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Maximizing measures for partially hyperbolic systems with compact center leaves

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In memory of Dan Rudolph

Abstract. We obtain the following dichotomy for accessible partially hyperbolic diffeomorphisms of three-dimensional manifolds having compact center leaves: either there is a unique entropy-maximizing measure, this measure has the Bernoulli property and its center Lyapunov exponent is 0, or there are a finite number of entropy-maximizing measures, all of them with non-zero center Lyapunov exponents (at least one with a negative exponent and one with a positive exponent), that are finite extensions of a Bernoulli system. In the first case of the dichotomy, we obtain that the system is topologically conjugated to a rotation extension of a hyperbolic system. This implies that the second case of the dichotomy holds for an open and dense set of diffeomorphisms in the hypothesis of our result. As a consequence, we obtain an open set of topologically mixing diffeomorphisms having more than one entropy-maximizing measure.

1. *Introduction*

Topological and metric entropy are among the most important invariants for measuring the degree of unpredictability in a dynamical system on an exponential scale. By the variational principle for homeomorphisms defined on compact spaces, the topological entropy is the supremum over the metric entropies of all invariant probability measures. The study of invariant measures that attain the highest entropy is a natural way of describing the behavior of most of the relevant orbits of a system.

Uniformly hyperbolic dynamical systems admit entropy-maximizing measures and topological transitivity implies the uniqueness of such measures. The construction of maximizing measures for hyperbolic systems is due to Bowen [**4**] and Margulis [**20**], in two different approaches. By means of their proofs, it comes out that maximizing measures equidistribute periodic orbits of the dynamics and also have a local product structure.

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Beyond uniform hyperbolicity, even the existence problem in general is not clear. On the one hand, Misiurewicz [**21**] has constructed a class of examples of diffeomorphisms without any entropy-maximizing measure. On the other hand, the regularity of the dynamics is relevant to the regularity of the function that attributes the metric entropy to each invariant measure. By a result of Newhouse [**23**], this is an upper semicontinuous function in the case of C^∞ diffeomorphisms of compact manifolds. Hence, any C^∞ diffeomorphism admits entropy-maximizing measures. However, if we are in the setting of partially hyperbolic diffeomorphisms, the results of Cowieson and Young in [**9**] (see also [**10**]) imply that there are always entropy-maximizing measures if the center bundle is one-dimensional, even if the diffeomorphism is only C^1.

In this paper, we address the problem of the existence and uniqueness (finiteness) of entropy-maximizing measures in the partially hyperbolic context. Although, as we have already mentioned above, it is known, we give a proof of the existence because it is almost straightforward in our setting.

Let us mention some previous related works. In [**24**], Newhouse and Young have shown that some partially hyperbolic examples on $\mathbb{T}^4$ have a unique entropy-maximizing measure (see for instance the examples in [**31**]). In three-dimensional manifolds, Buzzi *et al* [**7**] proved that a certain derived from Anosov construction due to Mañé (see [**19**]) also has this uniqueness property. Moreover, thanks to new techniques recently developed by Brin *et al* [**5**] and Hammerlindl [**16**] in the theory of partially hyperbolic dynamics, Ures [**34**] announced a proof of the same property for any absolutely partially hyperbolic diffeomorphism of the 3-torus homotopic to a hyperbolic automorphism.

THEOREM 1. *Let $f : M \to M$ be a $C^{1+\alpha}$ partially hyperbolic diffeomorphism of a three-dimensional closed manifold M. Assume that f is dynamically coherent with compact one-dimensional central leaves and has the accessibility property. Then f has finitely many ergodic measures of maximal entropy. There are two possibilities.*

(1) *f has a unique entropy-maximizing measure μ. The central Lyapunov exponent $\lambda_c(\mu)$ vanishes and (f, μ) is isomorphic to a Bernoulli shift.*
(2) *f has more than one ergodic entropy-maximizing measure, all of which have a non-vanishing central Lyapunov exponent. The central Lyapunov exponent $\lambda_c(\mu)$ is non-zero and (f, μ) is a finite extension of a Bernoulli shift for any such measure μ. Some of these measures have a positive central exponent and some have a negative central exponent.*

Moreover, the diffeomorphisms fulfilling the conditions of the second item form a C^1-open and C^∞-dense subset of the dynamically coherent partially hyperbolic diffeomorphisms with compact one-dimensional central leaves.

In the proof of this theorem, we consider a partially hyperbolic diffeomorphism with compact central leaves as a nonlinear cocycle over a uniformly hyperbolic homeomorphism and apply an invariance principle proved by Avila and Viana [**1**].

Let us mention that in the first item of the above theorem, based on ideas similar to the ones of Avila *et al* in [**2**], the dynamics will be shown to be conjugate to an isometric extension of a hyperbolic homeomorphism, implying that the subset of dynamics satisfying the first item is meager. Observe that, as a corollary, we obtain that for an open and

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dense subset of partially hyperbolic diffeomorphisms satisfying the conditions of the above theorem, the ergodic maximizing measures are hyperbolic, i.e., the Lyapunov exponents are non-vanishing.

In [**24**], Newhouse and Young introduced the definition of almost conjugacy. Let us recall that a system is intrinsically ergodic if it has a unique entropy-maximizing measure (see [**35**]). Let $f : X \to X$ and $g : Y \to Y$ be intrinsically ergodic systems (that is, having a unique entropy-maximizing measure) with maximal measures μ and $\overline{\mu}$, respectively. Say that f and g are almost conjugated if there are invariant sets $A \subset X$ and $B \subset Y$ such that $\mu(A) = \overline{\mu}(B) = 1$ and $f|_A$ is topologically conjugated to $g|_B$. They formulated the following question. Let $\mathcal{B}(M)$ denote the set of C^r diffeomorphisms f such that:

(1) f has only finitely many ergodic measures of maximal entropy; and
(2) on each support of an ergodic measure of maximal entropy, f is almost conjugate to the restriction of an Axiom A diffeomorphism to a topologically transitive basic set.

Is $\mathcal{B}(M)$ residual in $\mathrm{Diff}^r(M)$?

Observe that a positive answer to the Newhouse–Young question would imply that, generically, the supports of entropy-maximizing measures will not coincide.

Using Theorem 1 and the robustly (topologically) mixing diffeomorphisms obtained by Bonatti and Díaz [**3**], we get examples of robustly mixing systems with more than one maximizing measure.

