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Deterministic and stochastic inverse problems for the fractional Schrödinger equation

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Abstract. We are interested in an inverse problem for the fractional Schrödinger equation with a time-independent potential on closed Riemannian manifolds. The model is given by

$$i\partial_t^\alpha u + \big((-\Delta_g)^\beta + V\big)u = f,$$

with $\alpha, \beta \in (0,1)$ and $V \in L^\infty(M)$ real-valued. Our main goal is to determine the fractional powers α, β, the Riemannian metric g (up to an isometry), and the potential V simultaneously from the knowledge of the associated source-to-solution map. For disjoint source and observation regions, the map determines α, β, and the complete cross-spectral data of $(-\Delta_g)^\beta + V$ relative to these regions. For coinciding regions, if the potentials vanish there and the geometric and regularity hypotheses of Feizmohammadi et al. (2026, *Invent. Math.*) hold, the map determines α, β, the metric (up to an isometry), and the potential simultaneously. Our argument builds on the work of Li (2026, *Canad. Math. Bull.*, 1–15) by combining the high-frequency behavior of the resolvent, Weyl's law for the eigenvalues of the operator $(-\Delta_g)^\beta + V$, the unique continuation property of fractional operators with potentials, and the fractional Calderón theorem of Feizmohammadi et al. (2026). A probabilistic formulation of the inverse problem is also provided, and we prove that the corresponding uniqueness conclusions hold when the equation is driven by additive temporal white noise with a prescribed spatial profile.

1 Introduction

The study of fractional quantum mechanics dates back to Laskin's pioneering work [10, 11], where the classical Laplacian in the Schrödinger equation was replaced by the fractional Laplacian $(-\Delta)^\beta$, $\beta \in (0,1)$, via a generalization of the Feynman path integral from Brownian to Lévy paths. This framework captures anomalous diffusion and nonlocal interactions, including disordered media and turbulent plasmas. Subsequent developments introduced time-fractional (Caputo) derivatives to incorporate memory effects and non-Markovian dynamics, leading to space–time fractional Schrödinger equations. These fractional operators present challenges due to their nonlocality and singular kernels, and questions of well-posedness, regularity, spectral properties, and inverse problems have attracted considerable attention in

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recent years [3, 8]. Also see [19, 20] for the biharmonic case. The Calderón problem for the fractional Schrödinger equation, asking whether an unknown potential can be determined from exterior measurements, was initiated by Ghosh et al. [6]. Their work sparked a rapidly growing literature on inverse problems for nonlocal operators, including stability estimates, reconstruction algorithms, and extensions to anisotropic and variable-order settings [2, 18]. For space–time fractional parabolic equations, inverse coefficient problems have been studied in [9, 13, 15], demonstrating a unique determination of time-dependent coefficients from partial boundary data. A distinct line of inquiry concerns the recovery of geometric information from nonlocal equations. Feizmohammadi et al. [4] showed that the Riemannian metric on a closed manifold can be determined from the Dirichlet-to-Neumann map of a fractional elliptic equation, by employing a transmutation formula that reduces the problem to a classical hyperbolic inverse problem. This geometric determination problem was further extended to the space–time fractional diffusion equation by Helin et al. [7], who recovered the metric up to isometry from a single measurement. An important ingredient in our analysis is a unique continuation principle (UCP) for the spatial operator, which is an adaptation, to $H = (-\Delta_g)^\beta + V$, of the fractional UCP used in [14]. In contrast to the second-order case, where UCP follows from classical Carleman estimates for the Laplace–Beltrami operator, our spatial operator is the nonlocal pseudodifferential operator $(-\Delta_g)^\beta + V$ on a closed manifold. Writing $H = (-\Delta_g)^\beta + V$, we use this principle in a local form: if u and Hu both vanish on a nonempty open set W, then so does $(-\Delta_g)^\beta u = Hu - Vu$, and the rigidity of the fractional Laplace–Beltrami operator on closed manifolds [14, Proposition 2.1] forces $u \equiv 0$ on M (Proposition 2.3). This is precisely the form of UCP required in the recovery of the parameters from the source-to-solution map. Recently, Li [14] considered the uncoupled space–time fractional Schrödinger equation

$$i\partial_t^\alpha u + (-\Delta_g)^\beta u = f$$

on a closed manifold (M, g) and proved that the fractional orders α, β and the metric g (up to isometry) can be simultaneously determined from the source-to-solution map $L^{\alpha,\beta}_{W_1,W_2}$ under the spectral lower bound hypothesis imposed in the disjoint wave theorem of [12]. The proof combines the asymptotic behavior of Mittag-Leffler functions, Weyl's law for the Laplace–Beltrami eigenvalues, and disjoint data inverse spectral results for the wave equation. The present article studies the corresponding equation with a time-independent scalar potential $V(x)$.

1.1 Definition of the source-to-solution map

Let (M, g) be a closed Riemannian manifold, and let $W_1, W_2 \subset M$ be nonempty open subsets. For given fractional orders $\alpha, \beta \in (0, 1)$ and a potential $V \in L^\infty(M)$, consider the initial value problem

$$\begin{cases} i\partial_t^\alpha u + \big((-\Delta_g)^\beta + V\big)u = f, & \text{in } M \times (0, \infty), \\ u(0) = 0, & \text{in } M, \end{cases} \tag{1.1}$$

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where ∂_t^α denotes the Caputo derivative of order α and $(-\Delta_g)^\beta$ is the spectral fractional Laplacian.

Definition 1.1 (Source-to-solution map). For $f \in C_c^2\big((0,\infty); L^2(W_1)\big)$ (extended by zero to M), denote by u_f the unique solution to (1.1) given by Proposition 3.1. The source-to-solution map is the operator

$$L^{\alpha,\beta,V}_{W_1,W_2} : C_c^2\big((0,\infty); L^2(W_1)\big) \longrightarrow C\big([0,\infty); L^2(W_2)\big)$$

defined by

$$L^{\alpha,\beta,V}_{W_1,W_2} f := u_f|_{W_2\times(0,\infty)}.$$

In other words, $L^{\alpha,\beta,V}_{W_1,W_2}$ maps a source term f supported in $W_1 \times (0,\infty)$ to the restriction of the corresponding solution u_f to the observation region $W_2 \times (0,\infty)$.

Our main theorem asserts that from $L^{\alpha,\beta,V}_{W_1,W_2}$ one can recover α, β, and the complete cross-spectral data of $H = (-\Delta_g)^\beta + V$ relative to W_1, W_2; the recovery of g and V themselves is taken up in Theorem 1.3.

Theorem 1.2 *Let (M,g) and $(\widetilde{M},\widetilde{g})$ be smooth connected closed Riemannian manifolds of dimension $n \geq 2$, let $\alpha,\widetilde{\alpha},\beta,\widetilde{\beta} \in (0,1)$, and let $V \in L^\infty(M)$, $\widetilde{V} \in L^\infty(\widetilde{M})$ be real-valued potentials. Write $H = (-\Delta_g)^\beta + V$ and $\widetilde{H} = (-\Delta_{\widetilde{g}})^{\widetilde{\beta}} + \widetilde{V}$. Let $W_1, W_2 \subset M$ and $\widetilde{W}_1, \widetilde{W}_2 \subset \widetilde{M}$ be nonempty disjoint open subsets. Suppose there exists a diffeomorphism $\Phi : W_1 \cup W_2 \to \widetilde{W}_1 \cup \widetilde{W}_2$ with $\Phi(W_i) = \widetilde{W}_i$, $i = 1,2$, such that $\Phi^*\widetilde{g} = g$ on $W_1 \cup W_2$ and*

$$L^{\alpha,\beta,V}_{W_1,W_2}(f) = \Phi^*\Big(\widetilde{L}^{\widetilde{\alpha},\widetilde{\beta},\widetilde{V}}_{\widetilde{W}_1,\widetilde{W}_2}(\Phi_* f)\Big) \qquad \text{for all } f \in C_c^2\big((0,\infty); L^2(W_1)\big),$$

where $\widetilde{L}^{\widetilde{\alpha},\widetilde{\beta},\widetilde{V}}_{\widetilde{W}_1,\widetilde{W}_2}$ is the source-to-solution map associated with the parameters $\widetilde{\alpha},\widetilde{\beta}$ and the potential $\widetilde{V}$ on the manifold $(\widetilde{M},\widetilde{g})$. Then,

(1) *$\alpha = \widetilde{\alpha}$ and $\beta = \widetilde{\beta}$;*
(2) *the spectral data coincide: writing $\mu_1 < \mu_2 < \cdots$, $\mu_k \to +\infty$, for the distinct eigenvalues of H and $P^H_{W_1,W_2,k}$ for the associated cross projections (Section 4.2), one has $\mu_k = \widetilde{\mu}_k$ for all $k \geq 1$, with equal multiplicities, and*

$$P^H_{W_1,W_2,k}\,\xi = \Phi^*\Big(P^{\widetilde{H}}_{\widetilde{W}_1,\widetilde{W}_2,k}\,(\Phi_*\xi)\Big) \qquad \text{for all } \xi \in L^2(W_1),\ k \geq 1.$$

