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GROUPS WITH AT MOST 13 NONPOWER SUBGROUPS

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Abstract

For a group G and $m \geq 1$, G^m denotes the subgroup generated by the elements g^m, where g runs through G. The subgroups not of the form G^m are called nonpower subgroups. We extend the classification of groups with few nonpower subgroups from groups with at most nine nonpower subgroups to groups with at most 13 nonpower subgroups.

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1. Introduction

For a group G and $m \geq 1$, the *power* subgroup G^m is the subgroup generated by the elements g^m, where g runs through G. A subgroup that is not a power subgroup is a *nonpower* subgroup. Let $\mathrm{ps}(G)$ and $\mathrm{nps}(G)$ denote the number of power and nonpower subgroups of G. It is immediate that every power subgroup is a characteristic subgroup of G.

In 1956, Szász [12] proved that G is cyclic if and only if $\mathrm{nps}(G) = 0$. In 2006, Zhou *et al.* [16] showed that a noncyclic group is finite if and only if it has a finite number of nonpower subgroups, and it was observed that a finite noncyclic group must have at least three nonpower subgroups. Then, Anabanti *et al.* [1, 2] classified the groups with three or four nonpower subgroups. We summarise these results in the following theorem. (See Section 2 for definitions of the groups.)

THEOREM 1.1. *For $0 \leq k \leq 4$, a group has exactly k nonpower subgroups if and only if, up to isomorphism, it is one of the following:*

$k = 0$ *a cyclic group;*
$k = 1$ *no examples;*
$k = 2$ *no examples;*
$k = 3$ $C_2 \times C_2$, Q_8 *or* $G_{n,3}$ *for* $n \geq 1$*;*
$k = 4$ $C_3 \times C_3$.

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This theorem, combined with the following theorem from our previous paper [15], classifies groups with at most nine nonpower subgroups.

THEOREM 1.2. *For $5 \le k \le 9$, a group has exactly k nonpower subgroups if and only if, up to isomorphism, it is one of the following:*

$k = 5$ $C_2 \times C_4$ *or* $G_{n,5}$ *for* $n \ge 1$;
$k = 6$ $C_5 \times C_5$, $C_2 \times C_2 \times C_p$, $Q_8 \times C_p$, *where* $p > 2$ *is a prime, or* $G_{n,3} \times C_q$ *for* $n \ge 1$, *where* $q > 3$ *is a prime;*
$k = 7$ D_8, Alt(4), $C_2 \times C_8$, Q_{16}, $M_{4,2}$, $C_3 \times C_9$, $M_{3,3}$, $G_{n,7}$ *or* $F_{n,7}$ *for* $n \ge 1$;
$k = 8$ $C_7 \times C_7$ *or* $C_3 \times C_3 \times C_p$, *where* $p \ne 3$ *is a prime;*
$k = 9$ $C_2 \times C_{16}$, $M_{5,2}$, $C_2 \times C_2 \times C_{p^2}$, $Q_8 \times C_{p^2}$, *where* $p > 2$ *is a prime, or* $G_{n,3} \times C_{q^2}$, *where* $q > 3$ *is a prime.*

In the present paper, we extend the classification to groups with at most 13 nonpower subgroups.

THEOREM 1.3. *For $10 \le k \le 13$, a group has exactly k nonpower subgroups if and only if, up to isomorphism, it is one of the following:*

$k = 10$ $C_3 \times C_{27}$, $C_2 \times C_4 \times C_p$, $M_{4,3}$, $\mathrm{Sym}(3) \times C_3$, $A_2 = (C_2 \times C_2) \rtimes C_9$, $G^{(2)}_{2,n;\,5,1}$ *for* $n \ge 2$ *or* $G_{n,5} \times C_q$ *for* $n \ge 1$, *where* $p \ne 2$ *and* $q \ne 2, 5$ *are primes;*
$k = 11$ $C_2 \times C_{32}$, $C_5 \times C_{25}$, S_{16}, $M_{6,2}$, $M_{3,5}$, $\mathrm{SL}(2,3)$, $G_{n,11}$ *or* $G^{(3)}_{5,n;\,11,1}$ *for* $n \ge 1$;
$k = 12$ $C_4 \times C_4$, $C_{11} \times C_{11}$, $Q_8 \times C_{qr}$, $C_2 \times C_2 \times C_{qr}$, $C_3 \times C_3 \times C_{s^2}$, $C_5 \times C_5 \times C_p$, $B^2_{2,2}$, $G_{n,9}$ *or* $G_{n,3} \times C_{qr}$ *for* $n \ge 1$, *where* p, q, r *and* s *are primes such that* $p \ne 5$, $3 < q < r$ *and* $s \ne 3$;
$k = 13$ $C_2 \times C_{64}$, $C_3 \times C_{81}$, $M_{7,2}$, $M_{5,3}$, $\mathrm{Sym}(3) \times C_2 = D_{12}$, $C_3 \rtimes Q_8$, $A_3 = (C_2 \times C_2) \rtimes C_{27}$, $G_{n,13}$ *or* $F_{n,13}$ *for* $n \ge 1$.

2. Notation and definitions

All groups considered in this paper are finite. Let $\Phi(G)$ denote the Frattini subgroup of G. For subgroups H and K of G, let $[H, K]$ be the subgroup generated by the commutators $[x, y] = x^{-1}y^{-1}xy$ with $x \in H$ and $y \in K$.

Let G be a p-group. For all $i \ge 1$, $\Omega_i(G)$ denotes the subgroup $\langle x \in G \mid x^{p^i} = 1 \rangle$. (In general, the exponent of $\Omega_i(G)$ may be greater than p^i.)

We use Gorenstein [6, Ch. 5] as a reference for standard results about p-groups. Statements of most of the key lemmas from [6] used in the proof of the main theorem can be found in our previous paper [15]. Alternatively, see [9, Ch. III].

Let C_n denote the cyclic group of order n, and let $\mathrm{Alt}(n)$ and $\mathrm{Sym}(n)$ denote the alternating and symmetric groups of a set of size n. Let $\mathrm{SL}(2,3)$ denote the group of 2×2 matrices of determinant 1 over the field of three elements.

We use the notation $G = N \rtimes K$ to mean that G is a semidirect product of N by K. The action of K on N will be determined by the context. A group G is *metacyclic* if it has a normal cyclic subgroup N such that G/N is cyclic.

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Almost all the groups that occur in Theorems 1.1, 1.2 and 1.3 are metacyclic or the direct product of a metacyclic group and a cyclic group. Many of these groups can be described as a semidirect product $C_{q^m} \rtimes C_{p^n}$ of cyclic groups of prime power orders. Metacyclic groups have been classified by Hempel [8]. The special case that we need is given by the following definition.

DEFINITION 2.1. For primes p and q, positive integers m, n and an integer r such that $r^{p^n} \equiv 1 \pmod{q^m}$, let $G^{(r)}_{p,n;\,q,m}$ denote the group with presentation

$$\langle a, b \mid a^{p^n} = 1,\ b^{q^m} = 1,\ a^{-1}ba = b^r \rangle.$$

If p^k is the order of $r \pmod{q^m}$, the automorphism of $\langle b \rangle$ induced by conjugation by a has order p^k; therefore, if $p \neq q$, then p^k divides $q - 1$ and if the group is nonabelian, then p divides $q - 1$.

