A hyperbolic proportionality theorem and hyperbolic surveying problems

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Introduction

A surveyor often encounters problems like

Problem 1: Determine the length of side *AC* of the right triangle *ABC* with the right angle at *B*. A lake intersecting the sides *AC* and *CE* prevents the direct measurement of their sides.

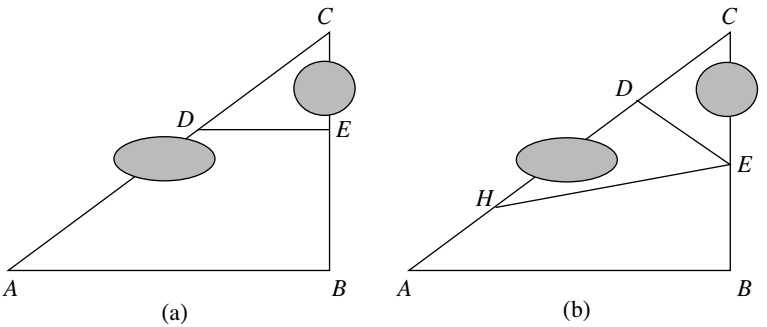


FIGURE 1

To solve this problem, the surveyor would first pick an appropriate point *E* on line *BC*. From *E*, he would draw the line *ED* parallel to *AB* intersecting the side *AC* in *D*. Next he would determine the lengths of the segments *ED*, *DC* and *AB* by direct measurements. Finally, he would substitute these lengths into

$$\frac{AC}{DC} = \frac{AB}{ED}$$

to obtain the length of *AC*. This ratio comes from the proportionality theorem of Euclidean geometry.

This prompts an interesting question: *How would a surveyor in a hyperbolic world solve this problem*? And that question raises another interesting question: *Is there a hyperbolic analogue to the proportionality theorem of Euclidean geometry*? The answer to this last question is 'yes'. Therefore the answer to the first question is that the hyperbolic surveyor would approach Problem 1 like his Euclidean counterpart. For example, he would first pick a point *E* on *BC* (see Figure 1(b)). From *E*, he would draw two parallel lines *ED* and *EH* intersecting the side *AC* in *D* and *H*, respectively. Next he would determine the lengths of the segments *EH*, *DC*, *AH* and *ED* by direct measurements. Finally, he would substitute these lengths into the equation

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$$\frac{\sinh ED}{\sinh DC} = \frac{\sinh EH}{\sinh HC} \tag{1}$$

to get the length of HC. (We shall prove this equation shortly.) Then $AC = AH + HC$.

The hyperbolic version of the Euclidean proportionality theorem has several properties associated with an arbitrary triangle ABC. See Figure 2(a). Half of these go with the parallel ED and the other half with the parallel EH. These properties are listed in the following theorem.

Theorem 1: (Hyperbolic Proportionality Theorem, see Figure 2(a))

Let E be any point on the line segment BC and draw the lines ED and EH through E parallel to line AB. Then for the parallel DE

$$\frac{\sinh AC \sinh DE}{\sinh AB \sinh DC} = \cosh BE + \sinh BE \cos B \tag{2}$$

$$\frac{\sinh BC \sinh DE}{\sinh AB \sinh EC} = \cosh AD - \sinh AD \cos A \tag{3}$$

and for the parallel HE

$$\frac{\sinh AC \sinh HE}{\sinh AB \sinh HC} = \cosh BE - \sinh BE \cos B \tag{4}$$

$$\frac{\sinh BC \sinh HE}{\sinh AB \sinh EC} = \cosh AH + \sinh AH \cos A. \tag{5}$$

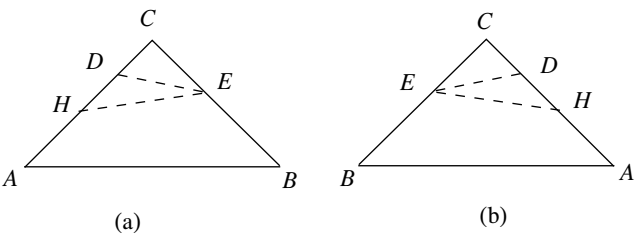


FIGURE 2

The converse of this theorem is

Theorem 2: (Converse of the Hyperbolic Proportionality Theorem)

If (2) and (4), or (3) and (5) are true, then the lines ED and EH are parallel to the line AB.