THEOREM 2. *There exist robustly mixing diffeomorphisms where the number of entropy-maximizing measures is larger than one.*

This contrasts with the uniformly hyperbolic case and gives a negative answer to a question of Buzzi and Fisher [**7**].

Here a natural question arises.

Question 1. Is the support of any of the measures given by Theorem 2 the whole manifold?

A positive answer to this question would imply the existence of an open set of diffeomorphisms having entropy-maximizing measures with full support, giving a negative answer to the Newhouse–Young question. Although this is the most likely situation, we obtain that in our setting the answer will be positive if we slightly modify the definition of almost conjugacy.

COROLLARY 1. *Let $\mathcal{C}$ be the set of partially hyperbolic diffeomorphisms of three-dimensional manifolds that are dynamically coherent with compact one-dimensional central leaves. Then there exists an open and dense subset $\tilde{\mathcal{C}}$ of $\mathcal{C}$ such that, for $f \in \tilde{\mathcal{C}}$, we have the following.*

(1) *f has a finite number of entropy-maximizing measures.*
(2) *If μ is an entropy-maximizing measure for f, there exist a transitive hyperbolic basic set of an Axiom A diffeomorphism g with entropy-maximizing measure m, a subset $A \subset \mathrm{supp}(\mu)$ with $\mu(A) = 1$ and a subset $B \subset \mathrm{supp}(m)$ with $m(B) = 1$ such that $f|_A$ is topologically conjugated to $g|_B$.*

We observe that the only difference with the Newhouse–Young question is that we do not require μ to be the unique measure maximizing entropy of f restricted to its support.

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We emphasize that in the category of smooth invariant measures μ, by Pesin's result, if all the Lyapunov exponents are non-zero and (f^n, μ) is ergodic for any $n \geq 1$, then (f, μ) is Bernoulli. Here we see that the entropy-maximizing measure with zero center Lyapunov exponent has the Bernoulli property by using profound results of Rudolph [**29**] about isometric extensions of Bernoulli shifts.

The paper is organized as follows. In §2, we present some preliminaries and definitions needed for the understanding of the paper. In §3, we describe the space of center leaves and, moreover, we show that the manifold M is (finitely covered by) a nilmanifold. In §4, we give a proof of the existence of entropy-maximizing measures and we show that any such measure projects onto a measure with local product structure. In §5, we prove the first part of the dichotomy of Theorem 1, and in §6, we prove the second part. In §7, we prove Theorem 2 and Corollary 1. Finally, in §8, the reader can find some questions and directions of development.

2. *Dynamic preliminaries*

2.1. *Partial hyperbolicity.* Throughout this paper, we will work with a *partially hyperbolic diffeomorphism* f, that is, a diffeomorphism admitting a non-trivial Tf-invariant splitting of the tangent bundle $TM = E^s \oplus E^c \oplus E^u$, such that all unit vectors $v^\sigma \in E^\sigma_x$ $(\sigma = s, c, u)$ with $x \in M$ satisfy

$$\|T_x f v^s\| < \|T_x f v^c\| < \|T_x f v^u\|$$

for some suitable Riemannian metric; f must also satisfy that $\|Tf|_{E^s}\| < 1$ and $\|Tf^{-1}|_{E^u}\| < 1$. We also want to introduce a stronger type of partial hyperbolicity. We will say that f is *absolutely partially hyperbolic* if it is partially hyperbolic and

$$\|T_x f v^s\| < \|T_y f v^c\| < \|T_z f v^u\|$$

for all $x, y, z \in M$.

For partially hyperbolic diffeomorphisms, it is well known that there are foliations $\mathcal{W}^\sigma$ tangent to the distributions E^σ for $\sigma = s, u$. The leaf of $\mathcal{W}^\sigma$ containing x will be called $W^\sigma(x)$, for $\sigma = s, u$.

2.2. *Dynamical coherence.* In general, it is not true that there is a foliation tangent to E^c. Sometimes there is no foliation tangent to E^c. Indeed, there may be no foliation tangent to E^c, even if $\dim E^c = 1$ (see [**28**]). We will say that f is *dynamically coherent* if there exist invariant foliations $\mathcal{W}^{c\sigma}$ tangent to $E^{c\sigma} = E^c \oplus E^\sigma$ for $\sigma = s, u$. Note that by taking the intersection of these foliations, we obtain an invariant foliation $\mathcal{W}^c$ tangent to E^c that subfoliates $\mathcal{W}^{c\sigma}$ for $\sigma = s, u$. In this paper, all partially hyperbolic diffeomorphisms will be dynamically coherent.

2.3. *Accessibility.* We will say that a set X is *σ-saturated* if it is a union of leaves of the strong foliations $\mathcal{W}^\sigma$ for $\sigma = s$ or u. We also say that X is *su-saturated* if it is both s- and u-saturated. The accessibility class of the point $x \in M$ is the minimal su-saturated set containing x. If there is some $x \in M$ whose accessibility class is M,

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then the diffeomorphism f is said to have the *accessibility property*. This is equivalent to saying that any two points of M can be joined by a path which is piecewise tangent to E^s or to E^u.

2.4. *Entropy.* Let $n \in \mathbb{N}$, $\delta > 0$ and $K \subset M$ a non-empty compact set. A finite subset E is (n, δ)-separated if for $x, y \in E$, $x \neq y$, we have $\max_{i=0,\dots,n} d(f^i(x), f^i(x)) \geq \delta$. Let

$$h_n(f, K, \delta) = \sup\{\#E;\ E \subset K \text{ is } (n, \delta)\text{-separated}\}$$

and

$$h(f, K) = \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log h_n(f, K, \delta).$$

When $K = M$, call $h(f, M) = h_{\text{top}}(f)$ the *topological entropy* of f. It is well known that $h(f, K) = 0$ if K is a subset of a curve having all its iterates with uniformly bounded length.

The variational principle states that $\sup\{h_\mu(f);\ \mu \text{ is } f\text{-invariant}\} = h_{\text{top}}(f)$ if f is a continuous map of a compact metric space. We will say that an *ergodic* probability μ satisfying $h_\mu(f) = h_{\text{top}}(f)$ is an *entropy-maximizing measure*. We remind the reader that, for simplicity of the exposition, we are supposing that entropy-maximizing measures are ergodic. In fact, if the metric entropy of a measure equals the topological entropy, the ergodic decomposition and the variational principle imply that almost all ergodic components are entropy maximizing. Conversely, the metric entropy of a convex combination of entropy-maximizing measures has maximal entropy. In other words, the measures whose metric entropies equal the topological entropy form a convex set.