REMARK 2.2. For certain values of the parameters, the groups $G^{(r)}_{p,n;q,m}$ have standard names (see [6]). Where possible, we prefer to use these names.

(i) $G^{(1)}_{p,n;\,q,m}$ is the direct product $C_{p^n} \times C_{q^m}$.

(ii) $G^{(-1)}_{2,1;\,q,m}$ is the *dihedral group* D_{2q^m} of order $2q^m$.

(iii) For $n \geq 4$, $G^{(-1+2^{n-2})}_{2,1;\,2,n-1}$ is the *semidihedral* group S_{2^n} of order 2^n.

(iv) For $n \geq 4$, when $p = 2$, and for $n \geq 3$, when p is an odd prime, $G^{(1+p^{n-2})}_{p,1;\,p,n-1}$ is the *quasidihedral* group $M_{n,p}$ of order p^n ($M_{3,p}$ is the *extraspecial* group of order p^3 and exponent p^2).

REMARK 2.3. Because of the frequency with which the groups occur in the theorems and for compatibility with [15], we use an abbreviated notation for the following special cases. For $n \geq 1$ and a prime p:

(i) G_{n,p^m} denotes $G^{(-1)}_{2,n;\,p,m}$;

(ii) $F_{n,p}$ denotes $G^{(r)}_{3,n;\,p,1}$, where the order of $r \pmod{p}$ is 3 and hence, $p \equiv 1 \pmod{3}$;

(iii) $B^2_{n,p}$ denotes $G^{(p+1)}_{p,n;\,p,2}$ and we have $B^2_{1,2} = D_8$, $B^2_{2,2}$ is the semidirect product $C_4 \rtimes C_4$ and, for p odd, $B^2_{1,p} = M_{3,p}$.

For completeness, we include definitions of two other families of groups with standard names and two families of groups (without standard names), which occur in the proof of the main theorem.

DEFINITION 2.4.

(i) For $n \geq 3$, $\langle a, b \mid a^{2^{n-1}} = b^2 = z,\ z^2 = 1,\ b^{-1}ab = a^{-1} \rangle$ is a presentation for the *generalised quaternion* group Q_{2^n} of order 2^n.

(ii) For an odd prime p, $\langle x, y, z \mid x^p = y^p = z^p = 1,\ [x, z] = [y, z] = 1,\ [x, y] = z \rangle$ is a presentation for the *extraspecial* group $M(p)$ of order p^3 and exponent p.

(iii) For $n \geq 1$, $\langle a, b, c \mid a^{3^n} = b^2 = 1,\ bc = cb,\ b^a = c,\ c^a = bc \rangle$ is a presentation for the group $A_n = (C_2 \times C_2) \rtimes C_{3^n}$ of order $2^2 3^n$ (for example, $A_1 \simeq \mathrm{Alt}(4)$).

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(iv) For $n \geq 1$ and a prime p,

$$\langle a, b, c \mid [a, b] = c,\ a^p = b^{p^n} = c^p = 1,\ [a, c] = [b, c] = 1 \rangle$$

is a presentation for the group $B^1_{n,p}$ of order p^{n+2}. Except for $B^1_{1,2} = D_8$, it is nonmetacyclic (see [4, Lemma 2.5]). The quotient mod $\langle c \rangle$ is $C_p \times C_{p^n}$ and for p odd, $B^1_{1,p} = M(p)$.

3. Preliminaries

First, we prove a general result about cyclic subgroups of prime power order.

THEOREM 3.1. *For all primes p and finite groups G, if m divides the exponent e of G and G^m is cyclic of order p^k, then $m = e/p^k$. In particular, for all k, G has at most one cyclic power subgroup of order p^k.*

PROOF. Suppose that for a divisor m of e, the power subgroup G^m is cyclic of order p^k. Let $q \neq p$ be a prime divisor of $|G|$ and let y be an element of order q^ℓ, where q^ℓ is the exponent of a Sylow q-subgroup. If q^h is the highest power of q that divides m, there exist integers s and t such that $q^h = sq^\ell + tm$. Therefore, $y^{q^h} = y^{tm} \in G^m$; thus, $y^{q^h} = 1$ and hence, $h = \ell$. This proves that $m = p^j d$ for some j, where d is the product of the exponents of the Sylow q-subgroups with $q \neq p$. Therefore, for all $g \in G$, g^d is a p-element and it follows that p^{k+j} is the exponent of the Sylow p-subgroups of G. Thus, $m = e/p^k$. □

COROLLARY 3.2. *Let p^f be the largest order of a p-element of G, where p is a prime divisor of $|G|$. Let p^k be the largest order of a cyclic power p-subgroup of G. If p is odd and a Sylow p-subgroup is not cyclic, then G contains at least $pf - k + 1$ cyclic nonpower p-subgroups.*

PROOF. Let P be a Sylow p-subgroup of G and suppose that P is not cyclic. By a theorem of Burnside [5, Section 105], for all m where $1 \leq m \leq f$, P contains more than one subgroup of order p^m. Miller [11] proved that for $m > 1$, the number of cyclic subgroups of order p^m is divisible by p and the number of subgroups of order p is congruent to $1 + p$ modulo p^2 (see also Berkovich [3, Theorem 1.10]). Therefore, P has at least p subgroups of order p^i for $2 \leq i \leq f$ and at least $p + 1$ subgroups of order p. Furthermore, from Theorem 3.1, P has exactly one cyclic power subgroup of order p^i for $1 \leq i \leq k$. Therefore, G has at least $p + (k-1)(p-1) + p(f-k) = pf - k + 1$ cyclic nonpower p-subgroups. □

The following two lemmas play an essential role in the proof of the main theorem.

LEMMA 3.3 [1, Lemma 3]. *If A and B are finite groups such that $|A|$ and $|B|$ are coprime, then $\mathrm{ps}(A \times B) = \mathrm{ps}(A)\,\mathrm{ps}(B)$ and $\mathrm{nps}(A \times B) = \mathrm{nps}(A)s(B) + \mathrm{ps}(A)\,\mathrm{nps}(B)$, where $s(B)$ is the number of subgroups of B.*

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LEMMA 3.4 [16, Lemma 2]. *Suppose N and H are subgroups of G such that $N \trianglelefteq G$. If HN/N is a nonpower subgroup of G/N, then H is a nonpower subgroup of G. Therefore,* $\operatorname{nps}(G) \geq \operatorname{nps}(G/N)$.