Theorem 1.3 *Let (M,g) and $(\widetilde{M},\widetilde{g})$ be smooth connected closed Riemannian manifolds of dimension $n \geq 2$ containing a common nonempty open set O with $g|_O = \widetilde{g}|_O$, $M \setminus \overline{O} \neq \varnothing$, and $\widetilde{M} \setminus \overline{O} \neq \varnothing$, and let $\alpha,\widetilde{\alpha},\beta,\widetilde{\beta} \in (0,1)$. Let $V \in C^\infty(M)$, $\widetilde{V} \in C^\infty(\widetilde{M})$ be real-valued with*

$$V|_O = \widetilde{V}|_O = 0.$$

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Assume that there exists a nonempty open connected set O' with $\overline{O'} \subset O$ such that (M, g, O') and $(\widetilde{M}, \widetilde{g}, O')$ satisfy Hypothesis (H) of [5]: the inaccessible parts $M \setminus O'$ and $\widetilde{M} \setminus O'$ are nontrapping, and there is a point $p \in O'$ whose antipodal sets in both manifolds are contained in O' (see [5] for the precise definitions). If

$$L_{O,O}^{\alpha,\beta,V}(f) = \widetilde{L}_{O,O}^{\widetilde{\alpha},\widetilde{\beta},\widetilde{V}}(f) \qquad \text{for all } f \in C_c^2\big((0,\infty); L^2(O)\big),$$

then $\alpha = \widetilde{\alpha}$, $\beta = \widetilde{\beta}$, and there exists an isometry $\Psi : (M, g) \to (\widetilde{M}, \widetilde{g})$ with $\Psi|_{O'} = \mathrm{Id}$ such that $V = \Psi^* \widetilde{V}$.

1.2 Mathematical challenges and strategy of proof

The presence of the potential V introduces several new difficulties compared to the potential-free case treated in [14]. First, while the eigenvalues and eigenfunctions of $H = (-\Delta_g)^\beta + V$ are not explicitly related to those of $(-\Delta_g)^\beta$, the min–max principle yields $|\mu_k - \lambda_k^\beta| \le \|V\|_{L^\infty}$ for the eigenvalues of H and of $(-\Delta_g)^\beta$, repeated according to multiplicity (Lemma 4.2); this ensures that Weyl's law retains the same exponent $n/(2\beta)$, which is important for recovering β from the asymptotic growth of eigenvalues. The unique continuation property needed for H is proved in Proposition 2.3, building on [14, Proposition 2.1]. There is also an earlier result by Rüland [18] for the fractional Schrödinger with rough potential in unbounded domains.

Another difficulty lies in disentangling the parameters. The order α is read off from the high-frequency behavior of the Laplace transform of the solution, which is governed by the resolvent of H and, to leading order, does not see the potential. This separation of scales – where the transform asymptotics depend only on α, while the spectral data determine β and, under the additional same-set hypotheses, lead to the recovery of g and V – is the key to the argument.

1.3 Organization of the article

Section 2 gives preliminaries: spectral theory for $(-\Delta_g)^\beta + V$, the unique continuation property, and Mittag-Leffler asymptotics. Section 3 establishes well-posedness for $f \in C_c^2\big((0,\infty); L^2(M)\big)$ via a spectral representation. Section 4 contains the proof of Theorem 1.2, divided into three steps: determination of α, recovery of spectral data, and determination of β; Section 4.4 proves Theorem 1.3 by reduction to the Calderón-type theorem of [5]. In Section 5, we provide a probabilistic formulation of the inverse problem, showing that the same uniqueness result holds when the equation is driven by additive temporal white noise with a prescribed spatial profile.

2 Preliminaries

2.1 Notation

Throughout, (M, g) denotes a smooth connected closed (compact without boundary) Riemannian manifold of dimension $n \ge 2$. The Laplace–Beltrami operator $-\Delta_g$ is nonnegative, with 0 as a simple eigenvalue, and its fractional power

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$(-\Delta_g)^\beta (0 < \beta < 1)$ is defined spectrally. For a function $u(t,x)$, $\partial_t^\alpha u$ denotes the Caputo derivative of order $\alpha \in (0,1)$. The L^2 inner product on M is written $\langle \cdot, \cdot \rangle$, and $\| \cdot \|$ denotes the corresponding norm. The symbol C denotes a generic constant that may change from line to line.

2.2 Spectral theory and function spaces

Let (M,g) be a closed Riemannian manifold and $V \in L^\infty(M)$ be real-valued. The operator

$$H := (-\Delta_g)^\beta + V$$

is self-adjoint on $L^2(M)$ with discrete spectrum $\{\mu_k\}_{k=1}^\infty$ and eigenbasis $\{\psi_k\}_{k=1}^\infty$.

Definition 2.1 (H-Sobolev scale) For $s \geq 0$, define

$$D_H^s := \Big\{ u \in L^2(M) : \sum_{k=1}^\infty (1 + |\mu_k|)^{s/\beta} |\langle u, \psi_k \rangle|^2 < \infty \Big\},$$

with norm $\|u\|_{D_H^s}^2 = \sum_{k=1}^\infty (1 + |\mu_k|)^{s/\beta} |\langle u, \psi_k \rangle|^2$.

Remark 2.2 Note that $D_H^0 = L^2(M)$ and $D_H^{2\beta} = D(H)$. The spaces $\{D_H^s\}_{s \geq 0}$ form a Hilbert scale interpolating between $L^2(M)$ and $D(H)$.

2.3 Unique continuation property

In this section, we discuss the UCP needed in the proof of Theorem 1.2. The operator appearing in our problem is

$$H = (-\Delta_g)^\beta + V,$$

where (M,g) is a smooth connected closed Riemannian manifold, $\beta \in (0,1)$, and $V \in L^\infty(M)$ is real-valued. The principal symbol of $(-\Delta_g)^\beta$ is the real, elliptic symbol $|\xi|_g^{2\beta}$; the potential term is of order zero and hence does not affect ellipticity. Ellipticity alone, however, does not yield unique continuation for a nonlocal operator, and we argue instead from the rigidity of $(-\Delta_g)^\beta$ itself.

We use the following *local* unique continuation result; part (ii) is the version applied to eigenfunctions.

Proposition 2.3 (Local UCP for H) *Let (M,g) be a smooth connected closed Riemannian manifold, $\beta \in (0,1)$, and $V \in L^\infty(M;\mathbb{R})$, and let $W \subset M$ be a nonempty open set.*

(i) *Let $u \in H^{2\beta}(M)$ satisfy*

$$u = 0 \text{ a.e. in } W \qquad \text{and} \qquad Hu = 0 \text{ a.e. in } W.$$

Then $u \equiv 0$ on M.

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(ii) *Let $\mu \in \mathbb{R}$ and let $u \in H^{2\beta}(M)$ satisfy $Hu = \mu u$ on M. If $u = 0$ a.e. in W, then $u \equiv 0$. In particular, no nontrivial element of an eigenspace of H vanishes on a nonempty open set, and the restrictions to W of a basis $\{\psi_{k,p}\}_{p=1}^{m_k}$ of any eigenspace are linearly independent.*

Proof (i) Since u vanishes a.e. in W and $V \in L^\infty(M)$, we have $Vu = 0$ a.e. in W, and hence

$$(-\Delta_g)^\beta u = Hu - Vu = 0 \quad \text{a.e. in } W.$$

Thus, $u \in H^{2\beta}(M)$ satisfies $u = (-\Delta_g)^\beta u = 0$ in W, and the unique continuation property of the fractional Laplace–Beltrami operator on closed manifolds yields $u \equiv 0$ on M; see [14, Proposition 2.1], proved there for $u \in H^{2r}(M)$, $r \in (0,\infty)\setminus\mathbb{N}$, with Euclidean antecedents [6, 18]. Note that $D(H) = D((-\Delta_g)^\beta) = H^{2\beta}(M)$, since V is a bounded perturbation.

(ii) Apply part (i) with V replaced by $V - \mu \in L^\infty(M;\mathbb{R})$, noting that $(-\Delta_g)^\beta + (V-\mu) = H - \mu$ and that $(H-\mu)u = 0$ on all of M. The last statement follows by applying the first to a linear combination $\sum_p a_p \psi_{k,p}$. ∎

Remark 2.4 This proposition transfers to H the fractional UCP used by Li in [14] and plays the role that the classical UCP for the Laplace–Beltrami operator plays in the second-order theory.