3. *The quotient dynamics*

One of the important issues in our work is the dynamics induced in the quotient space $M^* := M/\mathcal{W}^c$. Let $\pi : M \to M^*$ be the quotient map, M^* be equipped with the quotient topology and f^* be the dynamics induced on M^*. For a general partially hyperbolic diffeomorphism with not necessarily compact central foliation, this quotient space may be even non-Hausdorff. However, when all leaves are compact, we hope that M^* (this result was recently proved by Carrasco when the center bundle is one-dimensional, see [**8**]). For three-dimensional manifolds, Epstein proved that the volume of leaves of any one-dimensional foliation $\mathcal{W}$ by compact leaves is uniformly bounded and every leaf has an arbitrarily small saturated neighborhood [**13**]. Then the quotient is Hausdorff and, moreover, it is homeomorphic to a closed surface. In other words, $\mathcal{W}$ is a Seifert fibration without boundary.

3.1. *Simple example.* Before going into the more general case, let us present a simple example of diffeomorphisms satisfying the hypothesis of our theorem. Let

$$A := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

and $\phi : \mathbb{T}^2 \to S^1$. Then the Skew product $F : \mathbb{T}^2 \times S^1 \to \mathbb{T}^2 \times S^1$, $F(x, \theta) = (A(x), \theta + \phi(x))$ is a partially hyperbolic diffeomorphism with compact central leaves. By standard Hirsch–Pugh–Shub theory, any C^1-perturbation of F is partially hyperbolic with compact

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central leaves and leaf conjugated to F. In particular, if G is C^1-close to F, then $\mathbb{T}^3/\mathcal{F}^c(G)$ is homeomorphic to $\mathbb{T}^2$ and the dynamics induced in the quotient is conjugate to A.

We mention that, by a result of Hammerlindl [**16**], this leaf conjugacy can be obtained not only for perturbations but also for any other absolutely partially hyperbolic diffeomorphism having the same linear part. He has also announced a similar result for three-dimensional nilmanifolds. In particular, he obtained that any absolutely partially hyperbolic diffeomorphism of one of these manifolds, which is known to be dynamically coherent [**5, 26**], has compact center leaves.

The only hypothesis that *a priori* remains unsatisfied is the accessibility property, but accessibility is an open and dense property in this context (see [**6, 25**]).

3.2. *Orientability of bundles.* From now on, we will assume that M, E^u, E^s, E^c are orientable and that f preserves the orientation of these bundles. Let us show that these assumptions are not restrictive. Suppose that Theorem 1 is valid under these conditions. Now, let $g : N \to N$ be a diffeomorphism satisfying the hypothesis of Theorem 1. By taking a finite covering, we can obtain a diffeomorphism $\overline{g} : \overline{N} \to \overline{N}$ such that $\overline{N}$ and the corresponding invariant bundles are orientable, and $h = \overline{g}^2$ preserves their orientations. Observe that accessibility of g implies accessibility of $\overline{g}$ and so accessibility of h. Then h satisfies one of the two possibilities of the dichotomy given by Theorem 1.

Suppose that μ is an entropy-maximizing measure for h with zero center Lyapunov exponent. Uniqueness of the entropy-maximizing measure of h implies that μ is $\overline{g}$-invariant and, of course, it has null center Lyapunov exponent. Moreover, μ is the unique entropy-maximizing measure for $\overline{g}$. Thanks to the Rokhlin disintegration, μ projects onto a g-invariant measure ν. It is not difficult to see that ν is an entropy-maximizing measure of g. Uniqueness is a consequence of the uniqueness of the entropy-maximizing measure of $\overline{g}$. Indeed, let $\tilde{\nu}$ be entropy maximizing for g with null center Lyapunov exponent. Then there exists $\tilde{\mu}$, a $\overline{g}$-invariant lift of $\tilde{\nu}$. On the one hand, the finiteness of the covering gives that $h_{\tilde{\nu}} = h_{\tilde{\mu}}$ and $h_\nu = h_\mu$. On the other hand, since $\tilde{\nu}$ and ν are entropy maximizing, we have that $h_{\tilde{\nu}} = h_\nu$, leading to $h_\mu = h_{\tilde{\mu}}$. The uniqueness of the entropy-maximizing measure of $\overline{g}$ implies $\tilde{\mu} = \mu$. Bernoullicity of μ for h implies Bernoullicity of μ for $\overline{g}$, and (g, ν) is also Bernoulli since it is a factor of $(\overline{g}, \mu)$. Finally, if g has an entropy-maximizing measure with zero center exponent, then h also has an entropy-maximizing measure with zero center exponent. This implies uniqueness for the maximizing measure of h and the previous argument gives the uniqueness for g.

Assume now that h has two entropy-maximizing measures, μ^+ and μ^-, with positive and negative center exponent, respectively. Now, $(\mu^\sigma + \overline{g}_*\mu^\sigma)/2$, with $\sigma = +, -$, are entropy-maximizing measures for $\overline{g}$ with positive and negative center exponent, respectively. As in the previous argument, these measures project onto entropy-maximizing measures for g that are different, because the center exponent of one of them is positive and the other one is negative. This gives the second part of the dichotomy for g.

3.3. *M is a nilmanifold.* Let $f : M \to M$ be a partially hyperbolic diffeomorphism with $\dim(M) = 3$. Recall that we are assuming that f is dynamically coherent and

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that the center manifolds are compact, i.e., $W^c(x)$ is diffeomorphic to $\mathbb{S}^1$ for every $x \in M$. Moreover, as we have just shown, for our purposes it is enough to consider that M, E^u, E^s, E^c are orientable and that f preserves the orientation of these bundles.

$\mathcal{W}^c$ is a foliation by circles of a closed 3-manifold. As we have already said, Epstein's result [**13**] implies that $\mathcal{W}^c$ is a Seifert foliation. In our case, we can obtain more information from the partially hyperbolic structure. Since $W^c(x)$ is diffeomorphic to $\mathbb{S}^1$, we have that $W^{cs}(x) = W^s(W^c(x))$ is an immersed cylinder (observe that $\mathcal{W}^s$ is orientable, which prevents the presence of Möbius' strips) and thus the foliation $\mathcal{W}^c$ is locally trivial when restricted to $W^{cs}(x)$. Analogously, $\mathcal{W}^c$ is locally trivial when restricted to $W^{cu}(x)$. These facts and the transversality of the foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ imply that $\mathcal{W}^c$ is locally trivial, that is, for every leaf $W^c(x)$, there exist a saturated neighborhood V_x and a homeomorphism $\varphi_x : V_x \to \mathbb{D}^2 \times \mathbb{S}^1$ such that φ_x sends leaves of $\mathcal{W}^c$ to circles of the form $\{w\} \times \mathbb{S}^1$. This implies that $\mathcal{W}^c$ is a Seifert fibration (as we mentioned before this was already known for us by Epstein's result) without singular leaves. Moreover, we have the following result.