REMARK 3.5. This lemma was stated in [15] with the additional condition $N \subseteq H$. However, the same proof shows that it remains true without this restriction. It can often be used to show that $\operatorname{nps}(G) \geq 2\operatorname{nps}(G/N)$. For example, suppose that $|N|$ is coprime to $|G/N|$. From the Schur–Zassenhaus theorem [6, Theorem 6.2.1], G is a semidirect product $N \rtimes K$. In this case, if $H \subseteq K$ and if HN/N is a nonpower subgroup of G/N, then both H and HN are nonpower subgroups of G. Since $H \neq HN$, it follows that $\operatorname{nps}(G) \geq 2\operatorname{nps}(G/N)$.

LEMMA 3.6 [15, Theorem 2.12]. *There is no finite p-group G such that $G/N \simeq M_{n,p}$, where N is a central subgroup of G of order p contained in G'.*

The following lemma from [15] is a consequence of the work of Blackburn [4, Lemma 2.2 and Theorem 2.3].

LEMMA 3.7. *For a nonabelian p-group G generated by two elements, let $R = \Phi(G')G_3$, where $G_3 = [[G, G], G]$. Then:*

(i) *R is the only maximal subgroup of G' that is normal in G;*
(ii) *G is metacyclic if and only if G/R is metacyclic;*
(iii) *if the type of G/G' is (p, p^n) and G/R has no cyclic maximal subgroup, then G/R is isomorphic to $B^1_{n,p}$ or $B^2_{n,p}$.*

4. Nonpower values

Formulae for the number of nonpower subgroups for many families of groups are provided by [15, Section 3]. For convenience, we summarise this information in the following lemma, then prove formulae for the additional values of $\operatorname{nps}(G)$ needed in the proof of the main result. The formulae for $\operatorname{nps}(M_{n,p})$, $\operatorname{nps}(G_{n,p^k})$ and $\operatorname{nps}(F_{n,p})$ are special cases of Lemmas 4.2 and 4.3 below.

LEMMA 4.1. *For an integer n and a prime p:*

(i) *for $n \geq 3$, $\operatorname{nps}(D_{2^n}) = 2^n - 1$;*
(ii) *for $n \geq 3$, $\operatorname{nps}(Q_{2^n}) = 2^{n-1} - 1$;*
(iii) *for $n \geq 4$, $\operatorname{nps}(S_{2^n}) = 3 \cdot 2^{n-2} - 1$;*
(iv) *for $n \geq 3$, $\operatorname{nps}(M_{n,p}) = p(n-1) + 1$ (when $p = 2$, assume $n \geq 4$);*
(v) *$\operatorname{nps}(M(p)) = p^2 + 2p + 2$;*
(vi) *if $p > 2$, then $\operatorname{nps}(G_{n,p^k}) = p(p^k - 1)/(p - 1)$;*
(vii) *if $p \equiv 1 \pmod 3$, then $\operatorname{nps}(F_{n,p}) = p$;*
(viii) *for $n \geq 1$, $\operatorname{nps}(A_n) = 3n + 4$;*

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(ix) [13, Theorem 3.3] *for* $n_2 \geq n_1 \geq 1$ *and a prime* p *the value of* $\mathrm{nps}(C_{p^{n_1}} \times C_{p^{n_2}})$ *is*

$$\frac{(n_2 - n_1 + 1)p^{n_1+2} - (n_2 - n_1 - 1)p^{n_1+1} - (n_2 + n_1 + 3)p + (n_2 + n_1 + 1)}{(p-1)^2} - (n_2 + 1).$$

LEMMA 4.2. *For a prime* $p > 2$, $\mathrm{nps}(G^{(r)}_{p,n;p,m}) = \mathrm{nps}(C_{p^n} \times C_{p^m})$.

PROOF. Let $G = G^{(r)}_{p,n;p,m}$. Since G' is cyclic, from [9, III, Section 10], G is a regular p-group. Then, from [7, Theorem 4.21], G^{p^s} is the set $\{ g^{p^s} \mid g \in G \}$. Thus, there are exactly $\max(n, m) + 1$ power subgroups in G. This means that $\mathrm{ps}(G) = \mathrm{ps}(C_{p^n} \times C_{p^m})$. It follows from [10, Proposition 1] that G is lattice-isomorphic to $C_{p^n} \times C_{p^m}$. Thus, $s(G) = s(C_{p^n} \times C_{p^m})$ and, therefore, $\mathrm{nps}(G) = \mathrm{nps}(C_{p^n} \times C_{p^m})$. □

From items (i) and (ix) of Lemma 4.1, for $p = 2$, the conclusion of the Lemma 4.2 is not valid.

LEMMA 4.3. *Suppose that* $p \neq q$ *are primes, and that* m, n *and* r *are positive integers such that* $r \neq 1$ *and* $r^{p^n} \equiv 1 \pmod{q^m}$. *Let* $G = G^{(r)}_{p,n;q,m}$ *and let* p^k *be the order of* $r \pmod{q^m}$. *Then,* $p \mid q - 1$ *and* $\mathrm{nps}(G) = kq(q^m - 1)/(q - 1)$.

PROOF. For $0 \leq i < k$ and $0 \leq j < m$, let $P_i = \langle a^{p^i} \rangle$, $Q_j = \langle b^{q^{m-j}} \rangle$ and $H_{ij} = P_i Q_j$. From the presentation of G, we have $Q_j \trianglelefteq G$ and $Z(G) = \langle a^{p^k} \rangle$. Furthermore, $P_0 \in \mathrm{Syl}_p(G)$ and G is the semidirect product $Q \rtimes P_0$, where $Q \in \mathrm{Syl}_q(G)$. It is clear that $P_0 \subseteq N_G(P_i)$ and we claim that for $0 \leq i < k$, $N_G(P_i) = P_0$. Indeed, if $x \in Q \cap N_G(P_i)$, then $[P_i, \langle x \rangle] \subseteq P_i \cap Q = 1$; that is, $x \in C_G(P_i)$. It follows from [6, Theorem 5.2.4] that $x \in C_G(P_0)$ and hence, $x \in Z(G) \cap Q = 1$. This establishes the claim.

We also have $P_0 \subseteq N_G(H_{ij})$ and it follows from the Frattini argument that $N_G(H_{ij}) = Q_j N_G(P_i) = Q_j \rtimes P_0$. Therefore, H_{ij} has q^{m-j} conjugates in G and since for $0 \leq i < k$ and $0 \leq j < m$ every subgroup of order $p^{n-i}q^j$ is conjugate to H_{ij}, we have found $k \sum_{j=0}^{m-1} q^{m-j} = kq(q^m - 1)/(q - 1)$ subgroups, all of which are nonpower subgroups. For $1 \leq i \leq n$, the power subgroup $G^{p^{n-i}}$ is the unique subgroup of order $p^i q^m$. For $k \leq s \leq n$ and $0 \leq j < m$, the power subgroup $G^{p^s q^{m-j}}$ is the unique subgroup of order $p^{n-s}q^j$. This accounts for all subgroups of G. Therefore, $\mathrm{nps}(G) = kq(q^m - 1)/(q - 1)$. □

LEMMA 4.4. *For* $n \geq 2$*:*

(i) $\mathrm{nps}(B^1_{n,p}) = p^2(2n - 1) + p(n + 1) + 2$*; and*
(ii) $\mathrm{nps}(B^2_{n,p}) = p^2(n - 1) + p(n + 1) + 2$.

