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Concentrating standing waves for Davey–Stewartson systems

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In this paper, we study the existence and concentration behaviour of multi-peak standing waves for a singularly perturbed Davey–Stewartson system, which arises in the theory of shallow water waves. For this purpose, we first give a sharp threshold of the existence of ground-state solutions to the related limiting problem. Next, combining the penalization method and the regularity theory of elliptic equations, we construct a family of positive solutions concentrating around any prescribed finite set of local minima, possibly degenerate, of the potential. A feature of this analysis is that we do not need any uniqueness or non-degeneracy conditions for the limiting equation. To the best of our knowledge, this paper is the first study dealing with the study of concentrating solutions for Davey–Stewartson systems. We emphasize that with respect to the classical Schrödinger equation, the presence of a singular integral operator in the Davey–Stewartson system forces the implementation of new ideas to obtain the existence of multi-peak solutions.

Keywords: Concentration; Davey–Stewartson system; multi-peak solutions; singular integral operator

2020 Mathematics Subject Classification: Primary: 35J20 (primary)
Secondary: 35B25; 35J60; 76B15 (secondary)

1. Introduction

Nonlinear modulation of gravity-capillary waves travelling principally in one direction at the surface of a fluid leads to the Davey–Stewartson system for the wave amplitude and the induced mean flow. The Davey–Stewartson equations are a family of nonlinear Schrödinger type equations, which model the evolution of weakly nonlinear surface water waves travelling principally in one direction; see [**21**]. A rigorous derivation from the water wave problem in the modulational scaling regime is provided by Craig *et al.* [**18**]. As established in [**18**], according to the value of the ratio of the wavelength to the fired depth and of the normalized surface tension, the nonlinearity will be focusing with possible formation of solitons, or defocusing.

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This paper is a contribution to the mathematical theory of the water wave problem, in the framework of concentration phenomena for the Davey–Stewartson system. Our goal is a rigorous understanding of the multi-peak standing waves for a class of singularly perturbed Davey–Stewartson systems. A feature of the analysis developed in this paper is that we do not need any uniqueness or non-degeneracy conditions for the limiting equation. To the best of our knowledge, this study is the first paper dealing with the study of concentrating solutions for Davey–Stewartson systems.

1.1. Statement of the problem and previous results

We study the following singularly perturbed Davey–Stewartson system:

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u = bE_1(|u|^2)u - a|u|^\alpha u \text{ in } \mathbb{R}^N, \\ u > 0,\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where ε is a small positive parameter, $a \in \mathbb{R}$, $b > 0$, $0 < \alpha < 2^* - 2$ ($2^* = 2N/(N-2)$ if $N \geqslant 3$, $2^* = +\infty$ if $N = 2$), $N \in \{2, 3\}$, E_1 is the singular integral operator with symbol $\sigma_1(\xi) = \xi_1^2/|\xi|^2$, $\xi \in \mathbb{R}^N$, that is, $E_1(\psi)(x) = \mathcal{F}^{-1}(\sigma_1(\xi)\mathcal{F}(\psi)(\xi))(x)$, $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform operator and its inverse defined by

$$\mathcal{F}(\psi)(\xi) := \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \mathrm{e}^{-\mathrm{i}x\cdot\xi}\psi(x)\,\mathrm{d}x \quad \text{and}$$

$$\mathcal{F}^{-1}(\psi)(x) := \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \mathrm{e}^{\mathrm{i}\xi\cdot x}\psi(\xi)\,\mathrm{d}\xi$$

respectively. Moreover, the potential $V : \mathbb{R}^N \to \mathbb{R}$ satisfies:

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{x\in\mathbb{R}^N} V(x) = V_0 > 0$;

(V_2) There exist k mutually disjoint bounded domains Λ^i $(i = 1, 2, \ldots, k)$ such that

$$m_i := \inf_{\Lambda^i} V < \min_{\partial\Lambda^i} V$$

and we set $\mathcal{M}^i := \{x \in \Lambda^i :\ V(x) = m_i\}$. This kind of hypothesis on V was first introduced by del Pino and Felmer in [**23**] and Gui in [**29**].

Problem (1.1) arises from the study of standing waves for the following time-dependent Davey–Stewartson system:

$$\mathrm{i}\varepsilon\frac{\partial\psi(x,t)}{\partial t} = -\varepsilon^2\Delta\psi + W(x)\psi - bE_1(|\psi|^2)\psi + a|\psi|^\alpha\psi,\ x \in \mathbb{R}^N, \quad t > 0, \tag{1.2}$$

which appears as a mathematical model for the evolution of shallow water waves having one predominant direction of travel, see [**15**, **16**, **21**]. Here, t is the time, $0 < \varepsilon \ll 1$ denotes a small semi-classical parameter and $W(x)$ is an external trapping potential. The singular limiting regime where $\varepsilon \to 0$ yields the high-frequency

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asymptotics for

$$\begin{cases} \mathrm{i}\dfrac{\partial\psi(x,t)}{\partial t} = -\Delta\psi - bE_1(|\psi|^2)\psi + a|\psi|^{\alpha}\psi, \ (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\ \psi(0,x) = \psi_0(x) \end{cases} \tag{1.3}$$

in a weakly nonlinear scaling, see [**12**] for the details, in which the local well-posedness of (1.2) is proved when initial data is given by a superposition of ε-oscillatory plane waves. Furthermore, the unique solution to (1.2) can be approximated by (possibly countably many) slowly modulated plane waves.

In recent decades, a large amount of research has been devoted to the study of the Davey–Stewartson systems under different cases. The solvability in the Sobolev space $H^1(\mathbb{R}^2)$ of the Cauchy problem for the Davey–Stewartson systems (1.3) has been proved in [**27**] by using functional analytical methods. Guo and Wang [**30**] studied the local and global existence of solutions in the usual Sobolev space H^s with $1 \leqslant s \leqslant 2$ of problem (1.3). For the dynamics and analytic properties of solutions to the Davey–Stewartson systems, we refer the readers to [**15**, **16**, **26**, **36**–**38**] and references therein.

We are interested in high-frequency asymptotics of standing waves for Davey–Stewartson system (1.3). This leads to investigate the existence and concentration behaviour of peak solutions for an integro-differential equation involving a singular integral operator, namely (1.1).

When the singular integral operator E_1 in (1.1) vanishes, (1.1) becomes the following form of classical singularly perturbed Schrödinger equations, that is,

$$-\varepsilon^2\Delta u + V(x)u = f(u), \quad x \in \mathbb{R}^N, \ N \geqslant 1. \tag{1.4}$$

Many mathematicians proved the existence, concentration and multiplicity of peak solutions to (1.4). Floer and Weinstein [**25**] studied (1.4) in the case where $N = 1$, $f(u) = u^3$, $V \in L^{\infty}(\mathbb{R})$ with $\inf_{\mathbb{R}} V > 0$. They constructed a single-peak solution which concentrates around any given non-degenerate critical point of the potential V. Oh [**34**] extended this result in higher dimensions when $f(u) = u^p (1 < q < 2^* - 1)$ and the potential V belongs to a Kato class. Furthermore, Oh [**35**] proved the existence of multi-peak solutions which concentrate around any finite subsets of the non-degenerate critical points of V. Later, Ambrosetti *et al.* [**1**] lifted the non-degeneracy assumptions and obtained the existence of semiclassical standing wave solutions which concentrate around local minima or local maxima of V with non-degenerate nth-derivative, for some integer n. The arguments in [**1**, **25**, **34**, **35**] are mainly based on a Lyapunov–Schmidt reduction which requires uniqueness and non-degeneracy of the ground-state solutions for the following 'limiting equation':

$$-\Delta u + mu = f(u), \quad x \in \mathbb{R}^N, \ m > 0. \tag{1.5}$$

However, the uniqueness and non-degeneracy conditions are, in general, difficult to check. They are known only for a rather restricted class of nonlinearities f.

Rabinowitz [**40**] used the mountain pass theorem to show that problem (1.4) has a positive ground-state solution for $\varepsilon > 0$ small under the conditions:

(V_3) $V_\infty = \liminf_{|x|\to\infty} V(x) > V_0 = \inf_{x\in\mathbb{R}^N} V(x) > 0.$

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Rabinowitz's approach actually covers a broader class of nonlinearity. Then, Wang [**41**] proved that the positive ground-state solutions to (1.4) obtained in [**40**] must concentrate at global minima of V as $\varepsilon \to 0$. Under the same condition (V_3) on $V(x)$, Cingolani and Lazzo [**14**] proved the multiplicity of positive solutions to (1.4) by using Ljusternik–Schnirelmann theory. del Pino and Felmer [**22**] studied (1.4) with the conditions on V replaced by

(V_4) $\inf_{x\in\mathbb{R}^N} V(x) = \alpha_0 > 0$;

(V_5) There is a bounded domain Λ such that $V_0 := \inf_\Lambda V < \min_{\partial\Lambda} V$.

They proved that problem (1.4) has a positive bound state solution for $\varepsilon > 0$ small which concentrates around the local minima of V in Λ as $\varepsilon \to 0$. del Pino and Felmer [**23**] and Gui [**29**] obtained multi-peak solutions to (1.4) which exhibit concentration at any prescribed finite set of local minima, possibly degenerate, of the potential by gluing localized solutions due to Coti Zelati and Rabinowitz [**17**, proposition 3.4]. The authors of [**14**, **22**, **23**, **29**, **40**, **41**] mainly depend on the Ambrosetti–Rabinowitz condition: $\exists \mu > 2$, such that $0 < \mu \int_0^t f(s)\,\mathrm{d}s < tf(t)$ for $t > 0$ and the monotonicity condition: $f(t)/t (t > 0)$ is strictly increasing.

Byeon and Jeanjean [**6**] and Byeon *et al.* [**8**] showed the existence of positive solutions to (1.4) concentrating around local minima of V under the almost optimal conditions on f. More precisely, they assume that V satisfies (V_4)–(V_5) and f satisfies the celebrated Berestycki–Lions' type condition (see [**2**–**4**]). Under the same conditions as [**6**, **8**], Cingolani*et al.* [**13**] studied the multiplicity of positive bound state solutions to (1.4) and proved the existence of at least $\mathrm{cupl}(\mathcal{M}) + 1$ solutions concentrating around $\mathcal{M}$ as $\varepsilon \to 0$, where $\mathcal{M}$ is the set of minima of V in Λ. Byeon and Jeanjean [**7**] constructed multi-peak solutions to (1.4), which concentrate around prescribed finite set of local minima when the nonlinearity f satisfies only the Berestycki–Lions' type condition.

D'Aprile and Ruiz [**19**] constructed both positive and sign-changing multi-peak solutions to (1.4), which cluster around a non-degenerate saddle point of V via Lyapunov–Schmidt reduction when f is a subcritical nonlinearity and the uniqueness and non-degeneracy condition is required. D'Avenia *et al.* [**20**] used the penalty method to prove the existence of positive solutions to (1.4), which spike around saddle points of V when f is of subcritical growth and satisfies the Ambrosetti–Rabinowitz condition. Byeon and Tanaka [**9**] constructed positive solutions to (1.4) concentrating around an isolated saddle point or an isolated set of local maximum points of V under the Berestycki–Lions' type condition on $f \in C^1$. Byeon and Tanaka [**10**] developed a local variational and deformation argument to establish the existence of a family of positive solutions with peaks cluster near a local maximum of V when f satisfies the Berestycki–Lions' type condition.

To the best of our knowledge, the existence and concentration behaviour of multi-peak solutions to problem (1.1) has not ever been studied. The main difficulties lie in: (i) the appearance of the singular integral operator E_1 in Davey–Stewartson systems makes it much more complicated to prove the asymptotic relationship between a certain energy level ((3.4)) of the singularly perturbed problem and the

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ground-state level of the 'limiting equation', which is a key point in getting multi-peak solutions; and (ii) the indefinite sign of E_1 and a prevent us from obtaining a bounded Palais–Smale sequence and using the Nehari manifold respectively. As we shall see later, the above two aspects prevent us from using variational method in a standard way.

1.2. Main results

Before describing the organization of the paper and presenting our main results, we introduce the following set of admissible parameters

$$R_{m,b} := \{(\alpha, a) : 0 < \alpha < 2^* - 2 \text{ and } a < a^*_\alpha\},$$

where

$$a^*_\alpha = \begin{cases} +\infty, & \text{if } \alpha < 2, \\ b, & \text{if } \alpha = 2, \\ \left(\dfrac{b}{2}\right)^{\alpha/2} m^{(2-\alpha)/2} \left(\dfrac{\alpha+2}{\alpha}\right) \left(\dfrac{\alpha-2}{\alpha}\right)^{(\alpha-2)/2}, & \text{if } \alpha > 2. \end{cases}$$

In order to get multi-peak solutions to (1.1), we need to consider the existence of ground-state solutions to the associated 'limiting problem' of (1.1), which is given as

$$\begin{cases} -\Delta u + mu = bE_1(|u|^2)u - a|u|^\alpha u \text{ in } \mathbb{R}^N, \\ u > 0,\ u \in H^1(\mathbb{R}^N),\ m > 0 \end{cases} \tag{1.6}$$

with the corresponding functional

$$I_m(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{m}{2}\int_{\mathbb{R}^N} u^2 - \frac{b}{4}\int_{\mathbb{R}^N} E_1((u^+)^2)(u^+)^2 + \frac{a}{\alpha+2}\int_{\mathbb{R}^N} (u^+)^{\alpha+2}, \quad u \in H^1(\mathbb{R}^N).$$

If $(\alpha, a) \in R_{m,b}$, for technical reasons, we consider the following minimization problem

$$d_m := \inf\left\{\frac{1}{N}\int_{\mathbb{R}^N} |\nabla u|^2 : P_m(u) \leqslant 0, u \in H^1(\mathbb{R}^N)\backslash\{0\}\right\} \tag{1.7}$$

in the case of $N = 2$ and $N = 3$, where $P_m(u) = 0$ is the Pohozaev's identity of (1.6). Owing to the appearance of the factor $\sigma_1(\xi)$ in the definition of E_1, I_m is not invariant under rotations, which prevents the existence of radial solutions for problem (1.6). We use the concentration-compactness arguments due to Lions [**33**] to overcome the lack of compactness in the imbedding $H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ $(2 \leqslant p < 2^*)$ and obtain a minimizer of (1.7), after a scaling if necessary, the existence of ground-state solution to (1.6) follows. Denoting S_m the set of ground-state solutions U to (1.6) satisfying $U(0) = \max_{x\in\mathbb{R}^N} U(x)$, we then show that S_m is compact in $H^1(\mathbb{R}^N)$ (proposition 2.2). Moreover, we prove that problem (1.6) does not have any nontrivial solution for $(\alpha, a) \notin R_{m,b}$. The main result for the 'limiting equation' (1.6) is the following.

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THEOREM 1.1. (*i*) *Problem* (1.6) *has at least one positive ground-state solution for* $(\alpha, a) \in R_{m,b}$;

(*ii*) *Problem* (1.6) *does not have any nontrivial solution for* $(\alpha, a) \notin R_{m,b}$.

Note that Cipolatti in [**15**] has proved that if $a < a_\alpha$, (1.6) has at least one ground state for any $m > 0$, where

$$a_\alpha = \begin{cases} +\infty, & \text{if } 0 < \alpha < 2, \\ b, & \text{if } \alpha = 2, \\ b^{\alpha/2} m^{(2-\alpha)/2} \dfrac{2}{\alpha} \left(\dfrac{\alpha - 2}{\alpha} \right)^{(\alpha-2)/2}, & \text{if } 2 < \alpha < 2^* - 1. \end{cases}$$

Theorem 1.1 in the present paper gives a threshold a_α^* of the existence of ground-state solutions to (1.6), which is sharp comparing with the a_α in [**15**].

To study problem (1.1), we work with the following equivalent equation

$$\begin{cases} -\Delta v + V(\varepsilon x) v = b E_1(|v|^2) v - a|v|^\alpha v \text{ in } \mathbb{R}^N, \\ v > 0, \ v \in H_\varepsilon \end{cases} \tag{1.8}$$

with the functional

$$I_\varepsilon(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(\varepsilon x) v^2 - \frac{b}{4} \int_{\mathbb{R}^N} E_1((v^+)^2)(v^+)^2 + \frac{a}{\alpha + 2} \int_{\mathbb{R}^N} (v^+)^{\alpha+2}, \quad v \in H_\varepsilon,$$

where $(\alpha, a) \in R_{m,b}$ with $m := \max_{1 \leqslant i \leqslant k} m_i$ and

$$H_\varepsilon := \left\{ v \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\varepsilon x) v^2 < \infty \right\}.$$

Unlike [**40**], where the minimum of $V(x)$ is global and the nonlinear term $f(v)$ satisfies the Ambrosetti–Rabinowitz condition, the mountain pass theorem can be used globally, here in the present paper, the condition (V_2) is local and the Ambrosetti–Rabinowitz condition is no longer satisfied due to the indefinite sign of E_1 and a, we need to use a penalization method introduced by Byeon and Wang [**11**], which helps us to overcome the obstacle caused by the non-compactness due to the unboundedness of the domain $\mathbb{R}^N$ and the lack of Ambrosetti–Rabinowitz condition. To this end, we should modify the functional I_ε. Following [**6**, **7**, **11**], we define auxiliary functionals J_ε, J_ε^i $(i = 1, \ldots, k)$ on H_ε by $J_\varepsilon(v) := I_\varepsilon(v) + Q_\varepsilon(v)$ and $J_\varepsilon^i(v) := I_\varepsilon(v) + Q_\varepsilon^i(v)$, respectively (see § 3 for details). It will be shown that the functional Q_ε acts as a penalization to force the concentration phenomena to occur inside $\cup_{i=1}^k \Lambda^i$ (lemma 3.3).

Using a version of quantitative deformation lemma (lemma 3.6) to construct a special bounded and convergent Palais–Smale sequence of J_ε for $\varepsilon > 0$ small to get a critical point v_ε of J_ε. To verify the critical point v_ε of J_ε is indeed a solution to the original problem (1.8), we need to establish a uniform exponential decay of v_ε by using the classical 'bootstrap' technique (lemma B.1). Noting that v_ε consists essentially of k disjoints parts, each part is close to the compact set S_{m_i}

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$(i = 1, \dots, k)$. Namely, in our approach we take into account the shape and location of the solutions we expect to find. Thus, on the one hand, we benefit from the advantage of the Lyapunov–Schmidt reduction type approach, which is to discover the solution around a small neighbourhood of a well chosen first approximation. On the other hand, our approach, which is purely variational, does not require any uniqueness or non-degeneracy conditions.

Our main result is as follows:

THEOREM 1.2. *Assume that the potential V satisfies (V_1), (V_2) and $(\alpha, a) \in R_{m,b}$ with $m := \max_{1\leqslant i\leqslant k} m_i$, then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$, problem (1.1) has a positive bound state solution $u_\varepsilon \in H^1(\mathbb{R}^N)$. Moreover,*

(i) *there exist k local maximum points $x_\varepsilon^i \in \Lambda^i (i = 1, \dots, k)$ of u_ε such that for each i,*

$$\lim_{\varepsilon\to 0} \operatorname{dist}(x_\varepsilon^i, \mathcal{M}^i) = 0;$$

(ii) *$\exists C_1, C_2 > 0$, such that*

$$u_\varepsilon(x) \leqslant C_1 \exp\left(-\frac{C_2}{\varepsilon} \min_{1\leqslant i\leqslant k} |x - x_\varepsilon^i|\right),$$

where C_1, C_2 are independent of ε.

