103.20 Geometric problems leading to power means

Let p be a nonzero real number. The power mean (sometimes referred to as a generalised mean) of two positive numbers a and b is defined by

$$M_p(a, b) = \left(\frac{a^p + b^p}{2}\right)^{1/p}.$$

Some familiar means that occur as power means are the following:

arithmetic mean when $p = 1$: $M_1(a, b) = \dfrac{a + b}{2}$;

geometric mean when $p = 0$: $M_0(a, b) = \lim\limits_{p \to 0} M_p(a, b) = \sqrt{ab}$;

harmonic mean when $p = -1$: $M_{-1}(a, b) = \left(\dfrac{a^{-1} + b^{-1}}{2}\right)^{-1} = \dfrac{2ab}{a + b}$;

root mean square when $p = 2$: $M_2(a, b) = \sqrt{\dfrac{a^2 + b^2}{2}}$.

(We leave the elementary verification of the limit that appears in the geometric mean to the reader.) For the record, it is not difficult to verify that

$$\lim_{p \to -\infty} M_p(a, b) = \min\{a, b\} \quad \text{and} \quad \lim_{p \to \infty} M_p(a, b) = \max\{a, b\}.$$

In this Note, we consider some simple geometric problems from calculus that lead to power means.

As is evident from the definition, in informal terms, a power mean takes the usual average of the powers of a and b and then undoes the power. This is illustrated in Figure 1, where $\mu = \frac{1}{2}(a^p + b^b)$ is the average of a^p and b^p, and c represents the power mean $M_p(a, b)$.

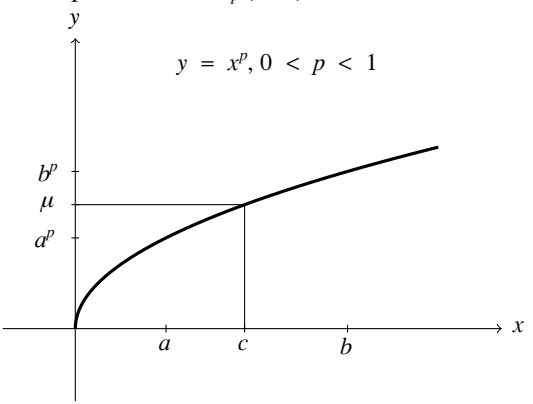


FIGURE 1

The figure illustrates how the Intermediate Value Theorem comes into play and thus indicates that the power mean expression can be applied to other continuous, strictly monotone, functions $f : (0, \infty) \to (0, \infty)$ via the

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expression $f^{-1}\left(\left(f(a) + f(b)\right)/2\right)$. However, we will only consider power functions.

For the record, since the shape of the graph of $y = x^r$ varies with the real number r, we consider two options in each of the figures that follow; the cases $r < 0$ and $r > 1$. We leave the figures for r values in the interval $[0, 1]$ for the reader to sketch; see Figure 1 for one example of these particular cases.

For our first problem, consider the curve $y = x^r$, where r is a given real number, and suppose that $0 < a < b$. Let R be the region that lies beneath the curve $y = x^r$ and above the x-axis on the interval $[a, b]$. We want to find a value for c so that the vertical line $x = c$ splits the region R into two parts of equal area. The situation is depicted in Figures 2 and 3; in each figure, the area of the darker region equals the area of the lighter region.

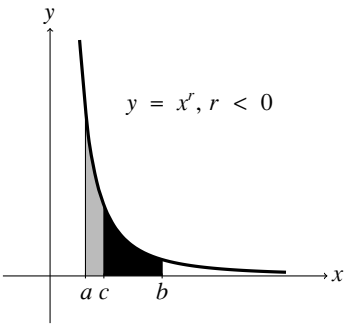


FIGURE 2

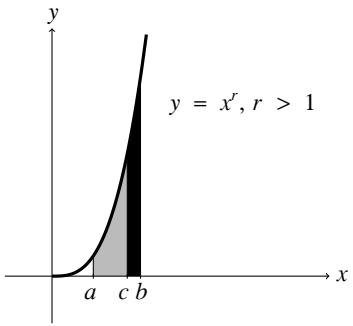


FIGURE 3

The number c can be determined as follows:

$$\int_a^c x^r\,dx = \int_c^b x^r\,dx \Rightarrow c^{r+1} - a^{r+1} = b^{r+1} - c^{r+1} \Rightarrow c = M_{r+1}(a, b).$$

The first implication above requires $r \neq -1$. It is easy to verify that $r = -1$ leads to the geometric mean; this may provide better clarification as to how the geometric mean corresponds to $p = 0$. Hence, this simple area problem leads to power means.