PROOF. There are $n + 1$ power subgroups of $B^1_{n,p}$ and $B^2_{n,p}$. We count all the subgroups of them by considering their exponents.

Notice that $\Omega_{n-1}(B^1_{n,p}) = \langle a \rangle \times \langle c \rangle \times \langle b^p \rangle \simeq C_p \times C_p \times C_{p^{n-1}}$. By [14, Corollary 2.2], the number of subgroups of exponent at most p^{n-1} is $s(C_p \times C_p \times C_{p^{n-1}}) = 2(n - 1)p^2 + np + n + 2$. Furthermore, there are $p^2 + p + 1$ subgroups of exponent p^n. Thus, $\mathrm{nps}(B^1_{n,p}) = p^2(2n - 1) + p(n + 1) + 2$.

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Similar to $B^1_{n,p}$, $\Omega_{n-1}(B^2_{n,p}) = \langle a\rangle \times \langle b^p\rangle \simeq C_{p^2} \times C_{p^{n-1}}$. From [14, Corollary 2.2], the number of subgroups of exponent at most p^{n-1} is $s(C_{p^2} \times C_{p^{n-1}}) = (n-2)p^2 + pn + n + 2$. There are also $p^2 + p + 1$ subgroups of exponent p^n. Thus, $\mathrm{nps}(B^2_{n,p}) = p^2(n-1) + p(n+1) + 2$. □

LEMMA 4.5. *Let* $G = G_{m,3} \times \underbrace{C_3 \times \cdots \times C_3}_{n}$. *For* $n \geq 1$, *we have* $\mathrm{nps}(G) \geq 4m + 6$ *and equality holds when* $n = 1$.

PROOF. Similar to the calculation of $\mathrm{nps}(G_{n,3})$, we have $\mathrm{nps}(G_{m,3} \times C_3) = 4m + 6$. As n increases, the number of subgroups will increase, but the number of power subgroups does not increase. Thus, $\mathrm{nps}(G) \geq 4m + 6$. This completes the proof. □

LEMMA 4.6. *Let* $G = Q_8 \times \underbrace{C_2 \times \cdots \times C_2}_{n}$. *For* $n \geq 1$, *we have* $\mathrm{nps}(G) \geq 16$ *and equality holds when* $n = 1$.

PROOF. For all $n \geq 1$, the exponent of G is 4 and the only proper nontrivial power subgroup is G^2 of order 2. The group $Q_8 \times C_2$ has 19 subgroups, consequently, $\mathrm{nps}(Q_8 \times C_2) = 16$ and $\mathrm{nps}(G) \geq 16$ for $n \geq 1$. □

LEMMA 4.7. *By direct calculation,* $\mathrm{nps}(\mathrm{Sym}(4)) = 26$, $\mathrm{nps}(\mathrm{SL}(2,3)) = 11$ *and* $\mathrm{nps}(C_3 \rtimes Q_8) = 13$.

LEMMA 4.8 [15, Lemma 3.3]. *Let* $G = D_{2p} \times \underbrace{C_2 \times \cdots \times C_2}_{n}$. *For* $n \geq 1$ *and a prime* $p > 2$, *we have* $\mathrm{nps}(G) \geq 3p + 4$ *and equality holds when* $n = 1$.

LEMMA 4.9 [15, Lemma 3.4]. *Let* $X_{n,p} = D_{2p} \times \underbrace{C_3 \times \cdots \times C_3}_{n}$. *For* $n \geq 1$ *and a prime* $p > 3$, *we have* $\mathrm{nps}(X_{n,p}) = (p+3)s(C_3^n) - 6 \geq 10$, *where* $C_3^n = \underbrace{C_3 \times \cdots \times C_3}_{n}$. *For* $n > 2$, $\mathrm{nps}(X_{n,3}) > \mathrm{nps}(X_{2,3}) = 48$ *and* $\mathrm{nps}(X_{1,3}) = 10$.

5. Proof of Theorem 1.3

LEMMA 5.1. *Let* P *be a Sylow* p*-subgroup of a group* G *such that* $P \neq N_G(P) \neq G$. *Then,* $10 \leq \mathrm{nps}(G) \leq 13$ *if and only if for some* $n \geq 1$, *one of the following holds:*

(1) $\mathrm{nps}(G) = 10$, $p = 2$ *and* G *is isomorphic to* $\mathrm{Sym}(3) \times C_3$ *or to* $G_{n,5} \times C_r$, *where* $r \neq 2, 5$ *is a prime;*
(2) $\mathrm{nps}(G) = 11$, $p = 3$ *and* G *is isomorphic to* $\mathrm{SL}(2,3)$*;*
(3) $\mathrm{nps}(G) = 12$, $p = 2$ *and* G *is isomorphic to* $G_{n,3} \times C_{qr}$, *where* q *and* r *are primes such that* $3 < q < r$.

In all cases, if $Q \in \mathrm{Syl}_q(G)$ *and* $q \neq p$, *then* $Q \trianglelefteq G$.

PROOF. Since $P \neq N_G(P) \neq G$, both P and $N_G(P)$ have at least $p + 1$ conjugates, and so $2(p+1) \leq \mathrm{nps}(G) \leq 13$; hence, p is 2, 3 or 5.

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First, we show that $p \neq 5$. Suppose to the contrary that $p = 5$. Then, the six conjugates of P and the six conjugates of $N_G(P)$ are nonpower subgroups; hence, every other subgroup is normal. In particular, if $Q \in \mathrm{Syl}_3(G)$, then $Q \trianglelefteq G$. The permutation action of G on $\mathrm{Syl}_5(G)$ defines a homomorphism $G \to \mathrm{Sym}(6)$ with kernel K. Then, $|KP/K| = 5$ and KQ/K is a normal 3-subgroup of G/K. However, Sym(6) does not contain a 3-subgroup normalised by a group of order 5; therefore, $p \neq 5$.

Claim. If $Q \in \mathrm{Syl}_q(G)$ and $q \neq p$, then $Q \trianglelefteq G$.

If $N_G(Q) \neq G$, then $Q = N_G(Q)$, otherwise, G would have least $2(p+1)+2(q+1) \geq 14$ nonpower subgroups.

There are at least $p+1$ conjugates of P and $N_G(P)$, and at least $q+1$ conjugates of Q; therefore, $2(p+1)+(q+1) \leq 13$ and so $q \leq 5$. If $q = 5$, then $p = 2$ and $|G : Q| = 6$. Then, $|P| = 2$ and G would have a normal subgroup H of index 2, which must contain the six conjugates of Q. However, $|H : Q| = 3$, which is a contradiction and, therefore, $q \neq 5$. It follows that $q \leq 3$, $G = PQ$ and $|P| = |G : Q|$.