2.4 Caputo derivative and Mittag-Leffler functions

We recall the definition of the Caputo derivative of order $\alpha \in (0,1)$:

$$\partial_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u'(\tau)}{(t-\tau)^\alpha}\, d\tau,$$

and its Laplace transform formula $\mathcal{L}(\partial_t^\alpha u)(s) = s^\alpha \mathcal{L}u(s)$ for $u(0) = 0$. The two-parameter Mittag-Leffler function is

$$E_{a,b}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak+b)}, \qquad a, b > 0.$$

We will use the following asymptotic estimate on the imaginary axis (see [14, 17]).

Proposition 2.5 *Let $0 < a < 1$ and $\dfrac{\pi a}{2} < \vartheta < \min\left\{\dfrac{\pi}{2}, \pi a\right\}$. Then, there exists $C > 0$ such that for $|z| \to \infty$ with $\vartheta \le |\arg z| \le \pi$,*

$$|E_{a,1}(z)| \le \frac{C}{1+|z|}, \qquad E_{a,1}(z) = -\frac{z^{-1}}{\Gamma(1-a)} + O(|z|^{-2}).$$

In the same sector, $E_{a,a}(z) = O(|z|^{-2})$ since the coefficient of z^{-1} in the corresponding expansion is $1/\Gamma(0) = 0$. Because $\vartheta < \pi/2$, both imaginary rays lie in the sector; together with continuity on bounded sets, this shows that $E_{a,1}$ and $E_{a,a}$ are bounded on the imaginary axis.

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3 Well-posedness

In this section, we establish the well-posedness of the system (1.1). We use the Hilbert scale generated by the operator $H = (-\Delta_g)^\beta + V$.

3.1 Function spaces generated by H

We use throughout the Hilbert scale D_H^s introduced in Definition 2.1. In particular, $D_H^0 = L^2(M)$ and $D_H^{2\beta} = D(H)$.

Proposition 3.1 (Well-posedness) *Let $f \in C_c^2\big((0,\infty); L^2(M)\big)$. Then, there exists a unique solution*

$$u \in C^1\big([0,\infty); D_H^{2\beta}\big) \cap L^\infty\big(0,\infty; D_H^{2\beta}\big), \qquad \partial_t u \in L^\infty\big(0,\infty; D_H^{2\beta}\big)$$

of problem (1.1). The solution is given by the spectral representation

$$u(t) = \sum_{k=1}^{\infty} u_k(t)\psi_k, \tag{3.1}$$

where $\{\psi_k\}_{k=1}^\infty$ is the orthonormal eigenbasis of H with eigenvalues μ_k, and the Fourier coefficients $u_k(t)$ satisfy the fractional ordinary differential equation

$$\begin{cases} (i\partial_t^\alpha + \mu_k)u_k = f_k, & t > 0, \\ u_k(0) = 0, \end{cases} \tag{3.2}$$

with $f_k(t) = \langle f(t), \psi_k\rangle$. Moreover,

$$u_k(t) = \int_0^t K_k(t-\tau) f_k(\tau)\, d\tau, \qquad K_k(t) = -i\, t^{\alpha-1} E_{\alpha,\alpha}(i\mu_k t^\alpha). \tag{3.3}$$

Proof We proceed as in [14], adapted to the scale D_H^s. For each $k \in \mathbb{N}$, the function f_k belongs to $C_c^2((0,\infty))$. Applying the Laplace transform to (3.2) and using the property $\mathcal{L}(\partial_t^\alpha u_k)(s) = s^\alpha \mathcal{L}u_k(s)$ (valid for $u_k(0) = 0$) yields

$$(is^\alpha + \mu_k)\mathcal{L}u_k(s) = \mathcal{L}f_k(s), \qquad \operatorname{Re} s > 0.$$

Hence,

$$\mathcal{L}u_k(s) = \frac{1}{is^\alpha + \mu_k}\mathcal{L}f_k(s).$$

The inverse Laplace transform of $(is^\alpha + \mu_k)^{-1}$ is known:

$$\mathcal{L}^{-1}\Big[\frac{1}{is^\alpha + \mu_k}\Big](t) = -it^{\alpha-1}E_{\alpha,\alpha}(i\mu_k t^\alpha) =: K_k(t).$$

Convolution in time gives the representation (3.3). Since $K_k \in L^1_{\mathrm{loc}}([0,\infty))$ and $f_k(0) = 0$, we may write $u_k(t) = \int_0^t K_k(\sigma) f_k(t-\sigma)\, d\sigma$ and differentiate under the integral sign, the boundary term $K_k(t)f_k(0)$ being absent; this gives

$$u_k'(t) = -i\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(i\mu_k(t-\tau)^\alpha\big) f_k'(\tau)\, d\tau. \tag{3.4}$$

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Integrating by parts in (3.3) gives the further representation

$$\mu_k u_k(t) = f_k(t) - \int_0^t E_{\alpha,1}\big(i\mu_k(t-\tau)^\alpha\big) f_k'(\tau)\, d\tau. \tag{3.5}$$

Now we establish convergence in $D_H^{2\beta}$. Let $T_0 > 0$ be such that $\operatorname{supp} f \subset (0, T_0) \times M$. From (3.5), we have

$$|\mu_k u_k(t)| \le |f_k(t)| + \int_0^t \big|E_{\alpha,1}\big(i\mu_k(t-\tau)^\alpha\big)\big|\, |f_k'(\tau)|\, d\tau.$$

Since $E_{\alpha,1}$ is bounded on the imaginary axis by Proposition 2.5, there exists $C_\alpha > 0$ such that

$$|\mu_k u_k(t)| \le (1 + C_\alpha) \int_0^{T_0} |f_k'(\tau)|\, d\tau.$$

Applying the Cauchy–Schwarz inequality yields

$$\mu_k^2 |u_k(t)|^2 \le (1 + C_\alpha)^2 T_0 \int_0^{T_0} |f_k'(\tau)|^2\, d\tau.$$

Summing over k and using Parseval's identity gives

$$\begin{aligned}\sum_{k=1}^\infty \mu_k^2 |u_k(t)|^2 &\le (1 + C_\alpha)^2 T_0 \sum_{k=1}^\infty \int_0^{T_0} |f_k'(\tau)|^2\, d\tau \\ &= (1 + C_\alpha)^2 T_0 \int_0^{T_0} \|f'(\tau)\|_{L^2(M)}^2\, d\tau < \infty.\end{aligned}$$

Recalling that $\|u(t)\|_{D_H^{2\beta}}^2 = \sum_{k=1}^\infty (1 + |\mu_k|)^2 |u_k(t)|^2$, that $(1 + |\mu_k|)^2 \le 2(1 + \mu_k^2)$, and that $\|u(t)\|_{L^2(M)}$ is bounded directly from (3.3): by $|K_k(r)| \le C_\alpha r^{\alpha-1}$ (Proposition 2.5), Minkowski's integral inequality, and $\int_0^{\min\{t,T_0\}} (t-\tau)^{\alpha-1} d\tau \le T_0^\alpha/\alpha$ for every $t \ge 0$,

$$\|u(t)\|_{L^2(M)} \le C_\alpha \int_0^{\min\{t,T_0\}} (t-\tau)^{\alpha-1} \|f(\tau)\|_{L^2(M)}\, d\tau \le \frac{C_\alpha T_0^\alpha}{\alpha} \|f\|_{C^0([0,T_0];L^2(M))};$$

altogether, we obtain

$$\sup_{t\ge 0} \|u(t)\|_{D_H^{2\beta}} \le C(T_0, \alpha) \|f\|_{C^1([0,T_0];L^2(M))}. \tag{3.6}$$