This paper is organized as follows. In § 2, we analyse the 'limiting problem' (1.6) and prove theorem 1.1. In § 3, we prove the main result theorem 1.2.

2. The limiting problem

The equation for $m > 0$

$$\begin{cases} -\Delta u + mu = bE_1(|u|^2)u - a|u|^\alpha u \text{ in } \mathbb{R}^N, \\ u > 0,\ u \in H^1(\mathbb{R}^N) \end{cases} \tag{2.1}$$

is the limiting problem of (1.1) with the corresponding functional

$$I_m(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{m}{2}\int_{\mathbb{R}^N} u^2 - \frac{b}{4}\int_{\mathbb{R}^N} E_1((u^+)^2)(u^+)^2 + \frac{a}{\alpha+2}\int_{\mathbb{R}^N} (u^+)^{\alpha+2}, \quad u \in H^1(\mathbb{R}^N).$$

In view of [**39**], if $u \in H^1(\mathbb{R}^N)$ is a weak solution to (2.1), the following Pohozaev's identity holds:

$$P_m(u) = \frac{(N-2)}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{m}{2}N\int_{\mathbb{R}^N} u^2 - \frac{b}{4}N\int_{\mathbb{R}^N} E_1((u^+)^2)(u^+)^2 + \frac{a}{\alpha+2}N\int_{\mathbb{R}^N} (u^+)^{\alpha+2} = 0.$$

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LEMMA 2.1. *If $(\alpha, a) \in R_{m,b}$, then I_m possesses the mountain pass geometry, that is,*

(i) *There exist $\rho_0, \alpha_0 > 0$ such that $I_m(u) \geqslant \alpha_0$ for all $u \in H^1(\mathbb{R}^N)$ with $\|u\|_{H^1(\mathbb{R}^N)} = \rho_0$;*

(ii) *$\exists \phi_0 \in H^1(\mathbb{R}^N)$ such that $I_m(\phi_0) < 0$.*

Proof. (i) By the definition of E_1 and Sobolev's imbedding theorem, we see that

$$I_m(u) \geqslant \frac{1}{2}\|u\|^2_{H^1(\mathbb{R}^N)} - C\|u\|^4_{H^1(\mathbb{R}^N)} - C\|u\|^{\alpha+2}_{H^1(\mathbb{R}^N)},$$

then taking $\rho_0, \alpha_0 > 0$ small, (i) holds.

(ii) We have the following cases:

Case 1: $\alpha < 2$ or $a \leqslant 0$. Fixing $u \in H^1(\mathbb{R}^N)\backslash\{0\}$, then for $t > 0$, $I_m(tu) \to -\infty$ as $t \to +\infty$. Choosing $t_0 > 0$ large to ensure that $I_m(t_0 u) < 0$, $t_0 u$ is the desired ϕ_0.

Case 2: $\alpha = 2$ and $a < a^*_\alpha = b$. Letting

$$C(u) := -b\int_{\mathbb{R}^N} E_1((u^+)^2)(u^+)^2 + a\int_{\mathbb{R}^N}(u^+)^4, \tag{2.2}$$

we select a positive $u \in H^1(\mathbb{R}^N)\backslash\{0\}$ and set $(u)^t(x) := t^{1/2}u(tx_1, x_2, \ldots, x_N)$, we see from Plancherel's theorem that

$$C((u)^t) := t\int_{\mathbb{R}^N}\left(-b\cdot\frac{t^2\xi_1^2}{t^2\xi_1^2 + |\xi'|^2} + a\right)\left|\mathcal{F}(|u|^2)\right|^2,$$

where $\xi' = (\xi_2, \ldots, \xi_N)$. Since

$$\lim_{t\to+\infty} -b\cdot\frac{t^2\xi_1^2}{t^2\xi_1^2 + |\xi'|^2} + a = a - b < 0,$$

we choose a large $t_0 > 0$ to ensure that $C((u)^{t_0}) < 0$ and then select a large $R_0 > 0$ such that $I_m(R_0(u)^{t_0}) < 0$, $R_0(u)^{t_0}$ is the desired ϕ_0.

Case 3: $\alpha > 2$ and $0 < a < a^*_\alpha = (b/2)^{\alpha/2}m^{(2-\alpha)/2}((\alpha+2)/\alpha)((\alpha-2)/\alpha)^{(\alpha-2)/2}$. Denoting

$$G_a(s) := \frac{m}{2}s^2 + \frac{a}{\alpha+2}s^{\alpha+2} - \frac{b}{4}s^4,$$

we see that there exists $s_a > 0$ such that $G_a(s_a) < 0$ if and only if $a < a^*_\alpha$.

Letting

$$D(u) := \frac{m}{2}\int_{\mathbb{R}^N} u^2 - \frac{b}{4}\int_{\mathbb{R}^N} E_1((u^+)^2)(u^+)^2 + \frac{a}{\alpha+2}\int_{\mathbb{R}^N}(u^+)^{\alpha+2} \tag{2.3}$$

and defining

$$w_R(x) := \begin{cases} s_a, & |x| \leqslant R, \\ s_a(R+1-|x|), & R \leqslant |x| \leqslant R+1, \\ 0, & |x| \geqslant R+1, \end{cases}$$

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for $t>0$, we denote $(u)_t(x):=u(tx_1,x_2,\dots,x_N)$, we see that

$$D((w_R)_t)=t^{-1}\left[\frac{m}{2}\int_{\mathbb{R}^N}|w_R|^2+\frac{a}{\alpha+2}\int_{\mathbb{R}^N}|w_R|^{\alpha+2}-\frac{b}{4}\int_{\mathbb{R}^N}\frac{t^2\xi_1^2}{t^2\xi_1^2+|\xi'|^2}\left|\mathcal{F}(|w_R|^2)\right|^2\right]. \tag{2.4}$$

Note that

$$\lim_{t\to+\infty}\int_{\mathbb{R}^N}\frac{t^2\xi_1^2}{t^2\xi_1^2+|\xi'|^2}\left|\mathcal{F}(|w_R|^2)\right|^2=\int_{\mathbb{R}^N}|w_R|^4,$$

then, for any $\varepsilon>0$, there exists $T=T(\varepsilon,R)>0$ such that $t>T$ implies

$$\int_{\mathbb{R}^N}\frac{t^2\xi_1^2}{t^2\xi_1^2+|\xi'|^2}\left|\mathcal{F}(|w_R|^2)\right|^2>\int_{\mathbb{R}^N}|w_R|^4-\varepsilon, \tag{2.5}$$

we see from (2.4) and (2.5) that for $R>1$,

$$D((w_R)_t)\leqslant t^{-1}\left[CR^NG_a(s_a)+CR^{N-1}+\frac{b}{4}\varepsilon\right].$$

Choosing $\varepsilon>0$ small and $R>0$ large to make sure that

$$CR^NG_a(s_a)+CR^{N-1}+\frac{b}{4}\varepsilon<0,$$

then there exists $t_R>0$ such that $D((w_R)_{t_R})<0$. Letting $u_R:=(w_R)_{t_R}$, we choose a large $R'>0$ such that $I_m(u_R(x/R'))<0$, $u_R(x/R')$ is the desired ϕ_0. □

We denote the mountain pass level of I_m by

$$c_m:=\inf_{\gamma\in\Gamma_m}\sup_{t\in[0,1]}I_m(\gamma(t)), \tag{2.6}$$

where the set of paths is defined as

$$\Gamma_m:=\left\{\gamma\in C([0,1],H^1(\mathbb{R}^N)):\gamma(0)=0 \text{ and } I_m(\gamma(1))<0\right\}. \tag{2.7}$$

By lemma 2.1(i), we see that $c_m>0$. Moreover, we denote

$$b_m:=\inf\{I_m(u):u\in H^1(\mathbb{R}^N)\backslash\{0\} \text{ and } P_m(u)=0\}.$$

Proof of theorem 1.1(i). We consider the following minimization problem

$$d_m:=\inf_{u\in\mathcal{P}_m}\frac{1}{N}\int_{\mathbb{R}^N}|\nabla u|^2, \tag{2.8}$$

where $\mathcal{P}_m:=\{u\in H^1(\mathbb{R}^N)\backslash\{0\}:P_m(u)\leqslant 0\}$. By lemma 2.1(ii), $\mathcal{P}_m\neq\emptyset$. We claim that $d_m>0$. Indeed, we divide the proof into two cases:

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Case 1: $N = 3$. From Plancherel's theorem, Young's and Sobolev's inequality, we see that for any $u \in \mathcal{P}_m$,

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2+\frac{3}{2}m\int_{\mathbb{R}^3}u^2 &\leqslant \frac{3}{4}b\int_{\mathbb{R}^3}E_1((u^+)^2)(u^+)^2+\frac{3|a|}{\alpha+2}\int_{\mathbb{R}^3}|u|^{\alpha+2}\\ &\leqslant \frac{3}{4}m\int_{\mathbb{R}^3}u^2+C\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)^3, \end{aligned} \tag{2.9}$$

from which we conclude that $d_m > 0$.

Case 2: $N = 2$. From Plancherel's theorem and Gagliardo–Nirenberg's inequality [**42**], we obtain that for any $u \in \mathcal{P}_m$, it holds that

$$\begin{aligned} m\int_{\mathbb{R}^2}u^2 &\leqslant \frac{b}{2}\int_{\mathbb{R}^2}E_1((u^+)^2)(u^+)^2+\frac{2|a|}{\alpha+2}\int_{\mathbb{R}^2}|u|^{\alpha+2}\\ &\leqslant C\int_{\mathbb{R}^2}|\nabla u|^2\cdot\int_{\mathbb{R}^2}u^2+C\left(\int_{\mathbb{R}^2}|\nabla u|^2\right)^{\alpha/2}\cdot\int_{\mathbb{R}^2}u^2, \end{aligned}$$

which implies that $d_m > 0$.

Next, we have the following cases:

Case 1: $N = 3$. Letting $\{u_n\}_{n=1}^{\infty}$ be a minimizing sequence of (2.8), arguing as (2.9), we see that $\{u_n\}_{n=1}^{\infty}$ is bounded in $H^1(\mathbb{R}^3)$. We claim that there exist a sequence $\{y_n\}_{n=1}^{\infty} \subset \mathbb{R}^3$ and $R, \beta > 0$ such that

$$\int_{B_R(y_n)}|u_n|^2 \geqslant \beta.$$

If not, by vanishing theorem [**33**, lemma I.1], it holds that

$$\int_{\mathbb{R}^3}|u_n|^s \to 0 \quad \text{as } n \to \infty \text{ for } 2 < s < 6.$$

$P_m(u_n) \leqslant 0$ implies that $\|u_n\|_{H^1(\mathbb{R}^3)} = o(1)$, which contradicts $d_m > 0$. Letting $\tilde{u}_n(x) = u_n(x + y_n)$, up to a subsequence, there is $\tilde{u} \in H^1(\mathbb{R}^3)\backslash\{0\}$ such that $\tilde{u}_n \rightharpoonup \tilde{u}$ in $H^1(\mathbb{R}^3)$ and $\tilde{u}_n \to \tilde{u}$ a.e. $\mathbb{R}^3$, then $(\tilde{u}_n^+)^2 \rightharpoonup (\tilde{u}^+)^2$ in $L^2(\mathbb{R}^3)$. Since the linear operator $v \mapsto (\sigma_1(\xi))^{1/2}\mathcal{F}(v)$ is bounded in $L^2(\mathbb{R}^3)$, we get $(\sigma_1(\xi))^{1/2}\mathcal{F}((\tilde{u}_n^+)^2) \rightharpoonup (\sigma_1(\xi))^{1/2}\mathcal{F}((\tilde{u}^+)^2)$ in $L^2(\mathbb{R}^3)$. Therefore, from Brézis–Lieb's lemma [**5**, theorem 1], we obtain

$$P_m(\tilde{u}_n) - P_m(\tilde{u}_n - \tilde{u}) - P_m(\tilde{u}) = o(1) \tag{2.10}$$

and

$$\int_{\mathbb{R}^3}|\nabla\tilde{u}_n|^2 - \int_{\mathbb{R}^3}|\nabla(\tilde{u}_n - \tilde{u})|^2 - \int_{\mathbb{R}^3}|\nabla\tilde{u}|^2 = o(1). \tag{2.11}$$

We see that $\tilde{u} \in \mathcal{P}_m$. Indeed, we suppose that $P_m(\tilde{u}) > 0$, it follows from (2.10) and the fact $\tilde{u}_n \in \mathcal{P}_m$ that $P_m(\tilde{u}_n - \tilde{u}) < 0$ for n large. Thus,

$$\frac{1}{3}\int_{\mathbb{R}^3}|\nabla(\tilde{u}_n - \tilde{u})|^2 \geqslant d_m. \tag{2.12}$$

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Since

$$\frac{1}{3}\int_{\mathbb{R}^3}|\nabla\tilde{u}_n|^2 \to d_m, \tag{2.13}$$

we deduce from (2.11) that $\|\nabla\tilde{u}\|_{L^2(\mathbb{R}^3)} \leqslant 0$, which contradicts $\tilde{u} \neq 0$. From the definition of d_m, we have

$$d_m \leqslant \frac{1}{3}\int_{\mathbb{R}^3}|\nabla\tilde{u}|^2 \leqslant \varliminf_{n\to\infty}\frac{1}{3}\int_{\mathbb{R}^3}|\nabla\tilde{u}_n|^2 = d_m, \tag{2.14}$$

hence, $\tilde{u}$ is a minimizer of (2.8).

Next, we claim that $P_m(\tilde{u}) = 0$. Indeed, if $P_m(\tilde{u}) < 0$, then there exists $\tau \in (0,1)$ such that $P_m(\tau\tilde{u}) = 0$, so

$$d_m \leqslant \frac{1}{3}\int_{\mathbb{R}^3}|\nabla(\tau\tilde{u})|^2 < \frac{1}{3}\int_{\mathbb{R}^3}|\nabla\tilde{u}|^2 = d_m, \tag{2.15}$$

which is a contradiction. Hence, (2.8) is equivalent to

$$d_m := \inf\left\{\frac{1}{3}\int_{\mathbb{R}^3}|\nabla u|^2 : u \in H^1(\mathbb{R}^3)\backslash\{0\} \text{ and } P_m(u) = 0\right\} = b_m.$$

Since $\tilde{u}$ is a minimizer of d_m, then there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that

$$-\frac{2}{3}\Delta\tilde{u} = \lambda\left[-\Delta\tilde{u} + 3m\tilde{u} - 3bE_1((\tilde{u}^+)^2)\tilde{u}^+ + 3a(\tilde{u}^+)^{\alpha+1}\right]. \tag{2.16}$$

From Pohozaev's identity [**39**], we have

$$\begin{aligned}&\left(\frac{2}{3}-\lambda\right)\frac{1}{2}\int_{\mathbb{R}^3}|\nabla\tilde{u}|^2\\ &\quad= 3\lambda\left[\frac{3}{2}m\int_{\mathbb{R}^3}\tilde{u}^2 - \frac{3}{4}b\int_{\mathbb{R}^3}E_1((\tilde{u}^+)^2)(\tilde{u}+)^2 + \frac{3a}{\alpha+2}\int_{\mathbb{R}^3}(\tilde{u}^+)^{\alpha+2}\right].\end{aligned} \tag{2.17}$$

Since $P_m(\tilde{u}) = 0$, it follows from (2.17) that

$$\left(\frac{1}{3}+\lambda\right)\int_{\mathbb{R}^3}|\nabla\tilde{u}|^2 = 0.$$

This implies that $\lambda = -1/3$, by (2.16), $\tilde{u}$ satisfies (2.1). Using standard elliptic estimates, we can verify that $\tilde{u} \in L^\infty(\mathbb{R}^3)\cap C^2(\mathbb{R}^3)$ and by strong maximum principle, we see that $\tilde{u}(x) > 0$ for all $x \in \mathbb{R}^3$. Moreover, by $d_m = b_m$, we see that $\tilde{u}$ is a positive ground-state solution to (2.1).

Case 2: $N = 2$. Letting $\{u_n\}_{n=1}^\infty$ be a minimizing sequence of (2.8), we define $v_n(x) = u_n(\lambda_n x)$, where $\lambda_n = \|u_n\|_{L^2(\mathbb{R}^2)}$ and we obtain

$$\|v_n\|_{L^2(\mathbb{R}^2)} = 1,\ \|\nabla v_n\|_{L^2(\mathbb{R}^2)} = \|\nabla u_n\|_{L^2(\mathbb{R}^2)} \quad\text{and}\quad P_m(v_n) = P_m(u_n)/\lambda_n^2 \leqslant 0. \tag{2.18}$$

From (2.18), we see that $\{v_n\}_{n=1}^\infty$ is a bounded minimizing sequence of (2.8) in $H^1(\mathbb{R}^2)$.

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We claim that there exist a sequence $\{z_n\}_{n=1}^{\infty} \subset \mathbb{R}^2$ and $R, \beta > 0$ such that

$$\int_{B_R(z_n)} |v_n|^2 \geqslant \beta.$$

Otherwise, by vanishing theorem,

$$\int_{\mathbb{R}^2} |v_n|^s \to 0 \quad \text{as } n \to \infty \text{ for } 2 < s < +\infty.$$

$P_m(v_n) \leqslant 0$ implies that $\|v_n\|_{L^2(\mathbb{R}^2)} = o(1)$, which contradicts (2.18). Letting $\tilde{v}_n(x) = v_n(x + z_n)$, up to a subsequence, there is a $\tilde{v} \in H^1(\mathbb{R}^2)\backslash\{0\}$ such that $\tilde{v}_n \rightharpoonup \tilde{v}$ in $H^1(\mathbb{R}^2)$. Arguing as (2.10)–(2.15), we see that (2.8) is equivalent to

$$d_m := \inf\left\{\frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 : u \in H^1(\mathbb{R}^2)\backslash\{0\} \text{ and } P_m(u) = 0\right\} = b_m$$

and $\tilde{v}$ is a minimizer of d_m, then there exists a Lagrange multiplier $\lambda \neq 0$ such that

$$-\Delta\tilde{v} = \lambda(m\tilde{v} - bE_1((\tilde{v}^+)^2)\tilde{v}^+ + a(\tilde{v}^+)^{\alpha+1}). \tag{2.19}$$

We claim that $\lambda < 0$. Choosing $\varphi \in H^1(\mathbb{R}^2)$ such that $\langle P_m'(\tilde{v}), \varphi\rangle < 0$, direct calculations show that

$$P_m(\tilde{v} + t\varphi) = P_m(\tilde{v}) + \int_0^t \langle P_m'(\tilde{v} + s\varphi), \varphi\rangle \, \mathrm{d}s \tag{2.20}$$

and

$$\frac{1}{2}\int_{\mathbb{R}^2} |\nabla(\tilde{v} + t\varphi)|^2 = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\tilde{v}|^2 + \frac{t}{2}\lambda\langle P_m'(\tilde{v}), \varphi\rangle + \frac{t^2}{2}\int_{\mathbb{R}^2} |\nabla\varphi|^2. \tag{2.21}$$

If $\lambda > 0$, we see from (2.20) and (2.21) that for $t > 0$ small,

$$\frac{1}{2}\int_{\mathbb{R}^2} |\nabla(\tilde{v} + t\varphi)|^2 < \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\tilde{v}|^2 = d_m \quad \text{and} \quad P_m(\tilde{v} + t\varphi) < 0,$$

which contradicts (2.8). Hence, $\lambda < 0$. Letting $\tilde{v}_\lambda = \tilde{v}(x/(-\lambda)^{1/2})$, we see that $\tilde{v}_\lambda$ satisfies (2.1). Using standard elliptic estimates, we can verify that $\tilde{v}_\lambda \in L^\infty(\mathbb{R}^2) \cap C^2(\mathbb{R}^2)$ and by strong maximum principle, we see that $\tilde{v}_\lambda > 0$ for all $x \in \mathbb{R}^2$. Moreover, since

$$I_m(\tilde{v}_\lambda) = I_m(\tilde{v}_\lambda) - \frac{1}{2}P_m(\tilde{v}_\lambda) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\tilde{v}_\lambda|^2 = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\tilde{v}|^2 = d_m = b_m,$$

we see that $\tilde{v}_\lambda$ is in fact a positive ground-state solution to (2.1). □

Letting S_m be the set of ground-state solutions U to (2.1) satisfying $U(0) = \max_{x\in\mathbb{R}^N} U(x)$. Then, we obtain the following compactness of S_m.