Suppose that $p = 2$. Then, $q = 3$, $|G : Q| = 4 = |P|$ and the permutation action of G on $\mathrm{Syl}_3(G)$ defines a homomorphism $G \to \mathrm{Sym}(4)$ with kernel $K \subseteq Q$. It follows that $G/K \simeq \mathrm{Alt}(4)$ and, therefore, $P \simeq C_2 \times C_2$. However, $N_G(P) \neq G$; therefore, $K \neq 1$ and so for each element $x \in P$ of order 2, it follows from Lemma 3.4 that $\langle x \rangle$ and $\langle x \rangle K$ are nonpower subgroups. Taking into account the three conjugates of P, the three conjugates of $N_G(P)$ and the four conjugates of Q, G would have at least 16 nonpower subgroups, contrary to our assumption.

This leaves the possibility that $p = 3$, $q = 4$ and hence, $|G : Q| = 3 = |P|$. Again, the permutation action of G on $\mathrm{Syl}_3(G)$ defines a homomorphism $G \to \mathrm{Sym}(4)$, this time with kernel $K \subseteq N_G(P)$. However, $P \nsubseteq K$; therefore, $K \subseteq Q$ and Lemma 4.7 implies $G/K \neq \mathrm{Sym}(4)$. Consequently, $G/K \simeq \mathrm{Alt}(4)$ and hence, $Q \trianglelefteq G$.

If $p = 2$, then $|G : N_G(P)| = 2k+1$ for some k and so $2(2k+1) \leq 13$. Therefore, $q = |G : N_G(P)|$ is either 3 or 5 and we choose $Q \in \mathrm{Syl}_q(G)$. Similarly, if $p = 3$, then $|G : N_G(P)| = 4$. In this case, we let $q = 2$ and choose $Q \in \mathrm{Syl}_2(G)$. In all cases $G = N_G(P)Q$.

Claim. $G = PQ \times A$, where A is a cyclic group of order 1, r, r^2 or rs, where r and s are primes different from p and q.

Let R be a Sylow r-subgroup of G, where $r \neq p, q$ is a prime. Then, R acts by conjugation on $\mathrm{Syl}_p(G)$ and in all cases, R fixes a conjugate of P. Thus, $R \subseteq N_G(P)$ and since $R \trianglelefteq G$, we have $[P, R] \subseteq P \cap R = 1$. Also $Q \trianglelefteq G$ and so $[Q, R] \subseteq Q \cap R = 1$. Let A be the product of the Sylow r-subgroups for $r \neq p, q$. Then, $G = PQ \times A$ and from Lemma 3.3,

$$\mathrm{nps}(G) = \mathrm{nps}(PQ)s(A) + \mathrm{ps}(PQ)\,\mathrm{nps}(A).$$

We have $\mathrm{nps}(PQ) \geq 3$ and $\mathrm{ps}(PQ) \geq 3$. Thus, A is cyclic, otherwise, $s(A) \geq 3$ and from Theorem 1.1, $\mathrm{nps}(A) \geq 3$ and so $\mathrm{nps}(G) > 13$. Since A is cyclic, we have $\mathrm{nps}(A) = 0$ and, therefore, $\mathrm{nps}(G) = \mathrm{nps}(PQ)s(A) \leq 13$. Thus, $s(A) \leq 4$ and $|A|$ is 1, r or rs, where $r \leq s$ are primes.

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Claim. If $p = 2$, then G is isomorphic to $\mathrm{Sym}(3) \times C_3$, $G_{n,5} \times C_r$, where $r \neq 2, 5$ is a prime, or $G_{n,3} \times C_{rs}$, where r and s are primes such that $3 < r < s$.

We have $G = PQ \times A$, where A is cyclic and $\mathrm{nps}(G) = \mathrm{nps}(PQ)s(A)$. If $A \neq 1$, then $s(A) \geq 2$ and, therefore, $\mathrm{nps}(G) \leq 6$. It follows from Theorems 1.1 and 1.2 that for some n, $G \simeq G_{n,3}$ or $G_{n,5}$. If $s(A) = 2$, then $A \simeq C_r$; if $s(A) = 3$, then $A \simeq C_{r^2}$; if $s(A) = 4$, then $A \simeq C_{rs}$ for some primes r and s.

This leaves the possibility that $A = 1$ and $G = PQ$. If $|G : N_G(P)| = 5$, then P and $N_G(P)$ each have five conjugates and if Q is not cyclic, it follows from Theorem 3.1 that Q contains at least five nonpower subgroups of order 5. This contradicts the assumption that $\mathrm{nps}(G) \leq 13$ and, therefore, $|G : N_G(P)| = 3$.

We have $[N_Q(P), P] \subseteq P \cap Q = 1$ and, therefore, $N_Q(P) = C_Q(P)$. Consequently, $N_G(P) = P \times C_Q(P)$. Let K be the kernel of the permutation action on $\mathrm{Syl}_2(G)$. Then, $G/K \simeq \mathrm{Sym}(3)$ and $K \subseteq N_G(P)$; hence, $K \cap Q = C_Q(P)$ and $|Q : C_Q(P)| = 3$.

If P is not cyclic, then $\mathrm{nps}(P) \geq 3$ and for each nonpower subgroup H of P, it follows from Lemma 3.4 that H, HQ and $HC_Q(P)$ are nonpower subgroups of G; hence, $\mathrm{nps}(G) \geq 15$, which contradicts the assumption $\mathrm{nps}(G) \leq 13$. Therefore, P is cyclic and so $N_G(P) = C_G(P)$.

Let $P = \langle a \rangle$. Then, $P \cap K = \langle a^2 \rangle$ and $K = \langle a^2 \rangle \times C_Q(P)$. Thus, $\langle a^2 \rangle$ is a characteristic subgroup of K and, therefore, $\langle a^2 \rangle \trianglelefteq G$. It follows that $z = a^2 \in Z(G)$. Let b be a conjugate of a such that $b \neq a$ and let $x = ab^{-1}$. Then, $b^2 = z$, $b^{-1} = bz^{-2}$ and $a^{-1}xa = x^{-1}$. Since $x \in \langle z \rangle Q$ and P is cyclic, $y = x^{2^k} \neq 1$ belongs to Q for some k. Furthermore, a has only three conjugates; therefore, $y^3 = 1$. The group $\langle y \rangle P$ is generated by the conjugates of P and so $\langle y \rangle P \trianglelefteq G$. Moreover, for some n, we have $\langle y \rangle P \simeq G_{n,3}$.

We now have $G = \langle y \rangle P \times C_Q(P)$. If $C_Q(P)$ is not cyclic, it contains an elementary abelian subgroup E of order 9. Then, $\langle y \rangle \times H$ is elementary abelian of order 27; it contains 13 subgroups of order 3, at most one of which can be a nonpower subgroup (Theorem 3.1). It follows that $C_Q(P)$ is cyclic. If the exponent of $C_Q(P)$ is 3^e, then Q^3 is a power subgroup of order 3^{e-1} and from Corollary 3.2, Q contains at least $2e + 2$ nonpower subgroups of G. If $e \geq 2$, then P, $P \times C_Q(P)$ and $P \times Q^3$ each have three conjugates and thus, $\mathrm{nps}(G) \geq 15$, which contradicts our assumption. Therefore, $G \simeq G_{n,3} \times C_3$ and from Lemma 4.5, we have $n = 1$; that is, $G \simeq \mathrm{Sym}(3) \times C_3$.