An identical estimate holds for u_k': differentiating (3.5) in the same way yields

$$\mu_k u_k'(t) = f_k'(t) - \int_0^t E_{\alpha,1}\big(i\mu_k(t-\tau)^\alpha\big) f_k''(\tau)\, d\tau,$$

and the argument just given applies verbatim with f' in place of f, at the cost of replacing $\|f\|_{C^1([0,T_0];L^2(M))}$ by $\|f\|_{C^2([0,T_0];L^2(M))}$ in (3.6). Let P_N denote the orthogonal projection onto $\operatorname{span}\{\psi_1, \dots, \psi_N\}$. Since P_N commutes with the solution operator, $(I - P_N)u$ is the solution with source $(I - P_N)f$; so, (3.6) and its analog for u' give, as $N \to \infty$,

$$\sup_{t\ge 0} \|(I - P_N)u(t)\|_{D_H^{2\beta}} \longrightarrow 0,$$

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$$\sup_{t\geq 0}\|(I-P_N)u'(t)\|_{D_H^{2\beta}}\longrightarrow 0,$$

the right-hand sides being controlled by the tails of f, f' in $C([0,T_0];L^2(M))$ and of f', f'' in $L^2(0,T_0;L^2(M))$. The latter tend to zero by dominated convergence; so do the former because f, f' are continuous on the compact interval $[0,T_0]$, and hence have compact range in $L^2(M)$, and $\|(I-P_N)v\|_{L^2(M)}\to 0$ for each fixed v with $\|I-P_N\|\leq 1$; so, the convergence is uniform on compact subsets of $L^2(M)$. Since the finite spectral sums are continuously differentiable, this proves convergence in $C^1([0,\infty);D_H^{2\beta})$. Moreover, for each $T>0$,

$$\sup_{0\leq t\leq T}\|\partial_t^\alpha(I-P_N)u(t)\|_{L^2(M)}\leq\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\sup_{0\leq t\leq T}\|(I-P_N)u'(t)\|_{L^2(M)}\longrightarrow 0,$$

and so, the Caputo derivative may be taken term by term. For each k,

$$i\partial_t^\alpha u_k(t)+\mu_k u_k(t)=f_k(t)$$

since $u_k\in C^1([0,\infty))$ is bounded with $u_k(0)=0$, so that $\mathcal{L}(\partial_t^\alpha u_k)=s^\alpha\mathcal{L}u_k$, and hence $\mathcal{L}\big(i\partial_t^\alpha u_k+\mu_k u_k\big)=(is^\alpha+\mu_k)\mathcal{L}u_k=\mathcal{L}f_k$ for $\operatorname{Re}s>0$; both sides being continuous in t, uniqueness of Laplace transforms gives the identity pointwise. Summing over k and using the boundedness of $H=(-\Delta_g)^\beta+V: D_H^{2\beta}\to L^2(M)$ yields

$$i\partial_t^\alpha u(t)+\big((-\Delta_g)^\beta+V\big)u(t)=\sum_{k=1}^{\infty}f_k(t)\psi_k=f(t).$$

The initial condition $u(0)=0$ follows from $u_k(0)=0$ for all k and the uniform convergence. Finally, if u and v are two solutions with the same source f, then $w=u-v$ satisfies

$$i\partial_t^\alpha w+\big((-\Delta_g)^\beta+V\big)w=0,\qquad w(0)=0.$$

Taking the Fourier coefficients $w_k(t)=\langle w(t),\psi_k\rangle$ gives

$$i\partial_t^\alpha w_k+\mu_k w_k=0,\qquad w_k(0)=0.$$

The unique solution of this fractional ODE is $w_k\equiv 0$. Hence, $w\equiv 0$, proving uniqueness. ∎

4 The inverse problem

4.1 Determination of the time-fractional power α

Proof of Theorem 1.2. We now recover α. Rather than the large-time asymptotics of a compactly supported pulse, we use the high-frequency behavior of the Laplace transform of the solution; this makes no use of the sign or the invertibility of H, and no mean-zero restriction on the sources is imposed here or in Section 4.2.

Choose $\xi\in C_c^\infty(W_1)$ with $\xi\neq 0$, write $q:=e(\xi)\in C^\infty(M)$ for its extension by zero from W_1 to M, and fix a nonnegative $a\in C_c^2((0,\infty))$, $a\not\equiv 0$, so that $\mathcal{L}a(s)>0$ for every $s>0$. Apply the source $f(t)=a(t)q$. Taking the Laplace transform of the

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spectral representation (3.1) exactly as in the proof of Proposition 3.1, and using $\mathcal{L}f_k(s) = \mathcal{L}a(s)\langle q, \psi_k\rangle$, we obtain, for $\operatorname{Re} s > 0$,

$$\mathcal{L}u_f(s) = \mathcal{L}a(s) \sum_{k\geq 1} \frac{\langle q, \psi_k\rangle}{\mu_k + is^\alpha}\, \psi_k = \mathcal{L}a(s)\left(H + is^\alpha\right)^{-1} q.$$

The a priori bound (3.6) guarantees that the transform converges for $\operatorname{Re} s > 0$; so, the source-to-solution map determines the $L^2(W_2)$-valued function

$$F(s) := \frac{1}{\mathcal{L}a(s)}\big(\mathcal{L}u_f(s)\big)\big|_{W_2} = \Big(\big(H + is^\alpha\big)^{-1} q\Big)\Big|_{W_2}, \qquad s > 0.$$

We claim that, with $\rho := s^\alpha$,

$$\Big\|\big(H + i\rho\big)^{-1} q - \frac{q}{i\rho} + \frac{Hq}{(i\rho)^2}\Big\|_{L^2(M)} = o\big(\rho^{-2}\big), \qquad \rho \to \infty. \tag{4.1}$$

Indeed, H is self-adjoint with real spectrum, $H + i\rho$ is boundedly invertible for $\rho > 0$, and for every $v \in L^2(M)$, the spectral theorem and dominated convergence give

$$\begin{aligned}\Big\|\big(H + i\rho\big)^{-1} v - \frac{v}{i\rho}\Big\|_{L^2(M)} &= \frac{1}{\rho}\big\|H\big(H + i\rho\big)^{-1} v\big\|_{L^2(M)} \\ &= \frac{1}{\rho}\Big(\sum_{k\geq 1} \frac{\mu_k^2}{\mu_k^2 + \rho^2}\big|\langle v, \psi_k\rangle\big|^2\Big)^{1/2} = o\big(\rho^{-1}\big).\end{aligned}$$

Since $q \in C^\infty(M) \subset D(H)$, the identity $(H + i\rho)^{-1} = \frac{1}{i\rho} I - \frac{1}{i\rho}(H + i\rho)^{-1} H$ on $D(H)$, followed by the estimate above applied to $v = Hq$, yields

$$\big(H + i\rho\big)^{-1} q = \frac{q}{i\rho} - \frac{1}{i\rho}\big(H + i\rho\big)^{-1} Hq = \frac{q}{i\rho} - \frac{Hq}{(i\rho)^2} + o\big(\rho^{-2}\big),$$

which is (4.1).

Because $W_1 \cap W_2 = \varnothing$, we have $q|_{W_2} = 0$, and the first-order term of (4.1) disappears on restriction to W_2; and since V acts by multiplication, $(Hq)|_{W_2} = \big((-\Delta_g)^\beta q\big)|_{W_2}$. As $(i)^{-2} = -1$, restricting (4.1) to W_2 therefore gives

$$F(s) = s^{-2\alpha}\big(\omega + o(1)\big) \quad \text{in } L^2(W_2), \qquad \omega := \Big((-\Delta_g)^\beta q\Big)\Big|_{W_2}. \tag{4.2}$$

We claim that $\omega \neq 0$. Indeed, if $\omega = 0$, then both q and $Hq = (-\Delta_g)^\beta q + Vq$ vanish a.e. on W_2, and so Proposition 2.3(i) forces $q \equiv 0$ on M, contradicting $\xi \neq 0$. Hence, (4.2) gives

$$\lim_{s\to\infty} \frac{\log \|F(s)\|_{L^2(W_2)}}{\log s} = -2\alpha.$$

Now run the same construction on $(\widetilde{M}, \widetilde{g})$ with the parameters $\widetilde{\alpha}, \widetilde{\beta}, \widetilde{V}$ and the source $\Phi_* f = a(t)\, e(\Phi_* \xi)$. Since $\Phi_* \xi \in C_c^\infty(\widetilde{W}_1)$ is nonzero and $\widetilde{W}_1 \cap \widetilde{W}_2 = \varnothing$, the argument just given applies verbatim on $(\widetilde{M}, \widetilde{g})$ and yields $\log \|\widetilde{F}(s)\|_{L^2(\widetilde{W}_2)} / \log s \to -2\widetilde{\alpha}$. Finally, the same a is used on both manifolds, and the same factor $\mathcal{L}a(s)$ is divided out; since $\Phi^* \widetilde{g} = g$ on W_2, Φ^* is an isometry of $L^2(\widetilde{W}_2)$ onto $L^2(W_2)$. Equality of the

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source-to-solution maps therefore gives $\|F(s)\|_{L^2(W_2)} = \|\widetilde{F}(s)\|_{L^2(\widetilde{W}_2)}$ for every $s > 0$, and comparing the two limits forces $\alpha = \widetilde{\alpha}$. This completes the determination of α.

