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Best approximation in $L^p(I, X)$

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0. *Introduction*

Let T be a measure space and m a finite measure on T. The space of p-Bochner integrable functions defined on T with values in a Banach space X is denoted by $L^p(T, X)$. It is well known (1) that $L^p(T, X)$ is a Banach space under the norm

$$\|f\|_{\alpha(p)} = \left(\int_T \|f(t)\|^p \, dm(t)\right)^{1/p}, \quad 1 \leqslant p < \infty.$$

A subspace E in a Banach space F is said to be proximinal if for each $x \in F$ there is at least one $y \in E$ such that

$$\|x-y\| = d(x, E) = \inf\{\|x-z\|, z \in E\}.$$

The element y is called best approximant of x in E.

In (3), Light and Cheney proved that if Y is finite-dimensional in $L^1(T)$ then $L^1(T, Y)$ is proximinal in $L^1(T, L^1(T))$. It should be remarked here that, since Y is finite-dimensional, it is reflexive, and a compactness argument shows that Y is proximinal in $L^1(T)$.

In this paper we prove a result which is stronger than the result in (3) in the following way: We let X be any Banach space. Then we determine a class of subspaces $L^p(T, Y)$ which are proximinal in $L^p(T, X)$, $1 \leqslant p < \infty$.

We refer to (1) for an excellent exposition on $L^p(T, X)$.

1. *Proximinal subspaces in* $L^p(T, X)$

If $u \in L^p(T)$ and $v \in X$, we let $u \otimes v$ denote the function $f(t) = u(t)v$. Clearly if Y is a proximinal subspace of X, then every function $u \otimes v$ in $L^p(T, X)$ has a best approximant in $L^p(T, Y)$, $1 \leqslant p < \infty$.

THEOREM 1·1. *Let Y be a closed proximinal subspace of X. Then every simple function $\theta = \Sigma_{i=1}^n 1_{A_i} \otimes v_i$, $(A_i \cap A_j = \varnothing$ if $i \neq j)$, in $L^p(T, X)$ has best approximant in $L^p(T, Y)$, $1 \leqslant p < \infty$.*

Proof. Let $\tilde{v}_i$ be a best approximant of v_i in Y, and set $\tilde{f} = \Sigma_{i=1}^n 1_{A_i} \otimes \tilde{v}_i$. Then

$$\|f-\tilde{f}\|_{\alpha(p)} = \left(\sum_{i=1}^n m(A_i) \cdot \|v_i - \tilde{v}_i\|^p\right)^{1/p}.$$

If $\psi \in L^p(T, Y)$ is an arbitrary element, then

$$\|f-\psi\|_{\alpha(p)} = \left(\int_T \|f(t)-\psi(t)\|^p \, dm(t)\right)^{1/p} = \left(\sum_{i=1}^n \int_{A_i} \|f(t)-\psi(t)\|^p \, dm(t)\right)^{1/p}$$

$$= \left(\sum_{i=1}^n \int_{A_i} \|v_i - \psi(t)\|^p \, dm(t)\right)^{1/p} \geqslant \left(\sum_{i=1}^n \int_{A_i} \|v_i - \tilde{v}_i\|^p \, dm(t)\right)^{1/p} = \|f-\tilde{f}\|_{\alpha(p)}.$$

Hence $\tilde{f}$ is a best approximant for f in $L^p(T, X)$. ∎

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Remark. It must be observed here that $\tilde{f}$ has the property that $\tilde{f}(t) \in Y$ is a best approximant for $f(t) \in X$ for all $t \in T$.

Now we assume that X is any Banach space.

THEOREM 1·2. *Let Y be a reflexive subspace of X. Then $L^1(T, Y)$ is proximinal in $L^1(T, X)$.*

Proof. Let $\psi \in L^1(T, X)$. Since simple functions are dense in $L^1(T, X)$ (1), let (S_n) be a sequence of simple functions such that (S_n) converges to ψ in $L^1(T, X)$. By Theorem 1·1, we let $\tilde{S}_n$ be the best approximant of S_n in $L^1(T, Y)$.

Since $S_n \to \psi$ in $L^1(T, X)$, the set $K = \{S_1, S_2, \ldots, \psi\}$ is compact in $L^1(T, X)$, and so weakly compact. We claim that $\tilde{K} = \{\tilde{S}_1, \tilde{S}_2, \ldots\}$ is weakly relatively compact. Since Y is assumed to be reflexive, the Dunford theorem (1, p. 101) implies that it is enough to prove that $\tilde{K}$ is bounded and uniformly integrable. That $\tilde{K}$ is bounded follows from the fact that K is bounded and

$$\|\tilde{S}_n\|_{\alpha(1)} \leqslant \|\tilde{S}_n - S_n\|_{\alpha(1)} + \|S_n\|_{\alpha(1)} \leqslant 2\|S_n\|_{\alpha(1)}.$$