THEOREM 3. *Let M be a closed orientable three-dimensional manifold and $f : M \to M$ a partially hyperbolic diffeomorphism. Suppose that f is dynamically coherent with compact center manifolds and that the bundles E^s, E^u, E^c are orientable. Then $\mathcal{W}^c$ is a Seifert fibration without singular leaves over $\mathbb{T}^2$. Moreover, f^* is conjugated to a hyperbolic automorphism of $\mathbb{T}^2$.*

Proof. On the one hand, the comments above imply that $\mathcal{W}^c$ is a Seifert fibration without singular leaves. On the other hand, center stable and center unstable manifolds are cylinders that go via the quotient projection to lines of M^*. Partial hyperbolicity implies that these lines are the stable and unstable 'manifolds' of a hyperbolic homeomorphism. By Franks' results [**14**] we obtain that the base surface of the Seifert fibration is a torus and the induced dynamics is conjugated to a hyperbolic automorphism of $\mathbb{T}^2$. □

We observe that, as a result of the classification of Seifert fibrations, an M satisfying the conclusions of Theorem 3 is diffeomorphic to a nilmanifold (including $\mathbb{T}^3$) or, equivalently, to a mapping torus of $\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ with $n \in \mathbb{N}$ (see for instance [**15**]).

As we have already mentioned, in case that $n \neq 0$, Hammerlindl has announced that he is able to prove that all center curves are compact if f is absolutely partially hyperbolic. Moreover, in [**27**], it is shown that any partially hyperbolic diffeomorphism of a three-dimensional nilmanifold has the accessibility property. These two facts lead to the following corollary.

COROLLARY 2. *Any absolutely partially hyperbolic diffeomorphism of a three-dimensional nilmanifold satisfies the dichotomy of Theorem 1.*

4. *Existence and structure of entropy-maximizing measures*

Take any invariant measure μ for f and let $\nu = \mu \circ \pi^{-1}$ (observe that ν is given by Rokhlin disintegration). By the Ledrappier–Walters variational principle [**18**],

$$\sup_{\hat{\mu}:\hat{\mu}\circ\pi^{-1}=\nu} h_{\hat{\mu}}(f) = h_\nu(f^*) + \int_{M^*} h(f, \pi^{-1}(y))\, d\nu(y).$$

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Since $\pi^{-1}(y)$, $y \in M^*$ is a circle and its iterates have bounded length, we have that $h(f, \pi^{-1}(y)) = 0$, that is, fibers do not contribute to the entropy. Hence, by the above equality and the well-known fact that $h_\mu(f) \geq h_\nu(f^*)$, we conclude that $h_\mu(f) = h_\nu(f^*)$. Using the usual variational principle, this implies that the topological entropies of f and f^* coincide. In particular, the set of entropy-maximizing measures of f coincides with the subset of ergodic measures which projects down to an entropy-maximizing measure for f^*.

In general, the existence of entropy-maximizing measures is a non-trivial question. However, by the comments above, in the setting of Theorem 1, it is easy to see that entropy-maximizing measures always exist. Let $\mathcal{M}$ be the set of probability measures on M and consider

$$\mathcal{M}_0 := \{\mu \in \mathcal{M} : \pi_*(\mu) = m\},$$

where m is the entropy-maximizing measure of f^*.

It is clear that $\mathcal{M}_0$ is a non-empty convex compact subset of $\mathcal{M}$ and the operator $f_* : \mathcal{M}_0 \to \mathcal{M}_0$ has a fixed point.

We finish this section by recalling some properties of entropy-maximizing measures of the quotient dynamics. An invariant probability measure m has local product structure if, for every x in the support of m, there exist measures m^s, m^u on $B^s(x)$, $B^u(x)$ such that $m|_{B(x)}$ is equivalent to $m^s \times m^u$. Let g be a linear Anosov diffeomorphism on $\mathbb{T}^2$. Then g is topologically mixing, and it is well known that it admits a unique maximizing measure m, which is the Lebesgue measure. Recall that, in general, by the Bowen and Margulis constructions of maximizing measures, any topologically mixing Anosov diffeomorphism admits a unique maximizing measure which has local product structure.

Now take any dynamics such as f^* (quotient to central foliation dynamics) which is topologically conjugate to a linear Anosov diffeomorphism. Let h be the conjugacy $f^* \circ h = h \circ g$. Then h_*m is the unique maximizing measure of f^*. Moreover, as h carries the stable (unstable) manifolds of g to stable (unstable) sets of f^*, we easily conclude that h_*m also has local product structure.

5. *Rigidity and zero Lyapunov exponents*

Here, we prove the first part of Theorem 1. Suppose that there exists an entropy-maximizing measure μ with $\lambda_c(\mu) = 0$ or, more generally, there exists a sequence of entropy-maximizing measures μ_n such that $\lambda_c(\mu_n) \to 0$. In both cases, we prove that (f, μ) is isomorphic to a rotation extension of an Anosov diffeomorphism A. In the second case, μ stands for a weak accumulation point of μ_n, which will also have maximal entropy.

To prove the above claim, we first apply the invariance principle of Avila–Viana [**1**] to conclude that μ admits a disintegration $[x \to \mu_x]$ which is s-invariant and u-invariant, and $x \to \mu_x$ varies continuously with x on $\operatorname{supp}(\pi_*(\mu)) = M^*$. We state the Avila–Viana result in a form adapted to our setting.