In this paper, we shall prove these two theorems and give two applications of their use. The proofs of these theorems depend on some well-known formulas from hyperbolic geometry. These formulas are given in the next section.

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Hyperbolic Geometric Formulas

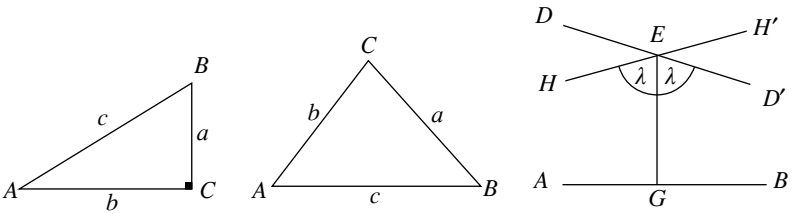
There are three sets of hyperbolic geometry formulas that play an important role in the proofs of Theorem 1 and Theorem 2. The proofs of these formulas can be found in Carslaw [1] and Sommerville [2].

The first set of formulas is associated with a right triangle ABC, see Figure 3(a); the right angle being at C and the sides being a, b and c. The relevant formulas are:

$$\cosh c = \cosh a \cosh b \tag{6}$$

$$\cos A = \frac{\tanh b}{\tanh c}, \sin A = \frac{\sinh a}{\sinh c}. \tag{7}$$

Formula (6) is known as the *Hyperbolic Pythagorean Theorem.* There are similar formulas for $\sin B$ and $\cos B$.



(a) (b) (c)

FIGURE 3

The second set of formulas is associated with the general triangle ABC in Figure 3(b). The relevant formulas are:

$$\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos A \tag{8}$$

$$\frac{\sin A}{\sinh a} = \frac{\sin B}{\sinh b} = \frac{\sin C}{\sinh c}. \tag{9}$$

The formula (8) is known as the hyperbolic *law of cosines* and formula (9) as the hyperbolic *law of sines*.

The third set of formulas is associated with parallel lines. Let the lines DD' and HH' in Figure 3(c), which pass through point E, be parallel to line AB. Let the line EG be perpendicular to AB. Let $\lambda = \angle D'EG$, the angle between the parallel ED' and the perpendicular EG. Then, the relevant formulas are:

$$\cos\lambda = \tanh EG, \sin\lambda = \frac{1}{\cosh EG}, \tan\lambda = \frac{1}{\sinh EG}. \tag{10}$$

Additionally, in order to prove Theorem 2, the following statement about parallel lines is needed: If $\cos\lambda = \tanh EG$, then DD' and HH' are parallel to AB. This statement is equivalent to Bolyai's construction [1, pp. 71-72] of drawing parallels to a given line from a given point.

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Proof of the Hyperbolic Proportionality Theorem: We start with equation (2).

In Figure 4, extend the parallel ED to D' and EH to H'. Draw EG perpendicular to AB. Let $\alpha = \angle CED, \beta = \angle BEG$ and $\lambda = \angle GED'$.

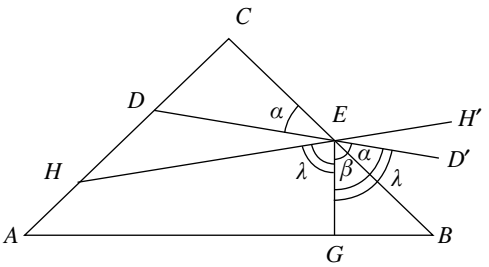


FIGURE 4

Apply the *law of sines* (9) to the triangles CED and ABC in Figure 4 to obtain

$$\frac{\sin C}{\sinh DE} = \frac{\sin \alpha}{\sinh CD}$$

$$\frac{\sin C}{\sinh AB} = \frac{\sin B}{\sinh AC}.$$

Now eliminate $\sin C$ from these two equations to obtain

$$\frac{\sinh AC \sinh DE}{\sinh AB \sinh DC} = \frac{\sin B}{\sin \alpha}.$$

The proof of (2) will be completed once we show that

$$\frac{\sin B}{\sin \alpha} = \cosh BE + \sinh BE \cos B.$$

Since the parallels ED', EH through E make equal acute angles with the perpendicular EG, we have $\angle HEG = \angle GED'$. Therefore, $\alpha + \beta = \lambda$. Now replace α by $\lambda - \beta$ in the left-hand side of the last equation, to get

$$\frac{\sin B}{\sin \alpha} = \frac{\sin B}{\sin \lambda \cos \beta - \cos \lambda \sin \beta}. \tag{11}$$