Claim. If $p = 3$, then $\mathrm{nps}(G) = 11$ and $G \simeq \mathrm{SL}(2, 3)$.

In this case, $|G : N_G(P)| = 4$ and the permutation action on $\mathrm{Syl}_3(G)$ defines a homomorphism $G \to \mathrm{Sym}(4)$ whose kernel K is a proper subgroup of $N_G(P)$. As $Q \trianglelefteq G$, we must have $G/K \simeq \mathrm{Alt}(4)$ and $QK/K \simeq C_2 \times C_2$. The assumption $10 \leq \mathrm{nps}(G)$ implies $K \neq 1$. Let $x_1, x_2, x_3 \in Q$ be elements such that x_iK is an involution in QK/K. From Lemma 3.4, $\langle x_i \rangle$ and $\langle x_i \rangle K$ are nonpower subgroups. The four conjugates of P and the four conjugates of $N_G(P)$ are nonpower subgroups, and since there are at most 13 nonpower subgroups in G, it follows that $\langle x_i \rangle = \langle x_i \rangle K$. That is, $K \subseteq \langle x_i \rangle$ and so $K = \langle x_i^2 \rangle$ for $1 \leq i \leq 3$. We now see that $|P| = 3$, $K \subseteq C_G(P)$ and P permutes the x_i; hence, $x_1^2 = x_2^2 = x_3^2$. It follows that $Q \simeq Q_8$ and thus, $G \simeq \mathrm{SL}(2, 3)$. □

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LEMMA 5.2. *Let P be a Sylow p-subgroup of G such that $P = N_G(P) \neq G$. Then, $10 \leq \mathrm{nps}(G) \leq 13$ if and only if for some $n \geq 1$, one of the following holds:*

(1) $\mathrm{nps}(G) = 10$, $p = 2$ *and G is isomorphic to $G^{(2)}_{n,5}$ for $n \geq 2$;*
(2) $\mathrm{nps}(G) = 10$, $p = 3$ *and G is isomorphic to $A_2 = (C_2 \times C_2) \rtimes C_9$;*
(3) $\mathrm{nps}(G) = 11$, $p = 2$ *and G is isomorphic to $G_{n,11}$;*
(4) $\mathrm{nps}(G) = 11$, $p = 5$ *and G is isomorphic to $G^{(3)}_{5,n;\,11,1}$;*
(5) $\mathrm{nps}(G) = 12$, $p = 2$ *and G is isomorphic to $G_{n,9}$;*
(6) $\mathrm{nps}(G) = 13$, $p = 2$ *and G is isomorphic to $G_{n,13}$; $\mathrm{Sym}(3) \times C_2 = D_{12}$ or $C_3 \rtimes Q_8$;*
(7) $\mathrm{nps}(G) = 13$, $p = 3$ *and G is isomorphic to $F_{n,13}$ or $A_3 = (C_2 \times C_2) \rtimes C_{27}$.*

PROOF. The Sylow subgroup P has $m = |G : N_G(P)|$ conjugates. Since $m \equiv 1 \pmod p$ and $N_G(P) \neq G$, we have $m \geq p + 1$ and the conjugates of P are nonpower subgroups. The assumption $\mathrm{nps}(G) \leq 13$ implies $p \in \{2, 3, 5, 7, 11\}$.

If $p = 2$, then $m \in \{3, 5, 7, 9, 11, 13\}$; if $p = 3$, then $m \in \{4, 7, 10, 13\}$; if $p = 5$, then $m \in \{6, 11\}$; if $p = 7$, then $m = 8$; if $p = 11$, then $m = 12$.

Suppose that $p = 3$ and $m = 10$ or $p = 5$ and $m = 6$. Then, G is a group of twice odd order and, therefore, has a subgroup H of index 2, which contains P. Then, $|H : P| = m/2$, which is impossible. Suppose that $p = 11$ and $m = 12$. The permutation action on $\mathrm{Syl}_{11}(G)$ defines a homomorphism $G \to \mathrm{Sym}(12)$ whose kernel $K = \bigcap_{g \in G} P^g$ is properly contained in P. Let $\overline{G} = G/K$. Then, $N_{\overline{G}}(P/K) = C_{\overline{G}}(P/K)$ and, therefore, by Burnside's normal p-complement theorem (see [6, Theorem 7.4.3]), P/K has a normal 11-complement in $\overline{G}$, which is a contradiction.

Thus, in all cases, m is a power of a prime q and for $Q \in \mathrm{Syl}_q(G)$, we have $G = PQ$ and $|Q| = m$. We shall show that $Q \trianglelefteq G$.

If $N_G(Q) \neq G$, it follows from Lemma 5.1 that $Q = N_G(Q)$, otherwise, $P \trianglelefteq G$. If the order of Q is q or q^2, then Q is abelian and it follows from Burnside's normal p-complement theorem that $P \trianglelefteq G$. Thus, the only possibility is $p = 7$, $m = 8$ and $q = 2$. However, then $|G : Q|$ must be a power of 7 and so $\mathrm{nps}(G) \geq 15$, in contrast to our assumption. This proves that $Q \trianglelefteq G$.

The permutation action on $\mathrm{Syl}_p(G)$ defines a homomorphism $G \to \mathrm{Sym}(m)$ whose kernel $K = \bigcap_{g \in G} P^g$ is properly contained in P. Since $Q \trianglelefteq G$, we have $[K, Q] \subseteq K \cap Q = 1$. Moreover, Q acts transitively on the m conjugates of P and, therefore, $K = C_P(Q)$.

Case 1: $p = 2$ and $m \in \{3, 5, 7, 9, 11, 13\}$.

We have $G = Q \rtimes P$, where P is a 2-group and Q is an abelian group of order m. We treat each value of m separately.

Case 1a: $p = 2$ and $|Q| = 3$.

In this case, $G/K \simeq \mathrm{Sym}(3)$ and $|P/K| = 2$. Therefore, $\Phi(P) \subseteq K = C_P(Q)$ and so $\Phi(P) \trianglelefteq G$. Thus,

$$G/\Phi(P) \simeq \mathrm{Sym}(3) \times \underbrace{C_2 \times \cdots \times C_2}_{n}.$$

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From Lemma 4.8, $n \leq 1$. If $n = 0$, then P is cyclic, $G \simeq G_{n,3}$ and $\operatorname{nps}(G) = 3$, which contradicts the assumption $10 \leq \operatorname{nps}(G)$. Thus, $n = 1$, $G/\Phi(P) \simeq \operatorname{Sym}(3) \times C_2 \simeq D_{12}$ and, from Lemma 4.8, $\operatorname{nps}(G/\Phi(P)) = 13$. The only nontrivial proper power subgroup of $\operatorname{Sym}(3) \times C_2$ is the subgroup of order 3. Therefore, for all $H \subseteq P$ such that $|H\Phi(P)/\Phi(P)| = 2$, it follows from Lemma 3.4 that both H and $H\Phi(P)$ are nonpower subgroups. Therefore, $H = H\Phi(P)$ and hence, $\Phi(P) \subseteq H$.