4.2 Recovery of the spectral data

Having identified α, we now recover the spectral data of the operator $H = (-\Delta_g)^\beta + V$. Fix a nonnegative function $a \in C_c^2((0,\infty))$ whose Laplace transform satisfies $\mathcal{L}a(s) > 0$ for all $s > 0$. For $\xi \in L^2(W_1)$, consider the source

$$f(t) = a(t)\, e(\xi).$$

Taking the Laplace transform in time of the solution u_f and using the convolution formula (see the proof of Proposition 3.1) gives

$$\mathcal{L}u_f(s) = \mathcal{L}a(s) \sum_{k=1}^{\infty} \frac{1}{\mu_k + is^\alpha} \langle e(\xi), \psi_k \rangle \psi_k, \qquad \operatorname{Re} s > 0.$$

Restricting to the observation set W_2 and regrouping the terms according to the *distinct* eigenvalues of H, which from here on we order as

$$\mu_1 < \mu_2 < \cdots, \qquad \mu_k \longrightarrow +\infty,$$

and continue to denote by $\mu_1, \mu_2, \ldots$, we obtain the operator-valued function

$$\mathcal{H}(s)\xi := \sum_{k=1}^{\infty} \frac{1}{\mu_k + is^\alpha} P^H_{W_1,W_2,k}\, \xi, \qquad \operatorname{Re} s > 0,$$

where $P^H_{W_1,W_2,k}$ denotes the projection onto the k-th eigenspace of H, restricted from W_1 to W_2:

$$P^H_{W_1,W_2,k}\xi = \sum_{p=1}^{m_k} \langle e(\xi), \psi_{k,p} \rangle \psi_{k,p}|_{W_2}.$$

Here, $\{\psi_{k,p}\}_{p=1}^{m_k}$ is an orthonormal basis of the eigenspace corresponding to μ_k, and m_k its multiplicity. In this section and in Section 4.4, the μ_k are enumerated without repetition, and Π_k denotes the orthogonal projection of $L^2(M)$ onto the μ_k-eigenspace, so that $\Pi_k F = \sum_{p=1}^{m_k} \langle F, \psi_{k,p} \rangle \psi_{k,p}$; its Schwartz kernel $\Pi_k(x,y) = \sum_{p=1}^{m_k} \psi_{k,p}(x)\overline{\psi_{k,p}(y)}$ does not depend on the choice of orthonormal basis of the eigenspace. The source-to-solution map $L^{\alpha,\beta,V}_{W_1,W_2}$ determines $\mathcal{H}(s)$ for all $s > 0$. Because the Π_k are mutually orthogonal and $P^H_{W_1,W_2,k}\xi = (\Pi_k e(\xi))|_{W_2}$, for every compact set $K \Subset \mathbb{C} \setminus \{-\mu_k : k \geq 1\}$,

$$\sup_{z \in K} \left\| \sum_{k > N} \frac{1}{\mu_k + z} P^H_{W_1,W_2,k} \right\| \leq \sup_{\substack{z \in K \\ k > N}} \frac{1}{|\mu_k + z|} \longrightarrow 0.$$

Thus, the series converges in operator norm locally uniformly away from its poles. Analytic continuation in the variable $z = is^\alpha$ yields a meromorphic operator-valued

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function

$$\mathcal{H}(z) = \sum_{k=1}^{\infty} \frac{1}{\mu_k + z} P^H_{W_1,W_2,k}, \qquad z \in \mathbb{C} \setminus \{-\mu_k\}_{k=1}^{\infty}. \tag{4.3}$$

The poles of $\mathcal{H}(z)$ are contained in the set $\{-\mu_k\}_{k\geq 1}$, are simple, and the residue at $-\mu_k$ equals $P^H_{W_1,W_2,k}$. The ray $\{is^\alpha : s > 0\}$ lies in the connected open set $\mathbb{C} \setminus \{-\mu_k\}_{k\geq 1}$ and has accumulation points there; so, by the identity theorem, the values of $\mathcal{H}$ on that ray – which the source-to-solution map supplies – determine $\mathcal{H}(z)$ for all z, and hence every pole with nonvanishing residue.

That every eigenvalue is indeed visible in the data, with the correct multiplicity – so that the poles are *exactly* the points $-\mu_k$ – is guaranteed by the following lemma.

Lemma 4.1 *For every $k \in \mathbb{N}$, the operator $P^H_{W_1,W_2,k} : L^2(W_1) \to L^2(W_2)$ is nonzero, and $\operatorname{rank} P^H_{W_1,W_2,k} = m_k$. The same statements hold when $W_1 = W_2 = O$ is a single nonempty open set.*

Proof Let $\{\psi_{k,p}\}_{p=1}^{m_k}$ be an orthonormal basis of the μ_k-eigenspace and let $e(\cdot)$ denote extension by zero. Since $\langle e(\xi), \psi_{k,p}\rangle_{L^2(M)} = \langle \xi, \psi_{k,p}|_{W_1}\rangle_{L^2(W_1)}$,

$$P^H_{W_1,W_2,k}\xi = \sum_{p=1}^{m_k} \big\langle \xi, \psi_{k,p}|_{W_1} \big\rangle_{L^2(W_1)} \psi_{k,p}|_{W_2}, \qquad \xi \in L^2(W_1).$$

By Proposition 2.3(ii), the restrictions $\psi_{k,p}|_{W_2}$, $1 \leq p \leq m_k$, are linearly independent, and their span has dimension m_k; by the same proposition, the $\psi_{k,p}|_{W_1}$ are linearly independent, and the map $\xi \mapsto (\langle \xi, \psi_{k,p}|_{W_1}\rangle)_{p=1}^{m_k}$ is onto $\mathbb{C}^{m_k}$ and the range of $P^H_{W_1,W_2,k}$ is that entire span. Hence, $\operatorname{rank} P^H_{W_1,W_2,k} = m_k \geq 1$, and in particular, $P^H_{W_1,W_2,k} \neq 0$. The case $W_1 = W_2 = O$ is identical. ∎

Therefore, from $\mathcal{H}(z)$, we recover the full spectral dataset

$$\mathcal{S} = \big\{(\mu_k, P^H_{W_1,W_2,k}) : k \in \mathbb{N}\big\}.$$

4.3 Determination of β

With the eigenvalues $\{\mu_k\}_{k=1}^{\infty}$ of $H = (-\Delta_g)^\beta + V$, their multiplicities, and the cross projections recovered from the source-to-solution map, we now complete the proof of Theorem 1.2.

Lemma 4.2 *Let $\lambda_1 \leq \lambda_2 \leq \cdots$ be the eigenvalues of $-\Delta_g$ and $\mu_1 \leq \mu_2 \leq \cdots$ those of H, both repeated according to multiplicity (in contrast with the enumeration by distinct eigenvalues used in Theorem 1.2(2)). Then,*

$$|\mu_k - \lambda_k^\beta| \leq \|V\|_{L^\infty(M)} \qquad \text{for all } k \geq 1.$$

Consequently, the counting function $N_H(\mu) := \#\{k : \mu_k \leq \mu\}$ satisfies

$$N_H(\mu) = \frac{\omega_n}{(2\pi)^n} \operatorname{vol}_g(M)\, \mu^{n/(2\beta)}\big(1 + o(1)\big) \qquad (\mu \to \infty), \tag{4.4}$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$.

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Proof By the spectral theorem, the eigenvalues of $(-\Delta_g)^\beta$ are λ_k^β, in the same order. By the Courant–Fischer min–max characterization, for a self-adjoint, semibounded operator A with discrete spectrum and a bounded self-adjoint perturbation B, one has $|\mu_k(A+B) - \mu_k(A)| \le \|B\|$: indeed, $\mu_k(A+B) = \min_{\dim S = k} \max_{u \in S,\, \|u\|=1} \langle (A+B)u, u\rangle$ and $|\langle Bu, u\rangle| \le \|B\|$ for every unit vector u, so each Rayleigh quotient – and hence the min–max value – moves by at most $\|B\|$. Apply this with $A = (-\Delta_g)^\beta$ and $B = V$, which is self-adjoint and bounded because $V \in L^\infty(M)$ is real-valued. Writing $c = \|V\|_{L^\infty}$, it follows that, for $\mu \ge c$,

$$N_{-\Delta_g}\big((\mu - c)^{1/\beta}\big) \le N_H(\mu) \le N_{-\Delta_g}\big((\mu + c)^{1/\beta}\big),$$

and Weyl's law for $-\Delta_g$ on the closed manifold (M, g), namely, $N_{-\Delta_g}(\lambda) = \frac{\omega_n}{(2\pi)^n} \mathrm{vol}_g(M) \lambda^{n/2}(1 + o(1))$, gives the stated asymptotics since $(\mu \pm c)^{n/(2\beta)} = \mu^{n/(2\beta)}(1 + o(1))$. ∎

Completion of the proof of Theorem 1.2. Conclusion (1) for α was obtained in Section 4.1. Since Φ is a diffeomorphism between nonempty open subsets of M and $\widetilde{M}$, we have $n := \dim M = \dim \widetilde{M}$. By Section 4.2 and Lemma 4.1, equality of the source-to-solution maps implies (after the identification by Φ) that the meromorphic families (4.3) coincide; hence, $\mu_k = \widetilde{\mu}_k$ for all k with equal multiplicities, together with the stated identification of the cross projections. This is conclusion (2). Applying Lemma 4.2 on both manifolds,

$$\frac{n}{2\beta} = \lim_{\mu \to \infty} \frac{\log N_H(\mu)}{\log \mu} = \lim_{\mu \to \infty} \frac{\log N_{\widetilde{H}}(\mu)}{\log \mu} = \frac{n}{2\widetilde{\beta}},$$

and so $\beta = \widetilde{\beta}$, completing conclusion (1).