THEOREM 4. [**1**, Theorem D] *Let $f : M \to M$ be a partially hyperbolic diffeomorphism with one-dimensional compact center leaves. Assume that, given $y \in W^\sigma(x)$, the naturally defined σ-holonomy between $W^c(y)$ and $W^c(x)$ is a homeomorphism for $\sigma = s, u$. Let $(m_k)_k$ be a sequence of f-invariant probability measures all having the same projection μ^*, and suppose that μ^* is a probability measure that has local*

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product structure. Assume the sequence converges to some probability measure μ in the weak topology and $\int |\lambda_c(x)|\, dm_k(x) \to 0$ when $k \to \infty$. Then μ admits a disintegration $\{\mu_{x^*} : x^* \in M^*\}$ which is s-invariant and u-invariant, and whose conditional probabilities μ_{x^*} vary continuously with x^* on the support of μ^*.*

By accessibility and the holonomy invariance of $[x \to \mu_x]$, we conclude that $\operatorname{supp}(\mu_x) = W^c(x)$ and μ_x is atom free. Suppose that $\mathcal{W}^c$ is orientable and choose an orientation for it. Now, using the Avila–Viana–Wilkinson method, we get an S^1-action on M that commutes with f, i.e., $\rho_\theta : M \to M$ such that $\rho_\theta \circ f = f \circ \rho_\theta$, $\theta \in S^1$. We define ρ in the following way: $\rho_\theta(x) = y$, where y is the point in $W^c(x)$ such that the arc of the center manifold joining x with y has conditional measure $\mu_x([x, y]_c) = \theta$ (we are identifying $\mathbb{S}^1$ with [0, 1] mod 1), and we are measuring in the positive direction.

5.1. *Rigidity.* We will show that this action is conjugated to an action that is isometric on fibers. This will imply that (f, μ) is conjugated to an isometric (rotation) extension of a Bernoulli shift.

Our next goal is to prove that f is conjugated to an isometric extension of a hyperbolic automorphism of $\mathbb{T}^2$ if the center Lyapunov exponent of an entropy-maximizing measure μ is 0.

Recall that we used the invariance principle of Avila–Viana [**1**] to conclude that μ admits a disintegration $[x \mapsto \mu_x]$ which is s-invariant and u-invariant, and such that $x \mapsto \mu_x$ varies continuously with x on $\operatorname{supp}(\pi_*(\mu)) = M^* (= M/\mathcal{W}^c)$. Moreover, we have concluded that $\operatorname{supp}(\mu_x) = W^c(x)$ and the measures μ_x are atom free. Now we want to obtain the desired conjugacy. The main difficulty here is the absence of a global section of $\mathcal{W}^c$ (except in the case $M = \mathbb{T}^3$).

PROPOSITION 1. *Let $\mathcal{W}$ be a one-dimensional foliation by compact leaves of a closed manifold M and $\rho : \mathbb{S}^1 \times M \to M$ an (at least continuous) action such that:*

(1) *$\mathcal{W}$ has no singular leaves, that is, every leaf has a neighborhood where the foliation is a product;*
(2) *the leaves of $\mathcal{W}$ are the orbits of ρ; and*
(3) *given $x \in M$, the map $g \mapsto \rho(g, x)$ is a homeomorphism between $\mathbb{S}^1$ and $W^c(x)$.*

Then ρ is conjugated to an action $\overline{\rho}$ of $\mathbb{S}^1$ in a fiber bundle with fiber $\mathbb{S}^1$ and base $M/\mathcal{W}$ (the principal bundle associated to the fiber bundle $\mathcal{W}$). Moreover, $\overline{\rho}$ acts by isometries in the fibers.

Remark 1. The regularity of $\overline{\rho}$ depends on the regularity of $\mathcal{W}$ and ρ. In the case we are interested in, both $\mathcal{W}$ and ρ are Hölder continuous, and then we obtain the same regularity for $\overline{\rho}$.

Proof. We give an outline of the proof since the arguments are classical in fiber bundle theory (see for instance [**32**]). Let $p : M \to M/\mathcal{W}$ be the natural projection. Take a covering $\{V_i\}$ of $M/\mathcal{W}$ by balls such that $p^{-1}(V_i)$ are trivial. Let $\varphi_i : V_i \times \mathbb{S}^1 \to p^{-1}(V_i)$ be coordinate functions chosen in such a way that $p \circ \varphi_i = \pi$ (where π is the projection on the first coordinate of $V_i \times \mathbb{S}^1$) and conjugate the restrictions of ρ to $p^{-1}(V_i)$ with the actions $((x, \theta), \alpha) \mapsto (x, \theta)$ on $V_i \times \mathbb{S}^1$.

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Now, define the new bundle by taking the disjoint union $\bigsqcup_i V_i \times \mathbb{S}^1$ and the quotient by the following relation of equivalence: (x, θ) with $x \in V_i$ is equivalent to $(\overline{x}, \overline{\theta})$ with $\overline{x} \in V_j$ if $\varphi_j \varphi_i^{-1}(x, \theta) = (\overline{x}, \overline{\theta})$. This defines our new fiber bundle and the identity map of $\bigsqcup_i V_i \times \mathbb{S}^1$ induces a homeomorphism between M and the new bundle. It is not difficult to show that this homeomorphism conjugates ρ and $\overline{\rho}$. □

Since f commutes with ρ, we obtain, using the conjugacy given by the proposition above, a homeomorphism $\overline{f}$ conjugated to f that commutes with $\overline{\rho}$. This implies that $\overline{f}$ is an isometric extension of the dynamics f^* of f on $M/\mathcal{W}^c$. As we have shown before, this quotient dynamics is conjugated to an Anosov dynamics of $\mathbb{T}^2$. Let us call $\overline{\mu}$ the measure of M formed by the entropy-maximizing measure of f^* in the base, and such that the conditional measures along the fibers are Lebesgue. Then $\overline{\mu}$ is a rotation extension of a Bernoulli measure.

Accessibility of f and Brin's theory on compact group extensions imply that $(\overline{f}, \overline{\mu})$ is weakly mixing (see for instance [**12**, §2.2]). A result of Rudolph [**29**] implies that $(\overline{f}, \overline{\mu})$ (and then (f, μ)) is Bernoulli.

5.2. *Uniqueness.* It only remains to show that μ is the unique entropy-maximizing measure for f. Of course, this is equivalent to showing that $\overline{\mu}$ is the unique entropy-maximizing measure for $\overline{f}$. Suppose that λ is an entropy-maximizing measure for $\overline{f}$. Then, as was shown before, λ projects onto the unique entropy-maximizing measure of f^*, and accessibility implies that the conditional measures along the center leaves are supported on the whole $\mathbb{S}^1$, atom free and invariant by rotations (again by the Avila–Viana invariance principle [**1**]). This yields that the conditional measures are Lebesgue and $\lambda = \overline{\mu}$.

Remark 2. In this paper, we present a proof for $\dim(M) = 3$. The reader can easily verify that the same proof works for higher dimensions if one asks for the following conditions.

(1) The center foliation is one-dimensional by compact curves and without singular leaves.
(2) The quotient dynamics f^* is transitive.