If $P \setminus \Phi(P)$ contains an element of order 2, then $\Phi(P) = 1$ and thus, $G \simeq \operatorname{Sym}(3) \times C_2$. Otherwise, $\Phi(P) = \langle x^2 \rangle$ for all $x \in P \setminus \Phi(P)$. Thus, P is nonabelian and it follows from Lemma 3.4 that $\operatorname{nps}(G) \geq 2 \operatorname{nps}(G/Q)$. Then, Theorems 1.1 and 1.2 show that the only possibility for $P \simeq G/Q$ is the quaternion group Q_8. Thus, G is the semidirect product

$$C_3 \rtimes Q_8 = \langle x, y, b \mid x^4 = y^4 = b^3 = [y, b] = 1,\ x^2 = y^2,\ [x, y] = x^2,\ b^x = b^{-1} \rangle.$$

Case 1b: $p = 2$ and $|Q| = 5$.

In this case, QK/K is a normal subgroup of $G/K \subseteq \operatorname{Sym}(5)$. Therefore, $|G/K|$ is either 10 or 20.

Suppose $|G/K| = 10$. If P is not cyclic, then $\Phi(P) \subseteq K$ and, therefore, $\Phi(P) \trianglelefteq G$. We have $G/\Phi(P) \simeq D_{10} \times C_2 \times \cdots \times C_2$, and Lemmas 3.4 and 4.8 imply $\operatorname{nps}(G) \geq 19$, whereas we assume that $\operatorname{nps}(G) \leq 13$; hence, P is cyclic. If a generates P and b generates Q, then $b^a = b^{-1}$ and $G \simeq G_{n,5}$. However, then $\operatorname{nps}(G) = 5$ and so G does not satisfy the assumption $10 \leq \operatorname{nps}(G)$.

Thus, $|G/K| = 20$, $G/K \simeq G^{(2)}_{2,n;\,5,1}$ and $\operatorname{nps}(G/K) = 10$. Choose $a \in P$ such that aK generates P/K. The five conjugates of $\langle a \rangle$ and the five conjugates of $\langle a^2 \rangle$ are nonpower subgroups of G. Similarly, the five conjugates of $\langle a \rangle K$ and the five conjugates of $\langle a^2 \rangle K$ are nonpower subgroups of G. Since $\operatorname{nps}(G) \leq 13$, we must have $K \subseteq \langle a \rangle$ and, therefore, P is cyclic. Therefore, $G \simeq G^{(2)}_{2,n;\,5,1}$.

Case 1c: $p = 2$ and $|Q| = 7$.

In this case, $G/K \simeq D_{14}$ and P has seven conjugates. Suppose that $a \in P \setminus K$. Then, P, $\langle a \rangle$ and their conjugates are nonpower subgroups. However, $\operatorname{nps}(G) \leq 13$ and so $P = \langle a \rangle$. Therefore, $G \simeq G_{n,7}$ whence $\operatorname{nps}(G) = 7$ and there are no examples with $10 \leq \operatorname{nps}(G)$.

Case 1d: $p = 2$ and $|Q| = 9$.

Suppose that there is a subgroup R of order 3 in Q such that $R \trianglelefteq G$. Then, RP is not normal in G, otherwise, the Frattini argument [6, Theorem 1.3.7] implies $G = RN_G(P) = RP$, which is a contradiction. The nine conjugates of P and the three conjugates of RP are nonpower subgroups. Therefore, all other subgroups are normal. Thus, R is the unique subgroup of Q of order 3 and hence, Q is cyclic. If $a \in P \setminus K$ and b generates Q, then $b^a = b^{-1}$. If $P \neq \langle a \rangle$, then $\langle a \rangle \trianglelefteq G$ and so $ab = ba$, which is a contradiction. Therefore, P is cyclic, $\operatorname{nps}(G) = 12$ and $G \simeq G_{n,9}$.

This leaves us with the possibility that no subgroup of order 3 is normal in G. This implies Q is not cyclic, otherwise the subgroup of order 3 in Q would be a normal subgroup of G. Consequently, $Q \simeq C_3 \times C_3$ and its four subgroups of order 3 are not normal in G. Since $\operatorname{nps}(G) \leq 13$ and the nine conjugates of P are nonpower subgroups,

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it follows that the proper subgroups of P are normal in G. As above, this implies that P is cyclic. Furthermore, if $P = \langle a \rangle$, then $\langle a^2 \rangle \trianglelefteq G$ and so $a^2 \in K$. Thus, if x is an element of order 3 in Q and $y = x^a \neq x$, then $(xy)^a = xy$ and $\langle xy \rangle \trianglelefteq G$, which is a contradiction.

Case 1e: $p = 2$ and $|Q| = 11$.
In this case, $G/K \simeq D_{22}$ and P has 11 conjugates. Thus, P is cyclic and $G \simeq G_{n,11}$.

Case 1f: $p = 2$ and $|Q| = 13$.
In this case, $G/K \simeq D_{26}$ and P has 13 conjugates. Thus, P is cyclic and $G \simeq G_{n,13}$.

Case 2: $p = 3$ and $m \in \{4, 7, 13\}$.
In this case, $G = Q \rtimes P$, where P is a 3-group and Q is a q-group with $q \in \{2, 7, 13\}$.

Case 2a: $p = 3$ and $|Q| = 4$.
Considering the homomorphism $G \to \mathrm{Sym}(4)$ (with kernel $K \subset P$), we have $G/K \simeq \mathrm{Alt}(4)$. Thus, $Q \simeq C_2 \times C_2$. From $\mathrm{nps}(\mathrm{Alt}(4)) = 7$ and $10 \leq \mathrm{nps}(G)$, it follows that $K \neq 1$. Then, G has seven nonpower subgroups that properly contain K and three nonpower subgroups that are nontrivial proper subgroups of Q. By $\mathrm{nps}(G) \leq 13$, we have $\mathrm{nps}(G/Q) \leq 3$. This implies $P \simeq C_{3^n}$ and hence, $G \simeq A_n$. From Lemma 4.1, we have $G \simeq A_2$ or A_3.

Case 2b: $p = 3$ and $|Q| = 7$.
From Theorems 1.1 and 1.2, Lemma 3.4, and the assumption $\mathrm{nps}(G) \leq 13$, P is either cyclic or isomorphic to $C_3 \times C_3$. If $P \simeq C_3 \times C_3$ and H is a noncentral subgroup of order 3, then H has seven conjugates. This contradicts $\mathrm{nps}(G) \leq 13$ because P has three subgroups of order 3 not in the centre of G. Thus, P is cyclic. Let a be a generator of P. Then, there exists a generator b of Q such that $a^{-1}ba = b^2$. Therefore, for some $n \geq 1$, $G \simeq F_{n,7}$. However, then $\mathrm{nps}(G) = 7$, whereas we assume that $10 \leq \mathrm{nps}(G)$.