We now move up a dimension and consider volume. Let S be the solid that is generated when the region R is rotated around the y-axis (resulting in a warped bagel of sorts). What value for c splits the solid S into two parts of equal volume? In this case, the number c satisfies

$$\int_a^c (2\pi x)x^r\,dx = \int_c^b (2\pi x)x^r\,dx \Rightarrow c^{r+2} - a^{r+2} = b^{r+2} - c^{r+2} \Rightarrow c = M_{r+2}(a, b),$$

where $r = -2$ is now the exceptional case leading to the geometric mean. Similarly, let T be the solid that is generated when the region R is rotated around the x-axis (resulting in a truncated volcano lying on its side). What value for c splits the solid T into two parts of equal volume? For this situation, the number c satisfies

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$$\int_a^c \pi\left(x^r\right)^2 dx = \int_c^b \pi\left(x^r\right)^2 dx \Rightarrow c^{2r+1} - a^{2r+1} = b^{2r+1} - c^{2r+1} \Rightarrow c = M_{2r+1}(a,b),$$

where $r = -\frac{1}{2}$ is the exceptional case leading to the geometric mean.

For these problems, we have explicitly seen the cases in which the solution is the geometric mean. The reader may find it interesting to identify the specific values of r that lead to the arithmetic mean, the harmonic mean and the root mean square in each of these scenarios.

We next consider another area problem for which the appearance of power means is not quite so obvious. As before, consider the curve $y = x^r$, where $r \neq 1$ is a given real number, and suppose that $0 < a < b$. Let R be the region bounded by the curves $y = x^r$, $y = ax$ and $y = bx$. We want to find a value for c so that the line $y = cx$ splits the region R into two parts of equal area. The situation is depicted in Figures 4 and 5, where $\alpha = a^{1/(r-1)}$ and $\beta = b^{1/(r-1)}$. (The relative sizes of α and β depend on whether or not $r > 1$.) As previously, the area of the darker region equals the area of the lighter region in each of the figures.

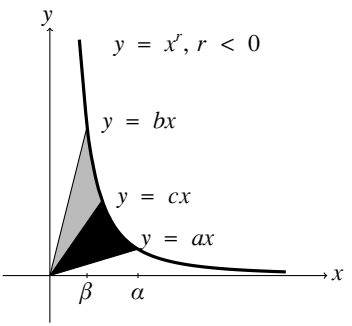


FIGURE 4

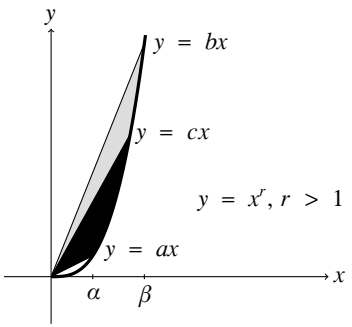


FIGURE 5

Referring to Figure 4 and assuming that $r \neq -1$, the area of R is given by

$$\int_0^\beta (bx - ax)\,dx + \int_\beta^\alpha \left(x^r - ax\right)dx = \frac{1}{2}b\beta^2 - \frac{1}{2}a\alpha^2 + \frac{\alpha^{r+1} - \beta^{r+1}}{r+1}$$

$$= \left(\frac{1}{2} - \frac{1}{r+1}\right)\left(\beta^{r+1} - \alpha^{r+1}\right).$$

(Note that this last quantity is the product of two positive numbers when $r < -1$ and the product of two negative numbers when $-1 < r < 0$.) To determine the area of the lighter shaded region, we simply need to replace the value of a with the value of c. Using $\gamma = c^{1/(r-1)}$, the number c must satisfy

$$\left(\frac{1}{2} - \frac{1}{r+1}\right)\left(\beta^{r+1} - \gamma^{r+1}\right) = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{r+1}\right)\left(\beta^{r+1} - \alpha^{r+1}\right)$$

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and thus

$$\gamma^{r+1} = \frac{\alpha^{r+1} + \beta^{r+1}}{2}.$$

In terms of a and b, this gives

$$c^{(r+1)/(r-1)} = \frac{a^{(r+1)/(r-1)} + b^{(r+1)/(r-1)}}{2}.$$

Letting $p = (r+1)/(r-1)$, we find that $c = M_p(a, b)$.

