Proc. Camb. Phil. Soc. (1966), **62**, 519
With 1 *text-figure*
PCPS 62–70
Printed in Great Britain

The two-dimensional jet

By WILLIAM E. CONWAY
West Texas State University

(*Received* 8 *December* 1965)

Consider the two-dimensional steady flow of an incompressible inviscid fluid under no body forces between two infinite horizontal planes. When an aperture is made in the lower plane, the fluid will issue as a jet bounded by free streamlines along which the pressure is constant. At infinity the flow in the jet will be uniform and parallel.

The situation is shown in the z-plane diagram in Fig. 1. The distance between the walls is h, the width of the aperture is W, and the origin is taken in the z-plane at C. We suppose the flow in the channel to result from two uniform opposite streams U_0 at A_∞ and A'_∞.

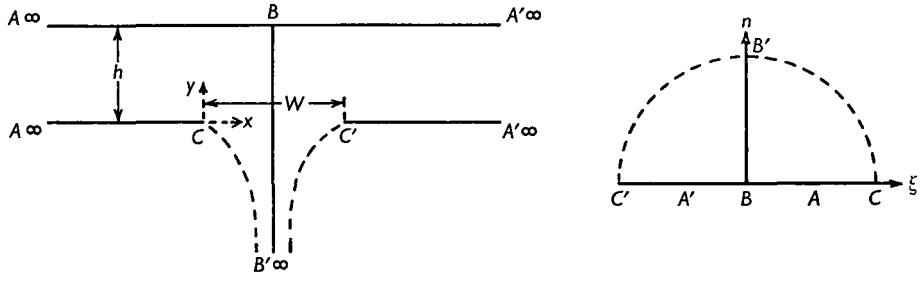


Fig. 1

The form of the free streamlines is unknown. However, we map these unknown curves on the circumference τ of the unit semicircle in the ζ-plane ($\zeta = \xi + i\eta$), while the region occupied by the fluid in the z-plane maps onto the interior of the unit semicircle. In the two planes corresponding points are denoted by the same letter. If $\zeta = re^{i\chi}$, we have $\chi = 0$ on BC.

The complex potential for the flow in the ζ-plane resulting from the flow in the z-plane is obtained by the circle theorem as (1).

$$w = -2m\ln(\zeta^2 - a^2) - 2m\ln(\zeta^2 - 1/a^2) + 4m\ln(\zeta^2 + 1).$$

By differentiation we find the complex velocity in the z-plane to be

$$u - iv = -\frac{dw}{dz} = \frac{4m(a+1/a)^2\zeta(\zeta^2-1)}{(\zeta^2-a^2)(\zeta^2-1/a^2)(\zeta^2+1)}\frac{d\zeta}{dz}. \tag{1}$$

Now $d\zeta/dz$ must remove the unwanted stagnation points and points of infinite velocity, and therefore

$$\frac{dz}{d\zeta} = \frac{(1-\zeta^2)f(\zeta)}{(\zeta^2-a^2)(\zeta^2+1)}, \tag{2}$$

where $f(\zeta)$ is holomorphic and $f(\zeta) \neq 0$ for $|\zeta| \leqslant 1$.

https://doi.org/10.1017/S0305004100040159 Published online by Cambridge University Press

To obtain an expression for $f(\zeta)$ we write

$$e^{\Omega(\zeta)} = \frac{1}{h}(1/a^2 - \zeta^2) f(\zeta), \tag{3}$$

where $\Omega(\zeta) = \omega(\zeta) + i\psi(\zeta)$. Then the complex velocity (1) for a particle on the free streamline is given by

$$q e^{-i\theta} = \frac{4m(a+1/a)^2}{h} e^{i(\chi-\psi)-\omega}. \tag{4}$$

On the free streamlines the pressure is constant, and therefore Bernoulli's theorem gives $q = U$, where U is the uniform velocity of the jet at B'_∞. Thus by (4) we find that ω is constant and hence $\psi \equiv 0$.

Let σ be the coefficient of contraction, so that $4m\pi = UW\sigma$. From (3) and (4) we find

$$f(\zeta) = \frac{W\sigma(a^2+1)^2}{\pi(1-a^2\zeta^2)}$$

and therefore the complex velocity (1) becomes

$$u - iv = U\zeta$$

and (2) becomes

$$\frac{dz}{d\zeta} = \frac{W\sigma(a^2+1)^2(1-\zeta^2)}{\pi(\zeta^2-a^2)(\zeta^2+1)(1-a^2\zeta^2)}. \tag{5}$$

Integration gives the mapping function

$$z = \frac{W\sigma}{2a\pi}\left\{\ln\frac{(\zeta-a)(1+a)}{(\zeta+a)(1-a)} + a^2\ln\frac{(1-a\zeta)(1+a)}{(1+a\zeta)(1-a)} + 2ai\ln\frac{i(\zeta-i)}{\zeta+i}\right\} \tag{6}$$

for $|\zeta| \leqslant 1$.

When $\zeta = 0$ (6) gives $h = W\sigma/(2a)$ and the coefficient of contraction as

$$\sigma = \pi\Big/\left[\pi + \frac{a^2+1}{a}\ln\frac{a+1}{1-a}\right]. \tag{7}$$

Combining these equations we have

$$\frac{W}{2h} = \frac{a}{\pi}\left[\pi + \frac{a^2+1}{a}\ln\frac{a+1}{1-a}\right]$$

which gives a when W and h are known.

Putting $\zeta = e^{i\chi}$ in (6) yields the equations of the free streamlines as

$$x = \frac{h}{\pi}\left\{\frac{a^2+1}{2}\ln\frac{(a+1)^2(a^2+1-2a\cos\chi)}{(a-1)^2(a^2+1+2a\cos\chi)} - 2a\arg(\cos\chi)\right\}$$

and

$$y = \frac{h}{\pi}\left\{(1-a^2)\tan^{-1}\frac{2a\sin\chi}{1-a^2} + 2a\ln\frac{|\cos\chi|}{1+\sin\chi}\right\},$$

where $0 \leqslant \chi \leqslant \pi$ and $\arg(\cos\chi)$ is 0 or $-\pi$ according to $\chi \leqslant \frac{1}{2}\pi$ or $\chi \geqslant \frac{1}{2}\pi$.

Consider the limiting case of flow through a jet when the height of the channel is infinite.

As $a \to 0$, $h \to \infty$ and (5) gives the equations for the left free streamline

$$x = \frac{W\sigma}{\pi}[1-\cos\chi], \quad y = -\frac{W\sigma}{\pi}[\ln\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\chi) - \sin\chi]$$

https://doi.org/10.1017/S0305004100040159 Published online by Cambridge University Press

for $0 \leqslant \chi \leqslant \frac{1}{2}\pi$. From (7) as $a \to 0$, the coefficient of contraction becomes

$$\sigma = \frac{\pi}{\pi + 2}.$$

These results agree with those given by Milne-Thomson ((2), p. 300).

REFERENCES

(1) Conway, William E. The two-dimensional jet under gravity from an aperture in the lower of two horizontal planes which bound a liquid. Ph.D. Dissertation, University of Arizona, 1965.

(2) Milne-Thomson, L. M. *Theoretical hydrodynamics*, 4th ed. (MacMillan Company; New York, 1960).

https://doi.org/10.1017/S0305004100040159 Published online by Cambridge University Press