PROPOSITION 2.2. *Letting $(\alpha, a) \in R_{m,b}$, then for each $m > 0$, S_m is compact in $H^1(\mathbb{R}^N)$.*

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Proof. For any sequence $\{U_k\}_{k=1}^{\infty} \subset S_m$, we see that

$$b_m = I_m(U_k) = I_m(U_k) - \frac{1}{N} P_m(U_k) = \frac{1}{N} \int_{\mathbb{R}^N} |\nabla U_k|^2.$$

Thus, $\{\nabla U_k\}_{k=1}^{\infty}$ is bounded in $L^2(\mathbb{R}^N)$. Next, we claim that $\{U_k\}_{k=1}^{\infty}$ is also bounded in $L^2(\mathbb{R}^N)$. We divide the proof into two cases:

Case 1: $N = 3$. Since $P_m(U_k) = 0$, arguing as (2.9), we see that $\{U_k\}_{k=1}^{\infty}$ is bounded in $L^2(\mathbb{R}^3)$.

Case 2: $N = 2$. We argue indirectly and assume that $\lambda_k := \|U_k\|_{L^2(\mathbb{R}^2)} \to \infty$ as $k \to \infty$. Letting $V_k(x) = U_k(\lambda_k x)$, we see that

$$\|V_k\|_{L^2(\mathbb{R}^2)} = 1 \quad \text{and} \quad \|\nabla V_k\|_{L^2(\mathbb{R}^2)} = \|\nabla U_k\|_{L^2(\mathbb{R}^2)}. \tag{2.22}$$

In particular, $\{V_k\}_{k=1}^{\infty}$ is bounded in $H^1(\mathbb{R}^2)$. Next, we claim that for any $\{y_k\}_{k=1}^{\infty} \subset \mathbb{R}^2$, after taking a subsequence,

$$\tilde{V}_k(x) := V_k(x + y_k) \rightharpoonup 0 \quad \text{in } H^1(\mathbb{R}^2). \tag{2.23}$$

Assuming on the contrary that there exists $\{y_k\}_{k=1}^{\infty} \subset \mathbb{R}^2$, up to a subsequence,

$$\tilde{V}_k(x) := V_k(x + y_k) \rightharpoonup \tilde{V}_0(x) \neq 0 \quad \text{in } H^1(\mathbb{R}^2). \tag{2.24}$$

For any $\varphi \in H^1(\mathbb{R}^2)$, letting $\varphi_k(x) = \varphi((x/\lambda_k) - y_k)$. From $\langle I'_m(U_k), \varphi_k \rangle = 0$, we have

$$\int_{\mathbb{R}^2} \nabla \tilde{V}_k \cdot \nabla \varphi + m\lambda_k^2 \int_{\mathbb{R}^2} \tilde{V}_k \varphi - b\lambda_k^2 \int_{\mathbb{R}^2} E_1((\tilde{V}_k)^2)\tilde{V}_k \varphi + a\lambda_k^2 \int_{\mathbb{R}^2} (\tilde{V}_k)^{\alpha+1} \varphi = 0. \tag{2.25}$$

Multiplying (2.25) by λ_k^{-2} and letting $k \to \infty$, we see that $\tilde{V}_0 \geqslant 0$ satisfies

$$m\tilde{V}_0 - bE_1((\tilde{V}_0)^2)\tilde{V}_0 + a(\tilde{V}_0)^{\alpha+1} = 0. \tag{2.26}$$

In view of [**39**], we have the following Pohozaev's type identity:

$$\frac{m}{2} \int_{\mathbb{R}^2} (\tilde{V}_0)^2 - \frac{b}{4} \int_{\mathbb{R}^2} E_1((\tilde{V}_0)^2)(\tilde{V}_0)^2 + \frac{a}{\alpha+2} \int_{\mathbb{R}^2} (\tilde{V}_0)^{\alpha+2} = 0, \tag{2.27}$$

that is, $P_m(\tilde{V}_0) = 0$. From the definition of b_m,

$$\int_{\mathbb{R}^2} |\nabla \tilde{V}_0|^2 \geqslant 2b_m. \tag{2.28}$$

Since $U_k \in S_m$, we have

$$\int_{\mathbb{R}^2} |\nabla \tilde{V}_k|^2 = \int_{\mathbb{R}^2} |\nabla V_k|^2 = \int_{\mathbb{R}^2} |\nabla U_k|^2 = 2b_m. \tag{2.29}$$

(2.27), (2.28) and (2.29) imply that $I_m(\tilde{V}_0) = b_m$. Using Ekeland's variational principle, there exist a Lagrange Multiplier $\lambda \in \mathbb{R}$ such that $I'_m(\tilde{V}_0) = \lambda P'_m(\tilde{V}_0)$,

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by (2.26), $\tilde{V}_0 = 0$, which contradicts (2.24). Thus, (2.23) holds. Since $\{y_k\}_{k=1}^{\infty}$ is arbitrary, we have

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_2(y)} |V_k|^2 = 0,$$

by vanishing theorem,

$$\int_{\mathbb{R}^2} |V_k|^p \to 0 \quad \text{as } k \to \infty \text{ for } 2 < p < +\infty. \tag{2.30}$$

Multiplying (2.25) by λ_k^{-2} and replacing φ by $\tilde{V}_k$, we see from (2.30) that $\|V_k\|_{L^2(\mathbb{R}^2)} = o(1)$ as $k \to \infty$, which contradicts (2.22). Therefore, $\{U_k\}_{k=1}^{\infty}$ is bounded in $L^2(\mathbb{R}^2)$. From the above arguments of *cases 1* and *2*, we obtain that $\{U_k\}_{k=1}^{\infty}$ is bounded in $H^1(\mathbb{R}^N)$.

Next, we claim that there exist a sequence $\{x_k\}_{k=1}^{\infty} \subset \mathbb{R}^N$ and $R, \beta > 0$ such that

$$\int_{B_R(x_k)} |U_k|^2 \geqslant \beta.$$

Indeed, if not, by vanishing theorem,

$$\int_{\mathbb{R}^N} |U_k|^p \to 0 \quad \text{as } k \to \infty \text{ for } 2 < p < 2^*. \tag{2.31}$$

(2.31) and $\langle I_m'(U_k), U_k\rangle = 0$ imply that $\|U_k\|_{H^1(\mathbb{R}^N)} = o(1)$, which contradicts $I_m(U_k) = b_m > 0$. Letting $\bar{U}_k(x) := U_k(x + x_k)$, up to a subsequence, there is a $\bar{U} \in H^1(\mathbb{R}^N)\backslash\{0\}$ such that $\bar{U}_k \rightharpoonup \bar{U}$ in $H^1(\mathbb{R}^N)$. We infer that $\bar{U}$ is a nontrivial solution to (2.1) and

$$\begin{aligned} b_m &\leqslant I_m(\bar{U}) = I_m(\bar{U}) - \frac{1}{N} P_m(\bar{U}) = \frac{1}{N} \int_{\mathbb{R}^N} |\nabla \bar{U}|^2 \\ &\leqslant \varliminf_{k\to\infty} \frac{1}{N} \int_{\mathbb{R}^N} |\nabla \bar{U}_k|^2 = \varliminf_{k\to\infty} \left[I_m(\bar{U}_k) - \frac{1}{N} P_m(\bar{U}_k) \right] = b_m. \end{aligned}$$

Hence, $\nabla \bar{U}_k \to \nabla \bar{U}$ in $L^2(\mathbb{R}^N)$. By Gagliardo–Nirenberg's inequality, we see that $\bar{U}_k \to \bar{U}$ in $L^q(\mathbb{R}^N)$ for $2 < q < 2^*$, noting that $\langle I_m'(\bar{U}_k) - I_m'(\bar{U}), \bar{U}_k - \bar{U}\rangle = 0$, we have

$$\bar{U}_k \to \bar{U} \quad \text{in } H^1(\mathbb{R}^N) \text{ as } k \to \infty. \tag{2.32}$$

By lemma B.1, we have

$$\|\bar{U}_k\|_{L^\infty(\mathbb{R}^N)} = \|U_k\|_{L^\infty(\mathbb{R}^N)} \leqslant C \text{ uniformly for } k \in \mathbb{N}. \tag{2.33}$$

By Hölder's inequality, (2.33) and $E_1 \in \mathcal{L}(L^p(\mathbb{R}^N), L^p(\mathbb{R}^N))(1 < p < +\infty)$ [**15**, lemma 2.1], we see that for any $y \in \mathbb{R}^N$,

$$\begin{aligned} &\|E_1(|\bar{U}_k|^2)\bar{U}_k + |\bar{U}_k|^\alpha \bar{U}_k\|_{L^N(B_2(y))} \\ &\quad \leqslant \|E_1(|\bar{U}_k|^2)\|_{L^{2N}(\mathbb{R}^N)} \|\bar{U}_k\|_{L^{2N}(B_2(y))} + \||\bar{U}_k|^{\alpha+1}\|_{L^N(B_2(y))} \\ &\quad \leqslant C \left(\|\bar{U}_k\|_{L^2(B_2(y))}^{1/N} + \|\bar{U}_k\|_{L^2(B_2(y))}^{2/N} \right), \end{aligned} \tag{2.34}$$

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where C is independent of $y \in \mathbb{R}^N$. Choosing $p = 2$, $q = 2N$ and $R = 1$ in theorem 8.17 of [**28**], by (2.34) and Young's inequality, we have

$$\sup_{B_1(y)} \bar{U}_k \leqslant C \left(\|\bar{U}_k\|_{L^2(B_2(y))} + \|\bar{U}_k\|^{1/N}_{L^2(B_2(y))} \right)$$

where C is independent of $y \in \mathbb{R}^N$ and $k \in N$. Letting $|y| \to +\infty$, we see from (2.32) that

$$\lim_{|x|\to\infty} \bar{U}_k(x) = 0 \text{ uniformly for } k \in \mathbb{N}. \tag{2.35}$$

Multiplying (2.1) by U_k and integrating, we see from Plancherel's theorem that

$$m \int_{\mathbb{R}^N} U_k^2 \leqslant C \left(\int_{\mathbb{R}^N} U_k^4 + \int_{\mathbb{R}^N} U_k^{\alpha+2} \right) \leqslant C(U_k^2(0) + U_k^\alpha(0)) \int_{\mathbb{R}^N} U_k^2,$$

which implies that $\bar{U}_k(-x_k) = U_k(0) \geqslant C > 0$, then by (2.35), $\{x_k\}_{k=1}^\infty$ must be bounded, that is, $|x_k| \leqslant r_0$. By (2.32) and Sobolev's imbedding theorem, we see that for any $\delta > 0$, there exists $R_\delta > 0$ such that for all $k \in \mathbb{N}$,

$$\int_{\mathbb{R}^N \setminus B_{R_\delta + r_0}} U_k^p \leqslant \int_{\mathbb{R}^N \setminus B_{R_\delta}} \bar{U}_k^p \leqslant \delta, \tag{2.36}$$

where $2 \leqslant p < 2^*$. By the boundedness of $\{U_k\}_{k=1}^\infty$ in $H^1(\mathbb{R}^N)$, there exists $U \in H^1(\mathbb{R}^N)$ such that $U_k \rightharpoonup U$ in $H^1(\mathbb{R}^N)$, $U_k \to U$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)(1 \leqslant p < 2^*)$. Combining with (2.36), we see that $U_k \to U$ in $L^p(\mathbb{R}^N)$ with $2 \leqslant p < 2^*$. Recalling that $\langle I'_m(U_k) - I'_m(U), U_k - U\rangle = 0$, we conclude that $U_k \to U$ in $H^1(\mathbb{R}^N)$. This completes the proof that S_m is compact in $H^1(\mathbb{R}^N)$. □

The next two lemmas show that b_m, the least energy level of (2.1), is equal to the mountain pass level c_m.

LEMMA 2.3. *Suppose that $(\alpha, a) \in R_{m,b}$. For each $U \in S_m$, there exists a $\bar{U} \in H^1(\mathbb{R}^N)$ such that for any $\delta > 0$, there is a path $\gamma_\delta \in \Gamma_m$ satisfying*

(i) *$\gamma_\delta(0) = 0$, $\gamma_\delta(t)(x) > 0$ for each $t \in (0, 1]$ and $x \in \mathbb{R}^N$, $\gamma_\delta(1) = \bar{U}$ with $I_m(\bar{U}) < -1$ and $\max_{t\in[0,1]} I_m(\gamma_\delta(t)) = I_m(U) = b_m$.*

(ii) *There exists a $t_0 \in (0, 1)$ such that $\gamma_\delta(t_0) = U$ and $I_m(\gamma_\delta(t)) < I_m(U) = b_m$ if $\|\gamma_\delta(t) - \gamma_\delta(t_0)\|_{H^1(\mathbb{R}^N)} \geqslant \delta$.*

LEMMA 2.4. *Suppose that $(\alpha, a) \in R_{m,b}$. $\gamma([0, 1]) \cap \{u \in H^1(\mathbb{R}^N)\setminus\{0\} : P_m(u) = 0\} \neq \emptyset$ for each $\gamma \in \Gamma_m$.*

Since we can prove lemmas 2.3 and 2.4 in the same way as the proof of proposition 2 in [**8**] and lemma 4.1 in [**31**] respectively, we omit the details.

As a corollary to lemmas 2.3 and 2.4, we have

COROLLARY 2.5. *$b_m = c_m$ for $(\alpha, a) \in R_{m,b}$.*

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Proof of theorem 1.1(ii). Letting $u \in H^1(\mathbb{R}^N)$ be a solution to (2.1), we divide the proof into the following cases:

Case 1: $\alpha = 2$ and $a \geqslant a^*_\alpha = b$. It follows from Plancherel's theorem that

$$C(u) = \int_{\mathbb{R}^N} (-b\sigma_1(\xi) + a)\left|\mathcal{F}(|u|^2)(\xi)\right|^2 \geqslant 0, \tag{2.37}$$

where $C(u)$ has been mentioned in (2.2). $\langle I'_m(u), u\rangle = 0$ and (2.37) imply that $u = 0$.

Case 2: $\alpha > 2$ and $a \geqslant a^*_\alpha = (b/2)^{\alpha/2} m^{(2-\alpha)/2}((\alpha+2)/\alpha)((\alpha-2)/\alpha)^{(\alpha-2)/2}$. Since

$$D(u) \geqslant \int_{\mathbb{R}^N} \frac{m}{2}|u|^2 + \frac{a}{\alpha+2}|u|^{\alpha+2} - \frac{b}{4}|u|^4, \tag{2.38}$$

where $D(u)$ has been mentioned in (2.3). We see that for all $s > 0$,

$$\frac{m}{2}s^2 + \frac{a}{\alpha+2}s^{\alpha+2} - \frac{b}{4}s^4 \geqslant 0 \quad \text{if and only if } a \geqslant a^*_\alpha.$$

Hence, $D(u) \geqslant 0$.

Case 2.1: $N = 3$. $P_m(u) = 0$ and $D(u) \geqslant 0$ imply that $u = 0$.

Case 2.2: $N = 2$. Obviously, $D(u) = 0$.

Case 2.2.1: $a = a^*_\alpha$. By (2.38), $u = 0$ or $(2 \cdot m/b \cdot \alpha/(\alpha-2))^{1/2}$ a.e. $\mathbb{R}^2$. Note that $u \in H^1(\mathbb{R}^2)$, then $u = 0$.

Case 2.2.2: $a > a^*_\alpha$. By (2.38), $u = 0$. □

3. The singularly perturbed problem

In this section, we always assume that $(\alpha, a) \in R_{m,b}$, where $m := \max_{1\leqslant i\leqslant k} m_i$.

(1.1) can be rewritten as

$$-\Delta v + V(\varepsilon x)v = bE_1(|v|^2)v - a|v|^\alpha v,\ v \in H^1(\mathbb{R}^N), \quad v > 0. \tag{3.1}$$

The corresponding energy functional to (3.1) is

$$I_\varepsilon(v) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)v^2 - \frac{b}{4}\int_{\mathbb{R}^N} E_1((v^+)^2)(v^+)^2 + \frac{a}{\alpha+2}\int_{\mathbb{R}^N}(v^+)^{\alpha+2}, \quad v \in H_\varepsilon,$$

where H_ε be a class of weighted Sobolev space as follows:

$$\left\{v \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\varepsilon x)v^2 < \infty\right\}$$

and the norm of the space H_ε is denoted by

$$\|v\|_{H_\varepsilon} := \left(\int_{\mathbb{R}^N} |\nabla v|^2 + \int_{\mathbb{R}^N} V(\varepsilon x)v^2\right)^{1/2} \quad \text{for } v \in H_\varepsilon.$$

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From now on, we define $\Lambda = \cup_{i=1}^k \Lambda^i$ and $\mathcal{M} = \cup_{i=1}^k \mathcal{M}^i$. Following [6, 7, 11], we define

$$\chi_\varepsilon(x) := \begin{cases} 0, & \text{if } x \in \Lambda/\varepsilon, \\ \varepsilon^{-1}, & \text{if } x \notin \Lambda/\varepsilon, \end{cases} \qquad Q_\varepsilon(v) := \left(\int_{\mathbb{R}^N} \chi_\varepsilon v^2 - 1 \right)_+^2$$

and

$$\chi_\varepsilon^i(x) := \begin{cases} 0, & \text{if } x \in \Lambda^i/\varepsilon, \\ \varepsilon^{-1}, & \text{if } x \notin \Lambda^i/\varepsilon, \end{cases} \qquad Q_\varepsilon^i(v) := \left(\int_{\mathbb{R}^N} \chi_\varepsilon^i v^2 - 1 \right)_+^2, \quad i = 1, \dots, k,$$

respectively. As we shall see, this type of modification will act as a penalization to force the concentration phenomena to occur inside Λ. Finally, we define auxiliary functionals J_ε, J_ε^i $(i = 1, \dots, k)$ on H_ε by

$$J_\varepsilon(v) := I_\varepsilon(v) + Q_\varepsilon(v) \quad \text{and} \quad J_\varepsilon^i(v) := I_\varepsilon(v) + Q_\varepsilon^i(v).$$

It is standard to see that the functionals J_ε, J_ε^i $(i = 1, \dots, k)$ are in $C^1(H_\varepsilon, \mathbb{R})$. To find solutions to (3.1) which concentrate around $\mathcal{M}$ as $\varepsilon \to 0$, we shall search critical points of J_ε for which Q_ε is zero.