The reader can verify that the case in which $r = -1$ does indeed lead to the geometric mean. It is also easy to see that the case $r = 0$ (leading to the harmonic mean) merely involves triangles on the same base; there is no need for integration. We leave the analogous details for $0 < r < 1$ to the reader. For $r > 1$, the area of R is given by

$$\int_0^{\alpha} (bx - ax)\,dx + \int_{\alpha}^{\beta} \left(bx - x^r\right) = \frac{1}{2}b\beta^2 - \frac{1}{2}a\alpha^2 - \frac{\beta^{r+1} - \alpha^{r+1}}{r+1}$$

$$= \left(\frac{1}{2} - \frac{1}{r+1}\right)\left(\beta^{r+1} - \alpha^{r+1}\right).$$

Continuing as above, we once again find that $c = M_p(a, b)$, where $p = (r+1)/(r-1)$.

It is interesting to note that the equation $p = (r+1)/(r-1)$ is equivalent to $r = (p+1)/(p-1)$; the function is its own inverse. These equations show that the arithmetic mean in this case corresponds to $r = \infty$. Playing loosely with the symbols, we see that $y = x^{\infty}$ is equivalent to $x = y^0$. Drawing the figure that gives the region bounded by the lines $x = 1$, $y = ax$ and $y = bx$ does reveal (using the areas of triangles) that the arithmetic mean makes an appearance.

Let us now modify the above problem slightly. Let R be the region bounded by the curve $y = x^r$, the line joining the origin to the point (a, a^r), and the line joining the origin to the point (b, b^r). We want to find a value for c so that the line joining the origin to the point (c, c^r) splits the region R into two parts of equal area. This situation is depicted in Figures 6 and 7.

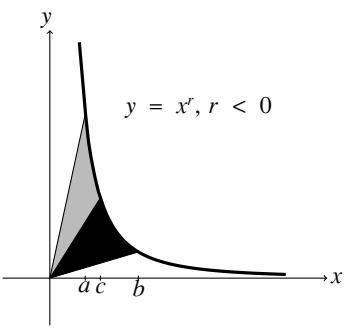


FIGURE 6

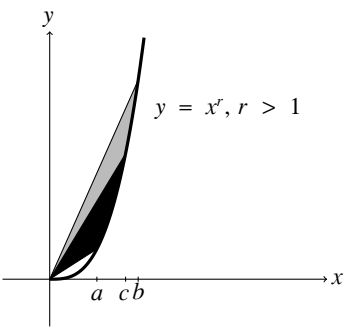


FIGURE 7

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In this case, referring to Figure 6 and assuming that $r \neq -1$, the area of R is given by

$$\int_0^a \left(a^{r-1}x - b^{r-1}x\right)dx + \int_a^b \left(x^r - b^{r-1}x\right)dx = \frac{1}{2}a^{r+1} - \frac{1}{2}b^{r+1} + \frac{b^{r+1} - a^{r+1}}{r+1}$$
$$= \left(\frac{1}{r+1} - \frac{1}{2}\right)\left(b^{r+1} - a^{r+1}\right).$$

It then follows (using steps as above) that $c = M_{r+1}(a, b)$. The result is the same when considering other r values; the case $0 < r < 1$ gives the same area formula as above whereas the case $1 < r < \infty$ introduces a negative sign. The most intriguing aspect of this conclusion is that this c value is the same as the one obtained using the ideas from Figures 2 and 3. A reason for this fact can be discovered by referring to Figure 8, where the various lines divide the region into eight distinct parts.