Under these hypotheses, f^* is a transitive hyperbolic homeomorphism and then it has a unique entropy-maximizing measure that is locally a product of measures of the stable and unstable manifolds (Bowen's measure [**4, 20**]).

In particular, our results remain valid for accessible perturbations of a diffeomorphism of the form $g \times \mathrm{id}|_{\mathbb{S}^1}$, where g is a transitive Anosov diffeomorphism of a manifold of any dimension. Recall that accessibility is abundant in this setting (see [**6**]).

6. *Second part of the dichotomy*

Now we suppose that there exists $c_0 > 0$ and that for all entropy-maximizing measures $|\lambda_c(\mu)| > c_0$. If this were not the case, we would have a sequence of entropy-maximizing measures $(\mu_n)_n$ weakly* convergent to μ, and such that $\lambda_c(\mu_n) \to 0$. The arguments of the previous section would imply that f is conjugate to a rotation extension and has a unique measure with null center Lyapunov exponent.

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6.1. *More than one measure.* Our first step is to show that, given an entropy-maximizing measure μ^+ such that $\lambda_c(\mu^+) > 0$, there exists a naturally associated measure μ^- with $\lambda_c(\mu)^- < 0$.

LEMMA 1. *There exist a set S and $k \in \mathbb{N}$ such that $\mu^+(S) = 1$ and, for every $x \in S$, we have $\#S \cap \pi^{-1}(\pi(x)) = k$.*

Proof. The lemma is a corollary of Theorem II of [**30**]. We observe that since f is $C^{1+\alpha}$ and the center manifolds also vary continuously in the $C^{1+\alpha}$ norm, we have the conditions to apply the Ruelle–Wilkinson theorem. □

Lemma 1 implies that the conditional measures of almost every center manifold are supported in k points. As a consequence of the ergodicity of μ^+, we obtain that the weight of each one of these points is $1/k$.

We can also suppose that S is contained in the set of Pesin's regular points. Then each point $x \in S$ has a two-dimensional Pesin unstable manifold $W^u_P(x)$, and we will denote by $W_{\lambda_c}(x)$ the intersection of this unstable manifold with the center manifold of x, $W^u_P(x) \cap W^c(x)$ the Pesin center manifold of x. Since $W^c(x)$ is diffeomorphic to $\mathbb{S}^1$, we have that $W_{\lambda_c}(x)$ is an arc. Let $\tilde{x}$ denote the extreme point of $W_{\lambda_c}(x)$ in the positive direction (recall that we have an orientation for E^c). Now we have the conditions to define the measure μ^-. We want μ^- to be an entropy-maximizing measure, so it must project onto the same measure as μ^+. It will be enough to define the conditional measures of μ^- along the center manifolds. Given $x \in S$, we will assign weight $1/k$ to $\tilde{x}$. This defines the conditional measures of μ^-. The invariance of μ^+ and unstable manifolds imply invariance of μ^-, and ergodicity of μ^+ implies ergodicity of μ^-.

LEMMA 2. *Let μ^- be as above then $\lambda_c(\mu^-) < 0$.*

Proof. First of all, observe that $\lambda_c(\mu^-) \neq 0$. Indeed, if this was not the case, the arguments of §5 would imply that μ^- is the unique entropy-maximizing measure. Now, if $\lambda_c(\mu^-) > 0$, the Pesin unstable manifolds of $\tilde{x}$ intersect the Pesin unstable manifolds of x, contradicting that $\tilde{x}$ is in the boundary of $W_{\lambda_c}(x)$. This implies the lemma. □

6.2. *Finiteness.* We have proved that we have more than one entropy-maximizing measure if we have an entropy-maximizing measure with non-zero center exponent. All these entropy-maximizing measures are trivially shown to be finite extensions of Bernoulli shifts (the entropy-maximizing measure of the quotient dynamics is equivalent to a Bernoulli shift). Then the only thing we have to prove to finish the proof of the second item of Theorem 1 is the finiteness of the entropy-maximizing measures.

LEMMA 3. *Suppose that there is a sequence of entropy-maximizing measures $(\mu_n^+)_n$ with $\mu_i^+ \neq \mu_j^+$ for $i \neq j$. Moreover, suppose that $\lambda_c(\mu_n^+) > 0$ for all n, and that the sequence converges in the weak* topology to a measure μ. Then the sequence $(\mu_n^-)_n$ also converges to μ.*

Proof. Since the μ_n^+ are infinitely many different measures, we have that, for almost every center manifold, the lengths of the Pesin center manifolds of the points that support the corresponding conditional measures $\mu_n^+(x)$ go to 0. For this, it is important to observe

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that the Birkhoff theorem implies that the supports of these conditional measures do not intersect. Then we have that, given $\varepsilon > 0$, there exists an $N > 0$ such that, for $n > N$, the length of the Pesin center manifold of x is less than ε for every x in a set of center manifolds of quotient measure greater than $1 - \varepsilon$. This implies that for any continuous $\phi : M \to \mathbb{R}$, we have that $\int \phi \, d\mu_n^+ - \int \phi \, d\mu_n^- \to 0$. Thus $\mu_n^- \to \mu$ if $\mu_n^+ \to \mu$. □

Now let us show that there exists just a finite number of ergodic entropy-maximizing measures. This means that the compact subset of measures that project onto the entropy-maximizing measure of f^* is a finite simplex.

Suppose, by contradiction, that we have infinitely many entropy-maximizing measures. As we have already shown, the center Lyapunov exponents of these measures are non-zero. Then we can take an infinite sequence having the center exponent with the same sign. Suppose that for this sequence the center exponent is positive (if not, take the inverse). Since the set of invariant probabilities is sequentially compact, we obtain a sequence of measures $(\mu_n^+)_n$ converging to a measure μ and satisfying the hypothesis of Lemma 3. Then $(\mu_n^-)_n$ also converges to μ. On the one hand, as we have observed at the beginning of this section, $|\lambda_c(\mu_n^\sigma)| > c_0$, $\sigma = +, -$, so

$$\int \log \|Df|_{E^c}\| \, d\mu = \lim \int \log \|Df|_{E^c}\| \, d\mu_n^+ = \lim \lambda_c(\mu_n^+) \geq c_0.$$

On the other hand, thanks to the same observation, we have

$$\int \log \|Df|_{E^c}\| \, d\mu = \lim \int \log \|Df|_{E^c}\| \, d\mu_n^- = \lim \lambda_c(\mu_n^-) \leq -c_0.$$

These two inequalities obviously yield a contradiction, finishing the proof of the second part of Theorem 1.