Letting $c_{m_i} = I_{m_i}(U)$ for $U \in S_{m_i}$ and

$$\bar{\beta} := \frac{1}{10} \min \left\{ \operatorname{dist}\{\mathcal{M}, \mathbb{R}^N \backslash \Lambda\}, \min_{i \neq j} \operatorname{dist}(\Lambda^i, \Lambda^j) \right\},$$

we fix a $\beta \in (0, \bar{\beta})$ and a cut-off function $\varphi \in C_c^\infty(\mathbb{R}^N, [0,1])$ such that $\varphi(x) = 1$ for $|x| \leqslant \beta$, $\varphi(x) = 0$ for $|x| \geqslant 2\beta$ and $|\nabla \varphi| \leqslant C/\beta$. For $\varepsilon > 0$ small, we will find a solution of (3.1) near the set

$$X_\varepsilon := \left\{ \sum_{i=1}^k \varphi(\varepsilon x - z^i) U^i(x - (z^i/\varepsilon)) : z^i \in (\mathcal{M}^i)^\beta \text{ and } U^i \in S_{m_i} \right\}$$

where $(\mathcal{M}^i)^\beta := \left\{ y \in \mathbb{R}^N : \inf_{z \in \mathcal{M}^i} |y - z| \leqslant \beta \right\}$. Similarly, for $A \subset H_\varepsilon$, we use the notation

$$A^a := \left\{ u \in H_\varepsilon : \inf_{v \in A} \|u - v\|_{H_\varepsilon} \leqslant a \right\}.$$

For each $1 \leqslant i \leqslant k$ and $\delta > 0$, choosing $U_*^i \in S_{m_i}$, then there is a path $\gamma_\delta^i(t)$ satisfying lemma 2.3(i), (ii) with m replaced by m_i. Letting $z_*^i \in \mathcal{M}^i$, then for each $\varepsilon > 0$, we set

$$\gamma_{\varepsilon,\delta}^i(t)(x) := \varphi(\varepsilon x - z_*^i) \gamma_\delta^i(t)(x - (z_*^i/\varepsilon)). \tag{3.2}$$

Noting that $\operatorname{supp} \gamma_{\varepsilon,\delta}^i(t) \subset \Lambda^i/\varepsilon$, direct calculations show that

$$J_\varepsilon^i(\gamma_{\varepsilon,\delta}^i(1)) = I_\varepsilon(\gamma_{\varepsilon,\delta}^i(1)) = I_{m_i}(\gamma_\delta^i(1)) + o(1) < -\frac{1}{2} \quad \text{for } \varepsilon > 0 \text{ small.} \tag{3.3}$$

We define

$$\tilde{c}_{\varepsilon,\delta} := \max_{s \in [0,1]^k} J_\varepsilon(\gamma_{\varepsilon,\delta}(s)), \tag{3.4}$$

where $\gamma_{\varepsilon,\delta}(s) := \sum_{i=1}^k \gamma_{\varepsilon,\delta}^i(s_i)$ for $s := (s_1, s_2, \dots, s_k) \in [0,1]^k$, we have the following estimates:

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LEMMA 3.1. *For each $\delta > 0$, the following holds:*

(i) $\lim_{\varepsilon\to 0} \tilde{c}_{\varepsilon,\delta} = \sum_{i=1}^{k} c_{m_i}$;

(ii) $\overline{\lim}_{\varepsilon\to 0} \max_{s\in\partial[0,1]^k} J_\varepsilon(\gamma_{\varepsilon,\delta}(s)) \leqslant \sum_{i=1}^{k} c_{m_i} - \sigma$, *where* $0 < \sigma < \min\{c_{m_i} : i = 1, 2, \ldots, k\}$ *is a fixed number.*

Proof. For the sake of simplicity, we only give the proof corresponding to the case $N = 2$, the case $N = 3$ is similar. For each $\delta > 0$, $s_i, s_j \in [0,1]$ with $1 \leqslant i, j \leqslant k$, $i \neq j$, we see from Appendix A that

$$\begin{aligned}
&\int_{\mathbb{R}^2} E_1(|\gamma^i_{\varepsilon,\delta}(s_i)|^2)|\gamma^j_{\varepsilon,\delta}(s_j)|^2 \\
&= \int_{\mathbb{R}^2} (\mathcal{K} * |\gamma^i_{\varepsilon,\delta}(s_i)|^2)|\gamma^j_{\varepsilon,\delta}(s_j)|^2 \\
&= \frac{1}{2\pi}\int_{\mathbb{R}^2} \left(\frac{-x_1^2 + x_2^2}{|x|^4} * |\gamma^i_{\varepsilon,\delta}(s_i)|^2\right)|\gamma^j_{\varepsilon,\delta}(s_j)|^2 + \frac{1}{2}\int_{\mathbb{R}^2} \left(\delta_0 * |\gamma^i_{\varepsilon,\delta}(s_i)|^2\right)|\gamma^j_{\varepsilon,\delta}(s_j)|^2 \\
&= (I) + (II), \qquad (3.5)
\end{aligned}$$

where δ_0 is the Dirac measure on $\mathbb{R}^2$ giving unit mass to the point 0. Choosing $\theta > 0$ small but fixed, we see from Hardy–Littlewood–Sobolev's inequality [**32**, theorem 4.3] that,

$$\begin{aligned}
|(I)| &\leqslant C\varepsilon^\theta \int_{B_{2\beta/\varepsilon}(z^j_*/\varepsilon)} \left[\int_{B_{2\beta/\varepsilon}(z^i_*/\varepsilon)} \frac{1}{|x-y|^{2-\theta}} |\gamma^i_\delta(s_i)|^2(x - (z^i_*/\varepsilon))\,\mathrm{d}y\right] \\
&\quad \times |\gamma^j_\delta(s_j)|^2(x - (z^j_*/\varepsilon))\,\mathrm{d}x \\
&\leqslant C\varepsilon^\theta \left\|\gamma^i_\delta(s_i)\right\|^2_{L^{8/(2+\theta)}(\mathbb{R}^2)} \left\|\gamma^j_\delta(s_j)\right\|^2_{L^{8/(2+\theta)}(\mathbb{R}^2)}. \qquad (3.6)
\end{aligned}$$

Since $\gamma^i_\delta([0,1])$ $(1 \leqslant i \leqslant k)$ is compact in $H^1(\mathbb{R}^2)$ and $2 < 8/2 + \theta < +\infty$ for $\theta > 0$ small, we get $(I) = o(1)$. Moreover,

$$\begin{aligned}
(II) &= \frac{1}{2}\int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} \delta_0(x-y)|\gamma^i_{\varepsilon,\delta}(s_i)|^2(y)\,\mathrm{d}y\right] |\gamma^j_{\varepsilon,\delta}(s_j)|^2(x)\,\mathrm{d}x \\
&= \frac{1}{2}\int_{\mathbb{R}^2} |\gamma^i_{\varepsilon,\delta}(s_i)|^2|\gamma^j_{\varepsilon,\delta}(s_j)|^2 = 0. \qquad (3.7)
\end{aligned}$$

Lemma 2.3(i) and (3.5)–(3.7) show that for each $\delta > 0$,

$$\begin{aligned}
\tilde{c}_{\varepsilon,\delta} &:= \max_{s\in[0,1]^k} J_\varepsilon(\gamma_{\varepsilon,\delta}(s)) = \sum_{i=1}^{k} \max_{s_i\in[0,1]} J_\varepsilon(\gamma^i_{\varepsilon,\delta}(s_i)) + o(1) \\
&= \sum_{i=1}^{k} \max_{t\in[0,1]} I_{m_i}(\gamma^i_\delta(t)) + o(1) = \sum_{i=1}^{k} c_{m_i} + o(1),
\end{aligned}$$

(i) holds. Moreover, by (3.3), (ii) is obvious. □

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Letting

$$c^i_{\varepsilon,\delta} := \inf_{\gamma\in\Gamma^i_\varepsilon}\max_{t\in[0,1]} J^i_\varepsilon(\gamma(t)),$$

where

$$\Gamma^i_\varepsilon := \{\gamma(t)\in C([0,1],H_\varepsilon) : \gamma(0)=0 \text{ and } \gamma(1)=\gamma^i_{\varepsilon,\delta}(1)\}.$$

We have the following estimates:

LEMMA 3.2. *For each $\delta>0$ and $i=1,2,\dots,k$,*

$$\underline{\lim}_{\varepsilon\to 0} c^i_{\varepsilon,\delta} \geqslant c_{m_i}.$$

Noting that the mountain pass level c_{m_i} corresponds to the least energy level b_{m_i} due to corollary 2.5, we can prove lemma 3.2 as in proposition 3 in [**6**]. We omit the proof.

In order to state the next lemma, we need some notations. For each $R>0$, we regard $H^1_0(B_R(0))$ as a subspace of H_ε. Namely, for any $u\in H^1_0(B_R(0))$, we extend u by defining $u(x)=0$ for $|x|>R$, then $\|\cdot\|_{H_\varepsilon}$ is equivalent to the standard norm of $H^1_0(B_R(0))$ for each $R>0$, $\varepsilon>0$. Using $\|\cdot\|_{H_\varepsilon}$, for each $T\in(H^1_0(B_R(0)))^{-1}$, we define

$$\|T\|_{*,\varepsilon,R} := \sup\left\{Tu : u\in H^1_0(B_R(0)), \|u\|_{H_\varepsilon}\leqslant 1\right\}.$$

Note also that $\|\cdot\|_{*,\varepsilon,R}$ is equivalent to the standard norm of $(H^1_0(B_R(0)))^{-1}$. We use the notation $J^\alpha_\varepsilon := \{u\in H_\varepsilon : J_\varepsilon(u)\leqslant\alpha\}$ and fix a $R_0>0$ such that $B_{R_0}(0)\supset\Lambda$.

We have the following lemma and this lemma is a key for the proof of theorem 1.2:

LEMMA 3.3. (i) *There exists a $d_0>0$ such that for any $\{\varepsilon_j\}_{j=1}^\infty$, $\{R_{\varepsilon_j}\}_{j=1}^\infty$, $\{u_{\varepsilon_j}\}_{j=1}^\infty$ satisfying*

$$\begin{cases} \lim_{j\to\infty}\varepsilon_j = 0,\ R_{\varepsilon_j}\geqslant R_0/\varepsilon_j,\ u_{\varepsilon_j}\in X^{d_0}_{\varepsilon_j}\cap H^1_0(B_{R_{\varepsilon_j}}(0)),\\ \lim_{j\to\infty} J_{\varepsilon_j}(u_{\varepsilon_j})\leqslant\sum_{i=1}^k c_{m_i} \text{ and } \lim_{j\to\infty}\|J'_{\varepsilon_j}(u_{\varepsilon_j})\|_{*,\varepsilon_j,R_{\varepsilon_j}}=0, \end{cases} \tag{3.8}$$

there exist, up to a subsequence, $\{y^i_{\varepsilon_j}\}_{j=1}^\infty\subset\mathbb{R}^N$, $z^i\in\mathcal{M}^i$, $U^i\in S_{m_i}$ such that

$$\lim_{j\to\infty}|\varepsilon_j y^i_{\varepsilon_j}-z^i|=0 \quad \text{and} \quad \lim_{j\to\infty}\left\|u_{\varepsilon_j}-\sum_{i=1}^k\varphi(\varepsilon_j x-\varepsilon_j y^i_{\varepsilon_j})U^i(x-y^i_{\varepsilon_j})\right\|_{H_{\varepsilon_j}}=0.$$

(ii) *If we drop $\{R_{\varepsilon_j}\}_{j=1}^\infty$ and replace (3.8) by*

$$\lim_{j\to\infty}\varepsilon_j=0,\quad u_{\varepsilon_j}\in X^{d_0}_{\varepsilon_j},\quad \lim_{j\to\infty}J_{\varepsilon_j}(u_{\varepsilon_j})\leqslant\sum_{i=1}^k c_{m_i}\quad \text{and}$$
$$\lim_{j\to\infty}\|J'_{\varepsilon_j}(u_{\varepsilon_j})\|_{(H_{\varepsilon_j})^{-1}}=0,$$

then the same conclusion holds.

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Proof. We only prove (i). (ii) follows with minor modifications. For notational simplicity, we write ε for ε_j and still use ε after taking a subsequence. By the definition of $X_\varepsilon^{d_0}$ and the compactness of S_{m_i} and $(\mathcal{M}^i)^\beta$, we see that there exist $W^i \in S_{m_i}$, $\{z_\varepsilon^i\}_{\varepsilon>0} \subset (\mathcal{M}^i)^\beta$ such that for $\varepsilon > 0$ small and $1 \leqslant i \leqslant k$,

$$\left\| u_\varepsilon - \sum_{i=1}^k \varphi(\varepsilon x - z_\varepsilon^i) W^i(x - (z_\varepsilon^i/\varepsilon)) \right\|_{H_\varepsilon} \leqslant 2d_0 \quad \text{and} \quad z_\varepsilon^i \to z^i \in (\mathcal{M}^i)^\beta \text{ as } \varepsilon \to 0. \tag{3.9}$$

Step 1: We claim that

$$\lim_{\varepsilon\to 0} \sup_{y\in A_\varepsilon} \int_{B_1(y)} |u_\varepsilon|^2 = 0, \tag{3.10}$$

where $A_\varepsilon = \cup_{i=1}^k (B_{3\beta/\varepsilon}(z_\varepsilon^i/\varepsilon) \backslash B_{\beta/2\varepsilon}(z_\varepsilon^i/\varepsilon))$.

Assuming on the contrary that there exists $r > 0$ such that

$$\varlimsup_{\varepsilon\to 0} \sup_{y\in A_\varepsilon} \int_{B_1(y)} |u_\varepsilon|^2 = 2r > 0,$$

then there exists $y_\varepsilon \in A_\varepsilon$ such that for $\varepsilon > 0$ small,

$$\int_{B_1(y_\varepsilon)} |u_\varepsilon|^2 \geqslant r > 0. \tag{3.11}$$

Letting $v_\varepsilon(x) := u_\varepsilon(x + y_\varepsilon)$, up to a subsequence, there exists $v \in H^1(\mathbb{R}^N)\backslash\{0\}$ such that $v_\varepsilon \rightharpoonup v$ in $H^1(\mathbb{R}^N)$ and $\varepsilon y_\varepsilon \to x_0$ with x_0 in the closure of $\cup_{i=1}^k (B_{3\beta}(z^i) \backslash B_{\beta/2}(z^i))$. Moreover, we see that v satisfies (2.1) with m replaced by $V(x_0)$. Since

$$c_{V(x_0)} \leqslant I_{V(x_0)}(v) - \frac{1}{N} P_{V(x_0)}(v) = \frac{1}{N} \int_{\mathbb{R}^N} |\nabla v|^2,$$

then for $R > 0$ large,

$$\begin{aligned} \lim_{\varepsilon\to 0} \frac{1}{N} \int_{B_R(y_\varepsilon)} |\nabla u_\varepsilon|^2 &= \lim_{\varepsilon\to 0} \frac{1}{N} \int_{B_R(0)} |\nabla v_\varepsilon|^2 \\ &\geqslant \frac{1}{N} \int_{B_R(0)} |\nabla v|^2 \geqslant \frac{1}{2N} \int_{\mathbb{R}^N} |\nabla v|^2 \geqslant \frac{1}{2} c_{V(x_0)} > 0. \end{aligned} \tag{3.12}$$

By (3.9) and Sobolev's imbedding theorem, we have

$$\begin{aligned} \frac{1}{N} \int_{B_R(y_\varepsilon)} |\nabla u_\varepsilon|^2 &\leqslant C \sum_{i=1}^k \int_{B_R(y_\varepsilon)} |\nabla(\varphi(\varepsilon x - z_\varepsilon^i) W^i(x - (z_\varepsilon^i/\varepsilon)))|^2 + Cd_0 \\ &\leqslant C \sum_{i=1}^k \int_{B_R(y_\varepsilon - (z_\varepsilon^i/\varepsilon))} |\nabla W^i|^2 + Cd_0 + o(1) = Cd_0 + o(1), \end{aligned}$$

where $o(1) \to 0$ as $\varepsilon \to 0$ and we have used $|y_\varepsilon - (z_\varepsilon^i/\varepsilon)| \geqslant \beta/2\varepsilon$. This leads to a contradiction for d_0 small. Hence, (3.10) holds.