Now from HE being parallel to AB, we have from (10)

$$\cos \lambda = \tanh EG, \qquad \sin \lambda = \frac{1}{\cosh EG}, \tag{12}$$

and from using the right triangle formulas (7) on ΔEGB, we get

$$\sin \beta = \frac{\sinh BG}{\sinh BE}, \qquad \cos \beta = \frac{\tanh EG}{\tanh BE}, \qquad \sin B = \frac{\sinh EG}{\sinh BE}.$$

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Next substitute the above values and the values from (12) into (11) to get

$$\frac{\sin B}{\sin \alpha} = \frac{\cosh EG}{\dfrac{\cosh BE}{\cosh EG} - \sinh BG}.$$

But $\cosh BE = \cosh EG \cosh BG$ (*Hyperbolic Pythagorean Theorem*), so

$$\begin{aligned}\frac{\sin B}{\sin \alpha} &= \frac{\cosh EG}{\cosh BG - \sinh BG} \\ &= \frac{\cosh EG}{\cosh BG - \sinh BG} \cdot \frac{\cosh BG + \sinh BG}{\cosh BG + \sinh BG} \\ &= \cosh EG \cosh BG + \cosh EG \sinh BG \\ &= \cosh BE + \cosh EG \sinh BG.\end{aligned}$$

The last step in the proof is to show that $\cosh EG \sinh BG = \sinh BE \cos B$.

Again from the right triangle EGB, we have

$$\cos B = \frac{\tanh BG}{\tanh EB}$$

or

$$\tanh BE \cos B = \tanh BG$$

and, after multiplying both sides of this equation by $\cosh EG \cosh BG$,

$$\cosh EG \cosh BG \tanh BE \cos B = \cosh EG \sinh BG.$$

But, from the *Hyperbolic Pythagorean Theorem*,

$$\cosh EG \cosh BG = \cosh BE.$$

Hence, after replacing $\cosh EG \cosh BG$ by $\cosh BE$ in the above equation,

$$\sinh BE \cos B = \cosh EG \sinh BG,$$

which establishes the equation

$$\frac{\sin B}{\sin \alpha} = \cosh BE + \sinh BE \cos B.$$

This completes the proof of (2).

Since the steps to prove (4) are the same ones we took to prove (2), we will let the reader supply those steps. The only difference between the proofs is that triangle CHE is used to prove (4) instead of triangle CDE.

To complete the proof, we will now prove the pair of equations (3) and (5). These results are easily established from the pair of equations (2) and (4) as follows: First flip triangle ABC in Figure 2(a) over to get the triangle in Figure 2(b). Comparing Figure 2(a) to Figure 2(b), we see that the side AC, the segments DC, BE, the angle B and the parallel DE in Figure 2(a) correspond to the side BC, the segments EC, AH, the angle A and the parallel

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HE in Figure 2(b). Equation (5) now follows from this correspondence. Replacing *AC*, *DE*, *DC*, *BE* and angle *B* in (2) by *BC*, *HE*, *EC*, *AH*, and angle *A* yields (5).

Similarly, to obtain (3) replace *AC*, *HC*, *BE*, the angle *B* and the parallel *HE* in (4) by *BC*, *EC*, *AD*, the angle *A* and the parallel *DE*, respectively. This replacement follows from the correspondence of side *AC*, the segments *HC*, *BE*, the angle *B* and the parallel *HE* in Figure 2(a) to the side *BC*, the segments *EC*, *AD*, the angle *A* and the parallel *DE* in Figure 2(b).

This completes the proof of (3) and (5), and the proof of Theorem 1 is now completed.

Here are two corollaries associated with a right triangle. The first corollary establishes the proportion (1) that was used to solve the hyperbolic version of Problem 1.

Corollary 1: If triangle *ABC* is a right triangle with the right angle at *B*, *E* a point on side *BC*, and *ED* and *EH* are parallel to side *AB*, then

$$\frac{\sinh ED}{\sinh DC} = \frac{\sinh EH}{\sinh HC}.$$

Proof: Set $\angle B = 90°$ in (2) and (4), and then rearrange the terms after dividing (4) by (2).

