This function has jump discontinuities at $x = 0, \pi, 2\pi, 3\pi, \ldots$.

For what is perhaps a more convincing example, consider an equilateral triangle ABC, inscribed in a circle of radius 3, where the chord AB is fixed.

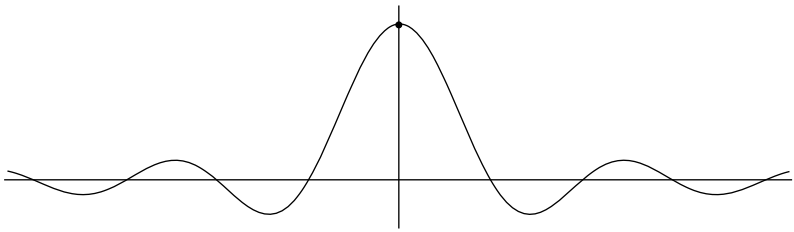


Again, a point P moves around the circumference of the circle in a clockwise direction, starting at A. We plot the size of the angle $\angle APB$ against the distance travelled by the point P. Now, since the opposite angles of a cyclic quadrilateral sum to π, this time the graph looks like this:

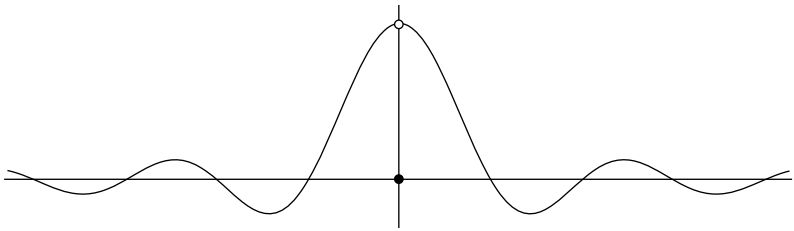
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Moving on to trigonometry, we consider the familiar function $f(x) = \dfrac{\sin x}{x}$, for $x \neq 0$. Since $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$, if we define $f(0)$ to be 1, we get a continuous function whose graph looks like this:



But if for some (perverse?) reason we define $f(0)$ to be 0, we get a discontinuous function with a jump discontinuity at $x = 0$. Now the graph looks like this:

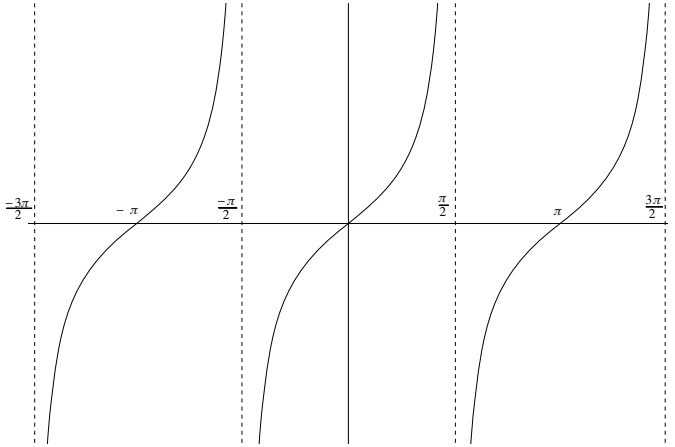


We include this example here because many students seem to believe that $\dfrac{\sin 0}{0}$ has value 1 rather than being a limit.

The trigonometric function $f(x) = \tan x$ (not defined for $x = \frac{1}{2}\pi$ for example) has an infinite jump discontinuity at $x = \frac{1}{2}\pi$ (and $x = \pm(2n+1)\frac{\pi}{2}$ and there is no way to define f (/2) to make the function continuous.

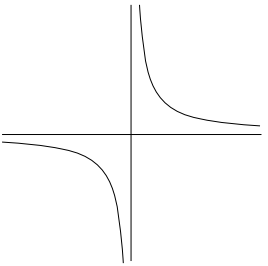
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The graph looks like this:



Here the function jumps from very large positive values to very large negative values, or from minus infinity to plus infinity, as some would say.

The graphs of $f(x) = \cot x$, $f(x) = \sec x$ and $f(x) = \operatorname{cosec} x$, should now be examined and their discontinuities described.

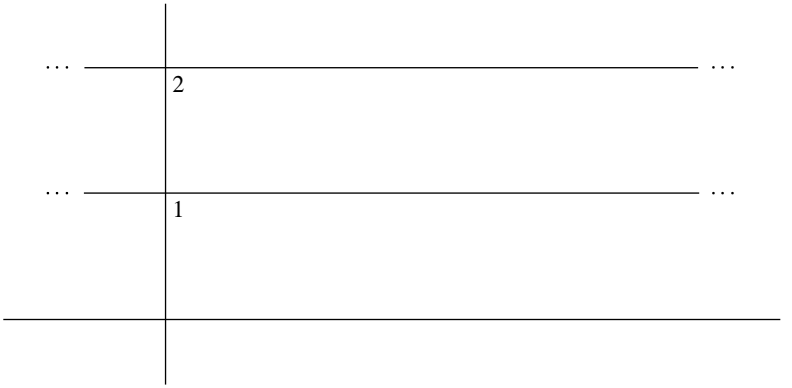


This is the graph of $y = \frac{1}{x}$ (a rectangular hyperbola) which is interesting because it appears to be discontinuous about the axes which are its asymptotes.

The function $f(x) = \frac{1}{x^2}$, $x \neq 0$ has an interesting discontinuity at $x = 0$ and this should be examined too.

Finally, the teacher should break the news gently to the class that there are perfectly natural and well-defined real functions that are discontinuous for *every* value of x. An example is $f(x)$ given by $f(x) = 2$ if x is a rational number and $f(x) = 1$ if x is an irrational number. Its graph is virtually impossible to draw, but to the naked eye it might look something like this:

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Feedback

On [1]: Paul Belcher writes: The Note talks about a *shear-like* transformation that preserves area. The term *shear-like* is not defined and actually the transformation given is a vertical stretch with a factor of 2. This does not preserve area but multiplies it by 2. However as only half of the circle has been transformed the integral given for π is correct.

To obtain the result given (not using the double angle formula), we do not have to use geometrical transformations. If we take the area enclosed between $y = +\sqrt{1 - x^2}$ and the x-axis it is half a circle, so $\frac{1}{2}\pi = \int_{-1}^{1} y\,dx$. Using parametric coordinates $x = \cos\theta$, $y = \sin\theta$ and integration by substitution with $\frac{dx}{d\theta} = -\sin\theta$ then

$$\frac{\pi}{2} = \int_{\pi}^{0} \sin\theta\,(-\sin\theta)\,d\theta = \int_{0}^{\pi} \sin^2\theta\,d\theta$$

The algebra is essentially that shown in the Teaching Note.

Reference

1. Colin Foster, Teaching Note $\int_0^{\pi} \sin^2\theta\,d\theta$ from a shear-like transformation of a circle, *Math. Gaz.* **108** (November 2024) pp. 541-542.

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