6.3. *End of the proof.* To finish the proof of Theorem 1, we only need the following observations.

(1) Accessibility is an open and dense property for systems with a one-dimensional center (see [**6, 11**]).

(2) The quotient dynamics is conjugated to a hyperbolic automorphism; in particular, it is expansive (for any metric compatible with its topology). This observation implies that there exists a constant α such that for any pair of center leaves, there is an iterate so that the distance between these leaves is greater than α. Now, we can easily conclude that the center foliation is plaque expansive. The Hirsch–Pugh–Shub theory on normally hyperbolic foliations implies that, in this setting, dynamical coherence is an open property.

(3) If we are in the first case of the dichotomy, f has no hyperbolic periodic points. It is easy to see that in this setting (there are periodic leaves diffeomorphic to $\mathbb{S}^1$), the diffeomorphisms having hyperbolic periodic points form an open and dense set.

7. *Proof of Theorem 2 and Corollary 1*

Proof of Theorem 2. Let $A : \mathbb{T}^2 \to \mathbb{T}^2$ be a hyperbolic automorphism and consider $F := A \times \mathbb{S}^1 : \mathbb{T}^2 \times \mathbb{S}^1 \to \mathbb{T}^2 \times \mathbb{S}^1$. Bonatti and Díaz [**3**] have shown that F can be approximated by robustly mixing diffeomorphisms. In other words, there exists an open set of mixing partially hyperbolic diffeomorphisms $\mathcal{M}$ such that F belongs to its closure.

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The set of accessible diffeomorphisms $\mathcal{A}$ is open and dense in the set of partially hyperbolic diffeomorphisms with a one-dimensional center. Then $\mathcal{N} = \mathcal{M} \cap \mathcal{A} \neq \emptyset$. The diffeomorphisms of $\mathcal{N}$ satisfy the hypothesis of Theorem 1, which implies that the diffeomorphisms of an open-ended dense subset of $\mathcal{N}$ satisfy the second item of the dichotomy. This proves Theorem 2. □

Proof of Corollary 1. If we neglect a set of null measure of fibers, the projection π induces a conjugation between the support of an entropy-maximizing measure and a linear automorphism of the torus times a finite permutation. □

8. *Questions and remarks*

8.1. *Noncompact center leaves.* As we mentioned in the introduction, there are some advances in the case that the diffeomorphism is homotopic to Anosov (see [**7, 34**]). In [**34**], Ures' result uses that the invariant foliations are quasi-isometric. Then the following questions remain open.

Question 2. Let f be a partially hyperbolic diffeomorphism of $\mathbb{T}^3$ homotopic to a hyperbolic automorphism. Is it always true that f has a unique entropy-maximizing measure? Or (which is presumably equivalent in this case), is its topological entropy equal to the entropy of its linear part?

A similar question, in the sense of generalizing Theorem 1, remains open for the case that f is homotopic to a hyperbolic diffeomorphism of $\mathbb{T}^2$ times the identity of $\mathbb{S}^1$ (or to an automorphism of a nilmanifold) in the case that f is dynamically incoherent. Observe that the dynamically incoherent examples of [**28**] are Axiom A and satisfy the conclusions of the second part of Theorem 1.

A different direction for future development is the case of perturbations of the time-one map of an Anosov flow. One of the difficulties in this case seems to be the fact that the entropy is not constant when the time varies. An interesting question would be the following.

Question 3. Is the topological entropy generically locally constant in a neighborhood of the time-one map of an Anosov flow?

In [**17**], Hua *et al* gave some topological conditions to obtain that the entropy is locally constant, but these conditions are not valid in the neighborhood of the preceding question.

8.2. *Symplectic case.* There are many interesting problems in the symplectic context (such as the n-body problem), where a partially hyperbolic invariant subset appears. By definition, a partially hyperbolic subset is a compact invariant subset with a partially hyperbolic decomposition of the tangent space satisfying similar conditions to the partially hyperbolic diffeomorphisms. As a simple and relevant model, consider a symplectic manifold N and $g : N \to N$, a symplectomorphism close to the identity. Now take $f : M \to M$, any diffeomorphism with a transitive hyperbolic invariant subset $\Lambda \subsetneq M$. Then $\Lambda \times N$ is a partially hyperbolic subset for $F := f \times g$. By the Hirsch–Pugh–Shub theory, this subset has a continuation after C^1-perturbations of F. (See [**22**] for mixing properties of these kind of subsets.) Let N be a surface with a symplectic form. Then, either both central Lyapunov exponents of F (for any invariant measure) vanish, or the sum of the central Lyapunov exponents is equal to zero.

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The invariant subsets under consideration have zero Lebesgue measure. The class of entropy-maximizing measures is a natural object of study for the dynamics of such subsets. Tahzibi and Nassiri conjectured that F is in the closure of an open set U such that any $G \in U$ has a partially hyperbolic set satisfying the following dichotomy: either there exists an ergodic entropy-maximizing measure with zero Lyapunov exponents (and in this case it is the unique maximizing measure) or there are finitely many (more than one) ergodic maximizing measures, all of them with non-zero Lyapunov exponents.

One of the difficulties in dealing with this conjecture is the lack of the accessibility property in this context. However, the strong transitivity property for the holonomy (stable and unstable) semigroup [**33**] is helpful in approaching the problem.

8.3. *Number of maximizing measures in Theorem 1.* In general, we can obtain any even number of entropy-maximizing measures by just taking the product of an Anosov diffeomorphism of $\mathbb{T}^2$ and a Morse–Smale diffeomorphism of $\mathbb{S}^1$ (an odd number of measures can be obtained if, for instance, we allow that f does not preserve the center orientation). Then the relevant question is as follows.

Question 4. Are there topologically mixing f satisfying the hypothesis of Theorem 1 and with more than two entropy-maximizing measures? Is it true that generically f has just two such measures?

8.4. *Conservative case.* According to [**2**], if f satisfies the hypothesis of Theorem 1 and is conservative (it preserves a smooth measure m), we have three possibilities (recall that accessibility implies K).

(1) m has non-zero center exponents, and then it is Bernoulli by Pesin's well-known results.
(2) The center exponent vanishes and the center foliation is absolutely continuous. It is proved that f is smoothly conjugated to a rotation extension of a Bernoulli system, and then it is Bernoulli thanks to a result of Rudolph [**29**].
(3) The center exponent vanishes but the center foliation is not absolutely continuous. The conditional measures along the center foliation are atomic and the projection of m is not locally a product in the quotient dynamics.