Case 2c: $p = 3$ and $|Q| = 13$.
As in previous cases, P is cyclic and thus, $G \simeq F_{n,13}$.

Case 3: $p = 5$ and $|Q| = 11$.
As in previous cases, P is cyclic and thus, $G \simeq G^{(3)}_{5,n;\,11,1}$.

Case 4: $p = 7$ and $|Q| = 8$.
The subgroup P acts transitively on the seven subgroups of order 2 of Q and on the seven subgroups of order 4, otherwise $Q \subseteq C_G(P)$. Thus, $\mathrm{nps}(G) \geq 14$ and so there are no examples with $p = 7$. □

Proof of Theorem 1.3. From the above discussion, we may suppose that G is nilpotent so that $G = P \times H$, where P is a noncyclic p-subgroup and $H \neq 1$ is a group whose order is not divisible by p. If H is also noncyclic, then H has at least three subgroups and from Lemma 3.3, we have $\mathrm{nps}(G) \geq 15$. Thus, H is cyclic and $\mathrm{nps}(P \times H) = \mathrm{nps}(P)s(H)$.

If $H \neq 1$, then $s(H) \geq 2$ and, therefore, the assumption $\mathrm{nps}(G) \leq 13$ implies $\mathrm{nps}(P) \leq 6$. Referring to Theorems 1.1 and 1.2, P is isomorphic to $C_2 \times C_2$, $C_3 \times C_3$,

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$C_2 \times C_4$ or $C_5 \times C_5$. If $\mathrm{nps}(G) = 10$, then $G \simeq C_2 \times C_4 \times C_p$, where $p > 3$ is a prime; if $\mathrm{nps}(G) = 12$, then $G \simeq Q_8 \times C_{qr}$, $C_2 \times C_2 \times C_{qr}$, $C_3 \times C_3 \times C_{s^2}$ or $C_5 \times C_5 \times C_p$, where p, q, r and s are primes such that $p \neq 5$, $3 < q < r$ and $s \neq 3$.

From now on, we may suppose that G is a noncyclic p-group. Then, $G/\Phi(G)$ is an elementary abelian group of order p^d and every proper nontrivial subgroup of $G/\Phi(G)$ is a nonpower subgroup. Since $\mathrm{nps}(C_p \times C_p \times C_p) \geq \mathrm{nps}(C_2 \times C_2 \times C_2) = 14$, it follows that $d = 2$ and G is generated by two elements. Thus, from Lemma 4.1(ix), we have $G/G' \simeq C_2 \times C_{2^n}$ for $1 \leq n \leq 6$, $C_3 \times C_{3^n}$ for $1 \leq n \leq 4$, $C_5 \times C_{5^n}$ for $n \in \{1, 2\}$, $C_7 \times C_7$, $C_{11} \times C_{11}$ or $C_4 \times C_4$. If G is abelian, the proof is complete. From now on, we assume that $G' \neq 1$.

If $G/G' = C_2 \times C_2$, it follows from [6, Theorem 5.4.5] that G is isomorphic to D_{2^m}, S_{2^m} or Q_{2^m}, and then from Lemma 4.1, the only possibility is S_{16}.

Since $G' \neq 1$, there exists $R \trianglelefteq G$ such that $|G'/R| = p$. We shall determine the structure of G/R for each choice of G/G'.

If $G/G' = C_p \times C_p$ and p is odd, it follows from [6, Theorem 5.5.1] that G/R is an extraspecial group of order p^3 and then from Lemma 4.1, the only possibility for G/R is $M_{3,5}$. We argue as in [15]. The group $M_{3,5}$ has a cyclic subgroup of order 25 and, therefore, it is metacyclic. It follows from Lemma 3.7 that G is metacyclic and so G has a cyclic normal subgroup that properly contains G'; that is, G has a cyclic subgroup of index 5 and, therefore, by [6, Theorem 5.4.4], we have $G \simeq M_{n,5}$. From Lemma 4.1, it follows that $n = 3$ and $G \simeq M_{3,5}$.

If $G/G' \simeq C_2 \times C_{2^6}$, $C_3 \times C_{3^4}$, $C_{11} \times C_{11}$ or $C_4 \times C_4$, then $\mathrm{nps}(G/G') \in \{12, 13\}$ and all subgroups of G that do not contain G' are normal. However, then every subgroup of G is normal because all subgroups that contain G' are normal. Thus, G is a Hamiltonian group whence $p = 2$ and $G \simeq Q_8 \times C_2 \times \cdots \times C_2$ (see [9, III, 7.12]). It follows from Lemmas 4.1 and 4.6 that no groups of this type satisfy $10 \leq \mathrm{nps}(G) \leq 13$. Similarly, because $\mathrm{nps}(C_5 \times C_{5^2}) = 11$, the same argument shows that there are no examples with $G/G' \simeq C_5 \times C_{5^2}$.

Finally, we consider the cases $G/G' \simeq C_2 \times C_{2^n}$ for $2 \leq n \leq 5$ and $C_3 \times C_{3^n}$ for $n = 2, 3$. If G/R has a cyclic maximal subgroup, it follows from [6, Theorem 5.4.4] that $G \simeq M_{n+2,2}$ or $M_{n+2,3}$. From Lemmas 3.6 and 4.1, G is isomorphic to $M_{6,2}$, $M_{7,2}$, $M_{4,3}$ or $M_{5,3}$. If the exponent of G/R is 2^n or 3^n, then from Lemmas 3.7 and 5.2, $G/R \simeq B^2_{2,2}$ and $\mathrm{nps}(G/R) = 12$.

We may choose generators a and b for G such that their images $\overline{a}$ and $\overline{b}$ in G/R satisfy the relations $\overline{a}^4 = 1$, $\overline{b}^4 = 1$ and $\overline{a}^{-1}\overline{b}\overline{a} = \overline{b}^{-1}$. The nonpower subgroup $\langle \overline{a} \rangle$ is the image in G/R of both $\langle a \rangle$ and $\langle a \rangle R$. Since $\langle a \rangle$ is not normal in G, we must have $R \subseteq \langle a \rangle$ and, therefore, $R = \langle a^4 \rangle$ is cyclic. The group G has at most one power subgroup of order 2 and since G is neither cyclic nor a generalised quaternion group, it has at least three subgroups of order 2. Therefore, there are elements $x, y \in G \setminus R$ of order 2 such that $\langle x \rangle$ and $\langle y \rangle$ are nonpower subgroups. However, then $\langle x \rangle R$ and $\langle y \rangle R$ are also nonpower subgroups and this is a contradiction. Therefore, $R = 1$, $G \simeq B^2_{2,2}$ and this completes the proof. □

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REMARK 5.3. All groups that occur in Theorems 1.1, 1.2 and 1.3 have the property that for at most one prime p, the group has more than one Sylow p-subgroup. The smallest example of a group that does not have this property is the semidirect product $C_7 \rtimes C_6$ (with trivial centre); it has 21 nonpower subgroups.

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