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Since

$$\sup_{y\in A_\varepsilon}\int_{B_1(y)}|u_\varepsilon|^2 \geqslant \sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|\eta_\varepsilon u_\varepsilon|^2,$$

where $\eta_\varepsilon \in C_c^\infty(\mathbb{R}^N,[0,1])$ such that $\eta_\varepsilon(x)=1$ for $x\in\cup_{i=1}^k(B_{(3\beta/\varepsilon)-2}(z_\varepsilon^i/\varepsilon)\backslash B_{(\beta/2\varepsilon)+2}(z_\varepsilon^i/\varepsilon))$, $\operatorname{supp}\eta_\varepsilon\subset\cup_{i=1}^k(B_{(3\beta/\varepsilon)-1}(z_\varepsilon^i/\varepsilon)\backslash B_{(\beta/2\varepsilon)+1}(z_\varepsilon^i/\varepsilon))$, $|\nabla\eta_\varepsilon|\leqslant C$ and $C>0$ is independent of ε. By (3.10) and the boundedness of $\{\eta_\varepsilon u_\varepsilon\}_{\varepsilon>0}$ in $H^1(\mathbb{R}^N)$, we derive from vanishing theorem that for any $q\in(2,2^*)$,

$$\lim_{\varepsilon\to0}\int_{\cup_{i=1}^k(B_{2\beta/\varepsilon}(z_\varepsilon^i/\varepsilon)\backslash B_{\beta/\varepsilon}(z_\varepsilon^i/\varepsilon))}|u_\varepsilon|^q\to0. \tag{3.13}$$

Step 2: Let $u_{\varepsilon,1}(x):=\sum_{i=1}^k u_{\varepsilon,1}^i(x):=\sum_{i=1}^k\varphi(\varepsilon x-z_\varepsilon^i)u_\varepsilon(x)$, $u_{\varepsilon,2}(x):=u_\varepsilon(x)-u_{\varepsilon,1}(x)$, by (3.13), we see that

$$\int_{\mathbb{R}^N}|\nabla u_\varepsilon|^2\geqslant\int_{\mathbb{R}^N}|\nabla u_{\varepsilon,1}|^2+\int_{\mathbb{R}^N}|\nabla u_{\varepsilon,2}|^2+o(1), \tag{3.14}$$

$$\int_{\mathbb{R}^N}V(\varepsilon x)|u_\varepsilon|^2\geqslant\int_{\mathbb{R}^N}V(\varepsilon x)|u_{\varepsilon,1}|^2+\int_{\mathbb{R}^N}V(\varepsilon x)|u_{\varepsilon,2}|^2, \tag{3.15}$$

$$\int_{\mathbb{R}^N}|u_\varepsilon|^{\alpha+2}=\int_{\mathbb{R}^N}|u_{\varepsilon,1}|^{\alpha+2}+\int_{\mathbb{R}^N}|u_{\varepsilon,2}|^{\alpha+2}+o(1), \tag{3.16}$$

$$Q_\varepsilon(u_{\varepsilon,1})=0,\quad Q_\varepsilon(u_{\varepsilon,2})=Q_\varepsilon(u_\varepsilon)\geqslant0. \tag{3.17}$$

Moreover, we deduce from $E_1\in\mathcal{L}(L^p(\mathbb{R}^N),L^p(\mathbb{R}^N))$ $(1<p<+\infty)$, the boundedness of $\{u_\varepsilon\}_{\varepsilon>0}$ in $H^1(\mathbb{R}^N)$, Hölder's inequality and (3.13) that

$$\int_{\mathbb{R}^N}E_1(u_{\varepsilon,1}u_{\varepsilon,2})|u_\varepsilon|^2=o(1) \tag{3.18}$$

and arguing as in the proof of lemma 3.1(i), we see from (3.13) that

$$\int_{\mathbb{R}^N}E_1(|u_{\varepsilon,1}|^2)|u_{\varepsilon,2}|^2=o(1). \tag{3.19}$$

Hence,

$$\int_{\mathbb{R}^N}E_1(|u_\varepsilon|^2)|u_\varepsilon|^2=\int_{\mathbb{R}^N}E_1(|u_{\varepsilon,1}|^2)|u_{\varepsilon,1}|^2+\int_{\mathbb{R}^N}E_1(|u_{\varepsilon,2}|^2)|u_{\varepsilon,2}|^2+o(1). \tag{3.20}$$

From (3.14)–(3.17), (3.20), we infer that

$$J_\varepsilon(u_\varepsilon)\geqslant J_\varepsilon(u_{\varepsilon,1})+J_\varepsilon(u_{\varepsilon,2})+o(1). \tag{3.21}$$

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From (3.9), it follows

$$\begin{aligned}
\|u_{\varepsilon,2}\|_{H_\varepsilon} &\leqslant \left\|u_{\varepsilon,1} - \sum_{i=1}^{k} \varphi(\varepsilon x - z_\varepsilon^i) W^i(x - (z_\varepsilon^i/\varepsilon))\right\|_{H_\varepsilon\left(\bigcup_{i=1}^k B_{2\beta/\varepsilon}(z_\varepsilon^i/\varepsilon)\right)} + 2d_0 \\
&\leqslant \|u_{\varepsilon,2}\|_{H_\varepsilon\left(\bigcup_{i=1}^k (B_{2\beta/\varepsilon}(z_\varepsilon^i/\varepsilon)\setminus B_{\beta/\varepsilon}(z_\varepsilon^i/\varepsilon))\right)} + 4d_0 \\
&\leqslant C\|u_\varepsilon\|_{H_\varepsilon\left(\bigcup_{i=1}^k (B_{2\beta/\varepsilon}(z_\varepsilon^i/\varepsilon)\setminus B_{\beta/\varepsilon}(z_\varepsilon^i/\varepsilon))\right)} + 4d_0 \\
&\leqslant C\left\|\sum_{i=1}^{k} \varphi(\varepsilon x - z_\varepsilon^i) W^i(x - (z_\varepsilon^i/\varepsilon))\right\|_{H_\varepsilon\left(\bigcup_{i=1}^k (B_{2\beta/\varepsilon}(z_\varepsilon^i/\varepsilon)\setminus B_{\beta/\varepsilon}(z_\varepsilon^i/\varepsilon))\right)} + Cd_0 \\
&\leqslant C\sum_{i=1}^{k} \|\varphi(\varepsilon x - z_\varepsilon^i) W^i(x - (z_\varepsilon^i/\varepsilon))\|_{H^1(B_{2\beta/\varepsilon}(z_\varepsilon^i/\varepsilon)\setminus B_{\beta/\varepsilon}(z_\varepsilon^i/\varepsilon))} + Cd_0 \\
&\leqslant C\sum_{i=1}^{k} \|W^i\|_{H^1(B_{2\beta/\varepsilon}(0)\setminus B_{\beta/\varepsilon}(0))} + Cd_0 = Cd_0 + o(1),
\end{aligned}$$

where $o(1) \to 0$ as $\varepsilon \to 0$. Thus, $\overline{\lim}_{\varepsilon\to 0} \|u_{\varepsilon,2}\|_{H_\varepsilon} \leqslant Cd_0$.

Moreover, since $\langle J'_\varepsilon(u_\varepsilon), u_{\varepsilon,2}\rangle \to 0$ as $\varepsilon \to 0$ and $\langle Q'_\varepsilon(u_\varepsilon), u_{\varepsilon,2}\rangle = \langle Q'_\varepsilon(u_{\varepsilon,2}), u_{\varepsilon,2}\rangle \geqslant 0$, we get

$$\int_{\mathbb{R}^N} \nabla u_\varepsilon \cdot \nabla u_{\varepsilon,2} + \int_{\mathbb{R}^N} V(\varepsilon x) u_\varepsilon u_{\varepsilon,2} \leqslant \frac{b}{4}\int_{\mathbb{R}^N} E_1(|u_\varepsilon|^2) u_\varepsilon u_{\varepsilon,2} + C\int_{\mathbb{R}^N} |u_\varepsilon|^\alpha u_\varepsilon u_{\varepsilon,2}.$$

Using Hölder's inequality and Sobolev's imbedding theorem, we see from (3.13), (3.18), (3.19) and $E_1 \in \mathcal{L}(L^p(\mathbb{R}^N), L^p(\mathbb{R}^N))(1 < p < +\infty)$ that

$$\|u_{\varepsilon,2}\|_{H_\varepsilon}^2 \leqslant C\|u_{\varepsilon,2}\|_{H_\varepsilon}^4 + C\|u_{\varepsilon,2}\|_{H_\varepsilon}^{\alpha+2} + o(1).$$

Choosing $d_0 > 0$ small, we see that $\|u_{\varepsilon,2}\|_{H_\varepsilon} = o(1)$, by (3.21),

$$J_\varepsilon(u_\varepsilon) \geqslant J_\varepsilon(u_{\varepsilon,1}) + o(1). \tag{3.22}$$

Step 3: For each $1 \leqslant i \leqslant k$, letting $\tilde{w}_\varepsilon^i(x) := u_{\varepsilon,1}^i(x + (z_\varepsilon^i/\varepsilon)) := \varphi(\varepsilon x)u_\varepsilon(x + (z_\varepsilon^i/\varepsilon))$, up to a subsequence, as $\varepsilon \to 0$, $\exists \tilde{w}^i \in H^1(\mathbb{R}^N)$ such that $\tilde{w}_\varepsilon^i \rightharpoonup \tilde{w}^i$ in $H^1(\mathbb{R}^N)$. Next, we claim that

$$\tilde{w}_\varepsilon^i \to \tilde{w}^i \text{ in } L^q(\mathbb{R}^N) \quad \text{for } q \in (2, 2^*). \tag{3.23}$$

If not, by vanishing theorem, $\exists r > 0$ such that

$$\varlimsup_{\varepsilon\to 0} \sup_{x\in\mathbb{R}^N} \int_{B_1(x)} |\tilde{w}_\varepsilon^i - \tilde{w}^i|^2 = 2r > 0,$$

then for $\varepsilon > 0$ small, $\exists x_\varepsilon^i \in \mathbb{R}^N$ such that

$$\int_{B_1(x_\varepsilon^i)} |\tilde{w}_\varepsilon^i - \tilde{w}^i|^2 \geqslant r > 0. \tag{3.24}$$

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There are two cases:

Case 1: $\{x_\varepsilon^i\}_{\varepsilon>0}$ is bounded, that is, $|x_\varepsilon^i| \leqslant R_i$ for some $R_i > 0$, then for $\varepsilon > 0$ small,

$$\int_{B_{R_i+1}(0)} |\tilde{w}_\varepsilon^i - \tilde{w}^i|^2 \geqslant r > 0,$$

which contradicts $\tilde{w}_\varepsilon^i \to \tilde{w}^i$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$.

Case 2: $\{x_\varepsilon^i\}_{\varepsilon>0}$ is unbounded, by (3.24),

$$\varliminf_{\varepsilon\to 0} \int_{B_1(x_\varepsilon^i)} |\varphi(\varepsilon x) u_\varepsilon(x + (z_\varepsilon^i/\varepsilon))|^2 \geqslant r > 0. \tag{3.25}$$

Since $\varphi(x) = 0$ for $|x| \geqslant 2\beta$, we see that $|x_\varepsilon^i| \leqslant 3\beta/\varepsilon$ for $\varepsilon > 0$ small. Furthermore, if $|x_\varepsilon^i| \geqslant \beta/2\varepsilon$, then $x_\varepsilon^i \in B_{3\beta/\varepsilon}(0)\backslash B_{\beta/2\varepsilon}(0)$, by (3.10),

$$\begin{aligned}
&\varliminf_{\varepsilon\to 0} \int_{B_1(x_\varepsilon^i)} |\varphi(\varepsilon x) u_\varepsilon(x + (z_\varepsilon^i/\varepsilon))|^2 \\
&\quad \leqslant \varliminf_{\varepsilon\to 0} \sup_{z\in B_{3\beta/\varepsilon}(0)\backslash B_{\beta/2\varepsilon}(0)} \int_{B_1(z)} |u_\varepsilon(x + (z_\varepsilon^i/\varepsilon))|^2 \\
&\quad = \varliminf_{\varepsilon\to 0} \sup_{y\in B_{3\beta/\varepsilon}(z_\varepsilon^i/\varepsilon)\backslash B_{\beta/2\varepsilon}(z_\varepsilon^i/\varepsilon)} \int_{B_1(y)} |u_\varepsilon|^2 \leqslant \varliminf_{\varepsilon\to 0} \sup_{y\in A_\varepsilon} \int_{B_1(y)} |u_\varepsilon|^2 = 0,
\end{aligned}$$

which contradicts (3.25). Thus, $|x_\varepsilon^i| \leqslant \beta/2\varepsilon$ for $\varepsilon > 0$ small. Up to a subsequence, $\varepsilon x_\varepsilon^i \to x^i \in \overline{B_{\beta/2}(0)}$ and $\bar{w}_\varepsilon^i(x) := \tilde{w}_\varepsilon^i(x + x_\varepsilon^i) \rightharpoonup \bar{w}^i$ in $H^1(\mathbb{R}^N)$, by (3.25), $\bar{w}^i \neq 0$ and satisfies (2.1) with m replaced by $V(z^i + x^i)$. Arguing as in *step 1*, we get a contradiction for $d_0 > 0$ small. Thus, (3.23) follows.

Similar to the argument in lemma 3.1(i), we have

$$J_\varepsilon(u_{\varepsilon,1}) = \sum_{i=1}^k J_\varepsilon(u_{\varepsilon,1}^i(x)) + o(1).$$

Recalling that $z_\varepsilon^i \to z^i$ and $\tilde{w}_\varepsilon^i(x) = u_{\varepsilon,1}^i(x + (z_\varepsilon^i/\varepsilon))$, by (3.22), (3.23), we obtain

$$\sum_{i=1}^k I_{V(z^i)}(\tilde{w}^i) \leqslant \sum_{i=1}^k c_{m_i}. \tag{3.26}$$

For any $\psi \in C_c^\infty(\mathbb{R}^N)$, letting $\psi(x - (z_\varepsilon^i/\varepsilon))$ as a test function for $J'_\varepsilon(u_\varepsilon)$. Since for $\varepsilon > 0$ small, $\operatorname{supp}\psi(x - (z_\varepsilon^i/\varepsilon)) \subset \Lambda/\varepsilon$, we observe that $\langle Q'_\varepsilon(u_\varepsilon), \psi(x - (z_\varepsilon^i/\varepsilon))\rangle = 0$ and recalling $\|u_{\varepsilon,2}\|_{H_\varepsilon} = o(1)$, we see that $\tilde{w}^i$ is a solution of (2.1) with m replaced by $V(z^i)$. Moreover, thanks to (3.19), (3.23), $E_1 \in \mathcal{L}(L^p(\mathbb{R}^N), L^p(\mathbb{R}^N))$ $(1 < p < +\infty)$ and $\langle J'_\varepsilon(u_\varepsilon), u_{\varepsilon,1}^i\rangle \to 0$, $\|u_{\varepsilon,2}\|_{H_\varepsilon} \to 0$ as $\varepsilon \to 0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla \tilde{w}^i|^2 + \int_{\mathbb{R}^N} V(z^i)|\tilde{w}^i|^2 \\
&\quad \leqslant \varliminf_{\varepsilon\to 0} \left[\int_{\mathbb{R}^N} |\nabla \tilde{w}_\varepsilon^i|^2 + \int_{\mathbb{R}^N} V(\varepsilon x + z_\varepsilon^i)|\nabla \tilde{w}_\varepsilon^i|^2\right]
\end{aligned}$$

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$$\leqslant \lim_{\varepsilon\to 0}\left[\frac{b}{4}\int_{\mathbb{R}^N} E_1(|\tilde{w}_\varepsilon^i|^2)|\tilde{w}_\varepsilon^i|^2 - \frac{a}{\alpha+2}\int_{\mathbb{R}^N}|\tilde{w}_\varepsilon^i|^{\alpha+2}\right]$$
$$= \frac{b}{4}\int_{\mathbb{R}^N} E_1(|\tilde{w}^i|^2)|\tilde{w}^i|^2 - \frac{a}{\alpha+2}\int_{\mathbb{R}^N}|\tilde{w}^i|^{\alpha+2}$$
$$= \int_{\mathbb{R}^N}|\nabla\tilde{w}^i|^2 + \int_{\mathbb{R}^N} V(z^i)|\tilde{w}^i|^2,$$

then as $\varepsilon\to 0$,

$$\int_{\mathbb{R}^N} V(\varepsilon x + z_\varepsilon^i)|\tilde{w}_\varepsilon^i|^2 \to \int_{\mathbb{R}^N} V(z^i)|\tilde{w}^i|^2 \quad\text{and}\quad \int_{\mathbb{R}^N}|\nabla\tilde{w}_\varepsilon^i|^2 \to \int_{\mathbb{R}^N}|\nabla\tilde{w}^i|^2. \tag{3.27}$$

By (3.9), (3.23) and $\|u_{\varepsilon,2}\|_{H_\varepsilon} = o(1)$, $\tilde{w}^i \neq 0$ for $d_0 > 0$ small. Thus

$$I_{V(z^i)}(\tilde{w}^i) \geqslant c_{V(z^i)}. \tag{3.28}$$

Since $z^i \in (\mathcal{M}^i)^\beta \subset \Lambda^i$, (3.26) and (3.28) imply that $V(z^i) = m_i$ and $z^i \in \mathcal{M}^i$. At this point, it is clear that $\exists U^i \in S_{m_i}$ and $y^i \in \mathbb{R}^N$ such that $\tilde{w}^i(x) = U^i(x - y^i)$. Since

$$m_i\int_{\mathbb{R}^N}|\tilde{w}_\varepsilon^i|^2 \leqslant \int_{\mathbb{R}^N} V(\varepsilon x + z_\varepsilon^i)|\tilde{w}_\varepsilon^i|^2,$$

by (3.27), $\tilde{w}_\varepsilon^i \to \tilde{w}^i$ in $H^1(\mathbb{R}^N)$. Therefore,

$$\left\|u_\varepsilon - \sum_{i=1}^k \varphi(\varepsilon x - (z_\varepsilon^i + \varepsilon y^i))U^i(x - ((z_\varepsilon^i/\varepsilon) + y^i))\right\|_{H_\varepsilon} \to 0 \quad \text{as } \varepsilon\to 0.$$

This completes the proof. □

Lemma 3.4. *Letting d_0 be the number given in lemma 3.3, then for any $d \in (0, d_0)$, there exist $\varepsilon_d > 0$, $\rho_d > 0$ and $\omega_d > 0$ such that*

$$\|J_\varepsilon'(u)\|_{*,\varepsilon,R} \geqslant \omega_d$$

for all $u \in J_\varepsilon^{\sum_{i=1}^k c_{m_i}+\rho_d} \cap (X_\varepsilon^{d_0}\backslash X_\varepsilon^d) \cap H_0^1(B_R(0))$ with $\varepsilon \in (0, \varepsilon_d)$ and $R \geqslant R_0/\varepsilon$.

Proof. Assuming on the contrary that, there exist $d \in (0, d_0)$, $\{\varepsilon_j\}_{j=1}^\infty$, $\{\rho_j\}_{j=1}^\infty$ with ε_j, $\rho_j \to 0$, $R_{\varepsilon_j} \geqslant R_0/\varepsilon_j$ and $u_j \in J_{\varepsilon_j}^{\sum_{i=1}^k c_{m_i}+\rho_j} \cap (X_{\varepsilon_j}^{d_0}\backslash X_{\varepsilon_j}^d) \cap H_0^1(B_{R_{\varepsilon_j}}(0))$ such that

$$\|J_{\varepsilon_j}'(u_j)\|_{*,\varepsilon_j,R_{\varepsilon_j}} \to 0 \quad \text{as } j\to\infty.$$

By lemma 3.3(i), for each $1 \leqslant i \leqslant k$, we find $\{y_j^i\}_{j=1}^\infty \subset \mathbb{R}^N$, $z^i \in \mathcal{M}^i$, $U^i \in S_{m_i}$ such that

$$\lim_{j\to\infty}|\varepsilon_j y_j^i - z^i| = 0 \quad\text{and}\quad \lim_{j\to\infty}\left\|u_j - \sum_{i=1}^k \varphi(\varepsilon_j x - \varepsilon_j y_j^i)U^i(x - y_j^i)\right\|_{H_{\varepsilon_j}} = 0,$$

which gives that $u_j \in X_{\varepsilon_j}^d$ for large j. This contradicts that $u_j \notin X_{\varepsilon_j}^d$. □

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LEMMA 3.5. *There exists $T_0 > 0$ with the following property: for any $\delta > 0$ small, there exist $\alpha_\delta > 0$ and $\varepsilon_\delta > 0$ such that if $J_\varepsilon(\gamma_{\varepsilon,\delta}(s)) \geqslant \sum_{i=1}^k c_{m_i} - \alpha_\delta$ and $\varepsilon \in (0, \varepsilon_\delta)$, then $\gamma_{\varepsilon,\delta}(s) \in X_\varepsilon^{T_0\delta}$.*

Proof. First, there is a $T_0' > 0$ such that for each $1 \leqslant i \leqslant k$, $z_*^i \in \mathcal{M}^i$ mentioned in (3.2) and $u \in H^1(\mathbb{R}^N)$,

$$\|\varphi(\varepsilon x - z_*^i)u(x - (z_*^i/\varepsilon))\|_{H_\varepsilon} \leqslant T_0'\|u(x)\|_{H^1(\mathbb{R}^N)}, \tag{3.29}$$

By lemma 2.3(ii), we define

$$\alpha_\delta = \frac{1}{4}\min\left\{\sum_{i=1}^k c_{m_i} - \sum_{i=1}^k I_{m_i}(\gamma_\delta^i(s_i)) : s_i \in [0,1], \sum_{i=1}^k \left\|\gamma_\delta^i(s_i) - U_*^i\right\|_{H^1(\mathbb{R}^N)} \geqslant k\delta\right\} > 0,$$

we have

$$\sum_{i=1}^k I_{m_i}(\gamma_\delta^i(s_i)) \geqslant \sum_{i=1}^k c_{m_i} - 2\alpha_\delta \text{ implies } \sum_{i=1}^k \left\|\gamma_\delta^i(s_i) - U_*^i\right\|_{H^1(\mathbb{R}^N)} \leqslant k\delta. \tag{3.30}$$

Moreover, since $\gamma_\delta^i([0,1])$ is compact in $H^1(\mathbb{R}^N)$, similar to the proof of lemma 3.1(i), we obtain

$$\max_{s\in[0,1]^k}\left|J_\varepsilon(\gamma_{\varepsilon,\delta}(s)) - \sum_{i=1}^k I_{m_i}(\gamma_\delta^i(s_i))\right| \to 0 \quad \text{as } \varepsilon \to 0.$$

Hence, there exists an $\varepsilon_\delta > 0$ such that

$$\max_{s\in[0,1]^k}\left|J_\varepsilon(\gamma_{\varepsilon,\delta}(s)) - \sum_{i=1}^k I_{m_i}(\gamma_\delta^i(s_i))\right| \leqslant \alpha_\delta \tag{3.31}$$

for all $\varepsilon \in (0, \varepsilon_\delta)$. Thus, if $\varepsilon \in (0, \varepsilon_\delta)$ and $J_\varepsilon(\gamma_{\varepsilon,\delta}(s)) \geqslant \sum_{i=1}^k c_{m_i} - \alpha_\delta$, by (3.30) and (3.31), we have

$$\sum_{i=1}^k \left\|\gamma_\delta^i(s_i) - U_*^i\right\|_{H^1(\mathbb{R}^N)} \leqslant k\delta,$$

by (3.29), we have

$$\begin{aligned}
&\left\|\sum_{i=1}^k \gamma_{\varepsilon,\delta}^i(s_i) - \sum_{i=1}^k \varphi(\varepsilon x - z_*^i)U_*^i(x - (z_*^i/\varepsilon))\right\|_{H_\varepsilon} \\
&\quad\leqslant \sum_{i=1}^k \left\|\varphi(\varepsilon x - z_*^i)((\gamma_\delta^i(s_i)(x - (z_*^i/\varepsilon))) - U_*^i(x - (z_*^i/\varepsilon)))\right\|_{H_\varepsilon} \\
&\quad\leqslant T_0' \sum_{i=1}^k \left\|\gamma_\delta^i(s_i)(x) - U_*^i(x)\right\|_{H^1(\mathbb{R}^N)} \leqslant kT_0'\delta := T_0\delta.
\end{aligned}$$

Thus, $\gamma_{\varepsilon,\delta}(s) \in X_\varepsilon^{T_0\delta}$. □

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Choosing $\delta_1 > 0$ to ensure that $T_0\delta_1 < d_0/4$, letting $\bar{\alpha} = \min\{\alpha_{\delta_1}, \sigma\}$ and fixing $d = d_0/4 := d_1$ in lemma 3.4. To prove the next lemma, we use an idea developed in [**24**]. However, for constructing multi-peak solutions, we give a more complicated proof than that in [**24**], where only the single-peak solution was considered.