In the third case, it is unknown if m is Bernoulli. A negative answer would be very interesting because there is no example of a conservative K-diffeomorphism that is not Bernoulli in dimension three.

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References

[1] A. Avila and M. Viana. Extremal Lyapunov exponents: an invariance principle and applications. *Invent. Math.* **181** (2010), 115–178.
[2] A. Avila, M. Viana and A. Wilkinson. Absolute continuity, Lyapunov exponents, and rigidity. *Preprint*.

https://doi.org/10.1017/S0143385711000757 Published online by Cambridge University Press

[3] C. Bonatti and L. J. Díaz. Persistent nonhyperbolic transitive diffeomorphisms. *Ann. of Math.* (2) **143**(2) (1996), 357–396.
[4] R. Bowen. Markov partitions for Axiom A diffeomorphisms. *Amer. J. Math.* **92** (1970), 725–747.
[5] M. Brin, D. Burago and S. Ivanov. Dynamical coherence of partially hyperbolic diffeomorphisms of the 3-torus. *J. Mod. Dyn.* **3** (2009), 1–11.
[6] K. Burns, F. Rodriguez Hertz, M. A. Rodriguez Hertz, A. Talitskaya and R. Ures. Density of accessibility for partially hyperbolic diffeomorphisms with one-dimensional center. *Discrete Contin. Dyn. Syst.* **22**(1–2) (2008), 75–88.
[7] J. Buzzi, T. Fisher, M. Sambarino and C. Vasquez. Maximal entropy measures for certain partially hyperbolic, derived from Anosov systems. *Ergod. Th. & Dynam. Sys.* to appear.
[8] P. Carrasco. Compact dynamical foliations. *PhD Thesis*, University of Toronto, 2010.
[9] W. Cowieson and L.-S. Young. SRB measures as zero-noise limits. *Ergod. Th. & Dynam. Sys.* **25**(4) (2005), 1115–1138.
[10] L. J. Díaz and T. Fisher. Symbolic extensions for partially hyperbolic diffeomorphisms. *Discrete Contin. Dyn. Syst.* **29** (2011), 1419–1441.
[11] P. Didier. Stability of accessibility. *Ergod. Th. & Dynam. Sys.* **23**(6) (2003), 1717–1731.
[12] D. Dolgopyat. On mixing properties of compact group extensions of hyperbolic systems. *Israel J. Math.* **130** (2002), 157–205.
[13] D. B. A. Epstein. Periodic flow on three-manifolds. *Ann. of Math.* (2) **95**(1) (1972), 66–82.
[14] J. Franks. Anosov diffeomorphisms. *Global Analysis (Proceedings of Symposia in Pure Mathematics, XIV, Berkeley, CA, 1968)*. American Mathematical Society, Providence, RI, 1970, pp. 61–93.
[15] A. Hatcher. Notes on basic 3-manifold topology. Hatcher's webpage at Cornell Math. Dep. http://www.math.cornell.edu/∼hatcher/3M/3M.pdf.
[16] A. Hammerlindl. Leaf conjugacies of the torus. *Ergod. Th. & Dynam. Sys.* to appear and *PhD Thesis*, University of Toronto, 2009.
[17] Y. Hua, R. Saghin and Z. Xia. Topological entropy and partially hyperbolic diffeomorphisms. *Ergod. Th. & Dynam. Sys.* **28**(3) (2008), 843–862.
[18] F. Ledrappier and P. Walters. A relativised variational principle for continuous transformations. *J. Lond. Math. Soc.* (2) **16**(3) (1977), 568–576.
[19] R. Mañé. Contributions to the stability conjecture. *Topology* **17** (1978), 383–396.
[20] G. Margulis. *On Some Aspects of the Theory of Anosov Systems (Springer Monographs in Mathematics, VII)*. Springer, New York, 2004, with a survey by Richard Sharp: periodic orbits of hyperbolic flows.
[21] M. Misiurewicz. Diffeomorphism without any measure with maximal entropy. *Bull. Acad. Polon. Sci., Sér. Sci. Math., Astr. et Phys.* **21** (1973), 903–910.
[22] M. Nassiri and E. Pujals. Robust transitivity in Hamiltonian dynamics. *Ann. Sci. ENS* to appear.
[23] S. Newhouse. Continuity properties of entropy. *Ann. of Math.* (2) **129** (1989), 215–235.
[24] S. Newhouse and L. S. Young. Dynamics of certain skew products. *Geometric Dynamics (Rio de Janeiro, 1981) (Lecture Notes in Mathematics, 1007)*. Springer, Berlin, 1983, pp. 611–629.
[25] V. Niţică and A. Török. An open dense set of stably ergodic diffeomorphisms in a neighborhood of a non-ergodic one. *Topology* **40** (2001), 259–278.
[26] K. Parwani. On 3-manifolds that support partially hyperbolic diffeomorphisms. *Nonlinearity* **23**(3) (2010), 589–606.
[27] F. Rodriguez Hertz, M. Rodriguez Hertz and R. Ures. Partial hyperbolicity and ergodicity in dimension three. *J. Mod. Dyn.* **2** (2008), 187–208.
[28] F. Rodriguez Hertz, M. A. Rodriguez Hertz and R. Ures. A non-dynamical coherent example in $\mathbb{T}^3$. in preparation.
[29] D. J. Rudolph. Classifying the isometric extensions of a Bernoulli shift. *J. Anal. Math.* **34** (1978), 36–60.
[30] D. Ruelle and A. Wilkinson. Absolutely singular dynamical foliations. *Comm. Math. Phys.* **219** (2001), 481–487.
[31] M. Shub. Topologically transitive diffeomorphisms on $\mathbb{T}^4$. *Lecture Notes on Math.* **206** (1971), 39.
[32] N. Steenrod. *The Topology of Fibre Bundles (Princeton Mathematical Series, 14)*. Princeton University Press, Princeton, NJ, 1951.
[33] M. Nassiri and A. Tahzibi. Ergodic properties of partially hyperbolic sets. in preparation.
[34] R. Ures. Intrinsic ergodicity of partially hyperbolic diffeomorphisms with hyperbolic linear part. *Proc. Amer. Math. Soc.* to appear.
[35] B. Weiss. Intrinsically ergodic systems. *Bull. Amer. Math. Soc.* **76** (1970), 1266–1269.

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