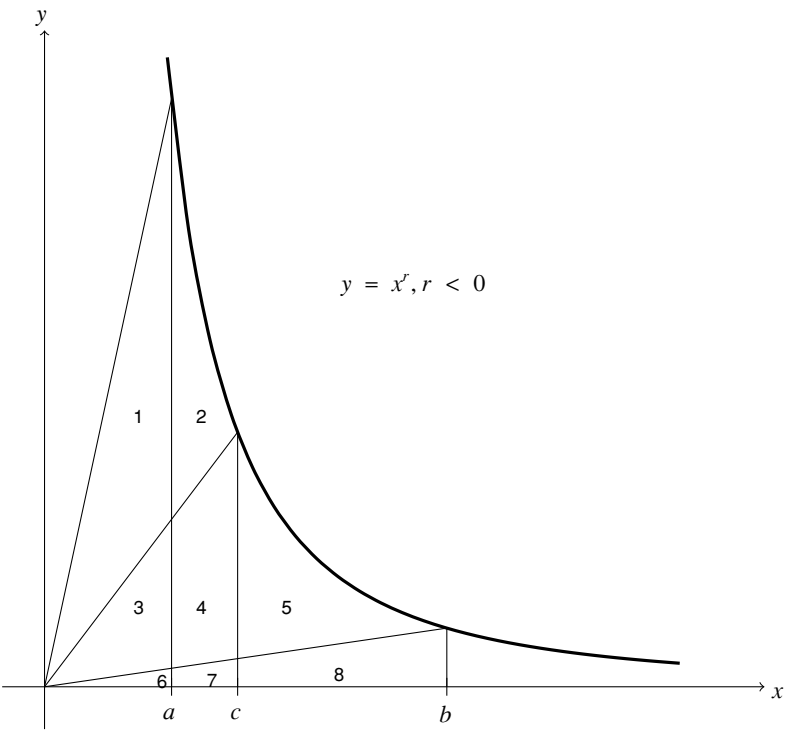


FIGURE 8

Given that the equation $A_2 + A_4 + A_7 = A_5 + A_8$ is satisfied (the result illustrated in Figure 2), we want to prove that $A_1 + A_2 = A_3 + A_4 + A_5$ (the result illustrated in Figure 6). From the first equality, it easily follows that $c^{r+1} - a^{r+1} = b^{r+1} - c_{r+1}$. Noting the appearance of the areas of triangles, we then have

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$$\begin{aligned} A_1 + A_2 &= (A_2 + A_4 + A_7) + (A_1 + A_3 + A_6) - (A_3 + A_4 + A_6 + A_7) \\ &= (A_5 + A_8) + \tfrac{1}{2}a^{r+1} - \tfrac{1}{2}c^{r+1} \\ &= (A_5 + A_8) + \tfrac{1}{2}c^{r+1} - \tfrac{1}{2}b^{r+1} \\ &= (A_5 + A_8) + (A_3 + A_4 + A_6 + A_7) - (A_6 + A_7 + A_8) \\ &= A_3 + A_4 + A_5. \end{aligned}$$

Hence, the c value is the same for these two cases. As a final comment regarding Figure 8, the reader may find it interesting to determine the values of r for which

$$(A_2 + A_4 + A_7) + (A_5 + A_8) = (A_1 + A_2) + (A_3 + A_4 + A_5)$$

that is, when the entire shaded region in Figure 2 equals the entire shaded region in Figure 6. (The reasoning is similar when $0 < r < 1$ and $r > 1$, but the number of regions differs in each case.)

With some thought and imagination, it is possible to create interesting and accessible problems for undergraduate students using the area and volume applications considered here. If nothing else, the examples presented in this Note provide some geometric motivation for power means associated with two positive numbers. We should mention that power means are defined for finite sets of non-negative numbers; see [1] for an extensive discussion of the properties of such means. (An Internet search also reveals a great deal of information about these generalised means.)

Reference

1. G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities* (2nd edn.), Cambridge University Press (1952).

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103.21 Closed form evaluation of a class of improper integrals

In the paper [1] which is part of a series of papers in a project to prove all the integrals in the book [2], the following integral is calculated ($n \geqslant 0$):

$$I = \int_0^{+\infty} \frac{\ln t}{(1 + t^2)^{n+1}}\, dt. \tag{1}$$

The calculation is based on properties of the gamma and digamma functions, and Feynman's trick of differentiating under the integral sign. In this Note we use a more straightforward method, based on properties of self-reciprocal polynomials, a topic that I first came across when still at school, and which is not very well-known.

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