LEMMA 3.6. *$\exists \bar{\varepsilon} > 0$ such that for each $\varepsilon \in (0, \bar{\varepsilon}]$ and $R > R_0/\varepsilon$, there exists a sequence $\{v_{n,\varepsilon}^R\}_{n=1}^{\infty} \subset J_\varepsilon^{\tilde{c}_{\varepsilon,\delta_1}+\varepsilon} \cap X_\varepsilon^{d_0} \cap H_0^1(B_R(0))$ such that $J'_\varepsilon(v_{n,\varepsilon}^R) \to 0$ in $(H_0^1(B_R(0)))^{-1}$ as $n \to \infty$.*

Proof. Assuming on the contrary that there always exist $\varepsilon > 0$ small, $R_\varepsilon > R_0/\varepsilon$ and $\gamma(\varepsilon, R_\varepsilon) > 0$ such that

$$\|J'_\varepsilon(u)\|_{*,\varepsilon,R_\varepsilon} \geqslant \gamma(\varepsilon, R_\varepsilon) > 0 \tag{3.32}$$

for $u \in J_\varepsilon^{\tilde{c}_{\varepsilon,\delta_1}+\varepsilon} \cap X_\varepsilon^{d_0} \cap H_0^1(B_{R_\varepsilon}(0))$.

Letting Y be a pseudo-gradient vector field for J'_ε in $H_0^1(B_{R_\varepsilon}(0))$, that is, $Y : H_0^1(B_{R_\varepsilon}(0)) \to H_0^1(B_{R_\varepsilon}(0))$ is a locally Lipschitz continuous vector field such that for every $u \in H_0^1(B_{R_\varepsilon}(0))$,

$$\|Y(u)\|_{H_\varepsilon} \leqslant 2\|J'_\varepsilon(u)\|_{*,\varepsilon,R_\varepsilon} \quad \text{and} \quad \langle J'_\varepsilon(u), Y(u)\rangle \geqslant \|J'_\varepsilon(u)\|^2_{*,\varepsilon,R_\varepsilon}. \tag{3.33}$$

Letting ψ_1, ψ_2 be locally Lipschitz continuous functions in $H_0^1(B_{R_\varepsilon}(0))$ such that $0 \leqslant \psi_1, \psi_2 \leqslant 1$ and

$$\psi_1(u) = \begin{cases} 1, & \displaystyle\sum_{i=1}^{k} c_{m_i} - \frac{1}{2}\bar{\alpha} \leqslant J_\varepsilon(u) \leqslant \tilde{c}_{\varepsilon,\delta_1}, \\ 0, & J_\varepsilon(u) \leqslant \displaystyle\sum_{i=1}^{k} c_{m_i} - \bar{\alpha} \text{ or } \tilde{c}_{\varepsilon,\delta_1} + \varepsilon \leqslant J_\varepsilon(u), \end{cases}$$

$$\psi_2(u) = \begin{cases} 1, & u \in X_\varepsilon^{3d_0/4}, \\ 0, & u \notin X_\varepsilon^{d_0}. \end{cases}$$

Considering the following ordinary differential equations:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}r}\eta(r,u) = -\dfrac{Y(\eta(r,u))}{\|Y(\eta(r,u))\|_{H_\varepsilon}}\psi_1(\eta(r,u))\psi_2(\eta(r,u)), \\ \eta(0,u) = u. \end{cases} \tag{3.34}$$

By (3.33) and (3.34), we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r}J_\varepsilon(\eta(r,u)) &= \left\langle J'_\varepsilon(\eta(r,u)), \frac{\mathrm{d}}{\mathrm{d}r}\eta(r,u)\right\rangle \\ &= \left\langle J'_\varepsilon(\eta(r,u)), -\frac{Y(\eta(r,u))}{\|Y(\eta(r,u))\|_{H_\varepsilon}}\psi_1(\eta(r,u))\psi_2(\eta(r,u))\right\rangle \\ &\leqslant -\frac{\psi_1(\eta(r,u))\psi_2(\eta(r,u))}{\|Y(\eta(r,u))\|_{H_\varepsilon}}\|J'_\varepsilon(\eta(r,u))\|^2_{*,\varepsilon,R_\varepsilon} \\ &\leqslant -\frac{1}{2}\psi_1(\eta(r,u))\psi_2(\eta(r,u))\|J'_\varepsilon(\eta(r,u))\|_{*,\varepsilon,R_\varepsilon} \end{aligned}$$

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and combining with lemma 3.1(i), lemma 3.4, (3.32), (3.34) and the definition of ψ_1, ψ_2, it is standard to show that $\eta \in C([0,+\infty)\times H_0^1(B_{R_\varepsilon}(0)), H_0^1(B_{R_\varepsilon}(0)))$ and satisfies that for $\varepsilon > 0$ small,

(i) $(\mathrm{d}/\mathrm{d}r)J_\varepsilon(\eta(r,u)) \leqslant 0$ for each $r \in [0,+\infty)$ and $u \in H_0^1(B_{R_\varepsilon}(0))$;

(ii) $(\mathrm{d}/\mathrm{d}r)J_\varepsilon(\eta(r,u)) \leqslant -\omega_{d_1}/2$ if $\eta(r,u) \in \overline{J_\varepsilon^{\tilde{c}_{\varepsilon,\delta_1}} \setminus J_\varepsilon^{\sum_{i=1}^k c_{m_i}-(1/2)\bar{\alpha}}} \cap \overline{X_\varepsilon^{3d_0/4} \setminus X_\varepsilon^{d_0/4}}$;

(iii) $(\mathrm{d}/\mathrm{d}r)J_\varepsilon(\eta(r,u)) \leqslant -\gamma(\varepsilon,R_\varepsilon)/2$ if $\eta(r,u) \in \overline{J_\varepsilon^{\tilde{c}_{\varepsilon,\delta_1}} \setminus J_\varepsilon^{\sum_{i=1}^k c_{m_i}-(1/2)\bar{\alpha}}} \cap X_\varepsilon^{3d_0/4}$;

(iv) $\eta(r,u) = u$ if $J_\varepsilon(u) \leqslant \sum_{i=1}^k c_{m_i} - \bar{\alpha}$.

Setting $r_1 := \omega_{d_1} d_0/\gamma(\varepsilon,R_\varepsilon)$ and $\xi_\varepsilon(s) := \eta(r_1, \gamma_{\varepsilon,\delta_1}(s))$, we have the following cases:

Case 1: $\gamma_{\varepsilon,\delta_1}(s) \in J_\varepsilon^{\sum_{i=1}^k c_{m_i}-\bar{\alpha}}$. By (iv), we see that

$$\eta(r, \gamma_{\varepsilon,\delta_1}(s)) = \gamma_{\varepsilon,\delta_1}(s). \tag{3.35}$$

Case 2: $\gamma_{\varepsilon,\delta_1}(s) \notin J_\varepsilon^{\sum_{i=1}^k c_{m_i}-\bar{\alpha}}$. By lemma 3.5 and the definition of $\tilde{c}_{\varepsilon,\delta_1}$, we see that

$$\gamma_{\varepsilon,\delta_1}(s) \in \overline{J_\varepsilon^{\tilde{c}_{\varepsilon,\delta_1}} \setminus J_\varepsilon^{\sum_{i=1}^k c_{m_i}-\bar{\alpha}}} \cap X_\varepsilon^{d_0/4}.$$

Moreover, we have

$$\eta(r, \gamma_{\varepsilon,\delta_1}(s)) \in X_\varepsilon^{d_0} \quad \text{for } r \in [0, r_1]. \tag{3.36}$$

Indeed, if not, $\exists r' \in [0,r_1]$ such that $\eta(r', \gamma_{\varepsilon,\delta_1}(s)) \notin X_\varepsilon^{d_0}$. Denote

$$r' := \sup\left\{ r \in [0,r'] : \eta(r, \gamma_{\varepsilon,\delta_1}(s)) \in X_\varepsilon^{d_0} \right\},$$

then by (3.34) and the definition of ψ_2, we see $\eta(r', \gamma_{\varepsilon,\delta_1}(s)) = \eta(r'', \gamma_{\varepsilon,\delta_1}(s)) \in X_\varepsilon^{d_0}$, which leads to a contradiction.

Next, we divide *case 2* into the following three subcases:

Case 2.1: $\eta(r_1, \gamma_{\varepsilon,\delta_1}(s)) \in J_\varepsilon^{\sum_{i=1}^k c_{m_i}-(1/2)\bar{\alpha}}$;

Case 2.2: $\eta(r_1, \gamma_{\varepsilon,\delta_1}(s)) \in J_\varepsilon^{\tilde{c}_{\varepsilon,\delta_1}} \setminus J_\varepsilon^{\sum_{i=1}^k c_{m_i}-(1/2)\bar{\alpha}}$ and $\eta(r, \gamma_{\varepsilon,\delta_1}(s)) \notin X_\varepsilon^{3d_0/4}$ for some $r \in [0,r_1]$;

Case 2.3: $\eta(r_1, \gamma_{\varepsilon,\delta_1}(s)) \in J_\varepsilon^{\tilde{c}_{\varepsilon,\delta_1}} \setminus J_\varepsilon^{\sum_{i=1}^k c_{m_i}-(1/2)\bar{\alpha}}$ and $\eta(r, \gamma_{\varepsilon,\delta_1}(s)) \in X_\varepsilon^{3d_0/4}$ for all $r \in [0,r_1]$.

In *case 2.2*, denote

$$r_2 := \inf\left\{ r \in [0,r_1] : \eta(r, \gamma_{\varepsilon,\delta_1}(s)) \notin X_\varepsilon^{3d_0/4} \right\}$$

and

$$r_3 := \sup\left\{ r \in [0,r_2] : \eta(r, \gamma_{\varepsilon,\delta_1}(s)) \in X_\varepsilon^{d_0/4} \right\},$$

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then by (3.34), $r_2 - r_3 \geqslant (1/2)d_0$ and $\eta(r, \gamma_{\varepsilon,\delta_1}(s)) \in \overline{X_\varepsilon^{3d_0/4} \backslash X_\varepsilon^{d_0/4}}$ for each $r \in [r_3, r_2]$. By (i), (ii) and lemma 3.1(i), we obtain

$$\begin{aligned} J_\varepsilon(\eta(r_1, \gamma_{\varepsilon,\delta_1}(s))) &= J_\varepsilon(\gamma_{\varepsilon,\delta_1}(s)) + \int_0^{r_1} \frac{\mathrm{d}}{\mathrm{d}r} J_\varepsilon(\eta(r, \gamma_{\varepsilon,\delta_1}(s)))\,\mathrm{d}s \\ &\leqslant \tilde{c}_{\varepsilon,\delta_1} + \int_{r_3}^{r_2} \frac{\mathrm{d}}{\mathrm{d}r} J_\varepsilon(\eta(r, \gamma_{\varepsilon,\delta_1}(s)))\,\mathrm{d}s \\ &\leqslant \tilde{c}_{\varepsilon,\delta_1} - \frac{1}{4}\omega_{d_1} d_0 = \sum_{i=1}^{k} c_{m_i} - \frac{1}{4}\omega_{d_1} d_0 + o(1), \end{aligned}$$

where $o(1) \to 0$ as $\varepsilon \to 0$.

In *case 2.3*, by (iii) and the definition of r_1, we have

$$\begin{aligned} J_\varepsilon(\eta(r_1, \gamma_{\varepsilon,\delta_1}(s))) &= J_\varepsilon(\gamma_{\varepsilon,\delta_1}(s)) + \int_0^{r_1} \frac{\mathrm{d}}{\mathrm{d}r} J_\varepsilon(\eta(r, \gamma_{\varepsilon,\delta_1}(s)))\,\mathrm{d}s \\ &\leqslant \tilde{c}_{\varepsilon,\delta_1} - \frac{1}{2}\omega_{d_1} d_0 = \sum_{i=1}^{k} c_{m_i} - \frac{1}{2}\omega_{d_1} d_0 + o(1). \end{aligned}$$

To sum up, choosing $\bar{\mu} = \min\{\bar{\alpha}/2, \omega_{d_1} d_0/4\} > 0$, we see that, for $s \in [0,1]^k$,

$$J_\varepsilon(\xi_\varepsilon(s)) \leqslant \sum_{i=1}^{k} c_{m_i} - \bar{\mu} + o(1). \tag{3.37}$$

From (3.35) and (3.36), we have

$$\|\xi_\varepsilon(s)\|_{H_\varepsilon} \leqslant C \quad \text{for } \varepsilon > 0 \text{ small and } s \in [0,1]^k. \tag{3.38}$$

Letting $\psi \in C_c^\infty(\mathbb{R}^N, [0,1])$ be such that $\psi(x) = 1$ for $x \in \Lambda$, $\psi(x) = 0$ for $x \notin \Lambda^\beta$ and $|\nabla\psi| \leqslant C/\beta$. We denote $\xi_{\varepsilon,1}(s) := \psi_\varepsilon \xi_\varepsilon(s)$ and $\xi_{\varepsilon,2}(s) := (1 - \psi_\varepsilon)\xi_\varepsilon(s)$, where $\psi_\varepsilon(x) = \psi(\varepsilon x)$. From (3.37) and (3.38), we see that $Q_\varepsilon(\xi_\varepsilon(s))$ is uniformly bounded, then for $\varepsilon > 0$ small and $s \in [0,1]^k$,

$$\int_{\mathbb{R}^N \backslash (\Lambda/\varepsilon)} |\xi_\varepsilon(s)|^2 \leqslant C\varepsilon. \tag{3.39}$$

By Gagliardo–Nirenberg's inequality, (3.38) and (3.39), we see that for each $2 < p < 2^*$,

$$\begin{aligned} \int_{\mathbb{R}^N} |\xi_{\varepsilon,2}(s)|^p &\leqslant C \|\nabla\xi_{\varepsilon,2}(s)\|_{L^2(\mathbb{R}^N)}^{(p-2)/2N} \|\xi_{\varepsilon,2}(s)\|_{L^2(\mathbb{R}^N)}^{2+(p-2)/2(2-N)} \\ &\leqslant C \|\xi_\varepsilon(s)\|_{L^2(\mathbb{R}^N \backslash (\Lambda/\varepsilon))}^{2+(p-2)/2(2-N)} = o(1), \end{aligned} \tag{3.40}$$

where $o(1) \to 0$ as $\varepsilon \to 0$ uniformly for $s \in [0,1]^k$. By (3.38), (3.40), Hölder's inequality and $E_1 \in \mathcal{L}(L^p(\mathbb{R}^N), L^p(\mathbb{R}^N))(1 < p < +\infty)$, we get

$$J_\varepsilon(\xi_\varepsilon(s)) \geqslant J_\varepsilon(\xi_{\varepsilon,1}(s)) + Q_\varepsilon(\xi_\varepsilon(s)) - Q_\varepsilon(\xi_{\varepsilon,1}(s)) + o(1).$$

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Since for $A, B \geqslant 0$, $(A+B-1)_+ \geqslant (A-1)_+ + (B-1)_+$, it follows that

$$\begin{aligned} Q_\varepsilon(\xi_\varepsilon(s)) &= \left(\varepsilon^{-1}\int_{\mathbb{R}^N\setminus(\Lambda/\varepsilon)} |\xi_{\varepsilon,1}(s)+\xi_{\varepsilon,2}(s)|^2 - 1\right)_+^2 \\ &\geqslant \left(\varepsilon^{-1}\int_{\mathbb{R}^N\setminus(\Lambda/\varepsilon)} |\xi_{\varepsilon,1}(s)|^2 + \varepsilon^{-1}\int_{\mathbb{R}^N\setminus(\Lambda/\varepsilon)} |\xi_{\varepsilon,2}(s)|^2 - 1\right)_+^2 \\ &\geqslant \left(\varepsilon^{-1}\int_{\mathbb{R}^N\setminus(\Lambda/\varepsilon)} |\xi_{\varepsilon,1}(s)|^2 - 1\right)_+^2 + \left(\varepsilon^{-1}\int_{\mathbb{R}^N\setminus(\Lambda/\varepsilon)} |\xi_{\varepsilon,2}(s)|^2 - 1\right)_+^2 \\ &= Q_\varepsilon(\xi_{\varepsilon,1}(s)) + Q_\varepsilon(\xi_{\varepsilon,2}(s)). \end{aligned} \tag{3.41}$$

Thus,

$$J_\varepsilon(\xi_\varepsilon(s)) \geqslant J_\varepsilon(\xi_{\varepsilon,1}(s)) + o(1). \tag{3.42}$$