Corollary 2: If triangle *ABC* is a right triangle with the right angle at *A*, *E* a point on side *BC*, and *ED* and *EH* are parallel to side *AB*, then

$$\frac{\sinh ED}{\cosh AD} = \frac{\sinh HE}{\cosh AH}.$$

Proof: Set $\angle A = 90°$ in (3) and (5), and then rearrange the terms after dividing (5) by (3).

Proof of the converse of the Hyperbolic Proportionality Theorem: Since the equation pair (3) and (5) are derived from the equation pair (2) and (4), and vice versa, we need only prove parallelism when the equation pair (2) and (4) is true. To simplify the proof even further, we will only prove the theorem when proportion (2) is assumed true. To prove parallelism, we need simply reverse the steps in the proof of (2).

Reversing these steps will take us from (2) back to

$$\sin\alpha = \frac{1}{\cosh EG}\cos\beta - \tanh EG \sin\beta.$$

And from Figure 4 we have $\lambda = \alpha + \beta$, which gives

$$\sin\alpha = \sin\lambda\cos\beta - \cos\lambda\sin\beta.$$

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Comparing this expression for $\sin\alpha$ with the one above yields

$$\sin\lambda = \frac{1}{\cosh EG} \qquad \text{and} \qquad \cos\lambda = \tanh EG.$$

Thus $ED \parallel AB$, which completes the proof of Theorem 2 when (2) is assumed true.

Because of the similarities in the proofs of (2) and (4), we will let the reader supply the proof of Theorem 2 when (4) is assumed true.

Two other surveyor problems

The problems below are ones the hyperbolic surveyors could be asked to solve. In these problems, the shaded areas are obstacles, like lakes, that prevent one from making direct measurements of distances.

Problem 2: (see Figure 5)

Determine the lengths of the sides AC, BC of ΔABC. A lake prevents the direct measurements of these lengths.

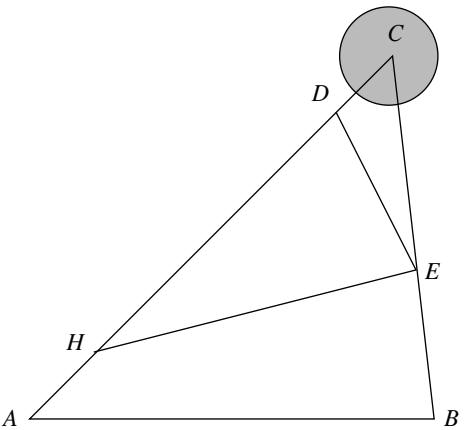


FIGURE 5

To solve this problem, the surveyor would pick a convenient point E on the side BC, from which to draw the two parallels EH and ED to AB. He would next determine the lengths of BE, AB, the sides of triangle EDH, and the size of angle B by direct measurements. After making these measurements, he would proceed as follows.

He would first find the length of DC, and then use that length in (2) to get the length of AC. To get DC, he would start by dividing (2) by (4) to get

$$\frac{\sinh DE \sinh HC}{\sinh DC \sinh HE} = \kappa, \tag{13}$$

where

$$\kappa = \frac{\cosh BE + \sinh BE \cos B}{\cosh BE - \sinh BE \cos B}.$$

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The value of κ is a known value since the lengths and angle appearing on the right-hand side of κ are known from direct measurements. The surveyor would next replace HC by $HD + DC$ in the ratio $\dfrac{\sinh HC}{\sinh DC}$ in (13) to get

$$\begin{aligned}\frac{\sinh HC}{\sinh DC} &= \frac{\sinh (HD + DC)}{\sinh DC}\\ &= \sinh HD \coth DC + \cosh HD.\end{aligned}$$

And after replacing the ratio $\dfrac{\sinh HC}{\sinh DC}$ in (13) by this value, he would then solve for $\tanh DC$, to get

$$\tanh DC = \frac{\sinh DE \sinh HD}{\kappa \sinh HE - \cosh HD \sinh DE}.$$

The right-hand side of $\tanh DC$ is expressed in known values, making the length of DC a known value. Hence AC follows from (2). With the length of AC known, the surveyor would next determine the length of BC.