For a general nonconstant potential, H need not commute with $-\Delta_g$, and the recovered H-spectral projectors cannot be identified with Laplace–Beltrami spectral projectors. Therefore, the disjoint-data argument stops at the cross-spectral data of H and does not recover (g, V); the next section treats coinciding measurements using [5].

4.4 Coinciding source and observation regions: Proof of Theorem 1.3

We place ourselves in the setting of Theorem 1.3: $W_1 = W_2 = O$, the potentials are smooth and vanish on O, and (M, g, O'), $(\widetilde{M}, \widetilde{g}, O')$ satisfy Hypothesis (H) of [5] for a fixed nonempty open connected O' with $\overline{O'} \subset O$. Definition 1.1 applies verbatim with $W_1 = W_2 = O$.

Step 1: Determination of α. The argument of Section 4.1 applies with $W_1 = W_2 = O$ and is in fact simpler: here, $q = e(\xi)$ satisfies $q|_O = \xi \ne 0$, and so the first-order term of (4.1) survives the restriction and neither the identification of ω nor the unique continuation step is needed. Thus,

$$\|F(s)\|_{L^2(O)} = s^{-\alpha}\big(\|\xi\|_{L^2(O)} + o(1)\big), \qquad \text{whence} \qquad \lim_{s \to \infty} \frac{\log \|F(s)\|_{L^2(O)}}{\log s} = -\alpha.$$

Comparing with the same limit computed on $(\widetilde{M}, \widetilde{g})$ gives $\alpha = \widetilde{\alpha}$.

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Step 2: Spectral data and β. Running Section 4.2 with $W_1 = W_2 = O$, the source-to-solution map determines the meromorphic family (4.3), and hence, by Lemma 4.1, the same-set spectral data

$$\mathcal{S}_O := \big\{(\mu_k,\ \Pi_k|_{O\times O}) : k \in \mathbb{N}\big\},$$

where $\Pi_k|_{O\times O}$ denotes the Schwartz kernel of Π_k restricted to $O \times O$ (equivalently, the operator $P^H_{O,O,k}$); here, the eigenfunctions, and hence the kernels Π_k, are smooth, by bootstrapping $(-\Delta_g)^\beta \psi = (\mu_k - V)\psi$ with $V \in C^\infty(M)$. Equality of the maps gives $\mu_k = \widetilde{\mu}_k$ and $\Pi_k|_{O\times O} = \widetilde{\Pi}_k|_{O\times O}$ for all k; the multiplicities agree as well since $m_k = \operatorname{rank} P^H_{O,O,k}$ by Lemma 4.1. Since M and $\widetilde{M}$ contain the common open set O, we have $\dim M = \dim \widetilde{M} =: n$, and Lemma 4.2 yields $\beta = \widetilde{\beta}$ exactly as in Section 4.3.

Step 3: Reduction to the Calderón problem for fractional Schrödinger operators. Following [5], define the Cauchy dataset on O':

$$\mathcal{C}^{O'}_{M,g,V} := \Big\{\big(u|_{O'}, ((-\Delta_g)^\beta u)|_{O'}\big) : u \in C^\infty(M),\ Hu = 0 \text{ in } M \setminus \overline{O'}\Big\}.$$

The reduction is accomplished by the following lemma.

Lemma 4.3 *Under the hypotheses of Theorem 1.3, the Cauchy dataset $\mathcal{C}^{O'}_{M,g,V}$ is determined by the spectral data $\mathcal{S}_O$ together with O, O', and $g|_O$.*

Proof Let

$$\mathcal{F} := \{F \in C^\infty(M) : \operatorname{supp} F \subset \overline{O'}\},$$

and let $\Pi_{\ker H}$ be the orthogonal projection onto $\ker H$, with $\Pi_{\ker H} = 0$ when $\ker H = \{0\}$. Define

$$\mathcal{F}^H_0 := \{F \in \mathcal{F} : \Pi_{\ker H} F = 0\}, \qquad \mathcal{N}^H_{O'} := \{w|_{O'} : w \in \ker H\}.$$

Because $\overline{O'} \subset O$, restriction to O identifies $\mathcal{F}$ with the smooth functions on O supported in $\overline{O'}$; thus, $\mathcal{F}$ is determined by O, O', and $g|_O$ alone.

Since H is self-adjoint with discrete spectrum, its Moore–Penrose inverse is

$$H^\dagger F = \sum_{\mu_k \neq 0} \mu_k^{-1} \Pi_k F.$$

For $F \in \mathcal{F}^H_0$,

$$\sum_{\mu_k \neq 0} \mu_k^2 \big\|\mu_k^{-1}\Pi_k F\big\|^2_{L^2(M)} = \sum_{\mu_k \neq 0} \|\Pi_k F\|^2_{L^2(M)} < \infty.$$

Hence, $H^\dagger F \in D(H)$ and $HH^\dagger F = (I - \Pi_{\ker H})F = F$. Writing $u := H^\dagger F$, the identity $(-\Delta_g)^\beta u = F - Vu$ and the boundedness of multiplication by the smooth function V on every $H^s(M)$, $s \geq 0$, bootstrap $u \in H^{2\beta}(M) \Rightarrow u \in H^{4\beta}(M) \Rightarrow \cdots$, whence $u \in \bigcap_{s\geq 0} H^s(M) = C^\infty(M)$.

We claim that

$$\mathcal{C}^{O'}_{M,g,V} = \left\{\left(\sum_{\mu_k \neq 0} \mu_k^{-1}\Pi_k F\Big|_{O'} + h,\ F|_{O'}\right) : F \in \mathcal{F}^H_0,\ h \in \mathcal{N}^H_{O'}\right\}. \tag{4.5}$$

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Indeed, if $u \in C^\infty(M)$ satisfies $Hu = 0$ in $M \setminus \overline{O'}$, then $F := Hu$ belongs to $\mathcal{F}$ and

$$\langle F, w\rangle = \langle Hu, w\rangle = \langle u, Hw\rangle = 0 \qquad (w \in \ker H).$$

Thus, $F \in \mathcal{F}_0^H$ and $u - H^\dagger F \in \ker H$. Conversely, every $u = H^\dagger F + w$, with $F \in \mathcal{F}_0^H$ and $w \in \ker H$, satisfies $Hu = F$, and hence $Hu = 0$ in $M \setminus \overline{O'}$. Since $V|_{O'} = 0$,

$$((-\Delta_g)^\beta u)|_{O'} = (Hu - Vu)|_{O'} = F|_{O'},$$

which proves (4.5).

It remains to verify that every object in (4.5) is determined by $\mathcal{S}_O$. For $F \in \mathcal{F}$, the restriction $(\Pi_{\ker H} F)|_O$ is computed from the zero-eigenspace kernel on $O \times O$. Moreover,

$$(\Pi_{\ker H} F)|_O = 0 \quad \Longleftrightarrow \quad \Pi_{\ker H} F = 0$$

because $\Pi_{\ker H} F \in \ker H$ and Proposition 2.3(ii) makes restriction to O injective on $\ker H$. Hence, $\mathcal{F}_0^H$ is determined. Let $e_O : L^2(O) \to L^2(M)$ denote extension by zero. The same local kernel determines $\mathcal{N}_{O'}^H$: indeed, $T := \Pi_{\ker H} e_O : L^2(O) \to \ker H$ has adjoint $T^* w = w|_O$. Proposition 2.3(ii) makes T^* injective, and since $\ker H$ is finite dimensional, T is surjective. Therefore,

$$\mathcal{N}_{O'}^H = \left\{ (\Pi_{\ker H} e_O G)|_{O'} : G \in L^2(O) \right\}.$$

Finally, for each $F \in \mathcal{F}_0^H$,

$$(H^\dagger F)|_{O'} = \sum_{\mu_k \neq 0} \mu_k^{-1} \Pi_k F \Big|_{O'}.$$

The series converges in $L^2(M)$, and its sum is smooth by the preceding elliptic-regularity argument; hence, its $L^2(O')$ restriction determines the smooth function $(H^\dagger F)|_{O'}$. Thus, it is determined from the kernels $\Pi_k|_{O\times O}$ because $\operatorname{supp} F \subset \overline{O'} \subset O$. Thus, the entire Cauchy dataset is determined. ∎

Completion of the proof of Theorem 1.3. By Steps 1 and 2, $\alpha = \widetilde{\alpha}$, $\beta = \widetilde{\beta}$, and $\mathcal{S}_O = \widetilde{\mathcal{S}}_O$. By Lemma 4.3 applied on both manifolds (with the same O, O', and $g|_O = \widetilde{g}|_O$), $\mathcal{C}_{M,g,V}^{O'} = \mathcal{C}_{\widetilde{M},\widetilde{g},\widetilde{V}}^{O'}$. All hypotheses of [5, Theorem 1.2] are met: $M \setminus \overline{O'} \supset M \setminus \overline{O} \neq \varnothing$ (and similarly on $\widetilde{M}$), the potentials are smooth and vanish on $O' \subset O$, O' is open and connected and satisfies $g|_{O'} = \widetilde{g}|_{O'}$ because $\overline{O'} \subset O$, Hypothesis (H) holds by assumption, and the fractional orders agree since $\beta = \widetilde{\beta}$. Hence, [5, Theorem 1.2] yields an isometry $\Psi : (M, g) \to (\widetilde{M}, \widetilde{g})$ with $\Psi|_{O'} = \mathrm{Id}$ and $V = \Psi^* \widetilde{V}$.