Next, defining $\xi^i_{\varepsilon,1}(s)(x) = \xi_{\varepsilon,1}(s)(x)$ for $x \in (\Lambda^i)^\beta/\varepsilon$, $\xi^i_{\varepsilon,1}(s)(x) = 0$ for $x \notin (\Lambda^i)^\beta/\varepsilon$ for each $1 \leqslant i \leqslant k$. Arguing as in the proof of lemma 3.1(i) and (3.42), we get

$$J_\varepsilon(\xi_{\varepsilon,1}(s)) \geqslant \sum_{i=1}^k J_\varepsilon(\xi^i_{\varepsilon,1}(s)) + o(1) = \sum_{i=1}^k J^i_\varepsilon(\xi^i_{\varepsilon,1}(s)) + o(1). \tag{3.43}$$

Next, we introduce some notations as in [**17**]. For $s = (s_1, \ldots, s_k) \in [0,1]^k$, let $0_i = (s_1, \ldots, s_{i-1}, 0, s_{i+1}, \ldots, s_k)$ and $1_i = (s_1, \ldots, s_{i-1}, 1, s_{i+1}, \ldots, s_k)$. We see from lemma 3.1(ii) and (iv) in the proof of lemma 3.6 that $\xi_\varepsilon(0_i) = \gamma_{\varepsilon,\delta_1}(0_i)$ and $\xi_\varepsilon(1_i) = \gamma_{\varepsilon,\delta_1}(1_i)$, then by the definition of $\xi^i_{\varepsilon,1}(s)$, we see that $J^i_\varepsilon(\xi^i_{\varepsilon,1}(0_i)) = J^i_\varepsilon(0) = 0$ and $J^i_\varepsilon(\xi^i_{\varepsilon,1}(1_i)) = J^i_\varepsilon(\gamma^i_{\varepsilon,\delta_1}(1)) < 0$ for $\varepsilon > 0$ small by (3.3). Using the celebrated gluing method due to Coti Zelati and Rabinowitz [**17**, proposition 3.4], there exists $\bar{s}_\varepsilon \in [0,1]^k$ such that

$$J^i_\varepsilon(\xi^i_{\varepsilon,1}(\bar{s}_\varepsilon)) \geqslant c^i_{\varepsilon,\delta_1} \quad \text{for each } 1 \leqslant i \leqslant k. \tag{3.44}$$

(3.42), (3.43), (3.44) and lemma 3.2 yield

$$\max_{s\in[0,1]^k} J_\varepsilon(\xi_\varepsilon(s)) \geqslant \sum_{i=1}^k c_{m_i} + o(1),$$

which contradicts (3.37) for $\varepsilon > 0$ small. □

Proof of theorem 1.2. *Step 1*: By lemma 3.6, $\exists \bar{\varepsilon} > 0$ such that for each $\varepsilon \in (0, \bar{\varepsilon}]$ and $R > R_0/\varepsilon$, there exists a sequence $\{v^R_{n,\varepsilon}\}_{n=1}^\infty \subset J_\varepsilon^{\tilde{c}_{\varepsilon,\delta_1}+\varepsilon} \cap X_\varepsilon^{d_0} \cap H_0^1(B_R(0))$ such that $J'_\varepsilon(v^R_{n,\varepsilon}) \to 0$ in $(H_0^1(B_R(0)))^{-1}$ as $n \to \infty$.

Since $\{v^R_{n,\varepsilon}\}_{n=1}^\infty$ is bounded in $H_0^1(B_R(0))$, up to a subsequence, as $n \to \infty$, we see that $v^R_{n,\varepsilon} \rightharpoonup v^R_\varepsilon$ in $H_0^1(B_R(0))$, $v^R_{n,\varepsilon} \to v^R_\varepsilon$ in $L^p(B_R(0))(1 \leqslant p < 2^*)$ and $v^R_{n,\varepsilon} \to v^R_\varepsilon$

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a.e. $B_R(0)$. Using standard argument, we verify that

$$v_{n,\varepsilon}^R \to v_\varepsilon^R \text{ in } H_0^1(B_R(0)) \quad \text{as } n \to \infty \tag{3.45}$$

and $v_\varepsilon^R \geqslant 0$ satisfies

$$\begin{cases} -\Delta v_\varepsilon^R + V(\varepsilon x)v_\varepsilon^R + 4\left(\displaystyle\int_{\mathbb{R}^N} \chi_\varepsilon (v_\varepsilon^R)^2 - 1\right)_+ \chi_\varepsilon v_\varepsilon^R \\ \quad = bE_1((v_\varepsilon^R)^2)v_\varepsilon^R - a(v_\varepsilon^R)^{\alpha+1} \text{ in } B_R(0),\ v_\varepsilon^R = 0 \text{ on } \partial B_R(0). \end{cases} \tag{3.46}$$

By (3.45), $v_\varepsilon^R \in J_\varepsilon^{\tilde{c}_{\varepsilon,\delta_1}+\varepsilon} \cap X_\varepsilon^{d_0}$, then

$$\|v_\varepsilon^R\|_{H_\varepsilon} \leqslant C \text{ uniformly for } \varepsilon \in (0, \bar{\varepsilon}] \text{ and } R > R_0/\varepsilon. \tag{3.47}$$

By lemma B.1, we see that

$$\|v_\varepsilon^R\|_{L^\infty(\mathbb{R}^N)} \leqslant C \text{ uniformly for } \varepsilon \in (0, \bar{\varepsilon}] \text{ and } R > R_0/\varepsilon. \tag{3.48}$$

Step 2: Next, we claim that $v_\varepsilon^R \to v_\varepsilon \in X_\varepsilon^{d_0} \cap J_\varepsilon^{\tilde{c}_{\varepsilon,\delta_1}+\varepsilon}$ as $R \to \infty$ in H_ε sense for $\varepsilon > 0$ small. Since $Q_\varepsilon(v_\varepsilon^R)$ is uniformly bounded for all $\varepsilon > 0$ small and $R > R_0/\varepsilon$, we have

$$\int_{\mathbb{R}^N \setminus (\Lambda/\varepsilon)} (v_\varepsilon^R)^2 \leqslant C\varepsilon. \tag{3.49}$$

For any $y \in \mathbb{R}^N$, by (3.47), (3.48) and $E_1 \in \mathcal{L}(L^p(\mathbb{R}^N), L^p(\mathbb{R}^N))$ $(1 < p < +\infty)$, we get

$$\begin{aligned} \left\| E_1((v_\varepsilon^R)^2)v_\varepsilon^R \right\|_{L^N(B_2(y))} &\leqslant C \left\| E_1((v_\varepsilon^R)^2) \right\|_{L^{2N}(\mathbb{R}^N)} \left\| v_\varepsilon^R \right\|_{L^{2N}(B_2(y))} \\ &\leqslant C \left\| v_\varepsilon^R \right\|_{L^2(B_2(y))}^{1/N} \end{aligned} \tag{3.50}$$

and

$$\left\| (v_\varepsilon^R)^{\alpha+1} \right\|_{L^N(B_2(y))} \leqslant C \left\| v_\varepsilon^R \right\|_{L^2(B_2(y))}^{2/N}. \tag{3.51}$$

Letting $p = 2$, $q = 2N$ and $R = 1$ in theorem 8.17 of [**28**], we see from (3.50) and (3.51) that

$$\begin{aligned} \sup_{B_1(y)} v_\varepsilon^R &\leqslant C \left(\|v_\varepsilon^R\|_{L^2(B_2(y))} + \left\| bE_1((v_\varepsilon^R)^2)v_\varepsilon^R - a(v_\varepsilon^R)^{\alpha+1} \right\|_{L^2(B_2(y))} \right) \\ &\leqslant C \left(\|v_\varepsilon^R\|_{L^2(B_2(y))} + \|v_\varepsilon^R\|_{L^2(B_2(y))}^{2/N} + \|v_\varepsilon^R\|_{L^2(B_2(y))}^{1/N} \right), y \in \mathbb{R}^N, \end{aligned} \tag{3.52}$$

then by (3.49), we see that for $\varepsilon > 0$ small,

$$v_\varepsilon^R(x) \leqslant C\varepsilon^{1/2N} \quad \text{for all } |x| \geqslant R_0/\varepsilon + 2 \text{ and } R > R_0/\varepsilon. \tag{3.53}$$

By (3.47), (3.48) and $E_1 \in \mathcal{L}(L^p(\mathbb{R}^N), L^p(\mathbb{R}^N))(1 < p < +\infty)$, we verify that $bE_1((v_\varepsilon^R)^2)v_\varepsilon^R - a(v_\varepsilon^R)^{\alpha+1}$ is bounded in $L^q_{\text{loc}}(\mathbb{R}^N)$ for every $q \geqslant 1$ and the L^p-estimate implies the boundedness of $\{v_\varepsilon^R\}$ in $W^{2,q}_{\text{loc}}(\mathbb{R}^N)(q \geqslant 1)$, then by Sobolev's

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imbedding theorem, $\exists \alpha \in (0,1)$ such that

$$\|v_\varepsilon^R\|_{C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)} \leqslant C \text{ uniformly for } \varepsilon \in (0, \bar{\varepsilon}] \text{ and } R > R_0/\varepsilon. \tag{3.54}$$

Next, we claim that for $\varepsilon > 0$ small,

$$bE_1((v_\varepsilon^R)^2)v_\varepsilon^R - a(v_\varepsilon^R)^{\alpha+1} \leqslant \frac{1}{2}V(\varepsilon x)v_\varepsilon^R \quad \text{for } |x| \geqslant 2R_0/\varepsilon \text{ and } R > R_0/\varepsilon. \tag{3.55}$$

Without loss of generality, we only treat the case $N = 2$. By the appendix,

$$\begin{aligned} E_1((v_\varepsilon^R)^2)v_\varepsilon^R &= (\mathcal{K} * (v_\varepsilon^R)^2)v_\varepsilon^R \\ &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{-(x_1-y_1)^2 + (x_2-y_2)^2}{|x-y|^4}(v_\varepsilon^R(y))^2 \,\mathrm{d}y v_\varepsilon^R(x) \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^2} \delta_0(x-y)(v_\varepsilon^R(y))^2 \,\mathrm{d}y v_\varepsilon^R(x) \\ &= (I) + (II). \end{aligned} \tag{3.56}$$

By symmetry, we see that the average of $-x_1^2 + x_2^2$ on $\partial B_r(x)$ $(r > 0)$ vanishes, that is

$$\int_{\partial B_r(0)} (-x_1^2 + x_2^2)\,\mathrm{d}S = 0. \tag{3.57}$$

Using differential mean value theorem, (3.53), (3.54) and (3.57) yield that for $|x| \geqslant 2R_0/\varepsilon$ and $R > R_0/\varepsilon$,

$$\begin{aligned} &\lim_{\delta\to 0}\left|\int_{B_1(x)\setminus B_\delta(x)} \frac{-(x_1-y_1)^2 + (x_2-y_2)^2}{|x-y|^4}(v_\varepsilon^R(y))^2 \,\mathrm{d}y\right| \\ &= \lim_{\delta\to 0}\left|\int_{B_1(x)\setminus B_\delta(x)} \frac{-(x_1-y_1)^2 + (x_2-y_2)^2}{|x-y|^4}\left((v_\varepsilon^R(y))^2 - (v_\varepsilon^R(x))^2\right)\mathrm{d}y\right| \\ &\leqslant C\|v_\varepsilon^R\|_{L^\infty(B_1(x))}\|\nabla v_\varepsilon^R\|_{L^\infty(B_1(x))}\int_{B_1(x)} \frac{1}{|x-y|}\mathrm{d}y \leqslant C\varepsilon^{1/4}. \end{aligned} \tag{3.58}$$

Recalling $\Lambda \subset B_{R_0}(0)$, we see from (3.47) that for $|x| \geqslant 2R_0/\varepsilon$,

$$\left|\int_{\Lambda/\varepsilon} \frac{-(x_1-y_1)^2 + (x_2-y_2)^2}{|x-y|^4}(v_\varepsilon^R(y))^2 \,\mathrm{d}y\right| \leqslant C\frac{\varepsilon^2}{R_0^2}\int_{\Lambda/\varepsilon}(v_\varepsilon^R(y))^2 \,\mathrm{d}y \leqslant C\varepsilon^2. \tag{3.59}$$

Moreover, (3.49) implies that for $|x| \geqslant 2R_0/\varepsilon$,

$$\begin{aligned} &\left|\int_{\mathbb{R}^2\setminus((\Lambda/\varepsilon)\cup B_1(x))} \frac{-(x_1-y_1)^2 + (x_2-y_2)^2}{|x-y|^4}(v_\varepsilon^R(y))^2 \,\mathrm{d}y\right| \\ &\quad \leqslant C\int_{\mathbb{R}^2\setminus(\Lambda/\varepsilon)} |v_\varepsilon^R(y)|^2 \,\mathrm{d}y \leqslant C\varepsilon. \end{aligned} \tag{3.60}$$

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Obviously,

$$(II) = \frac{1}{2}\big(v_\varepsilon^R(x)\big)^3. \tag{3.61}$$

Finally, from (3.53), (3.56), (3.58), (3.59), (3.60) and (3.61), we get (3.55). The maximum principle and (3.55) show that

$$0 \leqslant v_\varepsilon^R(x) \leqslant C_1(\varepsilon)\,\mathrm{e}^{-C_2(\varepsilon)|x|} \quad \text{for } |x| \geqslant 2R_0/\varepsilon \text{ and } R > R_0/\varepsilon, \tag{3.62}$$

where $C_1(\varepsilon)$ and $C_2(\varepsilon)$ are independent of R.

Choosing a cut-off function $\varphi_A \in C^\infty(\mathbb{R}^N)$ such that $0 \leqslant \varphi_A \leqslant 1$, $\varphi_A = 0$ for $|x| \leqslant A$, $\varphi_A = 1$ for $|x| \geqslant 2A$ and $|\nabla\varphi_A| \leqslant C/A$. It follows from (3.47), (3.62), $E_1 \in \mathcal{L}(L^p(\mathbb{R}^N), L^p(\mathbb{R}^N))(1 < p < +\infty)$ and $\langle J'_\varepsilon(v_\varepsilon^R), \varphi_A v_\varepsilon^R\rangle = 0$ that as $A \to \infty$,

$$\begin{aligned}
&\int_{\mathbb{R}^N\setminus B_{2A}(0)} |\nabla v_\varepsilon^R|^2 + V(\varepsilon x)\big(v_\varepsilon^R\big)^2 \\
&\quad\leqslant \frac{C}{A}\int_{\mathbb{R}^N\setminus B_A(0)} |\nabla v_\varepsilon^R|^2 + V(\varepsilon x)\big(v_\varepsilon^R\big)^2 + C\int_{\mathbb{R}^N\setminus B_A(0)} \big(v_\varepsilon^R\big)^{\alpha+2} \\
&\qquad + C\left(\int_{\mathbb{R}^N} |E_1((v_\varepsilon^R)^2)|^2\right)^{1/2}\left(\int_{\mathbb{R}^N\setminus B_A(0)} \big(v_\varepsilon^R\big)^4\right)^{1/2} \\
&\quad\leqslant \frac{C}{A} + C(\varepsilon)\int_{\mathbb{R}^N\setminus B_A(0)} \mathrm{e}^{-C(\varepsilon)|x|} + C(\varepsilon)\left(\int_{\mathbb{R}^N\setminus B_A(0)} \mathrm{e}^{-C(\varepsilon)|x|}\right)^{1/2} \to 0.
\end{aligned}$$

That is for $\varepsilon > 0$ small,

$$\lim_{A\to\infty}\int_{\mathbb{R}^N\setminus B_{2A}(0)} |\nabla v_\varepsilon^R|^2 + V(\varepsilon x)|v_\varepsilon^R|^2 = 0. \tag{3.63}$$

Since $\{v_\varepsilon^R\}$ is bounded in H_ε, we can assume that as $R \to \infty$, $v_\varepsilon^R \rightharpoonup v_\varepsilon$ in H_ε, $v_\varepsilon^R \to v_\varepsilon$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ $(1 \leqslant s < 2*)$ and $v_\varepsilon^R \to v_\varepsilon$ a.e. $\mathbb{R}^N$. By (3.63) and standard argument, we can prove the claim that $v_\varepsilon^R \to v_\varepsilon \in X_\varepsilon^{d_0} \cap J_\varepsilon^{\tilde{c}_{\varepsilon,\delta_1}+\varepsilon}$ in H_ε for $\varepsilon > 0$ small. Hence, $v_\varepsilon \in X_\varepsilon^{d_0} \cap J_\varepsilon^{\tilde{c}_{\varepsilon,\delta_1}+\varepsilon}$ is a nontrivial solution to

$$-\Delta u + V(\varepsilon x)u + 4\left(\int_{\mathbb{R}^N} \chi_\varepsilon u^2 - 1\right)_+ \chi_\varepsilon u = bE_1(u^2)u - au^{\alpha+1} \quad \text{in } \mathbb{R}^N. \tag{3.64}$$

Since $c_{m_i} > 0$ $(1 \leqslant i \leqslant k)$, we see that $0 \notin X_\varepsilon^{d_0}$ for $d_0 > 0$ small. Thus, $v_\varepsilon \neq 0$.

Step 3: For any sequence $\{\varepsilon_j\}_{j=1}^\infty$ with $\varepsilon_j \to 0$, by lemma 3.3(ii), there exist, up to a subsequence, $\{y_{\varepsilon_j}^i\}_{j=1}^\infty \subset \mathbb{R}^N$, $z^i \in \mathcal{M}^i$, $U^i \in S_{m_i}$ such that

$$\lim_{j\to\infty} |\varepsilon_j y_{\varepsilon_j}^i - z^i| = 0 \quad \text{and} \quad \lim_{j\to\infty}\left\| v_{\varepsilon_j} - \sum_{i=1}^k \varphi(\varepsilon_j x - \varepsilon_j y_{\varepsilon_j}^i)U^i(x - y_{\varepsilon_j}^i)\right\|_{H_{\varepsilon_j}} = 0. \tag{3.65}$$

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For each $R>0$, we have

$$\|v_{\varepsilon_j}\|_{L^2(\mathbb{R}^N\setminus\bigcup_{i=1}^k B_R(y^i_{\varepsilon_j}))}\leqslant\left\|v_{\varepsilon_j}-\sum_{i=1}^k\varphi(\varepsilon_j x-\varepsilon_j y^i_{\varepsilon_j})U^i(x-y^i_{\varepsilon_j})\right\|_{L^2(\mathbb{R}^N)}+\sum_{i=1}^k\|U^i\|_{L^2(\mathbb{R}^N\setminus B_R(0))}. \tag{3.66}$$

Since $v_{\varepsilon_j}\in X^{d_0}_{\varepsilon_j}$, then $\|v_{\varepsilon_j}\|_{H_{\varepsilon_j}}$ is bounded for $\varepsilon_j>0$ small, by lemma B.1, $\{v_{\varepsilon_j}\}_{j=1}^\infty$ is bounded in $L^\infty(\mathbb{R}^N)$. Similar to (3.50), (3.51) and (3.52), we see that

$$\sup_{B_1(y)}v_{\varepsilon_j}\leqslant C\|v_{\varepsilon_j}\|^{1/N}_{L^2(B_2(y))},\quad y\in\mathbb{R}^N. \tag{3.67}$$

We obtain from (3.65), (3.66) and (3.67) that for any $\delta>0$, there exists $R=R(\delta)>0$ such that

$$|v_{\varepsilon_j}(x)|<\delta\text{ uniformly for }x\in\mathbb{R}^N\bigg\backslash\bigcup_{i=1}^k B_R(y^i_{\varepsilon_j})\text{ and }\varepsilon_j>0\text{ small}. \tag{3.68}$$

Arguing as (3.55) and (3.62), we see from (3.68) that

$$v_{\varepsilon_j}(x)\leqslant C_1\sum_{i=1}^k \mathrm{e}^{-C_2\left|x-y^i_{\varepsilon_j}\right|}, \tag{3.69}$$

where C_1 and C_2 are independent of ε_j. As $j\to\infty$,

$$\begin{aligned}\varepsilon_j^{-1}\int_{\mathbb{R}^N\setminus(\Lambda/\varepsilon_j)}v^2_{\varepsilon_j}(x)&\leqslant C\varepsilon_j^{-1}\sum_{i=1}^k\int_{\mathbb{R}^N\setminus(\Lambda/\varepsilon_j-y^i_{\varepsilon_j})}\mathrm{e}^{-2C_2|x|}\\&\leqslant C\varepsilon_j^{-1}\sum_{i=1}^k\int_{\mathbb{R}^N\setminus B_{\beta/\varepsilon_j}(0)}\mathrm{e}^{-2C_2|x|}\to 0,\end{aligned}$$

that is, $Q_{\varepsilon_j}(v_{\varepsilon_j})=0$ for ε_j small. Therefore, v_{ε_j} is a solution to (3.1). Set $u_\varepsilon(x)=v_\varepsilon(x/\varepsilon)$, u_{ε_j} is a solution to (1.1).