To show that $\tilde{K}$ is uniformly integrable, we have first to realize that K is uniformly integrable, which is a consequence of the fact that K is the uniform closure of a converging sequence in $L^1(T, X)$. Thus

$$\lim_{m(A) \to 0} \int \|g(t)\| \, dm(t) = 0 \quad \text{uniformly for } g \text{ in } K.$$

Now, let $\epsilon > 0$ be given, and $\epsilon' = \epsilon/2$. By the uniform integrability of K, there exists $\delta > 0$ such that $\int_A \|g(t)\| \, dm(t) \leqslant \epsilon'$ whenever $m(A) \leqslant \delta$ for all $g \in K$. By Theorem 1·1 and the remark after, we have, for any set A, $m(A) \leqslant \delta$:

$$\begin{aligned} \int_A \|\tilde{S}_n(t)\| \, dm(t) &\leqslant 2 \int_A \|S_n(t)\| \, dm(t) \\ &\leqslant 2\epsilon' = \epsilon. \end{aligned}$$

Since ϵ and δ do not depend on n, $\tilde{K}$ is uniformly integrable, and hence weakly relatively compact in $L^1(T, Y)$. Consequently ((2), p. 311), there exists $\tilde{S} \in L^1(T, Y)$ and a subsequence of $(\tilde{S}_n)$ which converges to $\tilde{S}$ weakly. With no loss of generality we can assume $\tilde{S}_n \to \tilde{S}$ weakly, and we can assume that $(\|\tilde{S}_n - S_n\|_{\alpha(1)})$ converges in the real numbers.

For any $Q \in [L^1(T, X)]^*$, $\|Q\| \leqslant 1$, we have

$$\begin{aligned} |\langle \tilde{S} - \psi, Q \rangle| &= \lim_n |\langle \tilde{S}_n - \psi, Q \rangle| \\ &= \lim_n |\langle \tilde{S}_n - S_n, Q \rangle| + \lim_n |\langle S_n - \psi, Q \rangle| \\ &\leqslant \lim_n \|\tilde{S}_n - S_n\|_{\alpha(1)}. \end{aligned}$$

By theorem 1·1, $\|\tilde{S}_n - S_n\|_{\alpha(1)} \leqslant \|\theta - S_n\|_{\alpha(1)}$, $\theta \in L^1(T, Y)$. Hence

$$\begin{aligned} |\langle \tilde{S} - \psi, Q \rangle| &\leqslant \lim_n \|\theta - S_n\|_{\alpha(1)}, \\ &= \|\theta - \psi\|_{\alpha(1)}. \end{aligned}$$

Since this is true for all Q in the unit ball of $[L^1(T, X)]^*$, one has:

$$\|\tilde{S} - \psi\|_{\alpha(1)} \leqslant \|\theta - \psi\|_{\alpha(1)}.$$

Thus $\tilde{S}$ is a best approximant for ψ. |

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COROLLARY 1·3 ((3) theorem 1). *If Y is a finite-dimensional subspace of $L^1(\mu)$, then $L^1(T, Y)$ is proximinal in $L^1(T \times T)$.*

Proof. Since Y is finite-dimensional, it is reflexive. ∎

Remark. For $1 < p < \infty$, Theorem 1·2 is a consequence of corollary 2·4 ((4), p. 99).

2. *Further results*

A closed subspace Y of a Banach space X is called an L^p-summand, $1 \leqslant p < \infty$, if there is a bounded projection $P: X \to Y$ which is onto, and $\|x\|^p = \|P(x)\|^p + \|x - P(x)\|^p$. One can easily prove:

LEMMA 2·1. *If Y is an L^1-summand of X, then Y is proximinal in X.*

Now we prove

THEOREM 2·2. *If Y is an L^1-summand of X, then $L^1(T, Y)$ is an L^1-summand of $L^1(T, X)$ (hence proximinal).*

Proof. Let $P: X \to Y$ be the associated L^1-projection. If I is the identity operator on $L^1(T)$, then

$$I \mathbin{\widehat{\otimes}} P: L^1(T) \mathbin{\widehat{\otimes}} X \to L^1(T) \mathbin{\widehat{\otimes}} Y,$$

is a bounded projection (1), where $E \mathbin{\widehat{\otimes}} F$ is the completion of the projective tensor product of E with F. Let $f \in L^1(T, X) = L^1(T) \mathbin{\widehat{\otimes}} X$, (1), such that

$$f = \sum_{i=1}^{n} 1_{A_i} \otimes v_i, \quad A_i \cap A_j = \varnothing \quad \text{if} \quad i \neq j.$$

Then

$$f = \sum_{i=1}^{n} 1_{A_i} \otimes P(v_i) + \sum_{i=1}^{n} 1_{A_i} \otimes (v_i - P(v_i)).$$

Further:

$$\begin{aligned} \|f\| &= \|\Sigma\, 1_{A_i} \otimes P(v_i)\| + \|\Sigma\, 1_{A_i} \otimes (v_i - P(v_i))\| \\ &= \|(I \otimes P)(f)\| + \|f - (I \otimes P)(f)\|. \end{aligned}$$

Since simple functions are dense in $L^1(T, X)$, the result follows. ∎

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