5 Probabilistic formulation

Inverse problems for stochastic PDEs involving additive or multiplicative white noise have attracted considerable attention recently (see, for instance, [1, 16]). Here, we consider a stochastic version of the deterministic problem studied in previous sections. We show that the same uniqueness result holds when the equation is driven by additive temporal white noise with a prescribed spatial profile.

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5.1 Stochastic model and well-posedness

Throughout this section, we fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. We use $\mathbb{E}$ to denote the expectation and $\dot{B}$ to denote the formal derivative of a standard real-valued $(\mathcal{F}_t)$-Brownian motion B_t. For $\frac{1}{2} < \alpha < 1$ (the restriction $\alpha > \frac{1}{2}$ is what makes the integral $\int_0^t (t-\tau)^{2(\alpha-1)} d\tau$ appearing in the Itô isometry below finite), we consider the fractional stochastic problem

$$\begin{cases} i\partial_t^\alpha u + \big((-\Delta_g)^\beta + V\big)u = f + \sigma \dot{B}, & \text{in } M \times (0,\infty), \\ u(0) = 0. \end{cases} \tag{5.1}$$

Here, $f, \sigma \in C_c^2\big((0,\infty); L^2(M)\big)$ are deterministic, with $\operatorname{supp} f \cup \operatorname{supp} \sigma \subset (0, T_0)$ for some $T_0 > 0$; thus, the forcing $\sigma(t)\dot{B}_t$ is white in time and spatially correlated through the deterministic amplitude $\sigma(t,\cdot)$ and is not cylindrical space–time white noise. The spatial operator is

$$H = (-\Delta_g)^\beta + V, \qquad V \in L^\infty(M) \text{ real-valued},$$

with spectrum $\{\mu_k\}$ and eigenfunctions $\{\psi_k\}$ as in Section 2.

Definition 5.1 (Mild solution) For $f, \sigma \in C_c^2\big((0,\infty); L^2(M)\big)$, an $(\mathcal{F}_t)$-adapted process u is called a mild solution of (5.1) if

$$\sup_{t>0} \mathbb{E}\|u(t)\|_{L^2(M)}^2 < \infty, \tag{5.2}$$

and, for every $t \geq 0$,

$$u(t) = \sum_{k=1}^{\infty} u_k(t)\psi_k \quad \text{in } L^2(\Omega; L^2(M)),$$

where $u_k(t) = \langle u(t), \psi_k \rangle$ satisfies the fractional stochastic ODE

$$\begin{cases} (i\partial_t^\alpha + \mu_k)u_k = f_k + \sigma_k \dot{B}, \\ u_k(0) = 0, \end{cases} \tag{5.3}$$

with $f_k(t) = \langle f(t), \psi_k \rangle$, $\sigma_k(t) = \langle \sigma(t), \psi_k \rangle$, in the weak sense described below as an identity in $L^2(\Omega)$.

We see that equation (5.3) is interpreted in the weak form, understood as an identity in $L^2(\Omega)$ for each $t \geq 0$:

$$u_k(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \big(i\mu_k u_k(\tau) - i f_k(\tau)\big)\, d\tau + \frac{-i}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \sigma_k(\tau)\, dB_\tau. \tag{5.4}$$

We now give the stochastic representation of the solution.

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Proposition 5.2 (Representation formula) *The stochastic system (5.1) admits a unique mild solution*

$$u(t) = \sum_{k=1}^{\infty} u_k(t)\psi_k,$$

where

$$\begin{aligned} u_k(t) = &\int_0^t -i(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(i\mu_k(t-\tau)^\alpha\big)f_k(\tau)\,d\tau \\ &+ \int_0^t -i(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(i\mu_k(t-\tau)^\alpha\big)\sigma_k(\tau)\,dB_\tau. \end{aligned} \tag{5.5}$$

Moreover, the bound (5.2) holds.

Proof The argument is essentially the same as in the deterministic case by Proposition 3.1, combined with the Itô isometry. Write $g_\alpha(r) = r^{\alpha-1}/\Gamma(\alpha)$ and $K_k(r) = -ir^{\alpha-1}E_{\alpha,\alpha}(i\mu_k r^\alpha)$, so that (5.4) reads $u_k = i\mu_k(g_\alpha * u_k) - i(g_\alpha * f_k) - i\int_0^\cdot g_\alpha(\cdot-\tau)\sigma_k\,dB_\tau$, and let v_k denote the right-hand side of (5.5), that is, $v_k = K_k * f_k + \int_0^\cdot K_k(\cdot-\tau)\sigma_k\,dB_\tau$. Comparing Laplace transforms gives the resolvent identity

$$K_k = -ig_\alpha + i\mu_k(g_\alpha * K_k),$$

valid for every $\mu_k \in \mathbb{R}$. Substituting it into v_k and using stochastic Fubini to exchange $g_\alpha *$ with the Itô integral shows that v_k satisfies (5.4). The exchange is justified as follows. Proposition 2.5 gives

$$|K_k(r)| \le C_\alpha r^{\alpha-1}, \qquad \int_0^T |K_k(r)|^2\,dr \le \frac{C_\alpha^2 T^{2\alpha-1}}{2\alpha-1} < \infty,$$

uniformly in k, while

$$\int_\tau^t (t-s)^{\alpha-1}(s-\tau)^{\alpha-1}\,ds = B(\alpha,\alpha)(t-\tau)^{2\alpha-1}.$$

Consequently,

$$\begin{aligned} &\int_0^t \left(\int_\tau^t |g_\alpha(t-s)K_k(s-\tau)|\,ds\right)^2 |\sigma_k(\tau)|^2\,d\tau \\ &\qquad \le C_\alpha \int_0^{\min\{t,T_0\}} (t-\tau)^{4\alpha-2}|\sigma_k(\tau)|^2\,d\tau < \infty, \end{aligned}$$

which gives the square integrability required by stochastic Fubini.

Conversely, let $w := u_k - v_k$ be the difference of two solutions of (5.4) with the same data and set

$$W_T := \sup_{0\le t\le T} \big(\mathbb{E}|w(t)|^2\big)^{1/2},$$

which is finite by Definition 5.1 for u_k and by the estimate below for v_k. Iterating $w = i\mu_k(g_\alpha * w)$ gives $w = (i\mu_k)^N(g_{N\alpha} * w)$, and hence

$$W_T \le \frac{|\mu_k|^N T^{N\alpha}}{\Gamma(N\alpha+1)}\,W_T.$$

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The coefficient on the right tends to zero as $N \to \infty$; choosing N so that it is strictly smaller than one yields $W_T = 0$. Thus, $u_k = v_k$.

It remains to pass from finite spectral sums to the series in (5.5) and to prove (5.2). Let P_N be the projection onto the first N eigenfunctions and set $Q_N = I - P_N$. Write $u = u^D + u^I$ for the deterministic and Itô parts. The uniform kernel bound above, the Cauchy–Schwarz inequality, and the Itô isometry give

$$\sup_{t\geq 0} \|Q_N u^D(t)\|^2_{L^2(M)} \leq C_{\alpha,T_0} \sup_{0\leq\tau\leq T_0} \|Q_N f(\tau)\|^2_{L^2(M)},$$
$$\sup_{t\geq 0} \mathbb{E}\|Q_N u^I(t)\|^2_{L^2(M)} \leq C_{\alpha,T_0} \sup_{0\leq\tau\leq T_0} \|Q_N \sigma(\tau)\|^2_{L^2(M)}.$$

The right-hand sides tend to zero because f and σ have compact ranges in $L^2(M)$. Hence, the finite spectral sums converge in $L^\infty((0,\infty); L^2(\Omega; L^2(M)))$ to an adapted process. This also justifies the infinite series, Parseval's identity, the Bochner expectation, and the mean-zero identity used below.