Since $\{v_{\varepsilon_j}\}_{j=1}^\infty$ is bounded in $L^\infty(\mathbb{R}^N)$, by elliptic estimates, we see that for each $1\leqslant i\leqslant k$, $\{v_{\varepsilon_j}(x+y^i_{\varepsilon_j})\}_{j=1}^\infty$ is bounded in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ for some $0<\alpha<1$. It follows from Arzelá–Ascoli's theorem and (3.65) that,

$$v_{\varepsilon_j}(y^i_{\varepsilon_j})\to U^i(0)>0\quad\text{as }j\to\infty. \tag{3.70}$$

Letting $x^i_{\varepsilon_j}$ be a maximum point of u_{ε_j} in $\overline{\Lambda^i}$ and recalling that $\varepsilon_j y^i_{\varepsilon_j}\to z^i\in\mathcal{M}^i\subset\Lambda^i$, we obtain from (3.70) that for $\varepsilon_j>0$ small,

$$u_{\varepsilon_j}(x^i_{\varepsilon_j})=v_{\varepsilon_j}(x^i_{\varepsilon_j}/\varepsilon_j)\geqslant v_{\varepsilon_j}(y^i_{\varepsilon_j})\geqslant\frac{U^i(0)}{2}>0. \tag{3.71}$$

Moreover, choosing $\delta=\bar\delta:=\min_{1\leqslant i\leqslant k}\{U^i(0)/2\}$ in (3.68), then there exists $R_0>0$ such that $|v_{\varepsilon_j}(x)|<\bar\delta$ for all $x\in\mathbb{R}^N\setminus\cup_{i=1}^k B_{R_0}(y^i_{\varepsilon_j})$. Hence, $|(x^i_{\varepsilon_j}/\varepsilon_j)-y^i_{\varepsilon_j}|\leqslant R_0$,

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by (3.65), we get $x^i_{\varepsilon_j} \to z^i \in \mathcal{M}^i$ as $j \to \infty$. By the arbitrariness of the sequence $\{\varepsilon_j\}_{j=1}^{\infty}$, we obtain the existence and concentration results in theorem 1.2.

The exponential decay of u_ε follows immediately from (3.69). This completes the proof. □

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Appendix A. Distributional kernel of E_1

This section is devoted to giving an equivalent formulation for the singular integral operator E_1. For any $\psi \in L^p(\mathbb{R}^N)$, $(1 < p < +\infty, N \in \{2, 3\})$, we derive from Plancherel's theorem that

$$\begin{aligned} E_1(\psi)(x) &= \mathcal{F}^{-1}(\sigma_1(\xi)\mathcal{F}(\psi)(\xi))(x) \\ &= \mathcal{F}^{-1}(\mathcal{F}(\mathcal{F}^{-1}(\sigma_1(\xi)))\mathcal{F}(\psi)(\xi))(x) \\ &= \frac{1}{(2\pi)^{N/2}}(\mathcal{F}^{-1}(\sigma_1) * \psi)(x). \end{aligned} \tag{A.1}$$

From (A.1), we see that $\mathcal{K} := (1/(2\pi)^{N/2})\mathcal{F}^{-1}(\sigma_1)$ is the distributional kernel of the operator E_1. Next, we calculate the expression of $\mathcal{K}$. We only carry out a complete calculation for the case $N = 2$, the case $N = 3$ is similar. Recalling the fundamental solution of Laplacian equation on $\mathbb{R}^2$, the following distributional identity holds:

$$-\Delta\left(-\frac{1}{2\pi}\ln|x|\right) = \delta_0$$

with δ_0 being the Dirac measure on $\mathbb{R}^2$ giving unit mass to the point 0. Thus,

$$\mathcal{F}(\ln|x|)(\xi) = -\frac{1}{|\xi|^2}.$$

Since ∂x_1 is equivalent to multiplication by ξ_1 on the Fourier side, we have

$$\mathcal{F}^{-1}(\sigma_1(\xi))(x) = \mathcal{F}^{-1}\left(\frac{|\xi_1|^2}{|\xi|^2}\right)(x) = \frac{\partial^2 \ln|x|}{\partial x_1^2}.$$

For any $\delta > 0$, it is straightforward to check that

$$\frac{\partial^2 \ln|x|}{\partial x_1^2} = \frac{-x_1^2 + x_2^2}{|x|^4}, \quad x \in \mathbb{R}^2 \backslash B_\delta(0).$$

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Integrating by part, we see that for any $\varphi \in C_c^\infty(\mathbb{R}^2)$,

$$\begin{aligned}
&\int_{\mathbb{R}^2\setminus B_\delta(0)} \frac{-x_1^2+x_2^2}{|x|^4}\varphi(x)\\
&\quad= \int_{\mathbb{R}^2\setminus B_\delta(0)} \ln|x|\frac{\partial^2\varphi(x)}{\partial x_1^2} - \int_{\partial B_\delta(0)} \frac{\partial \ln|x|}{\partial x_1}\varphi(x)\frac{x_1}{|x|}\mathrm{d}S\\
&\qquad+ \int_{\partial B_\delta(0)} \ln|x|\frac{\partial\varphi(x)}{\partial x_1}\frac{x_1}{|x|}\mathrm{d}S\\
&\quad= \int_{\mathbb{R}^2\setminus B_\delta(0)} \ln|x|\frac{\partial^2\varphi(x)}{\partial x_1^2} - (I) + (II). \qquad (A.2)
\end{aligned}$$

Noting that

$$(I) = \int_{\partial B_\delta(0)} (\varphi(x)-\varphi(0))\frac{x_1^2}{|x|^3}\mathrm{d}S + \varphi(0)\int_{\partial B_\delta(0)} \frac{x_1^2}{|x|^3}\mathrm{d}S = (III) + (IV). \qquad (A.3)$$

By differential mean value theorem, we get

$$|(III)| \leqslant \|\nabla\varphi\|_{L^\infty(\mathbb{R}^2)} \int_{\partial B_\delta(0)} \frac{x_1^2}{|x|^2}\mathrm{d}S \leqslant C\delta. \qquad (A.4)$$

By symmetry, we obtain

$$(IV) = \frac{1}{2}\varphi(0)\int_{\partial B_\delta(0)} \frac{1}{|x|}\mathrm{d}S = \pi\varphi(0). \qquad (A.5)$$

Similarly,

$$|(II)| \leqslant C\|\nabla\varphi\|_{L^\infty(\mathbb{R}^2)} \int_{\partial B_\delta(0)} \ln|x|\,\mathrm{d}S \leqslant C\delta\ln\delta. \qquad (A.6)$$

Plugging (A.3), (A.4), (A.5), (A.6) into (A.2) and taking $\delta \to 0$ in (A.2), we obtain

$$\int_{\mathbb{R}^2} \frac{-x_1^2+x_2^2}{|x|^4}\varphi(x) + \pi\varphi(0) = \int_{\mathbb{R}^2} \ln|x|\frac{\partial^2\varphi(x)}{\partial x_1^2},$$

that is, in the distribution sense,

$$\frac{\partial^2\ln|x|}{\partial x_1^2} = \frac{-x_1^2+x_2^2}{|x|^4} + \pi\delta_0.$$

Hence,

$$\mathcal{K} = \frac{1}{2\pi}\frac{-x_1^2+x_2^2}{|x|^4} + \frac{1}{2}\delta_0. \qquad (A.7)$$

Similar to the arguments above, we see that for $N = 3$,

$$\mathcal{K} = \frac{1}{4\pi}\frac{-2x_1^2+x_2^2+x_3^2}{|x|^5} + \frac{1}{3}\delta_0. \qquad (A.8)$$

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Appendix B. L^∞-estimate for Davey–Stewartson equation

LEMMA B.1. *Suppose that there is a nonnegative bounded sequence* $\{u_n\}_{n=1}^\infty \subset H^1(\mathbb{R}^N)$ $(N=2,3)$ *and satisfies*

$$-\Delta u \leqslant C\left(E_1(|u|^2)u + |u|^\alpha u\right), \quad (0<\alpha<2^*-2) \tag{B.1}$$

in the weak sense, then $\{u_n\}_{n=1}^\infty$ *is uniformly bounded in* $L^\infty(\mathbb{R}^N)$.

Proof. Letting $u_L = \min(u, L)$ where $L>0$ and $\alpha_0 = \max\{\alpha, 2\}$. Taking $u_n(u_n)_L^{2(\beta-1)}$ for $\beta>1$ as a test function in (B.1), by Young's inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u_n|^2 (u_n)_L^{2(\beta-1)} + C(\beta-1)\int_{\mathbb{R}^N} |\nabla (u_n)_L|^2 (u_n)_L^{2(\beta-1)} \\
&\quad\leqslant C\int_{\mathbb{R}^N} E_1(|u_n|^2)|u_n|^2 (u_n)_L^{2(\beta-1)} + C\int_{\mathbb{R}^N} |u_n|^{\alpha_0+2}(u_n)_L^{2(\beta-1)} \\
&\qquad + C\int_{\mathbb{R}^N} |u_n|^2 (u_n)_L^{2(\beta-1)}.
\end{aligned} \tag{B.2}$$

Since $0<2(\beta-1)<\alpha_0+2(\beta-1)$, by Young's inequality and the boundedness of $\{u_n\}_{n=1}^\infty$ in $H^1(\mathbb{R}^N)$,

$$\begin{aligned}
\int_{\mathbb{R}^N} |u_n|^2 (u_n)_L^{2(\beta-1)} &\leqslant \int_{\mathbb{R}^N} |u_n|^2 + \int_{\mathbb{R}^N} |u_n|^2 (u_n)_L^{\alpha_0+2(\beta-1)} \\
&\leqslant C\left(1 + \int_{\mathbb{R}^N} |u_n|^{\alpha_0+2}(u_n)_L^{2(\beta-1)}\right).
\end{aligned} \tag{B.3}$$

Letting $W_L = u_n(u_n)_L^{(\beta-1)}$, we have the following cases:

Case 1: $N=3$. By Sobolev's inequality, (B.2) and (B.3), we see that

$$\begin{aligned}
\|W_L\|^2_{L^{2^*}(\mathbb{R}^3)} &\leqslant C\int_{\mathbb{R}^3} |\nabla W_L|^2 \\
&\leqslant C\int_{\mathbb{R}^3} |\nabla u_n|^2 (u_n)_L^{2(\beta-1)} + C(\beta-1)^2\int_{\mathbb{R}^3} |\nabla (u_n)_L|^2 (u_n)_L^{2(\beta-1)} \\
&\leqslant C\beta^2\left[\int_{\mathbb{R}^3} E_1(|u_n|^2)|u_n|^2 (u_n)_L^{2(\beta-1)} + \int_{\mathbb{R}^3} |u_n|^{\alpha_0+2}(u_n)_L^{2(\beta-1)} + 1\right].
\end{aligned} \tag{B.4}$$

Since $E_1 \in \mathcal{L}(L^p(\mathbb{R}^N), L^p(\mathbb{R}^N))$ $(1<p<+\infty)$ [**15**, lemma 2.1], we see from Hölder's inequality, Sobolev's imbedding theorem and the boundedness of $\{u_n\}_{n=1}^\infty$

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in $H^1(\mathbb{R}^3)$ that

$$\begin{aligned}
&\int_{\mathbb{R}^3} E_1(|u_n|^2)|u_n|^2(u_n)_L^{2(\beta-1)} \\
&\quad\leqslant \left(\int_{\mathbb{R}^3} |E_1(|u_n|^2)|^{2^*/\alpha_0}\right)^{\alpha_0/2^*} \\
&\qquad\times \left(\int_{\mathbb{R}^3} |u_n|^{2\cdot 2^*/(2^*-\alpha_0)}(u_n)_L^{(2\cdot 2^*/(2^*-\alpha_0))(\beta-1)}\right)^{(2^*-\alpha_0)/2^*} \\
&\quad\leqslant C\left(\int_{\mathbb{R}^3} |u_n|^{2\cdot 2^*/(2^*-\alpha_0)}(u_n)_L^{(2\cdot 2^*/(2^*-\alpha_0))(\beta-1)}\right)^{(2^*-\alpha_0)/2^*}
\end{aligned} \tag{B.5}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^3} |u_n|^{\alpha_0+2}(u_n)_L^{2(\beta-1)} \\
&\quad\leqslant \left(\int_{\mathbb{R}^3} |u_n|^{2^*}\right)^{\alpha_0/2^*} \left(\int_{\mathbb{R}^3} |u_n|^{2\cdot 2^*/(2^*-\alpha_0)}(u_n)_L^{(2\cdot 2^*/(2^*-\alpha_0))(\beta-1)}\right)^{(2^*-\alpha_0)/2^*} \\
&\quad\leqslant C\left(\int_{\mathbb{R}^3} |u_n|^{2\cdot 2^*/(2^*-\alpha_0)}(u_n)_L^{(2\cdot 2^*/(2^*-\alpha_0))(\beta-1)}\right)^{(2^*-\alpha_0)/2^*}.
\end{aligned} \tag{B.6}$$

Plugging (B.5) and (B.6) into (B.4), we have

$$\begin{aligned}
&\left(\int_{\mathbb{R}^3} |u_n|^{2^*}(u_n)_L^{2^*(\beta-1)}\right)^{2/2^*} \\
&\quad\leqslant C\beta^2\left[\left(\int_{\mathbb{R}^3} |u_n|^{2\cdot 2^*/(2^*-\alpha_0)}(u_n)_L^{(2\cdot 2^*/(2^*-\alpha_0))(\beta-1)}\right)^{(2^*-\alpha_0)/2^*} + 1\right].
\end{aligned}$$

Moreover, if $u_n \in L^{(2\cdot 2^*/(2^*-\alpha_0))\beta}(\mathbb{R}^3)$, letting $L \to +\infty$, Fatou's lemma yields

$$\|u_n\|_{L^{2^*\beta}(\mathbb{R}^3)} \leqslant C^{1/\beta}\beta^{1/\beta}\left(\|u_n\|^{2\beta}_{L^{(2\cdot 2^*/(2^*-\alpha_0))\beta}(\mathbb{R}^3)} + 1\right)^{1/2\beta}. \tag{B.7}$$

Hence,

$$\left(\|u_n\|^{(2^*-\alpha_0)\beta}_{L^{2^*\beta}(\mathbb{R}^3)} + 1\right)^{1/(2^*-\alpha_0)\beta} \leqslant C^{1/\beta}\beta^{1/\beta}\left(\|u_n\|^{2\beta}_{L^{(2\cdot 2^*/(2^*-\alpha_0))\beta}(\mathbb{R}^3)} + 1\right)^{1/2\beta} \tag{B.8}$$

with C changing from line to line, but remaining independent of β.

Letting $\chi = (2^* - \alpha_0)/2 > 1$ and $\beta = \chi^m (m = 1, 2, \ldots)$, we get

$$\left(\|u_n\|^{2\chi^{m+1}}_{L^{2^*\chi^m}(\mathbb{R}^3)} + 1\right)^{1/(2\chi^{m+1})} \leqslant C^{\chi^{-m}}\chi^{m\chi^{-m}}\left(\|u_n\|^{2\chi^m}_{L^{2^*\chi^{m-1}}(\mathbb{R}^3)} + 1\right)^{1/2\chi^m}. \tag{B.9}$$

Iterating by (B.9), we obtain

$$\begin{aligned}
\|u_n\|_{L^{2^*\chi^m}(\mathbb{R}^3)} &\leqslant \left(\|u_n\|^{2\chi^{m+1}}_{L^{2^*\chi^m}(\mathbb{R}^3)} + 1\right)^{1/2\chi^{m+1}} \\
&\leqslant C^{\sum_{i=1}^m \chi^{-i}}\chi^{\sum_{i=1}^m i\chi^{-i}}\left(\|u_n\|^{2\chi}_{L^{2^*}(\mathbb{R}^3)} + 1\right)^{1/2\chi}.
\end{aligned} \tag{B.10}$$

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Letting $m \to \infty$ in (B.10), we see that $\{u_n\}_{n=1}^{\infty}$ is uniformly bounded in $L^{\infty}(\mathbb{R}^3)$.

Case 2: $N = 2$. By Gagliardo–Nirenberg's inequality, it holds that for $p_0 > 0$ large but fixed,

$$\int_{\mathbb{R}^2} |W_L|^{p_0} \leqslant C\left(\int_{\mathbb{R}^2} |\nabla W_L|^2\right)^{(p_0-2)/2} \int_{\mathbb{R}^2} |W_L|^2. \tag{B.11}$$

Arguing as (B.2)–(B.6) and replacing 2^* by p_0, we see from (B.11) that for $\beta > 1$ and $L > 0$,

$$\left(\int_{\mathbb{R}^2} |u_n|^{p_0} (u_n)_L^{p_0(\beta-1)}\right)^{1/p_0\beta}$$
$$\leqslant C^{1/\beta}\beta^{1/\beta}\left[\left(\int_{\mathbb{R}^2} |u_n|^{2p_0/(p_0-\alpha_0)} (u_n)_L^{(2p_0/(p_0-\alpha_0))(\beta-1)}\right)^{(p_0-\alpha_0)/p_0} + 1\right]^{1/2\beta}.$$

If $u_n \in L^{(2p_0/(p_0-\alpha_0))\beta}(\mathbb{R}^2)$, similar to (B.7) and (B.8), we get

$$\left(\|u_n\|_{L^{p_0\beta}(\mathbb{R}^2)}^{(p_0-\alpha_0)\beta} + 1\right)^{1/(p_0-\alpha_0)\beta} \leqslant C^{1/\beta}\beta^{1/\beta}\left(\|u_n\|_{L^{(2p_0/(p_0-\alpha_0))\beta}(\mathbb{R}^2)}^{2\beta} + 1\right)^{1/2\beta}.$$

Iterating as (B.9), (B.10), we get the upper bound of $\|u_n\|_{L^{\infty}(\mathbb{R}^2)}$. □

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