To find the length of BC, the surveyor would first multiply (5) by $\sinh AD$ and (3) by $\sinh AH$ then he would add them together, and then he would solve for the ratio

$$\frac{\sinh BC}{\sinh EC} = \frac{\sinh AB \sinh (AH + AD)}{\sinh AD \sinh HE + \sinh AH \sinh DE}.$$

Next he would observe that the right-hand side is a known value. The lengths of $AD = AC - DC$, $AH = AC - HC$ and the other lengths are all known values.

Let the right-hand side of this ratio be denoted by λ. Then the length of BC will come from the solution of the equation

$$\frac{\sinh BC}{\sinh EC} = \lambda.$$

To solve this equation, the surveyor would replace BC by $BE + EC$ to get

$$\begin{aligned}\frac{\sinh BC}{\sinh EC} &= \frac{\sinh (BE + EC)}{\sinh EC}\\ &= \sinh BE \coth EC + \cosh BE\\ &= \lambda,\end{aligned}$$

from which it follows that

$$\tanh EC = \frac{\sinh BE}{\lambda - \cosh BE}.$$

The length of EC is now known, and therefore so is $BC = BE + EC$. This completes the solution of problem 2.

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Not all surveyor problems (Euclidean and Hyperbolic) can be solved using only the proportionality theorem. Some problems require the additional help from the *law of cosines* (8), and from the *law of sines* (9). The next problem illustrates this.

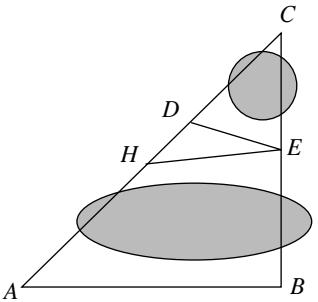


FIGURE 6

Problem 3: (see Figure 6)

Determine the lengths of the sides AC and BC of the right triangle ABC with the right angle at B. Lakes prevent the direct measurements of these quantities.

To solve this problem, the surveyor would first pick an appropriate point E on line BC and then he would draw the parallels HE and DE from E intersecting AC in H and D, respectively. He would next determine the lengths of HE, HD, DE and AB by direct measurements. After making these measurements, he would turn to Theorem 1 to determine the lengths of AC and BC.

His calculation begins with finding the sides DC, CE of ΔCDE and angle C. Since by hypothesis $\angle B = 90°$, the right-hand side of (2) and of (4) are equal, and therefore their left-hand sides are also equal, which yields

$$\frac{\sinh DE}{\sinh DC} = \frac{\sinh HE}{\sinh HC}.$$

And from Figure 6, $HC = HD + DC$. Now replace HC in the last equation by $HD + DC$, to obtain

$$\frac{\sinh DE}{\sinh DC} = \frac{\sinh HE}{\sinh (HD + DC)}.$$

The length of DC is the only unknown in this equation. Solving this equation for DC yields

$$\tanh DC = \frac{\sinh DE \sinh HD}{\sinh HE - \sinh DE \cosh HD}. \tag{14}$$

With the length of DC now known from (14), the next values to be computed are the length of CE and the angle C. These two values will follow from the *law of cosines* (8). This law applied to the triangles HCE

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and DCE gives

$$\cosh HE = \cosh HC \cosh CE - \sinh HC \sinh CE \cos C$$

$$\cosh DE = \cosh DC \cosh CE - \sinh DC \sinh CE \cos C.$$

The only unknowns in this system of equations are the length of CE and angle C. Now solve this system of equations for CE and C to get

$$\begin{aligned}\cosh CE &= \frac{\sinh HC \cosh DE - \cosh HE \sinh DC}{\sinh HC \cosh DC - \sinh DC \cosh HC} \\ &= \frac{\sinh HC \cosh DE - \cosh HE \sinh DC}{\sinh (HC - DC)} \\ &= \frac{\sinh HC \cosh DE - \cosh HE \sinh DC}{\sinh HD}\end{aligned}$$

$$\cos C = \frac{\cosh DC \cosh CE - \cosh DE}{\sinh DC \sinh CE}.$$

Finally, the lengths of AC and BC come from

$$\sin C = \frac{\sinh AB}{\sinh AC}$$

$$\cosh AC = \cosh AB \cosh BC.$$

The first equation comes from applying the second formula in (7) to the right triangle ABC and the second equation comes from the *Hyperbolic Pythagorean Theorem*. Use the first equation to get the length of AC and use the second equation to get the length of BC. This completes Problem 3.

Acknowledgements

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