For the uniform bound, set

$$I(t) := \int_0^{\min\{t,T_0\}} (t-\tau)^{2\alpha-2}\, d\tau.$$

Since $2\alpha - 2 \in (-1, 0)$,

$$\sup_{t\geq 0} I(t) = \frac{T_0^{2\alpha-1}}{2\alpha-1}.$$

The Cauchy–Schwarz inequality, the Itô isometry, and Parseval's identity now give

$$\mathbb{E}\|u(t)\|^2_{L^2(M)} \leq C_\alpha^2 I(t) \int_0^{T_0} \|f(\tau)\|^2_{L^2(M)}\, d\tau + C_\alpha^2 \int_0^{\min\{t,T_0\}} (t-\tau)^{2\alpha-2} \|\sigma(\tau)\|^2_{L^2(M)}\, d\tau.$$

Consequently,

$$\sup_{t\geq 0} \mathbb{E}\|u(t)\|^2_{L^2(M)} \leq C_{\alpha,T_0}\Big(\|f\|^2_{L^\infty(0,T_0;L^2(M))} + \|\sigma\|^2_{L^\infty(0,T_0;L^2(M))}\Big) < \infty. \qquad \blacksquare$$

5.2 Inverse problem with stochastic source

Fix nonempty disjoint open subsets $W_1, W_2 \subset M$ (or, in the setting of Theorem 1.3, $W_1 = W_2 = O$). For a prescribed deterministic $\sigma \in C_c^2\big((0,\infty); L^2(W_1)\big)$, define the stochastic source-to-solution map

$$\mathscr{L}^{\alpha,\beta,V,\sigma}_{W_1,W_2} : f \mapsto u_f|_{W_2\times(0,\infty)}, \qquad \operatorname{supp} f \subset W_1 \times (0,\infty),$$

where u_f is the mild solution of (5.1). Since $\mathbb{E}[u_f]$ solves the deterministic problem with source f, we have

$$\mathbb{E}[u_f(t)] = (u_f^{\mathrm{det}})(t),$$

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where $u_f^{\det}$ is the unique deterministic solution. Thus, all information in $\mathbb{E}[\mathscr{L}_{W_1,W_2}^{\alpha,\beta,V,\sigma} f]$ is identical to that of the deterministic source-to-solution map.

Theorem 5.3 (Stochastic uniqueness) *Let (M,g), $(\widetilde{M},\widetilde{g})$ be smooth connected closed Riemannian manifolds of common dimension $n \geq 2$, and let $V \in L^\infty(M)$ and $\widetilde{V} \in L^\infty(\widetilde{M})$ be real-valued potentials. Assume $\frac{1}{2} < \alpha, \widetilde{\alpha} < 1$ and $0 < \beta, \widetilde{\beta} < 1$. Let W_1, W_2 be nonempty open sets contained in both M and $\widetilde{M}$, and assume $g = \widetilde{g}$ on $W_1 \cup W_2$. Assume either (i) W_1, W_2 are disjoint or (ii) $W_1 = W_2 = O$ and the hypotheses of Theorem 1.3 hold. Fix $\sigma \in C_c^2\big((0,\infty); L^2(W_1)\big)$. Suppose*

$$\mathbb{E}\big[\mathscr{L}_{W_1,W_2}^{\alpha,\beta,V,\sigma} f\big] = \mathbb{E}\big[\widetilde{\mathscr{L}}_{W_1,W_2}^{\widetilde{\alpha},\widetilde{\beta},\widetilde{V},\sigma} f\big] \qquad \text{for all } f \in C_c^2\big((0,\infty); L^2(W_1)\big).$$

Then, in case (i), the conclusions (1) and (2) of Theorem 1.2 hold: $\alpha = \widetilde{\alpha}$, $\beta = \widetilde{\beta}$, and the cross spectral data of H and $\widetilde{H}$ associated with (W_1, W_2) coincide. In case (ii), the conclusions of Theorem 1.3 hold: $\alpha = \widetilde{\alpha}$, $\beta = \widetilde{\beta}$, and there exists an isometry $\Psi : (M,g) \to (\widetilde{M},\widetilde{g})$ with $\Psi|_{O'} = \mathrm{Id}$, such that $V = \Psi^* \widetilde{V}$.

Proof Since the expectation removes the noise – the stochastic integral in (5.5) has mean zero by Proposition 5.2 – and $\mathbb{E}[u_f]$ solves the deterministic equation with source f, the hypotheses give equality of the deterministic source-to-solution maps. The conclusion then follows directly from Theorem 1.2 (resp. Theorem 1.3). ∎

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References

[1] N. Bousquet, M. Blazère, and T. Cerbelaud, *Covariance constraints for stochastic inverse problems of computer models*. Electron. J. Stat. **19**(2025), no. 1, 1809–1854.
[2] G. Covi, K. Mönkkönen, J. Railo, and G. Uhlmann, *The higher order fractional Calderón problem for linear local operators: Uniqueness*. Adv. Math. **399**(2022), 108246.
[3] J. Dong and M. Xu, *Space-time fractional Schrödinger equation with time-independent potentials*. J. Math. Anal. Appl. **344**(2008), 1005–1017.
[4] A. Feizmohammadi, T. Ghosh, K. Krupchyk, and G. Uhlmann, *Fractional anisotropic Calderón problem on closed Riemannian manifolds*. J. Differential Geom. **131**(2025), no. 2, 401–414. https://doi.org/10.4310/jdg/1757353909
[5] A. Feizmohammadi, K. Krupchyk, and G. Uhlmann, *Calderón problem for fractional Schrödinger operators on closed Riemannian manifolds*. Invent. Math. (2026). arXiv:2407.16866. https://doi.org/10.1007/s00222-026-01432-6
[6] T. Ghosh, M. Salo, and G. Uhlmann, *The Calderón problem for the fractional Schrödinger equation*. Anal. PDE **13**(2020), 455–475.
[7] T. Helin, M. Lassas, L. Ylinen, and Z. Zhang, *Inverse problems for heat equation and space-time fractional diffusion equation with one measurement*. J. Differential Equations **269**(2020), no. 9, 7498–7528.
[8] P. D. Hislop and E. Soccorsi, *Edge currents for the time-fractional, half-plane, Schrödinger equation with constant magnetic field*. J. Math. Phys. **65**(2024), 102101. https://doi.org/10.1063/5.0198112
[9] R.-Y. Lai, Y.-H. Lin, and A. Rüland, *The Calderón problem for a space-time fractional parabolic equation*. SIAM J. Math. Anal. **52**(2020), 2655–2688.
[10] N. Laskin, *Fractional quantum mechanics*. Phys. Rev. E **62**(2000), 3135–3145.
[11] N. Laskin, *Fractional Schrödinger equation*. Phys. Rev. E **66**(2002), 56108.

https://doi.org/10.4153/S0008439526102604 Published online by Cambridge University Press

[12] M. Lassas, M. Nursultanov, L. Oksanen, and L. Ylinen, *Disjoint data inverse problem on manifolds with quantum chaos bounds*. SIAM J. Math. Anal. **56**(2024), no. 6, 7748–7779. https://doi.org/10.1137/23M1606897
[13] L. Li, *On inverse problems for uncoupled space-time fractional operators involving time-dependent coefficients*. Inverse Probl. Imaging **17**(2023), 890–906.
[14] L. Li, *An inverse problem for the space-time fractional Schrödinger equation on closed manifolds*. Canad. Math. Bull. (2026), 1–15. https://doi.org/10.4153/S0008439526102124
[15] L. Li and W. Wang, *A Calderón type inverse problem for the active scalar equations with fractional dissipation*. Inverse Probl. Imaging **22**(2026), 53–64. https://doi.org/10.3934/ipi.2025044
[16] Q. Li, M. Oprea, L. Wang, and Y. Yang, *Stochastic inverse problem: Stability, regularization and Wasserstein gradient flow*. Preprint, 2024. arXiv:2410.00229.
[17] I. Podlubny, Fractional differential equations, Academic Press, San Diego, CA, 1999.
[18] A. Rüland, *Unique continuation for fractional Schrödinger equations with rough potentials*. Comm. Partial Differential Equations **40**(2015), 77–114.
[19] L. Yan, *Inverse boundary problems for biharmonic operators in transversally anisotropic geometries*. SIAM J. Math. Anal. **53**(2021), no. 6, 6617–6653.
[20] L. Yan, *Reconstructing a potential perturbation of the biharmonic operator on transversally anisotropic manifolds*. Inverse Probl. Imaging **17**(2023), no. 1, 136–156.

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